Elite Mathematics
Edexcel International A Level

IAL PURE 3 · WMA13

WMA13 Pure 3 Expertise With Answers

Questions 6 and above with worked-solution pages and cross-topic placements.

299	2020–2026	Questions with worked solutions
topic placements	19 exam sessions	checked by Dr Eslam



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TOPIC

Rational Algebra

Every past-paper question on this topic, January 2020 to June 2026.

Question 6

9 marks | Rational Algebra | Main topic: Modulus

6.

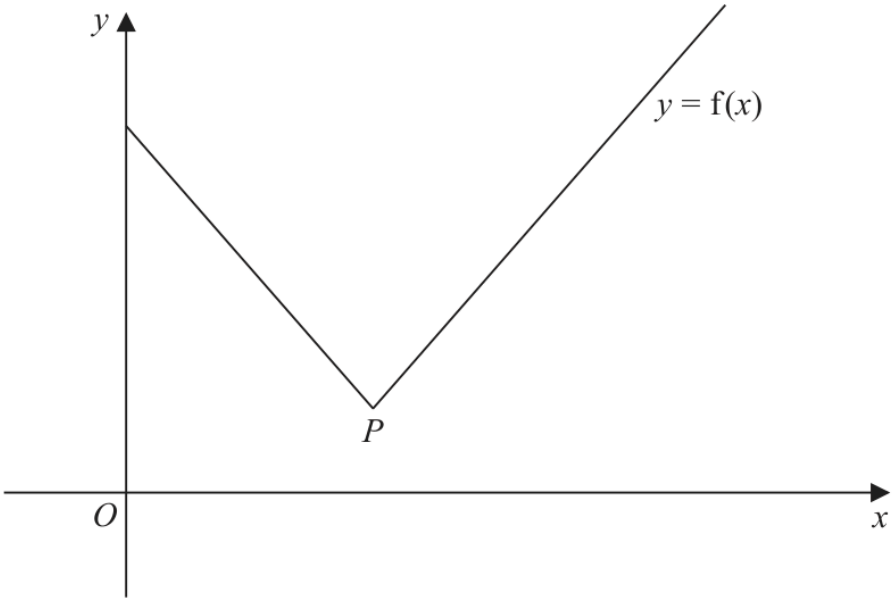


Figure 2

Figure 2 shows part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 2|2x - 5| + 3 \qquad x \geqslant 0$$

The vertex of the graph is at point P as shown.

(a) State the coordinates of P.

(2)

(b) Solve the equation $\mathrm{f}(x) = 3x - 2$

(4)

Given that the equation

$$\mathrm{f}(x) = kx + 2$$

where k is a constant, has exactly two roots,

(c) find the range of values of k.

(3)

1 Part (a)

At the vertex, the expression inside the modulus is zero:

$$2x - 5 = 0 \quad \Longrightarrow \quad x = \frac{5}{2}, \qquad f\left(\frac{5}{2}\right) = 3.$$

Hence $\boxed{P = \left(\frac{5}{2}, 3\right)}$.

2 Part (b)

Write the function in its two branches:

$$f(x) = \begin{cases} 13 - 4x, & 0 \le x \le \frac{5}{2}, \\ 4x - 7, & x \ge \frac{5}{2}. \end{cases}$$

On the left branch,

$$13 - 4x = 3x - 2 \quad \Longrightarrow \quad 7x = 15 \quad \Longrightarrow \quad x = \frac{15}{7}.$$

This lies in $0 \le x \le \frac{5}{2}$, so it is valid. On the right branch,

$$4x - 7 = 3x - 2 \quad \Longrightarrow \quad x = 5.$$

This satisfies $x \ge \frac{5}{2}$, so it is also valid. Thus

$$\boxed{x = \frac{15}{7} \text{ or } x = 5}.$$

3 Part (c)

Each straight branch can have at most one intersection with $y = kx + 2$. Two distinct roots therefore require one intersection strictly to each side of the vertex.

For the left branch,

$$13 - 4x = kx + 2 \quad \Longrightarrow \quad x_L = \frac{11}{k+4}.$$

The condition $0 \le x_L < \frac{5}{2}$ first requires $k + 4 > 0$, then gives

$$\frac{11}{k+4} < \frac{5}{2} \quad \Longrightarrow \quad 22 < 5(k+4) \quad \Longrightarrow \quad k > \frac{2}{5}.$$

For the right branch,

$$4x - 7 = kx + 2 \quad \Longrightarrow \quad x_R = \frac{9}{4-k}.$$

The condition $x_R > \frac{5}{2}$ requires $k < 4$, and then

$$\frac{9}{4-k} > \frac{5}{2} \quad \Longrightarrow \quad 18 > 5(4-k) \quad \Longrightarrow \quad k > \frac{2}{5}.$$

At $k = \frac{2}{5}$, both intersections coincide at the vertex, giving only one root. At $k = 4$, the line is parallel to the right branch and has no intersection with that branch, again leaving only one root. Hence

$$\boxed{\frac{2}{5} < k < 4}.$$

FINAL ANSWER

(a) $P = \left(\frac{5}{2}, 3\right)$ (b) $x = \frac{15}{7}$ or $x = 5$ (c) $\frac{2}{5} < k < 4$

WMA13/01 JANUARY 2020

Question 8

10 marks | Rational Algebra | Main topic: Integration

8. (i) Find, using algebraic integration, the exact value of

$$\int_3^{42} \frac{2}{3x-1}\,\mathrm{d}x$$

giving your answer in simplest form.

(4)

(ii) $$\mathrm{h}(x) = \frac{2x^3 - 7x^2 + 8x + 1}{(x-1)^2} \qquad x > 1$$

Given $\mathrm{h}(x) = Ax + B + \frac{C}{(x-1)^2}$ where A, B and C are constants to be found, find

$$\int \mathrm{h}(x)\,\mathrm{d}x$$

(6)

WORKED SOLUTION

Question 8 · WMA13/01 January 2020

1 **Part (i)**

Since $3x - 1 > 0$ throughout $3 \le x \le 42$, an antiderivative is

$$\int \frac{2}{3x-1}\,dx = \frac{2}{3}\ln(3x-1) + K.$$

The factor $1/3$ compensates for the derivative of $3x - 1$. Thus

$$\begin{aligned}\int_3^{42} \frac{2}{3x-1}\,dx &= \left[\frac{2}{3}\ln(3x-1)\right]_3^{42} \\ &= \frac{2}{3}\left(\ln 125 - \ln 8\right) \\ &= \frac{2}{3}\ln\left(\frac{125}{8}\right) = \frac{2}{3}\ln\left(\left(\frac{5}{2}\right)^3\right).\end{aligned}$$

Using $\ln(a^3) = 3\ln a$ for $a > 0$, the exact value in simplest form is

$$\boxed{2\ln\left(\frac{5}{2}\right).}$$

2 **Part (ii)**

Multiply the given decomposition by $(x-1)^2$:

$$\begin{aligned} 2x^3 - 7x^2 + 8x + 1 &= (Ax + B)(x-1)^2 + C \\ &= Ax^3 + (B - 2A)x^2 + (A - 2B)x + (B + C). \end{aligned}$$

Comparing coefficients gives

$$A = 2, \qquad B - 2A = -7 \implies B = -3, \qquad B + C = 1 \implies C = 4.$$

The coefficient of x also agrees, since $A - 2B = 2 - 2(-3) = 8$. Hence

$$\boxed{A = 2, \qquad B = -3, \qquad C = 4,} \qquad h(x) = 2x - 3 + 4(x-1)^{-2}.$$

Integrating each term,

$$\begin{aligned} \int h(x)\,dx &= \int \left(2x - 3 + 4(x-1)^{-2}\right) dx \\ &= x^2 - 3x - 4(x-1)^{-1} + K. \end{aligned}$$

Therefore, for $x > 1$,

$$\boxed{\int h(x)\,dx = x^2 - 3x - \frac{4}{x-1} + K,}$$

where K is an arbitrary constant of integration, distinct from the coefficient $C = 4$.

FINAL ANSWER

(i) $2\ln\left(\frac{5}{2}\right)$ (ii) $A = 2, \qquad B = -3, \qquad C = 4, \int h(x)\,dx = x^2 - 3x - \frac{4}{x-1} + K$

WMA13/01 OCTOBER 2020

Question 7

9 marks | Rational Algebra

Main topic: Trig Equations & R-Form

7. (a) Express $\cos x + 4\sin x$ in the form $R\cos(x - \alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

A scientist is studying the behaviour of seabirds in a colony.

She models the height above sea level, H metres, of one of the birds in the colony by the equation

$$H = \frac{24}{3 + \cos\left(\frac{1}{2}t\right) + 4\sin\left(\frac{1}{2}t\right)} \qquad 0 \leqslant t \leqslant 6.5$$

where t seconds is the time after it leaves the nest.

Find, according to the model,

(b) the minimum height of the seabird above sea level, giving your answer to the nearest cm,

(2)

(c) the value of t, to 2 decimal places, when $H = 10$

(4)

WORKED SOLUTION

Question 7 · WMA13/01 October 2020

1 **Part (a)**

Expand $R\cos(x-\alpha) = R\cos\alpha\cos x + R\sin\alpha\sin x$. Matching coefficients gives $R\cos\alpha = 1$, $R\sin\alpha = 4$. Thus $R = \sqrt{1^2 + 4^2} = \sqrt{17}$ and $\tan\alpha = 4$. Since $0 < \alpha < \pi/2$, $\alpha = \arctan 4 = 1.32581766\ldots$ radians.

$$\boxed{R = \sqrt{17}, \qquad \alpha = 1.326 \text{ radians}}$$

2 Part (b)

The model becomes $H = 24/[3 + \sqrt{17}\cos(t/2 - \alpha)]$. For $0 \le t \le 6.5$, the phase lies between $-\alpha$ and $3.25 - \alpha$; throughout this interval the denominator remains positive. Its maximum $3 + \sqrt{17}$ is attained at $t = 2\alpha = 2.651635\ldots$, inside the time interval. The height is then smallest:

$$H_{\min} = \frac{24}{3 + \sqrt{17}} = 3.369316876\ldots \text{ m}.$$

To the nearest centimetre this is 3.37 m (337 cm).

$$\boxed{H_{\min} = 3.37 \text{ m}}$$

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3 Part (c)

For $H = 10$, $3 + \sqrt{17}\cos(t/2 - \alpha) = 24/10$, so

$$\cos(t/2 - \alpha) = -\frac{3}{5\sqrt{17}}.$$

Let $\gamma = \arccos[-3/(5\sqrt{17})]$. In the permitted phase interval $[-\alpha, 3.25 - \alpha]$, only $+\gamma$ occurs: $-\gamma < -\alpha$, and shifts by 2π are outside it. Using the unrounded $\alpha = \arctan 4$,

$$t = 2(\alpha + \gamma) = 6.085307854\ldots \text{ s}.$$

$$\boxed{t = 6.09 \text{ s}}$$

FINAL ANSWER

(a) $R = \sqrt{17}, \qquad \alpha = 1.326$ radians (b) $H_{\min} = 3.37$ m (c) $t = 6.09$ s

Question 9

14 marks | Rational Algebra

Main topic: Applications of Differentiation

9. (a) Given that

$$\frac{x^4 - x^3 - 10x^2 + 3x - 9}{x^2 - x - 12} \equiv x^2 + P + \frac{Q}{x - 4} \qquad x > -3$$

find the value of the constant P and show that $Q = 5$

(4)

The curve C has equation $y = \mathrm{g}(x)$, where

$$\mathrm{g}(x) = \frac{x^4 - x^3 - 10x^2 + 3x - 9}{x^2 - x - 12} \qquad -3 < x < 3.5 \qquad x \in \mathbb{R}$$

(b) Find the equation of the tangent to C at the point where $x = 2$
Give your answer in the form $y = mx + c$, where m and c are constants to be found.

(5)

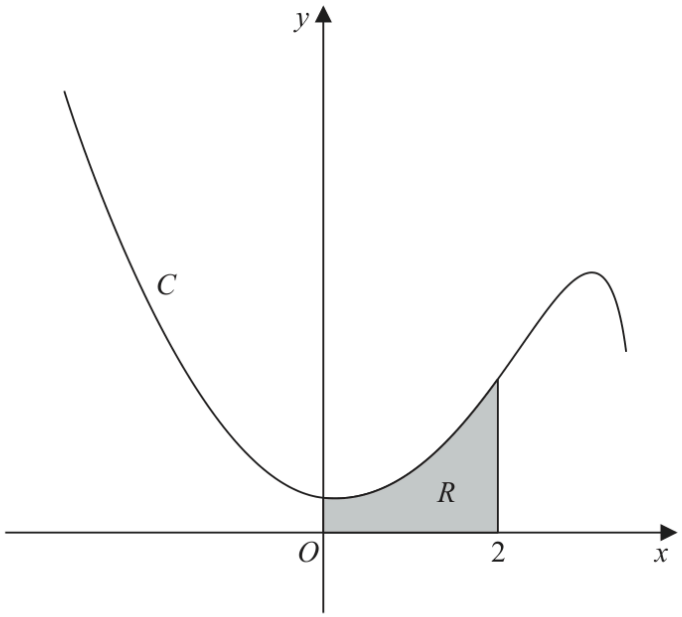


Figure 4

Figure 4 shows a sketch of the curve C.
The region R, shown shaded in Figure 4, is bounded by C, the y-axis, the x-axis and the line with equation $x = 2$

(c) Find the exact area of R, writing your answer in the form $a + b\ln 2$, where a and b are constants to be found.

(5)

1 Part (a)

Factor the denominator as $(x-4)(x+3)$ and multiply the proposed identity through by it:

$$x^4 - x^3 - 10x^2 + 3x - 9 = (x^2 + P)(x^2 - x - 12) + Q(x+3).$$

The right side expands to $x^4 - x^3 + (P-12)x^2 + (Q-P)x + 3Q - 12P$. Comparing coefficients gives $P - 12 = -10$, so $P = 2$, and $Q - P = 3$, so $Q = 5$. The constant check is $3(5) - 12(2) = -9$, as required. The identity holds where the original fraction is defined: the stated $x > -3$ still excludes $x = 4$.

$$\boxed{P = 2, \qquad Q = 5}$$

2 Part (b)

From (a), $g(x) = x^2 + 2 + 5/(x-4)$ on the curve domain. Thus $g'(x) = 2x - 5/(x-4)^2$. At $x = 2$, $g(2) = 4 + 2 - 5/2 = 7/2$ and $g'(2) = 4 - 5/4 = 11/4$. The tangent through $(2, 7/2)$ is

$$y - \frac{7}{2} = \frac{11}{4}(x-2) \quad \Longrightarrow \quad y = \frac{11}{4}x - 2.$$

$$\boxed{y = \frac{11}{4}x - 2}$$

3 Part (c)

The source shading is above the x-axis from $x = 0$ to $x = 2$, so integrate g over those bounds.

$$\text{Area}(R) = \int_0^2 \left(x^2 + 2 + \frac{5}{x-4}\right) dx = \left[\frac{x^3}{3} + 2x + 5\ln|x-4|\right]_0^2.$$

The absolute value is essential because $x - 4 < 0$ throughout this interval. Evaluating gives

$$\frac{8}{3} + 4 + 5\ln 2 - 5\ln 4 = \frac{20}{3} - 5\ln 2.$$

Thus $a = 20/3$ and $b = -5$ in the requested form.

$$\boxed{\text{Area}(R) = \frac{20}{3} - 5\ln 2}$$

FINAL ANSWER

(a) $P = 2, \qquad Q = 5$ (b) $y = \frac{11}{4}x - 2$ (c) $\text{Area}(R) = \frac{20}{3} - 5\ln 2$

WMA13/01 MAY/JUNE 2021

Question 8

13 marks | Rational Algebra

Main topic: Log Graphs & Exponential Models

8. A scientist is studying a population of fish in a lake. The number of fish, N, in the population, t years after the start of the study, is modelled by the equation

$$N = \frac{600e^{0.3t}}{2 + e^{0.3t}} \qquad t \geqslant 0$$

Use the equation of the model to answer parts (a), (b), (c), (d) and (e).

(a) Find the number of fish in the lake at the start of the study.

(1)

(b) Find the upper limit to the number of fish in the lake.

(1)

(c) Find the time, after the start of the study, when there are predicted to be 500 fish in the lake. Give your answer in years and months to the nearest month.

(4)

(d) Show that

$$\frac{dN}{dt} = \frac{Ae^{0.3t}}{(2 + e^{0.3t})^2}$$

where A is a constant to be found.

(3)

Given that when $t = T$, $\frac{dN}{dt} = 8$

(e) find the value of T to one decimal place.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(4)

WORKED SOLUTION

Question 8 · WMA13/01 May/June 2021

1 **Part (a)**

At the start, $t = 0$ and $e^{0.3t} = 1$. Thus $N(0) = 600/(2+1) = 200$.

$$\boxed{200 \text{ fish}}$$

2 Part (b)

Divide numerator and denominator by $e^{0.3t}$ to write $N = 600/(1 + 2e^{-0.3t})$. As $t \to \infty$, the denominator tends to 1, so the upper limit is 600 fish.

$$\boxed{600 \text{ fish}}$$

3 Part (c)

Set $N = 500$ and multiply through: $1000 + 500e^{0.3t} = 600e^{0.3t}$, so $e^{0.3t} = 10$. Hence $t = \ln 10/0.3 = 7.675283643\ldots$ years. The fractional year is $12(0.675283643\ldots) = 8.10340372\ldots$ months. Round only the months to the nearest whole month.

$$\boxed{7 \text{ years } 8 \text{ months}}$$

4 Part (d)

The quotient rule gives

$$\frac{dN}{dt} = \frac{180e^{0.3t}(2 + e^{0.3t}) - 600e^{0.3t}(0.3e^{0.3t})}{(2 + e^{0.3t})^2} = \frac{360e^{0.3t}}{(2 + e^{0.3t})^2}.$$

The two $e^{0.6t}$ terms cancel exactly.

$$\boxed{A = 360}$$

5 Part (e)

Let $z = e^{0.3T}$. Since $T \geq 0$, require $z \geq 1$. Then $360z/(2 + z)^2 = 8$ gives $45z = (2 + z)^2$, hence

$$z^2 - 41z + 4 = 0, \qquad z = \frac{41 \pm \sqrt{1665}}{2}.$$

The smaller root lies between 0 and 1 and would give $T < 0$, so reject it. For the larger root,

$$T = \frac{1}{0.3}\ln\left(\frac{41 + \sqrt{1665}}{2}\right) = 12.37061331\ldots.$$

$$\boxed{T = 12.4 \text{ years}}$$

FINAL ANSWER

(a) 200 fish (b) 600 fish (c) 7 years 8 months (d) $A = 360$ (e) $T = 12.4$ years

WMA13/01 OCTOBER 2021

Question 6

8 marks | Rational Algebra

Main topic: Differentiation Techniques

6. (i) The curve C_1 has equation

$$y = 3\ln(x^2 - 5) - 4x^2 + 15 \qquad x > \sqrt{5}$$

Show that C_1 has a stationary point at $x = \dfrac{\sqrt{p}}{2}$ where p is a constant to be found.

(4)

(ii) A different curve C_2 has equation

$$y = 4x - 12\sin^2 x$$

(a) Show that, for this curve,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = A + B\sin 2x$$

where A and B are constants to be found.

(b) Hence, state the maximum gradient of this curve.

(4)

WORKED SOLUTION

Question 6 · WMA13/01 October 2021

1 **Part (i)**

Differentiate to obtain $y' = 6x/(x^2 - 5) - 8x$. At a stationary point in $x > \sqrt{5}$, x is nonzero, so $6/(x^2 - 5) = 8$, giving $6 = 8x^2 - 40$ and $x^2 = 23/4$. The original positive domain selects $x = \sqrt{23}/2$, which exceeds $\sqrt{5}$. Hence $p = 23$.

$$\boxed{p = 23, \qquad x = \frac{\sqrt{23}}{2}}$$

2 **Part (ii)**

(a) Differentiate the squared sine with the chain rule: $y' = 4 - 24\sin x\cos x = 4 - 12\sin 2x$. Thus $A = 4$, $B = -12$. (b) Since $-1 \le \sin 2x \le 1$, the largest derivative is attained when $\sin 2x = -1$: $4 - 12(-1) = 16$.

$$\boxed{A = 4, \quad B = -12; \qquad \text{maximum gradient} = 16}$$

FINAL ANSWER

(i) $p = 23, \quad x = \frac{\sqrt{23}}{2}$ (ii) $A = 4, \quad B = -12; \quad$ maximum gradient $= 16$

Question 9

9 marks | Rational Algebra | Main topic: Numerical Methods

9.

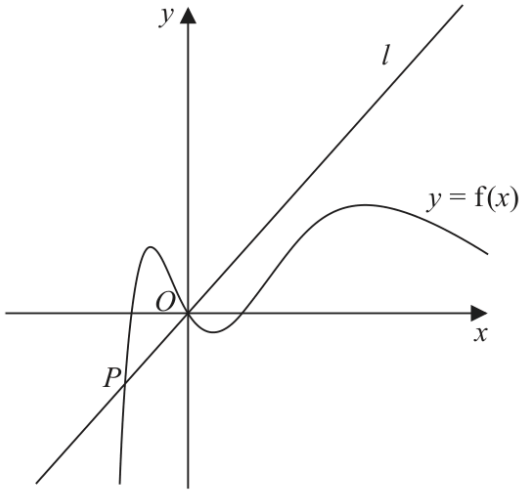


Figure 3

Figure 3 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x(x^2 - 4)\mathrm{e}^{-\frac{1}{2}x}$$

(a) Find $\mathrm{f}'(x)$.

(2)

The line l is the normal to the curve at O and meets the curve again at the point P.

The point P lies in the 3rd quadrant, as shown in Figure 3.

(b) Show that the x coordinate of P is a solution of the equation

$$x = -\frac{1}{2}\sqrt{16 + \mathrm{e}^{\frac{1}{2}x}}$$

(4)

(c) Using the iterative formula

$$x_{n+1} = -\frac{1}{2}\sqrt{16 + \mathrm{e}^{\frac{1}{2}x_n}} \quad \text{with } x_1 = -2$$

find, to 4 decimal places,

(i) the value of x_2

(ii) the x coordinate of P.

(3)

1 Part (a)

Write $f(x) = (x^3 - 4x)e^{-x/2}$. By the product rule,

$$f'(x) = (3x^2 - 4)e^{-x/2} - \frac{1}{2}(x^3 - 4x)e^{-x/2} = e^{-x/2}\left(-\frac{x^3}{2} + 3x^2 + 2x - 4\right).$$

$$\boxed{f'(x) = e^{-x/2}(-x^3/2 + 3x^2 + 2x - 4)}$$

2 Part (b)

At O, $f'(0) = -4$, so the normal gradient is $1/4$ and its equation is $y = x/4$. At the other intersection P, x is nonzero; equating curve and normal and dividing by x gives $(x^2 - 4)e^{-x/2} = 1/4$. Thus $x^2 = 4 + e^{x/2}/4 = (16 + e^{x/2})/4$. Since P lies in the third quadrant, select the negative square root.

$$\boxed{x = -\frac{1}{2}\sqrt{16 + e^{x/2}}}$$

3 Part (c)

Starting from $x_1 = -2$, $x_2 = -\frac{1}{2}\sqrt{16 + e^{-1}} = -2.0228617996\ldots$. Further unrounded iterates are $x_3 = -2.0226034084\ldots$, $x_4 = -2.0226063126\ldots$ and $x_5 = -2.0226062799\ldots$. The continuing sequence settles at $-2.0226062803\ldots$.

$$\boxed{\text{(i) } x_2 = -2.0229; \qquad \text{(ii) } x_P = -2.0226}$$

✓ FINAL ANSWER

(a) $f'(x) = e^{-x/2}(-x^3/2 + 3x^2 + 2x - 4)$ (b) $x = -\frac{1}{2}\sqrt{16 + e^{x/2}}$ (c) $\text{(i) } x_2 = -2.0229; \qquad \text{(ii) } x_P = -2.0226$

WMA13/01 JANUARY 2022

Question 6

11 marks | Rational Algebra

Main topic: Functions, Composition & Inverses

6. The function f is defined by

$$\mathrm{f}(x) = \frac{5x - 3}{x - 4} \qquad x > 4$$

(a) Show, by using calculus, that f is a decreasing function.

(3)

(b) Find f^{-1}

(3)

(c) (i) Show that $\mathrm{ff}(x) = \dfrac{ax + b}{x + c}$ where a, b and c are constants to be found.

(ii) Deduce the range of ff.

(5)

WORKED SOLUTION

Question 6 · WMA13/01 January 2022

1 **Part (a)**

By the quotient rule,

$$f'(x) = \frac{5(x-4) - (5x-3)}{(x-4)^2} = \frac{-17}{(x-4)^2}.$$

For every $x > 4$, the denominator is positive and the numerator is negative. Therefore $f'(x) < 0$ throughout its domain, so f is decreasing.

$$\boxed{f \text{ is decreasing for } x > 4}$$

2 **Part (b)**

Set $y = (5x - 3)/(x - 4)$. Then $yx - 4y = 5x - 3$, giving $x(y - 5) = 4y - 3$ and $x = (4y - 3)/(y - 5)$. Exchange the variable names. Since $f(x) = 5 + 17/(x - 4)$ on $x > 4$, its range is $(5, \infty)$, which becomes the inverse's domain.

$$\boxed{f^{-1}(x) = \frac{4x - 3}{x - 5}, \qquad x > 5}$$

3 **Part (c)**

(i) For $x > 4$, $f(x) > 5 > 4$, so the second application is defined.

$$ff(x) = \frac{5\left(\frac{5x-3}{x-4}\right) - 3}{\left(\frac{5x-3}{x-4}\right) - 4} = \frac{25x - 15 - 3x + 12}{5x - 3 - 4x + 16} = \frac{22x - 3}{x + 13}.$$

Thus $a = 22$, $b = -3$, $c = 13$. (ii) Rewrite $ff(x) = 22 - 289/(x + 13)$ with the original domain $x > 4$. It is strictly increasing, tends to 5 as $x \to 4^+$ and to 22 as $x \to \infty$. Neither limiting value is attained.

$$\boxed{ff(x) = \frac{22x - 3}{x + 13}, \qquad 5 < ff(x) < 22}$$

FINAL ANSWER

(a) f **is decreasing for** $x > 4$ (b) $f^{-1}(x) = \frac{4x-3}{x-5}, \qquad x > 5$ (c) $ff(x) = \frac{22x-3}{x+13}, \qquad 5 < ff(x) < 22$

Question 6

10 marks | Rational Algebra

Main topic: Applications of Differentiation

6.

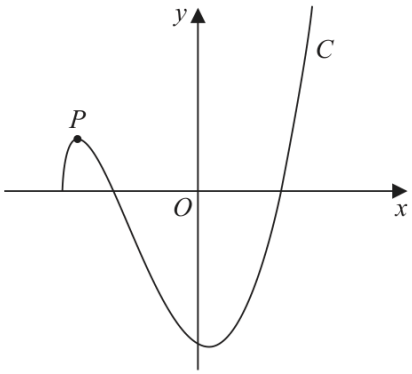


Figure 3

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

The function f is defined by

$$f(x) = 5(x^2 - 2)(4x + 9)^{\frac{1}{2}} \qquad x \geqslant -\frac{9}{4}$$

(a) Show that

$$f'(x) = \frac{k(5x^2 + 9x - 2)}{(4x + 9)^{\frac{1}{2}}}$$

where k is an integer to be found.

(4)

(b) Hence, find the values of x for which $f'(x) = 0$

(1)

Figure 3 shows a sketch of the curve C with equation $y = f(x)$.

The curve has a local maximum at the point P

(c) Find the exact coordinates of P

(2)

The function g is defined by

$$g(x) = 2f(x) + 4 \qquad -\frac{9}{4} \leqslant x \leqslant 0$$

(d) Find the range of g

(3)

1 Part (a)

Use product and chain rules:

$$f'(x) = 10x(4x+9)^{1/2} + 10(x^2-2)(4x+9)^{-1/2}.$$

With a common denominator,

$$f'(x) = \frac{10x(4x+9) + 10(x^2-2)}{\sqrt{4x+9}} = \frac{10(5x^2+9x-2)}{\sqrt{4x+9}}.$$

This derivative formula applies for $x > -9/4$; the endpoint derivative is not finite.

$$\boxed{k = 10}$$

2 Part (b)

The denominator is positive in the derivative domain. Thus $5x^2+9x-2 = (5x-1)(x+2) = 0$.

$$\boxed{x = -2, \quad \frac{1}{5}}$$

3 Part (c)

The derivative changes from positive to negative at $x = -2$, so this is the local maximum P. Substitute into f: $f(-2) = 5(4-2)\sqrt{-8+9} = 10$.

$$\boxed{P = (-2, 10)}$$

4 Part (d)

On $[-9/4, 0]$, f rises from $f(-9/4) = 0$ to $f(-2) = 10$, then decreases to $f(0) = -30$. The other stationary point $1/5$ is outside this restricted interval. Thus the minimum and maximum of f on it are -30 and 10. Applying the positive vertical transformation $g = 2f + 4$ gives endpoints $2(-30)+4 = -56$ and $2(10)+4 = 24$, both attained.

$$\boxed{-56 \le g(x) \le 24}$$

FINAL ANSWER

(a) $k = 10$ (b) $x = -2, \quad \frac{1}{5}$ (c) $P = (-2, 10)$ (d) $-56 \le g(x) \le 24$

WMA13/01 OCTOBER 2022

Question 9

9 marks | Rational Algebra | Main topic: Trig Equations & R-Form

9. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

Given that $\cos 2\theta - \sin 3\theta \neq 0$

(a) prove that

$$\frac{\cos^2\theta}{\cos 2\theta - \sin 3\theta} \equiv \frac{1+\sin\theta}{1-2\sin\theta-4\sin^2\theta}$$

(4)

(b) Hence solve, for $0 < \theta \leqslant 360°$

$$\frac{\cos^2\theta}{\cos 2\theta - \sin 3\theta} = 2\operatorname{cosec}\theta$$

Give your answers to one decimal place.

(5)

WORKED SOLUTION

Question 9 · WMA13/01 October 2022

1 Part (a)

Write $s = \sin\theta$. The compound-angle formula gives

$$\sin 3\theta = \sin(2\theta+\theta) = 2\sin\theta\cos^2\theta + (1-2\sin^2\theta)\sin\theta = 3s - 4s^3.$$

Hence

$$\cos 2\theta - \sin 3\theta = 1 - 2s^2 - 3s + 4s^3 = (1-s)(1-2s-4s^2), \quad \cos^2\theta = (1-s)(1+s).$$

The given nonzero denominator ensures that $1 - s \neq 0$ and the other factor is nonzero. Cancel the common factor to obtain the required identity.

$$\boxed{\frac{\cos^2\theta}{\cos 2\theta - \sin 3\theta} = \frac{1+\sin\theta}{1-2\sin\theta-4\sin^2\theta}}$$

2 Part (b)

Use (a) and $\csc\theta = 1/\sin\theta$. With $s = \sin\theta \neq 0$,

$$\frac{1+s}{1-2s-4s^2} = \frac{2}{s} \implies s + s^2 = 2 - 4s - 8s^2 \implies 9s^2 + 5s - 2 = 0.$$

Thus $s = (-5 \pm \sqrt{97})/18$, approximately 0.269380989 and -0.824936545. Both lie in $[-1, 1]$, are nonzero, and satisfy the original nonzero-denominator condition. The positive value gives $15.6274\ldots$° and $164.3726\ldots$°; the negative value gives $235.5821\ldots$° and $304.4179\ldots$°. All four angles lie in $0 < \theta \leq 360$°.

$$\boxed{\theta = 15.6^\circ,\ 164.4^\circ,\ 235.6^\circ,\ 304.4^\circ}$$

FINAL ANSWER

(a) $\frac{\cos^2\theta}{\cos 2\theta - \sin 3\theta} = \frac{1+\sin\theta}{1-2\sin\theta-4\sin^2\theta}$ (b) $\theta = 15.6^\circ,\ 164.4^\circ,\ 235.6^\circ,\ 304.4^\circ$

Question 9

11 marks | Rational Algebra | Main topic: Numerical Methods

9.

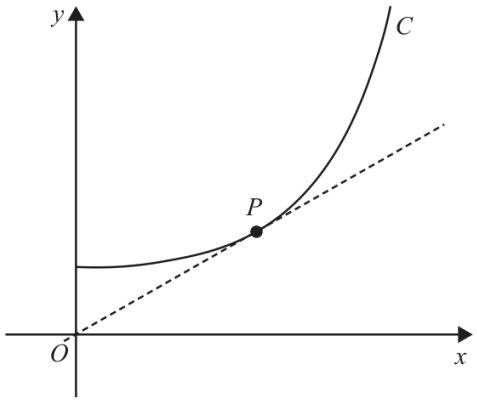


Figure 3

Figure 3 shows a sketch of part of the curve C with equation

$$y = \sqrt{3 + 4e^{x^2}} \qquad x \geqslant 0$$

(a) Find $\frac{dy}{dx}$, giving your answer in simplest form.

(2)

The point P with x coordinate α lies on C.

Given that the tangent to C at P passes through the origin, as shown in Figure 3,

(b) show that $x = \alpha$ is a solution of the equation

$$4x^2e^{x^2} - 4e^{x^2} - 3 = 0$$

(3)

(c) Hence show that α lies between 1 and 2

(2)

(d) Show that the equation in part (b) can be written in the form

$$x = \frac{1}{2}\sqrt{4 + 3e^{-x^2}}$$

(1)

The iteration formula

$$x_{n+1} = \frac{1}{2}\sqrt{4 + 3e^{-x_n^2}}$$

with $x_1 = 1$ is used to find an approximation for α.

(e) Use the iteration formula to find, to 4 decimal places, the value of

(i) x_3

(ii) α

(3)

1 Part (a)

Differentiate $y = (3 + 4e^{x^2})^{1/2}$ using the chain rule:

$$\frac{dy}{dx} = \frac{1}{2}(3 + 4e^{x^2})^{-1/2} \cdot 4e^{x^2} \cdot 2x = \frac{4xe^{x^2}}{\sqrt{3 + 4e^{x^2}}}.$$

$$\boxed{\frac{dy}{dx} = \frac{4xe^{x^2}}{\sqrt{3 + 4e^{x^2}}}}$$

2 Part (b)

Since the tangent at $P = (\alpha, \sqrt{3 + 4e^{\alpha^2}})$ passes through the origin, its gradient is y_P/α, with $\alpha > 0$. Equate this to the derivative:

$$\frac{4\alpha e^{\alpha^2}}{\sqrt{3 + 4e^{\alpha^2}}} = \frac{\sqrt{3 + 4e^{\alpha^2}}}{\alpha} \quad \Longrightarrow \quad 4\alpha^2 e^{\alpha^2} = 3 + 4e^{\alpha^2}.$$

Consequently $x = \alpha$ solves the displayed equation.

$$\boxed{4x^2e^{x^2} - 4e^{x^2} - 3 = 0}$$

3 Part (c)

Let $h(x) = 4x^2e^{x^2} - 4e^{x^2} - 3$, a continuous function. Then $h(1) = -3 < 0$ and $h(2) = 12e^4 - 3 = 652.1778\ldots > 0$, so a root lies between 1 and 2. Moreover $h'(x) = 8x^3e^{x^2} > 0$ for $x > 0$, so this is the unique positive root corresponding to the tangent abscissa α.

$$\boxed{1 < \alpha < 2}$$

4 Part (d)

From $4x^2e^{x^2} = 4e^{x^2} + 3$, divide by the positive e^{x^2} to get $4x^2 = 4 + 3e^{-x^2}$. The source domain has $x \geq 0$, so take the positive square root.

$$\boxed{x = \frac{1}{2}\sqrt{4 + 3e^{-x^2}}}$$

5 **Part (e)**

With $x_1 = 1$, the unrounded iterates start

$$x_2 = 1.1295616764\ldots, \quad x_3 = 1.0997193683\ldots, \quad x_4 = 1.1062486441\ldots, \quad x_5 = 1.1048012899\ldots.$$

Continue: $x_6 = 1.1051212304\ldots$, $x_7 = 1.1050504628\ldots$, $x_8 = 1.1050661137\ldots$, $x_9 = 1.1050626523\ldots$; further iteration settles at $1.1050632792\ldots$. Therefore (i) $x_3 = 1.0997$ and (ii) $\alpha = 1.1051$, both to four decimal places.

$$\boxed{x_3 = 1.0997, \qquad \alpha = 1.1051}$$

✓ **FINAL ANSWER**

(a) $\frac{dy}{dx} = \frac{4xe^{x^2}}{\sqrt{3+4e^{x^2}}}$ (b) $4x^2e^{x^2} - 4e^{x^2} - 3 = 0$ (c) $1 < \alpha < 2$ (d) $x = \frac{1}{2}\sqrt{4 + 3e^{-x^2}}$ (e) $x_3 = 1.0997, \qquad \alpha = 1.1051$

WMA13/01 JANUARY 2023

Question 10

9 marks | Rational Algebra

Main topic: Log Graphs & Exponential Models

10. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

A population of fruit flies is being studied.

The number of fruit flies, F, in the population, t days after the start of the study, is modelled by the equation

$$F = \frac{350e^{kt}}{9 + e^{kt}}$$

where k is a constant.

Use the equation of the model to answer parts (a), (b) and (c).

(a) Find the number of fruit flies in the population at the start of the study.

(1)

Given that there are 200 fruit flies in the population 15 days after the start of the study,

(b) show that $k = \frac{1}{15}\ln 12$

(3)

Given also that, when $t = T$, the number of fruit flies in the population is increasing at a rate of 10 per day,

(c) find the possible values of T, giving your answers to one decimal place.

(5)

WORKED SOLUTION

Question 10 · WMA13/01 January 2023

1 **Part (a)**

At the start $t = 0$, so $e^{kt} = 1$ and the model gives $F(0) = 350/(9 + 1) = 35$ fruit flies.

$$\boxed{F(0) = 35}$$

2 Part (b)

Using $F(15) = 200$,

$$200 = \frac{350e^{15k}}{9 + e^{15k}} \quad \Longrightarrow \quad 1800 + 200e^{15k} = 350e^{15k} \quad \Longrightarrow \quad e^{15k} = 12.$$

Take natural logs to obtain $15k = \ln 12$, as required.

$$\boxed{k = \frac{\ln 12}{15}}$$

---

3 Part (c)

By the quotient rule,

$$F'(t) = \frac{350ke^{kt}(9 + e^{kt}) - 350ke^{2kt}}{(9 + e^{kt})^2} = \frac{3150ke^{kt}}{(9 + e^{kt})^2}.$$

Set $F'(T) = 10$ and let $u = e^{kT} > 0$. Then

$$315ku = (9 + u)^2 \quad \Longrightarrow \quad u^2 + (18 - 315k)u + 81 = 0.$$

With the exact $k = \ln 12/15$, the quadratic formula gives

$$u = \frac{315k - 18 \pm \sqrt{(315k - 18)^2 - 324}}{2} = 2.5615490777\ldots \text{ or } 31.6214905678\ldots.$$

Both values are positive and at least 1, hence both correspond to $T \geq 0$. Finally $T = \ln u/k$ gives $5.6779528361\ldots$ or $20.8488936858\ldots$ days. To one decimal place the two times are as follows.

$$\boxed{T = 5.7 \text{ days}, \qquad T = 20.8 \text{ days}}$$

FINAL ANSWER

(a) $F(0) = 35$ (b) $k = \frac{\ln 12}{15}$ (c) $T = 5.7$ days, $\quad T = 20.8$ days

Question 8

9 marks | Rational Algebra

Main topic: Applications of Differentiation

8.

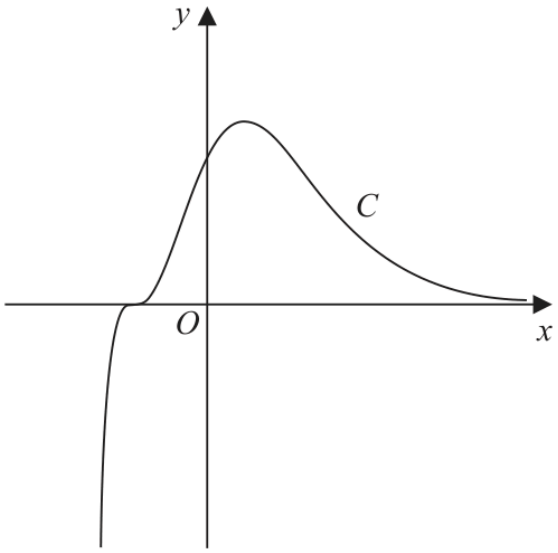


Figure 3

Figure 3 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = (2x + 1)^3 \mathrm{e}^{-4x}$$

(a) Show that

$$\mathrm{f}'(x) = A(2x + 1)^2 (1 - 4x)\, \mathrm{e}^{-4x}$$

where A is a constant to be found.

(4)

(b) Hence find the exact coordinates of the two stationary points on C.

(3)

The function g is defined by

$$\mathrm{g}(x) = 8\,\mathrm{f}(x - 2)$$

(c) Find the coordinates of the maximum stationary point on the curve with equation $y = \mathrm{g}(x)$.

(2)

1 Part (a)

The product rule gives

$$f'(x) = 6(2x+1)^2e^{-4x} - 4(2x+1)^3e^{-4x} = (2x+1)^2e^{-4x}[6-4(2x+1)] = 2(2x+1)^2(1-4x)e^{-4x}.$$

Comparison with the stated form gives $A = 2$.

$$\boxed{A = 2}$$

2 Part (b)

Since $e^{-4x} > 0$, $f'(x) = 0$ means $2x+1=0$ or $1-4x=0$. Thus $x = -1/2$ or $x = 1/4$.
Substitution gives $f(-1/2) = 0$ and $f(1/4) = (3/2)^3e^{-1} = 27/(8e)$.

$$\boxed{\left(-\frac{1}{2}, 0\right), \quad \left(\frac{1}{4}, \frac{27}{8e}\right)}$$

3 Part (c)

The derivative changes from positive to negative at $x = 1/4$, so this is the maximum; at $x = -1/2$ it does not change sign. The transformation $g(x) = 8f(x-2)$ sends (x, y) to $(x+2, 8y)$. Hence the maximum stationary point is $(1/4 + 2, 8 \cdot 27/(8e)) = (9/4, 27/e)$.

$$\boxed{\left(\frac{9}{4}, \frac{27}{e}\right)}$$

FINAL ANSWER

(a) $A = 2$ (b) $\left(-\frac{1}{2}, 0\right), \quad \left(\frac{1}{4}, \frac{27}{8e}\right)$ (c) $\left(\frac{9}{4}, \frac{27}{e}\right)$

Question 10

8 marks | Rational Algebra

Main topic: Differentiation Techniques

10.

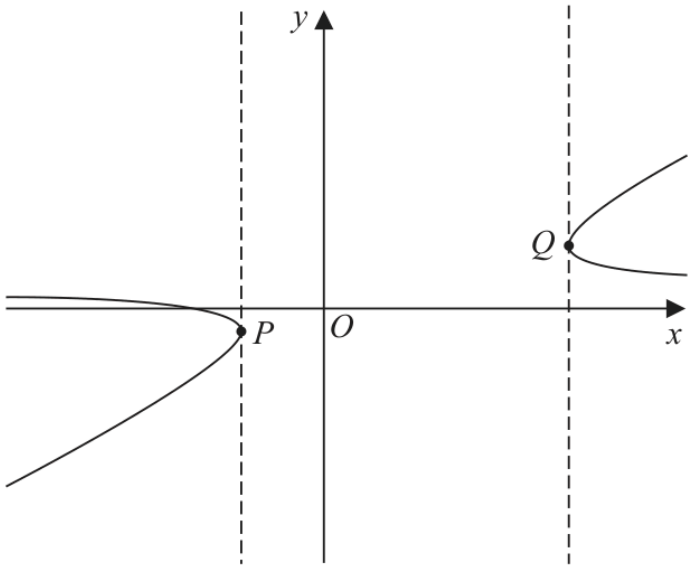


Figure 4

Figure 4 shows a sketch of the curve with equation

$$x = \frac{2y^2 + 6}{3y - 3}$$

(a) Find $\frac{dx}{dy}$ giving your answer as a fully simplified fraction.

(4)

The tangents at points P and Q on the curve are parallel to the y-axis, as shown in Figure 4.

(b) Use the answer to part (a) to find the equations of these two tangents.

(4)

1 Part (a)

The quotient rule gives, for $y \neq 1$,

$$\frac{dx}{dy} = \frac{4y(3y-3) - 3(2y^2+6)}{(3y-3)^2} = \frac{6y^2 - 12y - 18}{9(y-1)^2} = \frac{2(y-3)(y+1)}{3(y-1)^2}.$$

$$\boxed{\frac{dx}{dy} = \frac{2(y-3)(y+1)}{3(y-1)^2}, \quad y \neq 1}$$

2 Part (b)

A tangent parallel to the y-axis is vertical, so $dx/dy = 0$. The numerator gives $(y-3)(y+1) = 0$, hence $y = 3$ or $y = -1$; neither makes the denominator zero. Then

$$x(3) = \frac{18+6}{9-3} = 4, \qquad x(-1) = \frac{2+6}{-3-3} = -\frac{4}{3}.$$

Thus $Q = (4, 3)$ and $P = (-4/3, -1)$, with the vertical tangent equations below.

$$\boxed{x = -\frac{4}{3}, \qquad x = 4}$$

FINAL ANSWER

(a) $\frac{dx}{dy} = \frac{2(y-3)(y+1)}{3(y-1)^2}, \quad y \neq 1$ (b) $x = -\frac{4}{3}, \qquad x = 4$

Question 7

10 marks | Rational Algebra

Main topic: Functions, Composition & Inverses

7.

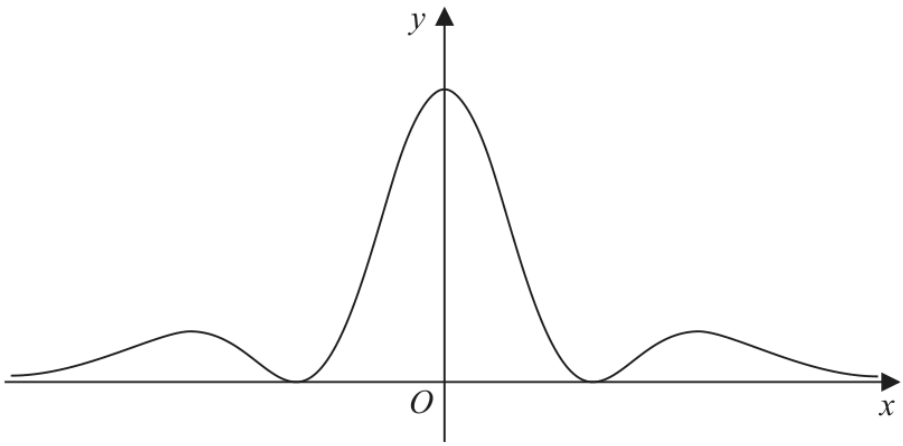


Figure 1

Figure 1 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \mathrm{e}^{-x^2}(2x^2 - 3)^2$$

(a) Find the range of f

(2)

(b) Show that

$$\mathrm{f}'(x) = 2x(2x^2 - 3)\mathrm{e}^{-x^2}(A - Bx^2)$$

where A and B are constants to be found.

(4)

Given that the line $y = k$, where k is a constant, $k > 0$, intersects the curve at exactly two distinct points,

(c) find the exact range of values of k

(4)

1 Part (a)

The exponential is positive and the squared factor is nonnegative, so $f(x) \geq 0$. The value zero is attained at $x = \pm\sqrt{3/2}$. The graph has its highest point at $x = 0$, where $f(0) = 9$; the outer maxima are lower. Thus both endpoints of the range are included.

$$\boxed{0 \leq f(x) \leq 9}$$

2 Part (b)

For $f(x) = e^{-x^2}(2x^2 - 3)^2$, product and chain rules give

$$f'(x) = -2xe^{-x^2}(2x^2 - 3)^2 + e^{-x^2}\,2(2x^2 - 3)\,4x.$$

Factoring yields

$$f'(x) = 2x(2x^2 - 3)e^{-x^2}\big(-(2x^2 - 3) + 4\big) = 2x(2x^2 - 3)e^{-x^2}(7 - 2x^2).$$

$$\boxed{A = 7, \qquad B = 2}$$

3 Part (c)

The stationary abscissae are 0, $\pm\sqrt{3/2}$ and $\pm\sqrt{7/2}$. At the outer local maxima,

$$f\left(\pm\sqrt{7/2}\right) = e^{-7/2}(7 - 3)^2 = 16e^{-7/2}.$$

The central maximum is 9 and the two minima are zero. Also $f(x) \to 0$ as $x \to \pm\infty$. A positive horizontal line above both outer maxima but below 9 meets only the two sides of the central peak. At $k = 16e^{-7/2}$ it also touches both outer peaks, giving four distinct intersections; at $k = 9$ there is only one. Both bounds must be excluded.

$$\boxed{16e^{-7/2} < k < 9}$$

FINAL ANSWER

(a) $0 \leq f(x) \leq 9$ (b) $A = 7, \qquad B = 2$ (c) $16e^{-7/2} < k < 9$

Question 10

9 marks | Rational Algebra

Main topic: Differentiation Techniques

10. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

A curve C has equation

$$x = \sin^2 4y \qquad 0 \leqslant y \leqslant \frac{\pi}{8} \qquad 0 \leqslant x \leqslant 1$$

The point P with x coordinate $\frac{1}{4}$ lies on C

(a) Find the exact y coordinate of P

(2)

(b) Find $\frac{\mathrm{d}x}{\mathrm{d}y}$

(2)

(c) Hence show that $\frac{\mathrm{d}y}{\mathrm{d}x}$ can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{q + r(x + s)^2}}$$

where q, r and s are constants to be found.

(3)

Using the answer to part (c),

(d) (i) state the x coordinate of the point where the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ is a minimum,

(ii) state the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at this point.

(2)

1 Part (a)

At $x = 1/4$, $\sin^2 4y = 1/4$. Since $0 \le 4y \le \pi/2$,

$$\sin 4y = \frac{1}{2} \quad \Longrightarrow \quad 4y = \frac{\pi}{6}.$$

Only this branch lies in the given range of y.

$$\boxed{y = \frac{\pi}{24}}$$

2 Part (b)

Differentiate $x = (\sin 4y)^2$ with respect to y:

$$\frac{dx}{dy} = 2\sin 4y\,(4\cos 4y).$$

$$\boxed{\frac{dx}{dy} = 8\sin 4y\cos 4y}$$

3 Part (c)

On the interior of the given interval, $\sin 4y = \sqrt{x}$ and $\cos 4y = \sqrt{1-x}$ are positive. Taking the reciprocal,

$$\frac{dy}{dx} = \frac{1}{8\sqrt{x}\sqrt{1-x}} = \frac{1}{\sqrt{64x - 64x^2}}.$$

Complete the square:

$$64x - 64x^2 = 16 - 64\left(x - \frac{1}{2}\right)^2.$$

Thus $dy/dx = (q + r(x+s)^2)^{-1/2}$ has the constants below, with $0 < x < 1$ for a finite derivative; the endpoints have vertical tangents.

$$\boxed{q = 16, \qquad r = -64, \qquad s = -\frac{1}{2}}$$

4 Part (d)

The derivative is positive. Its minimum occurs when the square root in the denominator is largest. Since $(x - 1/2)^2 \ge 0$, the largest radicand is 16, attained at $x = 1/2$. Hence the smallest gradient is $1/\sqrt{16}$.

$$\boxed{\text{(i) } x = \frac{1}{2}, \qquad \text{(ii) } \min \frac{dy}{dx} = \frac{1}{4}}$$

FINAL ANSWER

(a) $y = \frac{\pi}{24}$ (b) $\frac{dx}{dy} = 8\sin 4y \cos 4y$ (c) $q = 16, \qquad r = -64, \qquad s = -\frac{1}{2}$ (d)
(i) $x = \frac{1}{2}$, (ii) $\min \frac{dy}{dx} = \frac{1}{4}$

Question 7

12 marks | Rational Algebra

Main topic: Applications of Differentiation

7. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

The curve C has equation

$$y = \frac{16}{9(3x - k)} \qquad x \neq \frac{k}{3}$$

where k is a positive constant not equal to 3

(a) Find $\frac{dy}{dx}$ giving your answer in simplest form in terms of k.

(2)

The point P with x coordinate 1 lies on C.

Given that the gradient of the curve at P is -12

(b) find the two possible values of k

(3)

Given also that $k < 3$

(c) find the equation of the normal to C at P, writing your answer in the form $ax + by + c = 0$, where a, b and c are integers to be found.

(3)

(d) show, using algebraic integration that,

$$\int_1^3 \frac{16}{9(3x - k)} dx = \lambda \ln 10$$

where λ is a constant to be found.

(4)

1 Part (a)

Write $y = (16/9)(3x-k)^{-1}$. The chain rule gives

$$\frac{dy}{dx} = \frac{16}{9}(-1)(3x-k)^{-2}\cdot 3.$$

$$\boxed{\frac{dy}{dx} = -\frac{16}{3(3x-k)^2}}$$

2 Part (b)

At $x = 1$, the given gradient yields

$$-\frac{16}{3(3-k)^2} = -12 \quad\Longrightarrow\quad (3-k)^2 = \frac{4}{9}.$$

Hence $3-k = \pm 2/3$. Both resulting k-values are positive and different from 3.

$$\boxed{k = \frac{7}{3} \text{ or } \frac{11}{3}}$$

3 Part (c)

Since $k < 3$, use $k = 7/3$. At $x = 1$, $y = 16/[9(3-7/3)] = 8/3$. The tangent gradient is -12, so the normal gradient is $1/12$:

$$y - \frac{8}{3} = \frac{1}{12}(x-1), \qquad 12y - 32 = x - 1.$$

$$\boxed{x - 12y + 31 = 0}$$

4 Part (d)

With $k = 7/3$, the denominator does not vanish on $[1,3]$. Using the reciprocal-linear antiderivative,

$$\int_1^3 \frac{16}{9(3x-7/3)}\,dx = \left[\frac{16}{27}\ln|3x-7/3|\right]_1^3 = \frac{16}{27}\left(\ln\frac{20}{3} - \ln\frac{2}{3}\right) = \frac{16}{27}\ln\left(\frac{20/3}{2/3}\right).$$

$$\boxed{\frac{16}{27}\ln 10, \quad \lambda = \frac{16}{27}}$$

FINAL ANSWER

(a) $\frac{dy}{dx} = -\frac{16}{3(3x-k)^2}$ (b) $k = \frac{7}{3}$ or $\frac{11}{3}$ (c) $x - 12y + 31 = 0$ (d) $\frac{16}{27}\ln 10, \quad \lambda = \frac{16}{27}$

WMA13/01 JANUARY 2024

Question 9

8 marks | Rational Algebra

Main topic: Trig Equations & R-Form

9. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that the equation

$$\frac{3\sin\theta\cos\theta}{\cos\theta+\sin\theta} = (2+\sec 2\theta)(\cos\theta-\sin\theta)$$

can be written in the form

$$3\sin 2\theta - 4\cos 2\theta = 2$$

(3)

(b) Hence solve for $\pi < x < \frac{3\pi}{2}$

$$\frac{3\sin x\cos x}{\cos x+\sin x} = (2+\sec 2x)(\cos x-\sin x)$$

giving the answer to 3 significant figures.

(5)

WORKED SOLUTION

Question 9 · WMA13/01 January 2024

1 Part (a)

The original equation is defined only when $\cos\theta+\sin\theta\neq 0$ and $\cos 2\theta\neq 0$. Multiplying by $\cos\theta+\sin\theta$ gives

$$3\sin\theta\cos\theta = (2+\sec 2\theta)(\cos^2\theta-\sin^2\theta) = (2+\sec 2\theta)\cos 2\theta = 2\cos 2\theta+1.$$

Using $2\sin\theta\cos\theta=\sin 2\theta$, multiply through by 2: $3\sin 2\theta = 4\cos 2\theta+2$.

$$\boxed{3\sin 2\theta - 4\cos 2\theta = 2}$$

2 **Part (b)**

Write $3\sin 2x - 4\cos 2x = 5\sin(2x - \phi)$, where $\cos\phi = 3/5$, $\sin\phi = 4/5$ and $\phi = \arctan(4/3)$. Then $\sin(2x - \phi) = 2/5$. Because $\pi < x < 3\pi/2$, $2\pi - \phi < 2x - \phi < 3\pi - \phi$, approximately $(5.356, 8.497)$. With $\gamma = \arcsin(2/5)$, the only angle in this interval is $2\pi + \gamma$; $3\pi - \gamma \approx 9.013$ is outside it. Thus

$$x = \frac{2\pi + \arcsin(2/5) + \arctan(4/3)}{2} = 3.810998686\ldots.$$

At this value $\cos x + \sin x \neq 0$ and $\cos 2x \neq 0$; direct substitution satisfies the original equation.

$$\boxed{x = 3.81}$$

FINAL ANSWER

(a) $3\sin 2\theta - 4\cos 2\theta = 2$ (b) $x = 3.81$

WMA13/01 MAY/JUNE 2024

Question 7

8 marks | Rational Algebra

Main topic: Trig Identities & Exact Values

7. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Given that

$$\sqrt{2}\sin(x+45°) = \cos(x-60°)$$

show that

$$\tan x = -2-\sqrt{3}$$

(4)

(b) Hence or otherwise, solve, for $0 \leqslant \theta < 180°$

$$\sqrt{2}\sin(2\theta) = \cos(2\theta-105°)$$

(4)

WORKED SOLUTION

Question 7 · WMA13/01 May/June 2024

Part (a)

Expand both compound angles:

$$\sin x + \cos x = \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x.$$

Thus $(2-\sqrt{3})\sin x = -\cos x$. Here $\cos x \neq 0$: otherwise this equation would require $\sin x = 0$ as well, which is impossible. Dividing by $(2-\sqrt{3})\cos x$ gives

$$\tan x = -\frac{1}{2-\sqrt{3}} = -\frac{2+\sqrt{3}}{(2-\sqrt{3})(2+\sqrt{3})}.$$

$$\boxed{\tan x = -2-\sqrt{3}}$$

2 **Part (b)**

Set $x = 2\theta - 45^\circ$. Then $x + 45^\circ = 2\theta$ and $x - 60^\circ = 2\theta - 105^\circ$, so part (a) gives $\tan(2\theta - 45^\circ) = -2 - \sqrt{3}$. Since $\tan 75^\circ = 2 + \sqrt{3}$,

$$2\theta - 45^\circ = -75^\circ + 180^\circ k, \qquad \theta = -15^\circ + 90^\circ k.$$

In $0 \le \theta < 180^\circ$, only $k = 1, 2$ are allowed. Substitution in the original equation confirms both values.

$$\boxed{\theta = 75^\circ,\ 165^\circ}$$

FINAL ANSWER

(a) $\tan x = -2 - \sqrt{3}$ (b) $\theta = 75^\circ,\ 165^\circ$

WMA13/01 OCTOBER 2024

Question 6

8 marks | Rational Algebra

Main topic: Functions, Composition & Inverses

6. The functions f and g are defined by

$$\mathrm{f}(x) = 6 - \frac{21}{2x+3} \qquad x \geqslant 0$$

$$\mathrm{g}(x) = x^2 + 5 \qquad x \in \mathbb{R}$$

(a) Find $\mathrm{gf}(2)$

(2)

(b) Find f^{-1}

(3)

(c) Solve the equation

$$\mathrm{gg}(x) = 126$$

(3)

WORKED SOLUTION

Question 6 · WMA13/01 October 2024

1 **Part (a)**

First apply f: $f(2) = 6 - 21/(2\cdot 2+3) = 3$. Then apply g: $gf(2) = g(3) = 3^2 + 5$.

$$\boxed{gf(2) = 14}$$

2 **Part (b)**

Let $y = 6 - 21/(2x+3)$. Then

$$(6-y)(2x+3) = 21, \qquad x = \frac{1}{2}\left(\frac{21}{6-y} - 3\right) = \frac{3(y+1)}{2(6-y)}.$$

Interchange x, y. On the original domain $x \geq 0$, f increases from $f(0) = -1$ towards 6, never attaining 6. Thus the inverse domain is $-1 \leq x < 6$.

$$\boxed{f^{-1}(x) = \frac{3(x+1)}{2(6-x)}, \quad -1 \leq x < 6}$$

3 Part (c)

$$gg(x) = (x^2+5)^2 + 5 = 126 \quad \Longrightarrow \quad (x^2+5)^2 = 121.$$

Since $x^2 + 5 > 0$, take $x^2 + 5 = 11$, not -11. Therefore $x^2 = 6$.

$$\boxed{x = \pm\sqrt{6}}$$

FINAL ANSWER

(a) $gf(2) = 14$ (b) $f^{-1}(x) = \frac{3(x+1)}{2(6-x)}, \quad -1 \le x < 6$ (c) $x = \pm\sqrt{6}$

Question 7

10 marks | Rational Algebra

Main topic: Graph Transformations

7.

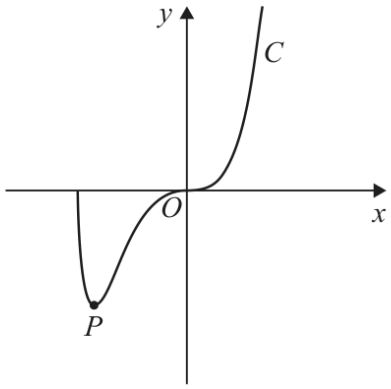


Figure 3

The curve C has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x^3\sqrt{4x+7} \qquad x \geqslant -\frac{7}{4}$$

(a) Show that

$$\mathrm{f}'(x) = \frac{kx^2(2x+3)}{\sqrt{4x+7}}$$

where k is a constant to be found.

(4)

The point P, shown in Figure 3, is the minimum turning point on C.

(b) Find the coordinates of P.

(2)

(c) Hence find the range of the function g defined by

$$\mathrm{g}(x) = -4\,\mathrm{f}(x) \qquad x \geqslant -\frac{7}{4}$$

(2)

The point Q with coordinates $\left(\frac{1}{2}, \frac{3}{8}\right)$ lies on C.

(d) Find the coordinates of the point to which Q is mapped when C is transformed to the curve with equation

$$y = 40\,\mathrm{f}\left(x - \frac{3}{2}\right) - 8$$

(2)

1 Part (a)

For $x > -7/4$, apply the product and chain rules:

$$f'(x) = 3x^2\sqrt{4x+7} + x^3\frac{2}{\sqrt{4x+7}} = \frac{3x^2(4x+7)+2x^3}{\sqrt{4x+7}} = \frac{7x^2(2x+3)}{\sqrt{4x+7}}.$$

$$\boxed{k = 7}$$

2 Part (b)

The interior stationary values satisfy $x^2(2x+3)=0$, so $x=0$ or $x=-3/2$. Near $x=-3/2$, the derivative changes from negative to positive; near $x=0$ it stays positive on both sides. Thus the minimum turning point is at $x=-3/2$.

$$f(-3/2) = (-3/2)^3\sqrt{-6+7} = -27/8.$$

$$\boxed{P = \left(-\frac{3}{2}, -\frac{27}{8}\right)}$$

3 Part (c)

The endpoint has $f(-7/4)=0$, the global minimum is $-27/8$ and $f(x)$ tends to infinity as x increases. Hence the range of f is $[-27/8, \infty)$. Multiplication by -4 reverses the inequality and makes the upper endpoint $27/2$.

$$\boxed{g(x) \le \frac{27}{2}}$$

4 Part (d)

The transformation is a shift right by $3/2$, vertical stretch by 40, then shift down by 8:

$$(x, y) \mapsto (x + 3/2,\, 40y - 8).$$

Therefore $Q = (1/2, 3/8)$ maps to $(1/2 + 3/2,\ 40\cdot 3/8 - 8)$.

$$\boxed{Q \mapsto (2, 7)}$$

FINAL ANSWER

(a) $k = 7$ (b) $P = \left(-\frac{3}{2}, -\frac{27}{8}\right)$ (c) $g(x) \le \frac{27}{2}$ (d) $Q \mapsto (2, 7)$

Question 9

14 marks | Rational Algebra | Main topic: Integration

9.

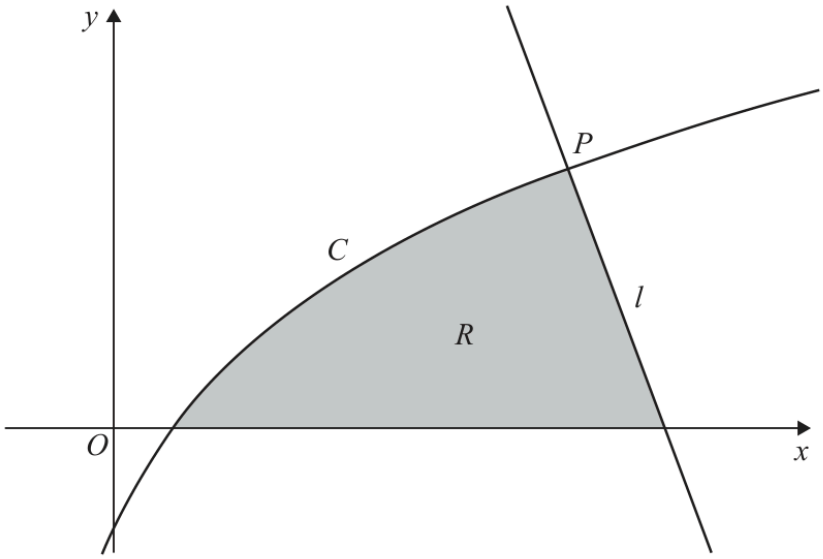


Figure 5

Figure 5 shows a sketch of part of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{6x^2 + 4x - 2}{2x + 1} \qquad x > -\frac{1}{2}$$

(a) Find $\mathrm{f}'(x)$, giving the answer in simplest form.

(3)

The line l is the normal to C at the point $P(2, 6)$

(b) Show that an equation for l is

$$16y + 5x = 106$$

(3)

(c) Write $\mathrm{f}(x)$ in the form $Ax + B + \frac{D}{2x + 1}$ where A, B and D are constants.

(3)

The region R, shown shaded in Figure 5, is bounded by C, l and the x-axis.

(d) Use algebraic integration to find the exact area of R, giving your answer in the form $P + Q\ln 3$, where P and Q are rational constants.

(*Solutions based entirely on calculator technology are not acceptable.*)

(5)

1 Part (a)

With $u = 6x^2 + 4x - 2$ and $v = 2x + 1$, the quotient rule gives

$$f'(x) = \frac{(12x + 4)(2x + 1) - 2(6x^2 + 4x - 2)}{(2x + 1)^2} = \frac{12x^2 + 12x + 8}{(2x + 1)^2}.$$

$$\boxed{f'(x) = \frac{4(3x^2 + 3x + 2)}{(2x + 1)^2}}$$

2 Part (b)

At $x = 2$, $f'(2) = 80/25 = 16/5$, so the normal gradient is $-5/16$. Through $P = (2, 6)$,

$$y - 6 = -\frac{5}{16}(x - 2) \quad \Longrightarrow \quad 16y - 96 = -5x + 10.$$

$$\boxed{16y + 5x = 106}$$

3 Part (c)

Multiplying the proposed identity by $2x + 1$ gives

$$6x^2 + 4x - 2 = (Ax + B)(2x + 1) + D = 2Ax^2 + (A + 2B)x + B + D.$$

Compare coefficients: $2A = 6$, $A + 2B = 4$, $B + D = -2$. Thus $f(x) = 3x + \frac{1}{2} - \frac{5}{2(2x+1)}$.

$$\boxed{A = 3, \quad B = \frac{1}{2}, \quad D = -\frac{5}{2}}$$

4 Part (d)

The curve meets the x-axis when $6x^2 + 4x - 2 = 2(3x - 1)(x + 1) = 0$. The root $x = -1$ is outside $x > -1/2$, leaving $x = 1/3$. The normal meets the axis at $x = 106/5$. Split the shaded region at P: a curve area from $1/3$ to 2, then a triangle.

$$\int_{1/3}^{2} f(x)\,dx = \left[\frac{3}{2}x^2 + \frac{1}{2}x - \frac{5}{4}\ln(2x + 1)\right]_{1/3}^{2} = 7 - \frac{1}{3} - \frac{5}{4}\ln\left(\frac{5}{5/3}\right) = \frac{20}{3} - \frac{5}{4}\ln 3.$$

The triangle has area $\frac{1}{2}(106/5 - 2) \cdot 6 = 288/5$. Adding gives the exact area of R.

$$\boxed{\text{Area}(R) = \frac{964}{15} - \frac{5}{4}\ln 3}$$

FINAL ANSWER

(a) $f'(x) = \frac{4(3x^2+3x+2)}{(2x+1)^2}$ (b) $16y + 5x = 106$ (c) $A = 3, \quad B = \frac{1}{2}, \quad D = -\frac{5}{2}$ (d) $\text{Area}(R) = \frac{964}{15} - \frac{5}{4}\ln 3$

WMA13/01 JANUARY 2025

Question 6

8 marks | Rational Algebra

Main topic: Functions, Composition & Inverses

6. The function f is defined by

$$\mathrm{f}(x) = \frac{4x + 3}{x - 2} \qquad x \neq 2$$

(a) Find f^{-1}

(3)

(b) Show that

$$\mathrm{ff}(x) = \frac{ax + b}{cx + d}$$

where a, b, c and d are integers to be found.

(3)

The point $P(3, 15)$ lies on the curve with equation $y = \mathrm{f}(x)$.

(c) Find the point to which P is mapped when $y = \mathrm{f}(x)$ is transformed to the curve with equation $y = 2\mathrm{f}(3x) + 8$

(2)

WORKED SOLUTION

Question 6 · WMA13/01 January 2025

1 **Part (a)**

Let $y = (4x + 3)/(x - 2)$. Then $yx - 2y = 4x + 3$, so

$$x(y - 4) = 2y + 3 \quad \Longrightarrow \quad x = \frac{2y + 3}{y - 4}.$$

Swap the input and output names. Since $f(x) = 4 + 11/(x - 2)$ can never equal 4, the inverse domain excludes 4.

$$\boxed{f^{-1}(x) = \frac{2x + 3}{x - 4}, \quad x \neq 4}$$

2 Part (b)

Substitute $f(x)$ into both occurrences of the argument:

$$ff(x) = \frac{4\left(\frac{4x+3}{x-2}\right)+3}{\left(\frac{4x+3}{x-2}\right)-2} = \frac{4(4x+3)+3(x-2)}{4x+3-2(x-2)} = \frac{19x+6}{2x+7}.$$

Thus $a=19$, $b=6$, $c=2$, $d=7$. The composite retains $x \neq 2$ from the inner function and also requires $f(x) \neq 2$, that is $x \neq -7/2$.

$$\boxed{ff(x) = \frac{19x+6}{2x+7}, \quad (a,b,c,d) = (19,6,2,7)}$$

---

3 Part (c)

For $y = 2f(3x)+8$, the point mapping is $(x,y) \mapsto (x/3, 2y+8)$. Hence $(3,15) \mapsto (3/3, 2(15)+8) = (1,38)$.

$$\boxed{(1,38)}$$

FINAL ANSWER

(a) $f^{-1}(x) = \frac{2x+3}{x-4}, \quad x \neq 4$ (b) $ff(x) = \frac{19x+6}{2x+7}, \quad (a,b,c,d) = (19,6,2,7)$ (c) $(1,38)$

WMA13/01 OCTOBER 2025

Question 6

7 marks | Rational Algebra | Main topic: Integration

6. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(i) Given $y = \ln(2x^2 + 5)$ find $\dfrac{dy}{dx}$

(2)

(ii) A curve C has equation $y = f(x)$ where

$$f(x) = \frac{21x}{3x^2 + k} \qquad x \in \mathbb{R}$$

where k is a positive integer, $k > 1$

Given that

$$\int_1^k f(x)\,dx < 7\ln 8$$

find the greatest possible value of k

(5)

WORKED SOLUTION

Question 6 · WMA13/01 October 2025

1 **Part (i)**

Using the chain rule,

$$\frac{dy}{dx} = \frac{1}{2x^2 + 5}(4x) = \frac{4x}{2x^2 + 5}.$$

2 Part (ii)

Treat k as a constant when integrating:

$$\int_1^k \frac{21x}{3x^2+k}\,dx = \left[\frac{7}{2}\ln(3x^2+k)\right]_1^k = \frac{7}{2}\ln\left(\frac{3k^2+k}{k+3}\right).$$

The stated inequality therefore implies

$$\ln\left(\frac{3k^2+k}{k+3}\right) < 2\ln 8 = \ln 64.$$

Since the exponential is increasing and $k+3>0$,

$$3k^2+k < 64(k+3) \Longrightarrow k^2-21k-64<0.$$

The roots are $(21\pm\sqrt{697})/2$, so the positive upper bound is $k<(21+\sqrt{697})/2 = 23.700378782\ldots$. With $k>1$ an integer, the greatest possible value is $k=23$.

FINAL ANSWER

(i) $4*x/(2*x^2+5)$ (ii) 23

WMA13/01 OCTOBER 2025

Question 7

7 marks | Rational Algebra | Main topic: Applications of Differentiation

7. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

A curve C has equation

$$y = 2x\mathrm{e}^{x^2+(3k-2)x}$$

where k is a constant.

Given that C has two distinct turning points, find the range of possible values of k.

(7)

WORKED SOLUTION

Question 7 · WMA13/01 October 2025

1 Solution

\paragraph{Solution} Put $u = x^2 + (3k-2)x$, so $u' = 2x + 3k - 2$. By the product and chain rules,

$$\frac{dy}{dx} = 2e^u + 2xe^u(2x + 3k - 2) = 2e^u\big(2x^2 + (3k-2)x + 1\big).$$

As $2e^u > 0$, stationary points satisfy

$$2x^2 + (3k-2)x + 1 = 0.$$

For two distinct real roots, the discriminant must be positive:

$$(3k-2)^2 - 8 > 0 \iff |3k-2| > 2\sqrt{2}.$$

Consequently

$$k < \frac{2-2\sqrt{2}}{3} \quad \text{or} \quad k > \frac{2+2\sqrt{2}}{3}.$$

For these values the quadratic changes sign at both simple roots, so both stationary points are genuine turning points. Equality gives a repeated root with no derivative sign change and is excluded.

FINAL ANSWER

(main) $(-infinity, (2 - 2 * sqrt(2))/3) union ((2 + 2 * sqrt(2))/3, infinity)$

Question 8

13 marks | Rational Algebra | Main topic: Modulus

8.

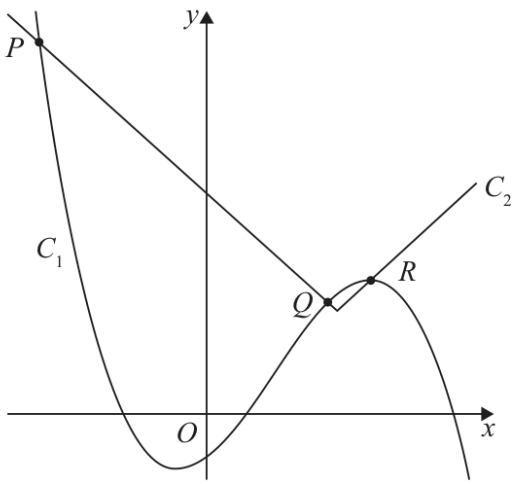


Figure 2

Figure 2 shows a sketch of the graph C_1 with equation

$$y = -2x^3 + 5x^2 + 4x - 3$$

and a sketch of the graph C_2 with equation

$$y = a + |5x + b|$$

where a and b are constants.

Graphs C_1 and C_2 intersect at point P, point Q and point R, as shown in Figure 2.

Given that P has coordinates $(-2, 25)$

(a) show that

$$a = 15 + b$$

(2)

Given also that R has coordinates $(2, 9)$

(b) find the value of a and the value of b

(3)

Using the answer to part (b),

(c) state the coordinates of the vertex of C_2

(2)

(d) Find, using algebra, the coordinates of Q. Show each stage of your working.

(*Solutions relying on calculator technology are not acceptable.*)

(6)

1 Part (a)

Point P lies on the descending branch of C_2, where $5x + b < 0$. At $P = (-2, 25)$,

$$25 = a - (-10 + b) = a + 10 - b,$$

hence $a = 15 + b$.

2 Part (b)

Point $R = (2, 9)$ lies on the ascending branch, so

$$9 = a + (10 + b) \Longrightarrow a + b = -1.$$

Combining with $a - b = 15$ gives $2a = 14$, hence $a = 7$ and $b = -8$.

3 Part (c)

The vertex occurs at $5x - 8 = 0$, giving $(x, y) = (8/5, 7)$.

4 Part (d)

The diagram places Q on the descending branch $y = 7 - (5x - 8) = 15 - 5x$, with $x < 8/5$. Equate this with the cubic:

$$-2x^3 + 5x^2 + 4x - 3 = 15 - 5x \Longrightarrow 2x^3 - 5x^2 - 9x + 18 = 0.$$

Since $x = -2$ is the known intersection P, divide by $x + 2$:

$$2x^3 - 5x^2 - 9x + 18 = (x + 2)(2x^2 - 9x + 9) = (x + 2)(2x - 3)(x - 3).$$

The candidates are $-2, 3/2, 3$. Reject $x = 3$ because it is outside the descending branch, and $x = -2$ is P, not Q. Thus $x_Q = 3/2$ and $y_Q = 15 - 5(3/2) = 15/2$, giving $Q = (3/2, 15/2)$.

FINAL ANSWER

(a) Shown, as required. (b) 7; -8 (c) $(8/5, 7)$ (d) $(3/2, 15/2)$

WMA13/01 JANUARY 2026

Question 6

7 marks | Rational Algebra

Main topic: Trig Identities & Exact Values

6. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(a) Show that the equation

$$2\cos(\theta - 60°) = 3\sin\theta$$

can be written in the form

$$\tan\theta = \frac{1}{6}\left(3 + \sqrt{3}\right)$$

(4)

(b) Hence solve, for $0 \leqslant x < 180°$, the equation

$$2\cos(2x - 70°) = 3\sin(2x - 10°)$$

giving your solutions to one decimal place.

(3)

WORKED SOLUTION

Question 6 · WMA13/01 January 2026

1 Part (a)

Expanding the compound angle,

$$2\left(\cos\theta\cos 60° + \sin\theta\sin 60°\right) = 3\sin\theta,$$

so $\cos\theta + \sqrt{3}\sin\theta = 3\sin\theta$ and $\cos\theta = (3 - \sqrt{3})\sin\theta$. If $\cos\theta = 0$, this equation would require $\sin\theta = 0$, which is impossible. We may therefore divide by $\cos\theta$:

$$\tan\theta = \frac{1}{3 - \sqrt{3}} = \frac{3 + \sqrt{3}}{9 - 3} = \frac{3 + \sqrt{3}}{6}.$$

2 Part (b)

Set $\theta = 2x - 10°$, so $2x - 70° = \theta - 60°$ and $-10° \leq \theta < 350°$.

$$\theta = \arctan\left(\frac{3 + \sqrt{3}}{6}\right) + 180°n = 38.2619662\ldots° + 180°n.$$

The admissible values are $38.2619662\ldots°$ and $218.2619662\ldots°$. Hence $x = (\theta + 10°)/2$, giving $x = 24.1°$ or $114.1°$ (1 decimal place).

FINAL ANSWER

(a) $\frac{3+\sqrt{3}}{6}$ (b) $x = 24.1° \text{ or } 114.1°$

Question 7

12 marks | Rational Algebra

Main topic: Log Graphs & Exponential Models

7. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The selling price of a car is monitored over a 15-year period.

The selling price P_C of the car, in thousands of pounds, t years after monitoring began, is modelled by the equation

$$P_C = 7 + 16e^{-0.1t} \qquad 0 \leqslant t \leqslant 15$$

Given that when $t = T$ the selling price of the car is £15 000

(a) find, according to the model, the value of T. Give the answer to 2 decimal places.

(3)

(b) Find the value of $\frac{dP_C}{dt}$ when $t = 5$. Give the answer to 2 decimal places.

(2)

The selling price of a van is monitored over the same 15-year period.

The selling price P_V of the van, in thousands of pounds, t years after monitoring began, is modelled by the equation

$$P_V = 4 + me^{-0.2t} \qquad 0 \leqslant t \leqslant 15$$

where m is a constant.

Use the equations of these models to answer parts (c) and (d).

Given that $P_V = 36$ when $t = 0$

(c) (i) state the value of m

(ii) find the difference between the initial selling price of the car and the initial selling price of the van.

(3)

When $t = \alpha$, the selling price of the car is equal to the selling price of the van.

(d) Find, using algebra and showing your working, the value of α, giving the answer to 3 significant figures.

(4)

1 Part (a)

Prices in the model are in thousands of pounds, so

$$15 = 7 + 16e^{-0.1T} \quad \Longrightarrow \quad e^{-0.1T} = \frac{1}{2}.$$

Taking logarithms gives $T = 10\ln 2 = 6.9314718\ldots$, hence $\boxed{T = 6.93 \text{ years}}$.

2 Part (b)

$$\frac{dP_C}{dt} = -1.6e^{-0.1t}, \qquad \left.\frac{dP_C}{dt}\right|_{t=5} = -1.6e^{-0.5} = -0.970449\ldots.$$

Thus the answer is $\boxed{-0.97 \text{ thousand pounds per year}}$ to 2 decimal places.

3 Part (c)(i)

At $t = 0$, $36 = 4 + m$, so $\boxed{m = 32}$.

4 Part (c)(ii)

The initial car price is $7 + 16 = 23$ thousand pounds. The difference is $36 - 23 = 13$ thousand pounds, or $\boxed{\pounds 13\,000}$.

5 Part (d)

Equating the models with $m = 32$ gives

$$7 + 16e^{-0.1\alpha} = 4 + 32e^{-0.2\alpha}.$$

Let $u = e^{-0.1\alpha} > 0$. Then $32u^2 - 16u - 3 = 0$, so

$$u = \frac{16 \pm \sqrt{256 + 384}}{64} = \frac{2 \pm \sqrt{10}}{8}.$$

Reject the negative root because an exponential is positive. Hence

$$\alpha = -10\ln\left(\frac{2 + \sqrt{10}}{8}\right) = 4.3806365\ldots, \qquad \boxed{\alpha = 4.38 \text{ years}}.$$

This value is inside the model interval $0 \le t \le 15$.

FINAL ANSWER

(a) $T = 6.93$ years (b) -0.97 thousand pounds per year (c)(i) $m = 32$ (c)(ii) $\pounds 13\,000$ (d) $\alpha = 4.38$ years

WMA13/01 MAY/JUNE 2026

Question 8

9 marks | Rational Algebra

Main topic: Trig Identities & Exact Values

8. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(a) Show that

$$\frac{\sec\theta\,\operatorname{cosec}\theta}{4\operatorname{cosec}4\theta} \equiv \cos 2\theta$$

(2)

(b) Hence show that the equation

$$4(5\cos^2\theta - 9)\operatorname{cosec}4\theta = 3(1+\tan^2 2\theta)\sec\theta\operatorname{cosec}\theta$$

can be written as

$$a\cos^2 2\theta + b\cos 2\theta + c = 0$$

where a, b and c are integers to be found.

(3)

(c) Hence, solve for $0 < x < 180°$, $x \neq (45n)°$, $n \in \mathbb{N}$, the equation

$$4(5\cos^2 x - 9)\operatorname{cosec}4x = 3(1+\tan^2 2x)\sec x\operatorname{cosec}x$$

giving the answers to one decimal place.

(4)

WORKED SOLUTION

Question 8 · WMA13/01 May/June 2026

1 **Part (a)**

For angles where the original expression is defined,

$$\frac{\sec\theta\ \operatorname{cosec}\theta}{4\operatorname{cosec}4\theta} = \frac{\sin 4\theta}{4\sin\theta\cos\theta} = \frac{2\sin 2\theta\cos 2\theta}{2\sin 2\theta} = \boxed{\cos 2\theta}.$$

2 Part (b)

Divide the given equation by $4\operatorname{cosec} 4\theta$ and use (a):

$$5\cos^2\theta - 9 = 3(1+\tan^2 2\theta)\cos 2\theta = \frac{3}{\cos 2\theta}.$$

Use $\cos^2\theta = (1+\cos 2\theta)/2$ and multiply by $2\cos 2\theta$:

$$(5\cos 2\theta - 13)\cos 2\theta = 6.$$

Therefore

$$\boxed{5\cos^2 2\theta - 13\cos 2\theta - 6 = 0}, \qquad a = 5,\ b = -13,\ c = -6.$$

The divisions are valid on the domain of the original equation.

---

3 Part (c)

Set $u = \cos 2x$:

$$5u^2 - 13u - 6 = (5u+2)(u-3) = 0.$$

Reject $u = 3$ because cosine lies between -1 and 1. Thus $\cos 2x = -2/5$. Since $0 < 2x < 360°$,

$$2x = 113.5781785\ldots° \quad \text{or} \quad 246.4218215\ldots°.$$

Hence $\boxed{x = 56.8° \text{ or } 123.2°}$ to one decimal place. Neither is an excluded multiple of $45°$; these are all the solutions in the interval.

FINAL ANSWER

(a) $\cos 2\theta$ (b) $5\cos^2 2\theta - 13\cos 2\theta - 6 = 0$ (c) $x = 56.8° \text{ or } 123.2°$

TOPIC

Functions, Composition & Inverses

Every past-paper question on this topic, January 2020 to June 2026.

WMA13/01 JANUARY 2022

Question 6

11 marks | Functions, Composition & Inverses

6. The function f is defined by

$$\mathrm{f}(x) = \frac{5x - 3}{x - 4} \qquad x > 4$$

(a) Show, by using calculus, that f is a decreasing function.

(3)

(b) Find f^{-1}

(3)

(c) (i) Show that $\mathrm{ff}(x) = \dfrac{ax + b}{x + c}$ where a, b and c are constants to be found.

(ii) Deduce the range of ff.

(5)

WORKED SOLUTION

Question 6 · WMA13/01 January 2022

1 Part (a)

By the quotient rule,

$$f'(x) = \frac{5(x-4) - (5x-3)}{(x-4)^2} = \frac{-17}{(x-4)^2}.$$

For every $x > 4$, the denominator is positive and the numerator is negative. Therefore $f'(x) < 0$ throughout its domain, so f is decreasing.

$$\boxed{f \text{ is decreasing for } x > 4}$$

2 Part (b)

Set $y = (5x - 3)/(x - 4)$. Then $yx - 4y = 5x - 3$, giving $x(y - 5) = 4y - 3$ and $x = (4y - 3)/(y - 5)$. Exchange the variable names. Since $f(x) = 5 + 17/(x - 4)$ on $x > 4$, its range is $(5, \infty)$, which becomes the inverse's domain.

$$\boxed{f^{-1}(x) = \frac{4x - 3}{x - 5}, \qquad x > 5}$$

3 **Part (c)**

(i) For $x > 4$, $f(x) > 5 > 4$, so the second application is defined.

$$ff(x) = \frac{5\left(\frac{5x-3}{x-4}\right) - 3}{\left(\frac{5x-3}{x-4}\right) - 4} = \frac{25x - 15 - 3x + 12}{5x - 3 - 4x + 16} = \frac{22x - 3}{x + 13}.$$

Thus $a = 22$, $b = -3$, $c = 13$. (ii) Rewrite $ff(x) = 22 - 289/(x + 13)$ with the original domain $x > 4$. It is strictly increasing, tends to 5 as $x \to 4^+$ and to 22 as $x \to \infty$. Neither limiting value is attained.

$$\boxed{ff(x) = \frac{22x - 3}{x + 13}, \qquad 5 < ff(x) < 22}$$

FINAL ANSWER

(a) f **is decreasing for** $x > 4$ (b) $f^{-1}(x) = \frac{4x-3}{x-5}, \qquad x > 5$ (c) $ff(x) = \frac{22x-3}{x+13}, \qquad 5 < ff(x) < 22$

Question 7

10 marks | Functions, Composition & Inverses

7.

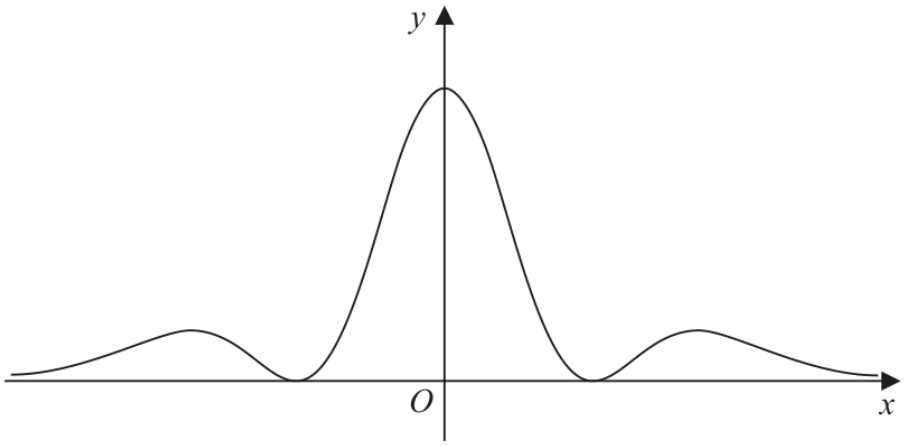


Figure 1

Figure 1 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \mathrm{e}^{-x^2}(2x^2 - 3)^2$$

(a) Find the range of f

(2)

(b) Show that

$$\mathrm{f}'(x) = 2x(2x^2 - 3)\mathrm{e}^{-x^2}(A - Bx^2)$$

where A and B are constants to be found.

(4)

Given that the line $y = k$, where k is a constant, $k > 0$, intersects the curve at exactly two distinct points,

(c) find the exact range of values of k

(4)

1 Part (a)

The exponential is positive and the squared factor is nonnegative, so $f(x) \geq 0$. The value zero is attained at $x = \pm\sqrt{3/2}$. The graph has its highest point at $x = 0$, where $f(0) = 9$; the outer maxima are lower. Thus both endpoints of the range are included.

$$\boxed{0 \leq f(x) \leq 9}$$

2 Part (b)

For $f(x) = e^{-x^2}(2x^2 - 3)^2$, product and chain rules give

$$f'(x) = -2xe^{-x^2}(2x^2 - 3)^2 + e^{-x^2}\,2(2x^2 - 3)\,4x.$$

Factoring yields

$$f'(x) = 2x(2x^2 - 3)e^{-x^2}\big(-(2x^2 - 3) + 4\big) = 2x(2x^2 - 3)e^{-x^2}(7 - 2x^2).$$

$$\boxed{A = 7, \qquad B = 2}$$

3 Part (c)

The stationary abscissae are 0, $\pm\sqrt{3/2}$ and $\pm\sqrt{7/2}$. At the outer local maxima,

$$f\left(\pm\sqrt{7/2}\right) = e^{-7/2}(7 - 3)^2 = 16e^{-7/2}.$$

The central maximum is 9 and the two minima are zero. Also $f(x) \to 0$ as $x \to \pm\infty$. A positive horizontal line above both outer maxima but below 9 meets only the two sides of the central peak. At $k = 16e^{-7/2}$ it also touches both outer peaks, giving four distinct intersections; at $k = 9$ there is only one. Both bounds must be excluded.

$$\boxed{16e^{-7/2} < k < 9}$$

FINAL ANSWER

(a) $0 \leq f(x) \leq 9$ (b) $A = 7, \qquad B = 2$ (c) $16e^{-7/2} < k < 9$

WMA13/01 OCTOBER 2024

Question 6

8 marks | Functions, Composition & Inverses

6. The functions f and g are defined by

$$\mathrm{f}(x) = 6 - \frac{21}{2x+3} \qquad x \geqslant 0$$

$$\mathrm{g}(x) = x^2 + 5 \qquad x \in \mathbb{R}$$

(a) Find $\mathrm{gf}(2)$

(2)

(b) Find f^{-1}

(3)

(c) Solve the equation

$$\mathrm{gg}(x) = 126$$

(3)

WORKED SOLUTION

Question 6 · WMA13/01 October 2024

1 **Part (a)**

First apply f: $f(2) = 6 - 21/(2\cdot 2+3) = 3$. Then apply g: $gf(2) = g(3) = 3^2 + 5$.

$$\boxed{gf(2) = 14}$$

2 **Part (b)**

Let $y = 6 - 21/(2x+3)$. Then

$$(6-y)(2x+3) = 21, \qquad x = \frac{1}{2}\left(\frac{21}{6-y} - 3\right) = \frac{3(y+1)}{2(6-y)}.$$

Interchange x, y. On the original domain $x \geq 0$, f increases from $f(0) = -1$ towards 6, never attaining 6. Thus the inverse domain is $-1 \leq x < 6$.

$$\boxed{f^{-1}(x) = \frac{3(x+1)}{2(6-x)}, \quad -1 \leq x < 6}$$

3 Part (c)

$$gg(x) = (x^2+5)^2 + 5 = 126 \quad \Longrightarrow \quad (x^2+5)^2 = 121.$$

Since $x^2 + 5 > 0$, take $x^2 + 5 = 11$, not -11. Therefore $x^2 = 6$.

$$\boxed{x = \pm\sqrt{6}}$$

FINAL ANSWER

(a) $gf(2) = 14$ (b) $f^{-1}(x) = \frac{3(x+1)}{2(6-x)}, \quad -1 \le x < 6$ (c) $x = \pm\sqrt{6}$

WMA13/01 JANUARY 2025

Question 6

8 marks | Functions, Composition & Inverses

6. The function f is defined by

$$\mathrm{f}(x) = \frac{4x + 3}{x - 2} \qquad x \neq 2$$

(a) Find f^{-1}

(3)

(b) Show that

$$\mathrm{ff}(x) = \frac{ax + b}{cx + d}$$

where a, b, c and d are integers to be found.

(3)

The point $P(3, 15)$ lies on the curve with equation $y = \mathrm{f}(x)$.

(c) Find the point to which P is mapped when $y = \mathrm{f}(x)$ is transformed to the curve with equation $y = 2\mathrm{f}(3x) + 8$

(2)

WORKED SOLUTION

Question 6 · WMA13/01 January 2025

1 **Part (a)**

Let $y = (4x + 3)/(x - 2)$. Then $yx - 2y = 4x + 3$, so

$$x(y - 4) = 2y + 3 \quad \Longrightarrow \quad x = \frac{2y + 3}{y - 4}.$$

Swap the input and output names. Since $f(x) = 4 + 11/(x - 2)$ can never equal 4, the inverse domain excludes 4.

$$\boxed{f^{-1}(x) = \frac{2x + 3}{x - 4}, \quad x \neq 4}$$

2 Part (b)

Substitute $f(x)$ into both occurrences of the argument:

$$ff(x) = \frac{4\left(\frac{4x+3}{x-2}\right)+3}{\left(\frac{4x+3}{x-2}\right)-2} = \frac{4(4x+3)+3(x-2)}{4x+3-2(x-2)} = \frac{19x+6}{2x+7}.$$

Thus $a=19$, $b=6$, $c=2$, $d=7$. The composite retains $x \neq 2$ from the inner function and also requires $f(x) \neq 2$, that is $x \neq -7/2$.

$$\boxed{ff(x) = \frac{19x+6}{2x+7}, \quad (a,b,c,d) = (19,6,2,7)}$$

---

3 Part (c)

For $y = 2f(3x)+8$, the point mapping is $(x,y) \mapsto (x/3, 2y+8)$. Hence $(3,15) \mapsto (3/3, 2(15)+8) = (1,38)$.

$$\boxed{(1,38)}$$

FINAL ANSWER

(a) $f^{-1}(x) = \frac{2x+3}{x-4}, \quad x \neq 4$ (b) $ff(x) = \frac{19x+6}{2x+7}, \quad (a,b,c,d) = (19,6,2,7)$ (c) $(1,38)$

Question 9

7 marks | Functions, Composition & Inverses | Main topic: Trig Equations & R-Form

9. $$f(\theta) = 5\cos\theta - 4\sin\theta \qquad \theta \in \mathbb{R}$$

(a) Express $f(\theta)$ in the form $R\cos(\theta + \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

The curve with equation $y = \cos\theta$ is transformed onto the curve with equation $y = f(\theta)$ by a sequence of two transformations.

Given that the first transformation is a stretch and the second a translation,

(b) (i) describe fully the transformation that is a stretch,

(ii) describe fully the transformation that is a translation.

(2)

Given

$$g(\theta) = \frac{90}{4 + (f(\theta))^2} \qquad \theta \in \mathbb{R}$$

(c) find the range of g.

(2)

1 Part (a)

Expanding the required form gives

$$R\cos(\theta+\alpha) = R\cos\alpha\cos\theta - R\sin\alpha\sin\theta.$$

Comparing coefficients with $5\cos\theta - 4\sin\theta$,

$$R\cos\alpha = 5, \qquad R\sin\alpha = 4.$$

Squaring and adding, using $R > 0$, gives

$$R^2 = 5^2 + 4^2 = 41 \quad \Longrightarrow \quad \boxed{R = \sqrt{41}}.$$

Also, since $0 < \alpha < \frac{\pi}{2}$,

$$\tan\alpha = \frac{4}{5}, \qquad \alpha = \arctan\left(\frac{4}{5}\right) = 0.6747409\ldots \quad \Longrightarrow \quad \boxed{\alpha = 0.675 \text{ radians (3 d.p.)}}.$$

Thus the exact form is

$$\boxed{f(\theta) = \sqrt{41}\cos\left(\theta + \arctan\frac{4}{5}\right)}.$$

2 Part (b)(i)

A stretch parallel to the y-axis with scale factor $\boxed{\sqrt{41}}$, taking $y = \cos\theta$ to $y = \sqrt{41}\cos\theta$.

3 Part (b)(ii)

A translation by the vector

$$\boxed{\begin{pmatrix} -\alpha \\ 0 \end{pmatrix}} = \begin{pmatrix} -\arctan(4/5) \\ 0 \end{pmatrix} \approx \begin{pmatrix} -0.675 \\ 0 \end{pmatrix}.$$

This moves the curve α units to the left, giving $y = \sqrt{41}\cos(\theta+\alpha)$.

4

Part (c)

Since θ can be any real number,

$$-\sqrt{41} \le f(\theta) \le \sqrt{41} \quad \Longrightarrow \quad 0 \le (f(\theta))^2 \le 41.$$

Hence $4 \le 4 + (f(\theta))^2 \le 45$. Taking reciprocals of these positive bounds and multiplying by 90 gives

$$\boxed{2 \le g(\theta) \le \frac{45}{2}}.$$

Both endpoints are attained: the minimum occurs when $f(\theta) = \pm\sqrt{41}$, and the maximum when $f(\theta) = 0$.

FINAL ANSWER

(a) $R = \sqrt{41}$, $\alpha = 0.675$ radians (3 d.p.), $f(\theta) = \sqrt{41}\cos\left(\theta + \arctan\frac{4}{5}\right)$ (b)(i) $\sqrt{41}$ (b)(ii) $\begin{pmatrix} -\alpha \\ 0 \end{pmatrix}$ (c) $2 \le g(\theta) \le \frac{45}{2}$

WMA13/01 OCTOBER 2020

Question 7

9 marks | Functions, Composition & Inverses | Main topic: Trig Equations & R-Form

7. (a) Express $\cos x + 4\sin x$ in the form $R\cos(x - \alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

A scientist is studying the behaviour of seabirds in a colony.

She models the height above sea level, H metres, of one of the birds in the colony by the equation

$$H = \frac{24}{3 + \cos\left(\frac{1}{2}t\right) + 4\sin\left(\frac{1}{2}t\right)} \qquad 0 \leqslant t \leqslant 6.5$$

where t seconds is the time after it leaves the nest.

Find, according to the model,

(b) the minimum height of the seabird above sea level, giving your answer to the nearest cm,

(2)

(c) the value of t, to 2 decimal places, when $H = 10$

(4)

WORKED SOLUTION

Question 7 · WMA13/01 October 2020

1 **Part (a)**

Expand $R\cos(x-\alpha) = R\cos\alpha\cos x + R\sin\alpha\sin x$. Matching coefficients gives $R\cos\alpha = 1$, $R\sin\alpha = 4$. Thus $R = \sqrt{1^2 + 4^2} = \sqrt{17}$ and $\tan\alpha = 4$. Since $0 < \alpha < \pi/2$, $\alpha = \arctan 4 = 1.32581766\ldots$ radians.

$$\boxed{R = \sqrt{17}, \qquad \alpha = 1.326 \text{ radians}}$$

2 Part (b)

The model becomes $H = 24/[3 + \sqrt{17}\cos(t/2 - \alpha)]$. For $0 \le t \le 6.5$, the phase lies between $-\alpha$ and $3.25 - \alpha$; throughout this interval the denominator remains positive. Its maximum $3 + \sqrt{17}$ is attained at $t = 2\alpha = 2.651635\ldots$, inside the time interval. The height is then smallest:

$$H_{\min} = \frac{24}{3 + \sqrt{17}} = 3.369316876\ldots \text{ m}.$$

To the nearest centimetre this is 3.37 m (337 cm).

$$\boxed{H_{\min} = 3.37 \text{ m}}$$

---

3 Part (c)

For $H = 10$, $3 + \sqrt{17}\cos(t/2 - \alpha) = 24/10$, so

$$\cos(t/2 - \alpha) = -\frac{3}{5\sqrt{17}}.$$

Let $\gamma = \arccos[-3/(5\sqrt{17})]$. In the permitted phase interval $[-\alpha, 3.25 - \alpha]$, only $+\gamma$ occurs: $-\gamma < -\alpha$, and shifts by 2π are outside it. Using the unrounded $\alpha = \arctan 4$,

$$t = 2(\alpha + \gamma) = 6.085307854\ldots \text{ s}.$$

$$\boxed{t = 6.09 \text{ s}}$$

FINAL ANSWER

(a) $R = \sqrt{17}, \qquad \alpha = 1.326$ radians (b) $H_{\min} = 3.37$ m (c) $t = 6.09$ s

WMA13/01 MAY/JUNE 2021

Question 9

8 marks | Functions, Composition & Inverses | Main topic: Trig Equations & R-Form

9. (a) Express $12\sin x - 5\cos x$ in the form $R\sin(x - \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the exact value of R and give the value of α in radians, to 3 decimal places.

(3)

The function g is defined by

$$g(\theta) = 10 + 12\sin\left(2\theta - \frac{\pi}{6}\right) - 5\cos\left(2\theta - \frac{\pi}{6}\right) \qquad \theta > 0$$

Find

(b) (i) the minimum value of $g(\theta)$

(ii) the smallest value of θ at which the minimum value occurs.

(3)

The function h is defined by

$$h(\beta) = 10 - (12\sin\beta - 5\cos\beta)^2$$

(c) Find the range of h.

(2)

WORKED SOLUTION

Question 9 · WMA13/01 May/June 2021

1 **Part (a)**

Expand $R\sin(x-\alpha) = R\cos\alpha\sin x - R\sin\alpha\cos x$. Therefore $R\cos\alpha = 12$, $R\sin\alpha = 5$. Thus $R = \sqrt{12^2 + 5^2} = 13$ and $\alpha = \arctan(5/12) = 0.3947911197\ldots$ radians, in the required first quadrant.

$$\boxed{13\sin(x - 0.395) \quad (R = 13,\ \alpha = 0.395 \text{ rad})}$$

2 Part (b)

Write $g(\theta) = 10 + 13\sin(2\theta - \pi/6 - \alpha)$. (i) The minimum is $10 - 13 = -3$. (ii) For sine to equal -1, $2\theta - \pi/6 - \alpha = -\pi/2 + 2\pi n$. Hence $\theta = -\pi/6 + \alpha/2 + \pi n$. For $n = 0$ this is negative, so the smallest permitted value uses $n = 1$:

$$\theta = \frac{5\pi}{6} + \frac{1}{2}\arctan(5/12) = 2.815389438\ldots \text{ radians}.$$

Use the unrounded phase angle for the calculation.

$$\boxed{\min g = -3; \qquad \theta_{\min} = \frac{5\pi}{6} + \frac{1}{2}\arctan(5/12)}$$

3 Part (c)

As β varies over the reals, $12\sin\beta - 5\cos\beta$ covers $[-13, 13]$. Its square therefore covers $[0, 169]$, not $[169, 169]$. Subtracting from 10 reverses the bounds, giving $[-159, 10]$; both endpoints are attained.

$$\boxed{-159 \le h(\beta) \le 10}$$

FINAL ANSWER

(a) $13\sin(x - 0.395) \quad (R = 13,\ \alpha = 0.395\ \text{rad})$ (b) $\min g = -3; \qquad \theta_{\min} = \frac{5\pi}{6} + \frac{1}{2}\arctan(5/12)$ (c) $-159 \le h(\beta) \le 10$

Question 8

7 marks | Functions, Composition & Inverses | Main topic: Differentiation Techniques

8. A curve C has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \arcsin\left(\frac{1}{2}x\right) \qquad -2 \leqslant x \leqslant 2 \qquad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

(a) Sketch C.

(1)

(b) Given $x = 2\sin y$, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{A - x^2}}$$

where A is a constant to be found.

(3)

The point P lies on C and has y coordinate $\frac{\pi}{4}$

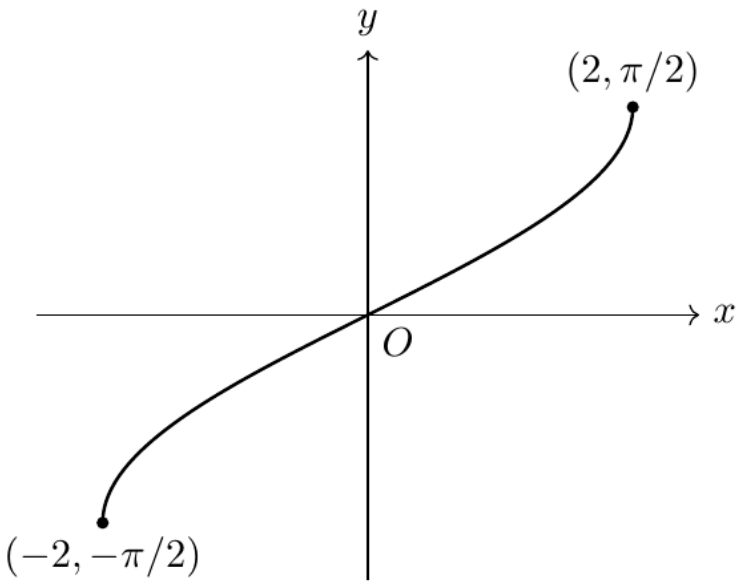
(c) Find the equation of the tangent to C at P. Write your answer in the form $y = mx + c$, where m and c are constants to be found.

(3)

1 Part (a)

The principal arcsine curve increases from $(-2, -\pi/2)$ through $(0, 0)$ to $(2, \pi/2)$. It is concave down on the negative-x half, concave up on the positive-x half, and vertical at both ends.

$$\boxed{y = \arcsin(x/2), \quad -2 \le x \le 2}$$



2 Part (b)

Differentiate the supplied inverse relation: $dx/dy = 2\cos y$. For $-\pi/2 < y < \pi/2$, cosine is positive, so $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2/4}$. Taking the reciprocal gives

$$\frac{dy}{dx} = \frac{1}{2\cos y} = \frac{1}{\sqrt{4 - x^2}}, \qquad -2 < x < 2.$$

Thus $A = 4$; the endpoints have vertical tangents, not finite derivatives.

$$\boxed{A = 4, \qquad \frac{dy}{dx} = \frac{1}{\sqrt{4 - x^2}}}$$

3 Part (c)

At $y = \pi/4$, $x = 2\sin(\pi/4) = \sqrt{2}$. The gradient is $1/\sqrt{4 - 2} = 1/\sqrt{2}$. Hence $y - \pi/4 = (x - \sqrt{2})/\sqrt{2}$, so $y = x/\sqrt{2} + \pi/4 - 1$.

$$\boxed{y = \frac{x}{\sqrt{2}} + \frac{\pi}{4} - 1}$$

FINAL ANSWER

(a) $y = \arcsin(x/2), \quad -2 \le x \le 2$ (b) $A = 4, \quad \frac{dy}{dx} = \frac{1}{\sqrt{4-x^2}}$ (c) $y = \frac{x}{\sqrt{2}} + \frac{\pi}{4} - 1$

Question 10

7 marks | Functions, Composition & Inverses | Main topic: Differentiation Techniques

10.

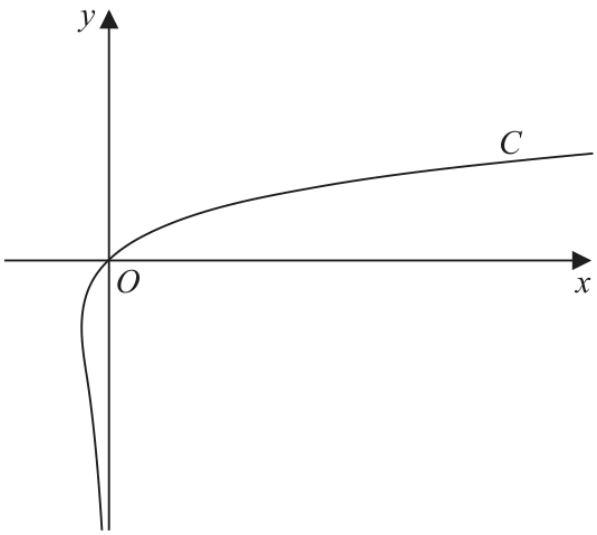


Figure 3

Figure 3 shows a sketch of the curve C with equation

$$x = ye^{2y} \qquad y \in \mathbb{R}$$

(a) Show that

$$\frac{dy}{dx} = \frac{y}{x(1+2y)}$$

(4)

Given that the straight line with equation $x = k$, where k is a constant, cuts C at exactly two points,

(b) find the range of possible values for k.

(3)

1 Part (a)

Differentiate x with respect to y by the product rule:

$$\frac{dx}{dy} = e^{2y} + 2ye^{2y} = e^{2y}(1+2y).$$

Where $dx/dy \neq 0$, take the reciprocal:

$$\frac{dy}{dx} = \frac{e^{-2y}}{1+2y}.$$

For $y \neq 0$, the original relation gives $e^{2y} = x/y$, hence the required form $dy/dx = y/[x(1+2y)]$. At the origin, that last quotient is $0/0$; the preceding uncancelled derivative gives $dy/dx = 1$. At $y = -1/2$ there is no finite dy/dx.

$$\boxed{\frac{dy}{dx} = \frac{y}{x(1+2y)} \quad \text{where defined}}$$

2 Part (b)

Consider $x = h(y) = ye^{2y}$. Its derivative $h'(y) = e^{2y}(1+2y)$ vanishes at $y = -1/2$. It is negative before this y value and positive after it, so the minimum x value is $h(-1/2) = -1/(2e)$. As $y \to -\infty$, $h(y) \to 0$ from below; also $h(0) = 0$ and h then increases without bound. Thus a level $x = k$ meets both negative-y branches precisely between the minimum and zero. At the minimum the two intersections coincide; at zero there is just the finite point $y = 0$.

$$\boxed{-\frac{1}{2e} < k < 0}$$

FINAL ANSWER

(a) $\frac{dy}{dx} = \frac{y}{x(1+2y)}$ where defined (b) $-\frac{1}{2e} < k < 0$

Question 6

10 marks | Functions, Composition & Inverses | Main topic: Applications of Differentiation

6.

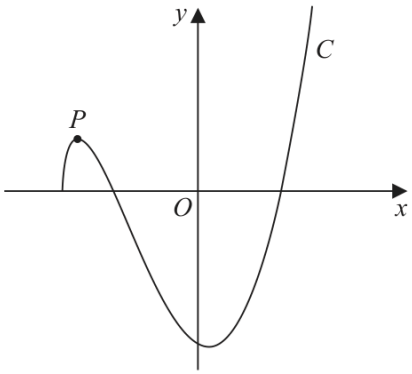


Figure 3

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

The function f is defined by

$$f(x) = 5(x^2 - 2)(4x + 9)^{\frac{1}{2}} \qquad x \geqslant -\frac{9}{4}$$

(a) Show that

$$f'(x) = \frac{k(5x^2 + 9x - 2)}{(4x + 9)^{\frac{1}{2}}}$$

where k is an integer to be found.

(4)

(b) Hence, find the values of x for which $f'(x) = 0$

(1)

Figure 3 shows a sketch of the curve C with equation $y = f(x)$.

The curve has a local maximum at the point P

(c) Find the exact coordinates of P

(2)

The function g is defined by

$$g(x) = 2f(x) + 4 \qquad -\frac{9}{4} \leqslant x \leqslant 0$$

(d) Find the range of g

(3)

1 Part (a)

Use product and chain rules:

$$f'(x) = 10x(4x+9)^{1/2} + 10(x^2-2)(4x+9)^{-1/2}.$$

With a common denominator,

$$f'(x) = \frac{10x(4x+9) + 10(x^2-2)}{\sqrt{4x+9}} = \frac{10(5x^2+9x-2)}{\sqrt{4x+9}}.$$

This derivative formula applies for $x > -9/4$; the endpoint derivative is not finite.

$$\boxed{k = 10}$$

2 Part (b)

The denominator is positive in the derivative domain. Thus $5x^2+9x-2 = (5x-1)(x+2) = 0$.

$$\boxed{x = -2, \quad \frac{1}{5}}$$

3 Part (c)

The derivative changes from positive to negative at $x = -2$, so this is the local maximum P. Substitute into f: $f(-2) = 5(4-2)\sqrt{-8+9} = 10$.

$$\boxed{P = (-2, 10)}$$

4 Part (d)

On $[-9/4, 0]$, f rises from $f(-9/4) = 0$ to $f(-2) = 10$, then decreases to $f(0) = -30$. The other stationary point $1/5$ is outside this restricted interval. Thus the minimum and maximum of f on it are -30 and 10. Applying the positive vertical transformation $g = 2f + 4$ gives endpoints $2(-30)+4 = -56$ and $2(10)+4 = 24$, both attained.

$$\boxed{-56 \le g(x) \le 24}$$

FINAL ANSWER

(a) $k = 10$ (b) $x = -2, \quad \frac{1}{5}$ (c) $P = (-2, 10)$ (d) $-56 \le g(x) \le 24$

Question 7

12 marks | Functions, Composition & Inverses | Main topic: Modulus

7.

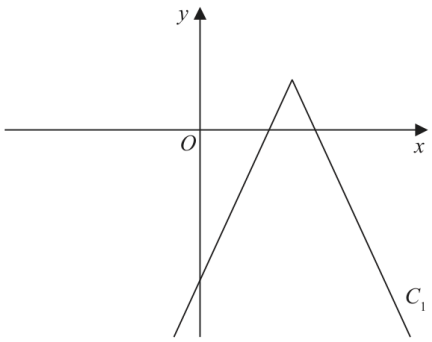


Figure 3

Figure 3 shows a sketch of the graph of C_1 with equation

$$y = 5 - |3x - 22|$$

(a) Write down the coordinates of

(i) the vertex of C_1

(ii) the intersection of C_1 with the y-axis.

(2)

(b) Find the x coordinates of the intersections of C_1 with the x-axis.

(2)

Diagram 1, shown on page 21, is a copy of Figure 3.

(c) On Diagram 1, sketch the curve C_2 with equation

$$y = \frac{1}{9}x^2 - 9$$

Identify clearly the coordinates of any points of intersection of C_2 with the coordinate axes.

(3)

(d) Find the coordinates of the points of intersection of C_1 and C_2
(*Solutions relying entirely on calculator technology are not acceptable.*)

(5)

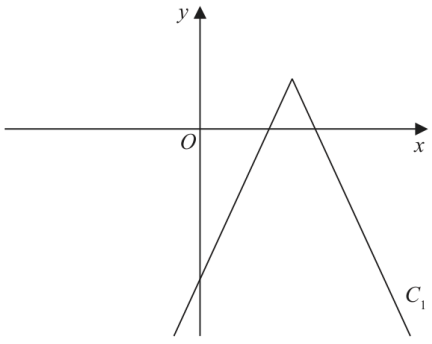


Diagram 1

1 Part (a)

(i) The modulus is zero at $3x - 22 = 0$, giving the vertex $(22/3, 5)$. (ii) At $x = 0$, $y = 5 - 22 = -17$, so the y-axis intersection is $(0, -17)$.

$$\boxed{(i)\ (22/3, 5), \qquad (ii)\ (0, -17)}$$

2 Part (b)

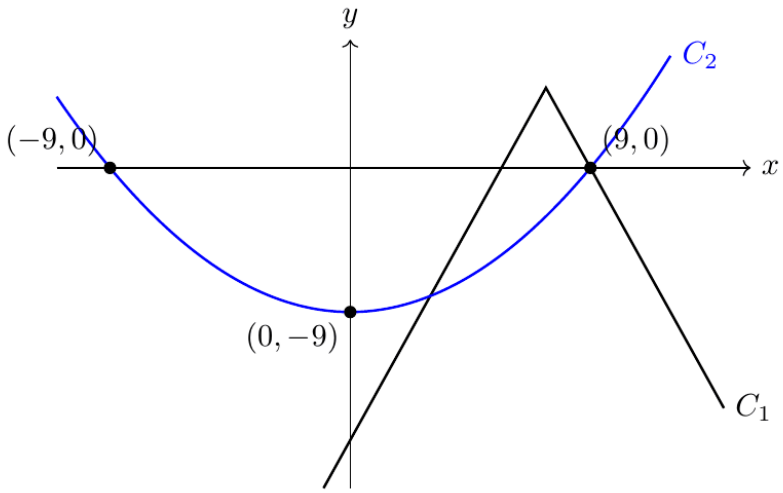
Set $y = 0$: $|3x - 22| = 5$. Thus $3x - 22 = 5$ or -5, giving $x = 9$ or $17/3$.

$$\boxed{x = \frac{17}{3}, \quad 9}$$

3 Part (c)

The curve $C_2 : y = x^2/9 - 9$ is an upward-opening parabola, symmetric about the y-axis. Its vertex and y-intercept are $(0, -9)$; setting y=0 gives $x^2 = 81$, hence x-intercepts $(-9, 0)$ and $(9, 0)$. The sketch below includes C_1 for context and labels all three axis intersections of C_2.

$$\boxed{(-9, 0), \quad (9, 0), \quad (0, -9)}$$



4 Part (d)

For $x \le 22/3$, C_1 is $y = 3x - 17$. Equating to C_2 gives

$$x^2/9 - 9 = 3x - 17 \implies x^2 - 27x + 72 = (x - 3)(x - 24) = 0.$$

Only $x = 3$ belongs to this branch; $y = -8$. For $x \ge 22/3$, C_1 is $y = 27 - 3x$. Then

$$x^2/9 - 9 = 27 - 3x \implies x^2 + 27x - 324 = (x - 9)(x + 36) = 0.$$

Only $x = 9$ belongs to this branch; $y = 0$. The rejected candidates 24 and -36 lie outside their assumed branch regions.

$$\boxed{(3, -8), \qquad (9, 0)}$$

FINAL ANSWER

(a) $(i)\ (22/3, 5), \quad (ii)\ (0, -17)$ (b) $x = \frac{17}{3}, \quad 9$ (c) $(-9, 0), \quad (9, 0), \quad (0, -9)$ (d) $(3, -8), \quad (9, 0)$

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Question 8

9 marks | Functions, Composition & Inverses | Main topic: Trig Equations & R-Form

8. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Express $8\sin x - 15\cos x$ in the form $R\sin(x-\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R, and give the value of α, in radians, to 4 significant figures.

(3)

$$f(x) = \frac{15}{41 + 16\sin x - 30\cos x} \qquad x > 0$$

(b) Find

(i) the minimum value of $f(x)$

(ii) the smallest value of x at which this minimum value occurs.

(4)

(c) State the y coordinate of the minimum points on the curve with equation

$$y = 2f(x) - 5 \qquad x > 0$$

(1)

(d) State the smallest value of x at which a maximum point occurs for the curve with equation

$$y = -f(2x) \qquad x > 0$$

(1)

WORKED SOLUTION

Question 8 · WMA13/01 October 2022

1 **Part (a)**

Expand $R\sin(x-\alpha) = R\cos\alpha\sin x - R\sin\alpha\cos x$. Thus $R\cos\alpha = 8$, $R\sin\alpha = 15$, so $R = \sqrt{64+225} = 17$ and $\alpha = \arctan(15/8) = 1.0808390005\ldots$ radians, in the required quadrant.

$$\boxed{R = 17, \qquad \alpha = 1.081 \text{ radians}}$$

2 Part (b)

Using the exact phase from (a),

$$f(x) = \frac{15}{41 + 34\sin(x - \alpha)}.$$

The denominator lies between 7 and 75, so it is always positive. (i) The smallest fraction occurs at its largest denominator: $f_{\min} = 15/75 = 1/5$. (ii) This occurs when $x - \alpha = \pi/2 + 2n\pi$. The smallest positive value is $x = \alpha + \pi/2 = 2.6516353273\ldots$, or 2.65 to three significant figures.

$$\boxed{f_{\min} = \frac{1}{5}, \qquad x = 2.65}$$

---

3 Part (c)

The transformation $2f(x) - 5$ preserves the locations of minima and maps their y coordinate to $2(1/5) - 5 = -23/5$.

$$\boxed{y = -\frac{23}{5}}$$

---

4 Part (d)

Reflection in the x-axis changes minima of f into maxima of $-f$. The argument $2x$ halves their x coordinates, so the first positive maximum is at

$$x = \frac{\alpha + \pi/2}{2} = 1.3258176637\ldots \approx 1.33.$$

$$\boxed{x = 1.33}$$

FINAL ANSWER

(a) $R = 17, \qquad \alpha = 1.081$ radians (b) $f_{\min} = \frac{1}{5}, \qquad x = 2.65$ (c) $y = -\frac{23}{5}$ (d) $x = 1.33$

Question 6

9 marks | Functions, Composition & Inverses | Main topic: Modulus

6.

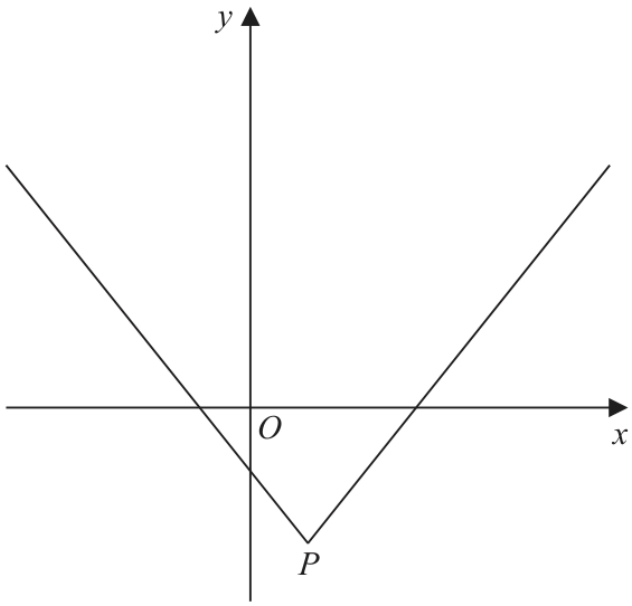


Figure 2

Figure 2 shows a sketch of the graph $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 3|x - 2| - 10$$

The vertex of the graph is at point P, shown in Figure 2.

(a) Find the coordinates of P

(2)

(b) Find $\mathrm{ff}(0)$

(2)

(c) Solve the inequality

$$3|x - 2| - 10 < 5x + 10$$

(2)

(d) Solve the equation

$$\mathrm{f}(|x|) = 0$$

(3)

1 **Part (a)**

The modulus is smallest when $x - 2 = 0$, giving $x = 2$ and $f(2) = -10$. The two straight-line branches meet there, so this is the vertex of the V-shaped graph.

$$\boxed{P = (2, -10)}$$

2 **Part (b)**

First $f(0) = 3|0 - 2| - 10 = -4$. Then $ff(0) = f(-4) = 3|-4 - 2| - 10 = 18 - 10 = 8$. In particular $ff(0)$ means $f(f(0))$, not the square of $f(0)$.

$$\boxed{ff(0) = 8}$$

3 **Part (c)**

For $x < 2$, the inequality becomes $3(2 - x) - 10 < 5x + 10$, so $-4 - 3x < 5x + 10$, giving $x > -7/4$. This yields $-7/4 < x < 2$. For $x \geq 2$, it becomes $3x - 16 < 5x + 10$, giving $x > -13$; every $x \geq 2$ satisfies this. The union is $x > -7/4$.

$$\boxed{x > -\frac{7}{4}}$$

4 **Part (d)**

Let $u = |x| \geq 0$. Then $f(|x|) = 0$ gives

$$3|u - 2| - 10 = 0 \quad \Longrightarrow \quad u - 2 = \pm\frac{10}{3}.$$

Thus $u = 16/3$ or $u = -4/3$. Reject $u = -4/3$ because $u \geq 0$; $|x| = 16/3$ has the two solutions below.

$$\boxed{x = -\frac{16}{3}, \quad \frac{16}{3}}$$

✓ **FINAL ANSWER**

(a) $P = (2, -10)$ (b) $ff(0) = 8$ (c) $x > -\frac{7}{4}$ (d) $x = -\frac{16}{3}, \quad \frac{16}{3}$

Question 9

9 marks | Functions, Composition & Inverses | Main topic: Exponentials & Logarithms

9. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

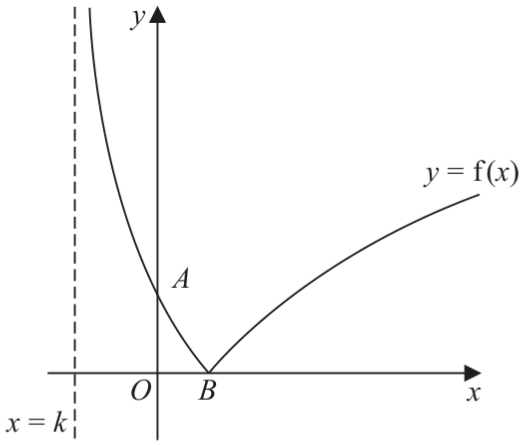


Figure 2

Figure 2 shows a sketch of the curve with equation

$$y = |2 - 4\ln(x + 1)| \qquad x > k$$

where k is a constant.

Given that the curve

- has an asymptote at $x = k$
- cuts the y-axis at point A
- meets the x-axis at point B

as shown in Figure 2,

(a) state the value of k

(1)

(b) (i) find the y coordinate of A

(ii) find the exact x coordinate of B

(3)

(c) Using algebra and showing your working, find the set of values of x such that

$$|2 - 4\ln(x + 1)| > 3$$

(5)

1 Part (a)

The logarithm requires $x+1>0$, so the domain is $x>-1$. As $x\to -1^{+}$, $\ln(x+1)\to -\infty$ and the ordinate tends to infinity.

$$\boxed{k=-1}$$

2 Part (b)

(i) At the y-axis, $x=0$, giving $y=|2-4\ln 1|=2$.
(ii) At the x-axis, $|2-4\ln(x+1)|=0$, so

$$\ln(x+1)=\frac{1}{2}\quad\Longrightarrow\quad x=e^{1/2}-1.$$

Thus $A=(0,2)$ and $B=(e^{1/2}-1,0)$.

$$\boxed{y_A=2,\qquad x_B=e^{1/2}-1}$$

3 Part (c)

On $x>-1$, split the absolute-value inequality into two alternatives:

$$2-4\ln(x+1)>3\quad\text{or}\quad 2-4\ln(x+1)<-3.$$

Dividing by -4 reverses the inequality in each branch:

$$\ln(x+1)<-\frac{1}{4}\quad\text{or}\quad\ln(x+1)>\frac{5}{4}.$$

Since the exponential is increasing, this becomes $x+1<e^{-1/4}$ or $x+1>e^{5/4}$. Intersect with the logarithmic domain; neither boundary is included because the source inequality is strict.

$$\boxed{-1<x<e^{-1/4}-1\quad\text{or}\quad x>e^{5/4}-1}$$

FINAL ANSWER

(a) $k=-1$ (b) $y_A=2,\qquad x_B=e^{1/2}-1$ (c) $-1<x<e^{-1/4}-1\quad\text{or}\quad x>e^{5/4}-1$

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Question 10

9 marks | Functions, Composition & Inverses | Main topic: Differentiation Techniques

10. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

A curve C has equation

$$x = \sin^2 4y \qquad 0 \leqslant y \leqslant \frac{\pi}{8} \qquad 0 \leqslant x \leqslant 1$$

The point P with x coordinate $\frac{1}{4}$ lies on C

(a) Find the exact y coordinate of P

(2)

(b) Find $\frac{\mathrm{d}x}{\mathrm{d}y}$

(2)

(c) Hence show that $\frac{\mathrm{d}y}{\mathrm{d}x}$ can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{q + r(x + s)^2}}$$

where q, r and s are constants to be found.

(3)

Using the answer to part (c),

(d) (i) state the x coordinate of the point where the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ is a minimum,

(ii) state the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at this point.

(2)

1 Part (a)

At $x = 1/4$, $\sin^2 4y = 1/4$. Since $0 \le 4y \le \pi/2$,

$$\sin 4y = \frac{1}{2} \quad \Longrightarrow \quad 4y = \frac{\pi}{6}.$$

Only this branch lies in the given range of y.

$$\boxed{y = \frac{\pi}{24}}$$

2 Part (b)

Differentiate $x = (\sin 4y)^2$ with respect to y:

$$\frac{dx}{dy} = 2\sin 4y\,(4\cos 4y).$$

$$\boxed{\frac{dx}{dy} = 8\sin 4y\cos 4y}$$

3 Part (c)

On the interior of the given interval, $\sin 4y = \sqrt{x}$ and $\cos 4y = \sqrt{1-x}$ are positive. Taking the reciprocal,

$$\frac{dy}{dx} = \frac{1}{8\sqrt{x}\sqrt{1-x}} = \frac{1}{\sqrt{64x - 64x^2}}.$$

Complete the square:

$$64x - 64x^2 = 16 - 64\left(x - \frac{1}{2}\right)^2.$$

Thus $dy/dx = (q + r(x+s)^2)^{-1/2}$ has the constants below, with $0 < x < 1$ for a finite derivative; the endpoints have vertical tangents.

$$\boxed{q = 16, \qquad r = -64, \qquad s = -\frac{1}{2}}$$

4 Part (d)

The derivative is positive. Its minimum occurs when the square root in the denominator is largest. Since $(x - 1/2)^2 \ge 0$, the largest radicand is 16, attained at $x = 1/2$. Hence the smallest gradient is $1/\sqrt{16}$.

$$\boxed{\text{(i) } x = \frac{1}{2}, \qquad \text{(ii) } \min \frac{dy}{dx} = \frac{1}{4}}$$

FINAL ANSWER

(a) $y = \frac{\pi}{24}$ (b) $\frac{dx}{dy} = 8\sin 4y \cos 4y$ (c) $q = 16, \qquad r = -64, \qquad s = -\frac{1}{2}$ (d)
(i) $x = \frac{1}{2}$, (ii) $\min \frac{dy}{dx} = \frac{1}{4}$

Question 8

11 marks | Functions, Composition & Inverses | Main topic: Modulus

8.

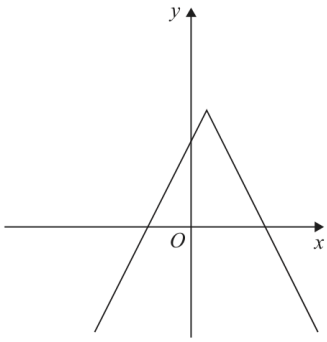


Figure 2

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

The graph shown in Figure 2 has equation

$$y = a - |2x - b|$$

where a and b are positive constants, $a > b$

(a) Find, giving your answer in terms of a and b,

(i) the coordinates of the maximum point of the graph,

(ii) the coordinates of the point of intersection of the graph with the y-axis,

(iii) the coordinates of the points of intersection of the graph with the x-axis.

(5)

On page 24 there is a copy of Figure 2 called Diagram 1.

(b) On Diagram 1, sketch the graph with equation

$$y = |x| - 1$$

(2)

Given that the graphs $y = |x| - 1$ and $y = a - |2x - b|$ intersect at $x = -3$ and $x = 5$

(c) find the value of a and the value of b

(4)

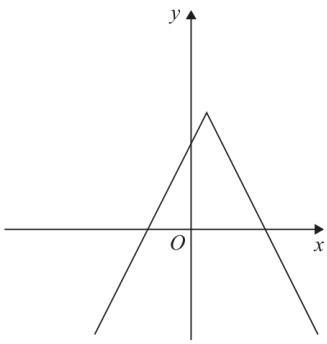


Diagram 1

1 Part (a)

(i) Since $|2x-b| \geq 0$, the maximum occurs at $2x-b=0$: $x=b/2$ and $y=a$. (ii) At $x=0$, $y=a-|-b|=a-b$ because $b>0$. (iii) At an x-axis crossing, $|2x-b|=a$, hence $2x-b=\pm a$. Since $a>b>0$, $(b-a)/2<0$ and $(b+a)/2>0$, matching the sketch.

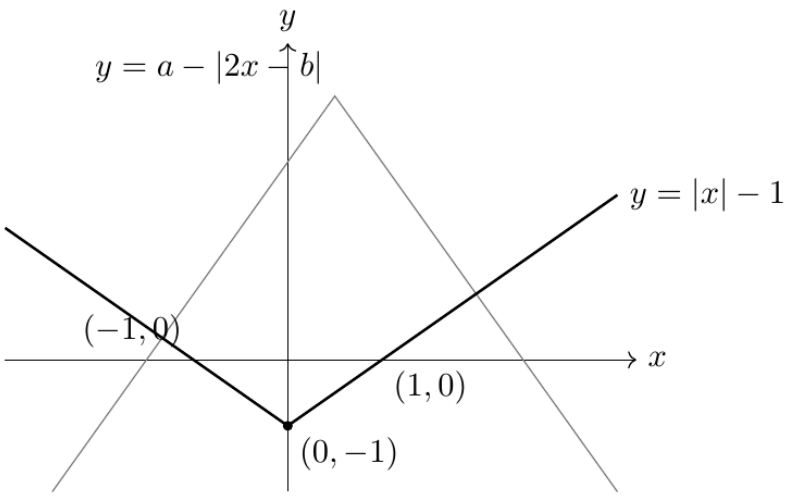
$$\boxed{\left(\frac{b}{2}, a\right); \quad (0, a-b); \quad \left(\frac{b-a}{2}, 0\right), \ \left(\frac{b+a}{2}, 0\right)}$$

2 Part (b)

The graph $y=|x|-1$ is a V with vertex $(0,-1)$ and intercepts $(-1,0)$ and $(1,0)$. Its left branch is $y=-x-1$ for $x<0$ and its right branch is $y=x-1$ for $x \geq 0$. Add this V to the supplied axes; the original inverted V remains unchanged.

The grey original curve is schematic; the three labelled points belong to the added graph.

$$\boxed{\text{Vertex } (0,-1); \quad \text{zeros } x=\pm 1}$$



3 Part (c)

At $x=-3$, the common ordinate is $|-3|-1=2$. Since $b>0$, $|-6-b|=6+b$, so $a-(6+b)=2$, giving $a=b+8$. At $x=5$, the common ordinate is $|5|-1=4$, so $a-|10-b|=4$. Substitute $a=b+8$ to obtain $|10-b|=b+4$. If $b \leq 10$, then $10-b=b+4$, giving $b=3$, $a=11$. If $b>10$, then $b-10=b+4$, impossible. Both accepted values satisfy $a>b>0$.

$$\boxed{a=11, \quad b=3}$$

FINAL ANSWER

(a) $\left(\frac{b}{2}, a\right); \quad (0, a-b); \quad \left(\frac{b-a}{2}, 0\right), \ \left(\frac{b+a}{2}, 0\right)$ (b) $\text{Vertex } (0,-1); \quad \text{zeros } x=\pm 1$ (c) $a=11, \quad b=3$

Question 7

10 marks | Functions, Composition & Inverses | Main topic: Graph Transformations

7.

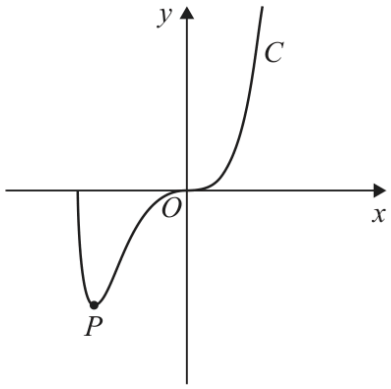


Figure 3

The curve C has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x^3\sqrt{4x+7} \qquad x \geqslant -\frac{7}{4}$$

(a) Show that

$$\mathrm{f}'(x) = \frac{kx^2(2x+3)}{\sqrt{4x+7}}$$

where k is a constant to be found.

(4)

The point P, shown in Figure 3, is the minimum turning point on C.

(b) Find the coordinates of P.

(2)

(c) Hence find the range of the function g defined by

$$\mathrm{g}(x) = -4\,\mathrm{f}(x) \qquad x \geqslant -\frac{7}{4}$$

(2)

The point Q with coordinates $\left(\frac{1}{2}, \frac{3}{8}\right)$ lies on C.

(d) Find the coordinates of the point to which Q is mapped when C is transformed to the curve with equation

$$y = 40\,\mathrm{f}\left(x - \frac{3}{2}\right) - 8$$

(2)

1 Part (a)

For $x > -7/4$, apply the product and chain rules:

$$f'(x) = 3x^2\sqrt{4x+7} + x^3\frac{2}{\sqrt{4x+7}} = \frac{3x^2(4x+7)+2x^3}{\sqrt{4x+7}} = \frac{7x^2(2x+3)}{\sqrt{4x+7}}.$$

$$\boxed{k = 7}$$

2 Part (b)

The interior stationary values satisfy $x^2(2x+3)=0$, so $x=0$ or $x=-3/2$. Near $x=-3/2$, the derivative changes from negative to positive; near $x=0$ it stays positive on both sides. Thus the minimum turning point is at $x=-3/2$.

$$f(-3/2) = (-3/2)^3\sqrt{-6+7} = -27/8.$$

$$\boxed{P = \left(-\frac{3}{2}, -\frac{27}{8}\right)}$$

3 Part (c)

The endpoint has $f(-7/4)=0$, the global minimum is $-27/8$ and $f(x)$ tends to infinity as x increases. Hence the range of f is $[-27/8, \infty)$. Multiplication by -4 reverses the inequality and makes the upper endpoint $27/2$.

$$\boxed{g(x) \le \frac{27}{2}}$$

4 Part (d)

The transformation is a shift right by $3/2$, vertical stretch by 40, then shift down by 8:

$$(x, y) \mapsto (x + 3/2,\, 40y - 8).$$

Therefore $Q = (1/2, 3/8)$ maps to $(1/2 + 3/2,\ 40\cdot 3/8 - 8)$.

$$\boxed{Q \mapsto (2, 7)}$$

FINAL ANSWER

(a) $k = 7$ (b) $P = \left(-\frac{3}{2}, -\frac{27}{8}\right)$ (c) $g(x) \le \frac{27}{2}$ (d) $Q \mapsto (2, 7)$

Question 9

8 marks | Functions, Composition & Inverses | Main topic: Applications of Differentiation

9.

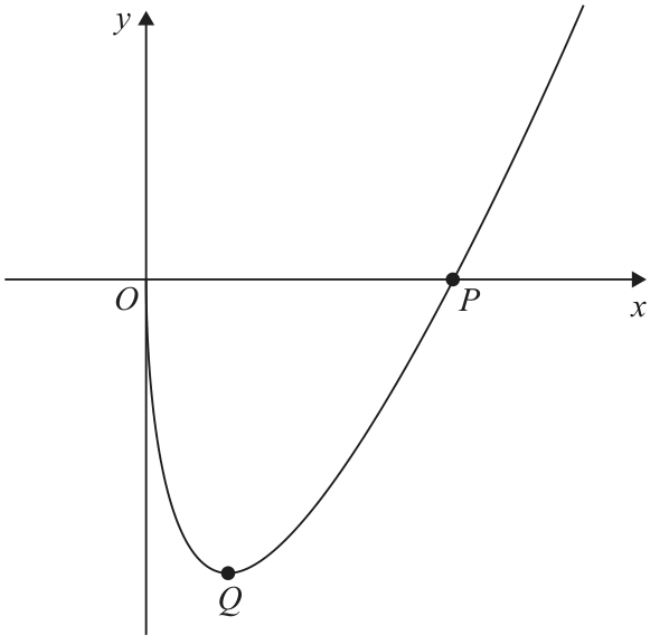


Figure 1

Figure 1 shows a sketch of part of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 6\sqrt{x}\ln(4x) \qquad x > 0$$

The curve cuts the x-axis at point P

(a) State the x coordinate of P

(1)

The point Q, shown in Figure 1, is the stationary point on C

(b) Use calculus to find the exact coordinates of Q

(5)

(c) Hence find the range of the function $\mathrm{g}(x)$ where

$$\mathrm{g}(x) = -2\,\mathrm{f}(x)$$

(2)

1 Part (a)

Since $x > 0$, $6\sqrt{x}$ is positive. Therefore $f(x) = 0$ requires $\ln(4x) = 0$, so $4x = 1$ and $x = 1/4$.

$$\boxed{x_P = \frac{1}{4}}$$

2 Part (b)

Apply the product and chain rules:

$$f'(x) = \frac{3}{\sqrt{x}}\ln(4x) + 6\sqrt{x}\frac{1}{x} = \frac{3}{\sqrt{x}}[\ln(4x) + 2].$$

At Q, set $f'(x) = 0$. Since $3/\sqrt{x} > 0$,

$$\ln(4x) + 2 = 0 \quad \Longrightarrow \quad 4x = e^{-2} \quad \Longrightarrow \quad x = \frac{1}{4e^2}.$$

Then $\sqrt{x} = 1/(2e)$ and $\ln(4x) = -2$, giving $f(x) = 6(1/(2e))(-2) = -6/e$.

$$\boxed{Q = \left(\frac{1}{4e^2}, -\frac{6}{e}\right)}$$

3 Part (c)

The factor $3/\sqrt{x}$ is positive and $\ln(4x) + 2$ increases through zero at Q, so f' changes from negative to positive: Q is the global minimum. As $x \to \infty$, $f(x) \to \infty$, hence the range of f is $[-6/e, \infty)$. Multiplication by -2 reverses the inequality and includes the maximum $12/e$.

$$\boxed{g(x) \leq \frac{12}{e}}$$

FINAL ANSWER

(a) $x_P = \frac{1}{4}$ (b) $Q = \left(\frac{1}{4e^2}, -\frac{6}{e}\right)$ (c) $g(x) \leq \frac{12}{e}$

Question 10

10 marks | Functions, Composition & Inverses | Main topic: Modulus

10:

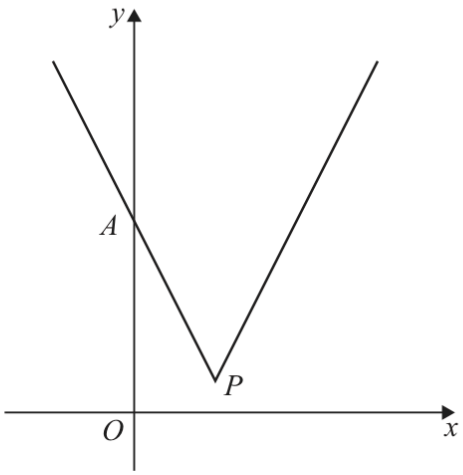


Figure 3

Figure 3 shows a sketch of part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = |kx - 10| + k \qquad x \in \mathbb{R}$$

and k is a positive constant.

The graph

- cuts the y-axis at the point A
- has a vertex at the point P

(a) Find, in simplest form in terms of k,

(i) the y coordinate of A

(ii) the coordinates of P

(3)

(b) Find, in terms of k, the range of values of x which satisfy

$$|kx - 10| + k \geqslant 2k$$

(3)

Given that the line with equation $y = 3x + 1$ intersects the graph of $y = \mathrm{f}(x)$ at 2 distinct points,

(c) find the range of values of k.

(4)

1 **Part (i)**

At $x = 0$, $f(0) = |-10| + k$, so $\boxed{y_A = 10 + k}$.

2 **Part (ii)**

At the vertex the expression inside the modulus vanishes:

$$kx - 10 = 0 \quad \Longrightarrow \quad \boxed{P = \left(\frac{10}{k}, k\right)}.$$

3 **Part (b)**

$$|kx - 10| + k \geq 2k \quad \Longleftrightarrow \quad |kx - 10| \geq k.$$

Since $k > 0$, the two branches give

$$kx - 10 \leq -k \quad \text{or} \quad kx - 10 \geq k,$$

$$\boxed{x \leq \frac{10}{k} - 1 \quad \text{or} \quad x \geq \frac{10}{k} + 1}.$$

4 **Part (c)**

Consider the vertical difference $d(x) = f(x) - (3x + 1)$:

$$d(x) = \begin{cases} 9 + k - (k+3)x, & x \leq 10/k, \\ (k-3)x + k - 11, & x \geq 10/k. \end{cases}$$

The left branch is strictly decreasing. If $0 < k < 3$, the right branch also decreases, so there cannot be two roots. If $k = 3$, the right branch is the nonzero constant -8, again giving no second root. For $k > 3$ the right branch increases and both outer ends tend to $+\infty$. There are two distinct roots precisely when the vertex lies below zero:

$$d(10/k) = k - 1 - \frac{30}{k} < 0.$$

Multiplying by $k > 0$ gives

$$k^2 - k - 30 = (k-6)(k+5) < 0 \quad \Longrightarrow \quad 0 < k < 6.$$

Combining with $k > 3$ yields $\boxed{3 < k < 6}$. At $k = 6$ the two roots merge at the vertex, so the endpoint is excluded.

FINAL ANSWER

(i) $y_A = 10 + k$ (ii) $P = \left(\frac{10}{k}, k\right)$ (b) $x \leq \frac{10}{k} - 1 \quad \text{or} \quad x \geq \frac{10}{k} + 1$ (c) $3 < k < 6$ (a)(i) $y_A = 10 + k$ (a)(ii) $P = \left(\frac{10}{k}, k\right)$

Question 8

13 marks | Functions, Composition & Inverses | Main topic: Modulus

8.

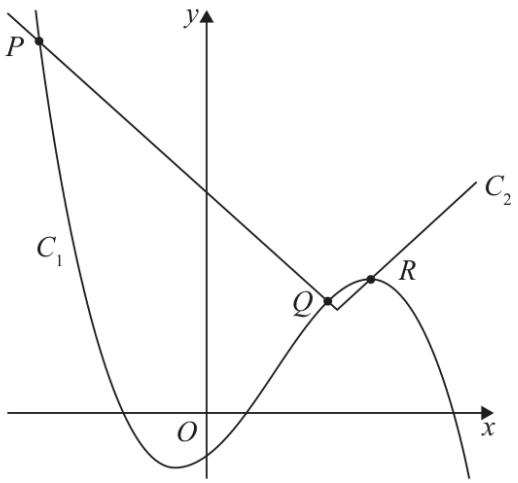


Figure 2

Figure 2 shows a sketch of the graph C_1 with equation

$$y = -2x^3 + 5x^2 + 4x - 3$$

and a sketch of the graph C_2 with equation

$$y = a + |5x + b|$$

where a and b are constants.

Graphs C_1 and C_2 intersect at point P, point Q and point R, as shown in Figure 2.

Given that P has coordinates $(-2, 25)$

(a) show that

$$a = 15 + b$$

(2)

Given also that R has coordinates $(2, 9)$

(b) find the value of a and the value of b

(3)

Using the answer to part (b),

(c) state the coordinates of the vertex of C_2

(2)

(d) Find, using algebra, the coordinates of Q. Show each stage of your working.

(*Solutions relying on calculator technology are not acceptable.*)

(6)

1 Part (a)

Point P lies on the descending branch of C_2, where $5x + b < 0$. At $P = (-2, 25)$,

$$25 = a - (-10 + b) = a + 10 - b,$$

hence $a = 15 + b$.

2 Part (b)

Point $R = (2, 9)$ lies on the ascending branch, so

$$9 = a + (10 + b) \Longrightarrow a + b = -1.$$

Combining with $a - b = 15$ gives $2a = 14$, hence $a = 7$ and $b = -8$.

3 Part (c)

The vertex occurs at $5x - 8 = 0$, giving $(x, y) = (8/5, 7)$.

4 Part (d)

The diagram places Q on the descending branch $y = 7 - (5x - 8) = 15 - 5x$, with $x < 8/5$. Equate this with the cubic:

$$-2x^3 + 5x^2 + 4x - 3 = 15 - 5x \Longrightarrow 2x^3 - 5x^2 - 9x + 18 = 0.$$

Since $x = -2$ is the known intersection P, divide by $x + 2$:

$$2x^3 - 5x^2 - 9x + 18 = (x + 2)(2x^2 - 9x + 9) = (x + 2)(2x - 3)(x - 3).$$

The candidates are $-2, 3/2, 3$. Reject $x = 3$ because it is outside the descending branch, and $x = -2$ is P, not Q. Thus $x_Q = 3/2$ and $y_Q = 15 - 5(3/2) = 15/2$, giving $Q = (3/2, 15/2)$.

FINAL ANSWER

(a) Shown, as required. (b) 7; -8 (c) $(8/5, 7)$ (d) $(3/2, 15/2)$

Question 7

10 marks | Functions, Composition & Inverses | Main topic: Modulus

7.

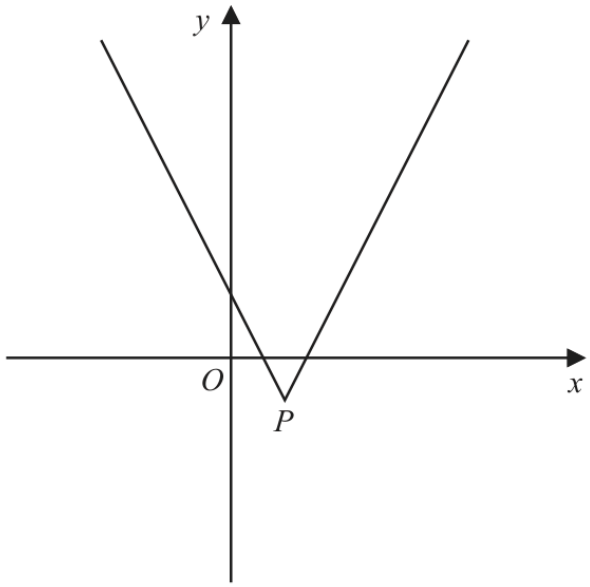


Figure 1

Figure 1 shows a sketch of part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = |3x - 11| - 4$$

The vertex of the graph is at the point P, as shown in Figure 1.

(a) State the coordinates of P.

(2)

(b) Solve the equation $\mathrm{f}(x) = 8$

(3)

The line l has equation $y = mx$, where m is a constant.

Given that l intersects the graph of $y = \mathrm{f}(x)$ at exactly one point,

(c) find the possible values of m.

(3)

The graph with equation $y = \mathrm{f}(x)$ is transformed onto the graph with equation $y = a\,\mathrm{f}(x - b)$, where a and b are constants.

Given that the vertex of the graph with equation $y = a\,\mathrm{f}(x - b)$ is $(5, 16)$

(d) find the value of a and the value of b.

(2)

1 Part (a)

At the vertex, $3x - 11 = 0$, so $P = (\frac{11}{3}, -4)$.

2 Part (b)

$$|3x - 11| - 4 = 8 \quad \Longrightarrow \quad |3x - 11| = 12.$$

Thus $3x - 11 = 12$ or $3x - 11 = -12$, giving $x = \frac{23}{3}$ or $x = -\frac{1}{3}$.

3 Part (c)

Write the two branches explicitly:

$$f(x) = \begin{cases} 7 - 3x, & x \leq \frac{11}{3}, \\ 3x - 15, & x \geq \frac{11}{3}. \end{cases}$$

On the left branch, $mx = 7 - 3x$ gives $x = 7/(m + 3)$ for $m \neq -3$. This lies on the left branch precisely when $m < -3$ or $m \geq -12/11$. On the right branch, $mx = 3x - 15$ gives $x = 15/(3 - m)$ for $m \neq 3$. This lies on the right branch precisely when $-12/11 \leq m < 3$. At $m = -12/11$, the two branch intersections coincide at the vertex, so there is only one distinct point. For $m < -3$ or $m \geq 3$, only the left intersection exists. At $m = -3$ there is no intersection; at $m = 3$ there is one. Consequently the complete answer is

$$m < -3 \quad \text{or} \quad m = -\frac{12}{11} \quad \text{or} \quad m \geq 3.$$

4 Part (d)

Under $y = af(x - b)$, the vertex becomes $(\frac{11}{3} + b, -4a)$. Equating this with $(5, 16)$ gives

$$b = 5 - \frac{11}{3} = \frac{4}{3}, \qquad a = \frac{16}{-4} = -4.$$

FINAL ANSWER

(a) $P = (\frac{11}{3}, -4)$ (b) $x = \frac{23}{3}$ or $-\frac{1}{3}$ (c) $m < -3 \quad \text{or} \quad m = -\frac{12}{11} \quad \text{or} \quad m \geq 3$ (d) $b = 5 - \frac{11}{3} = \frac{4}{3}, \qquad a = \frac{16}{-4} = -4$

Question 9

11 marks | Functions, Composition & Inverses | Main topic: Modulus

9.

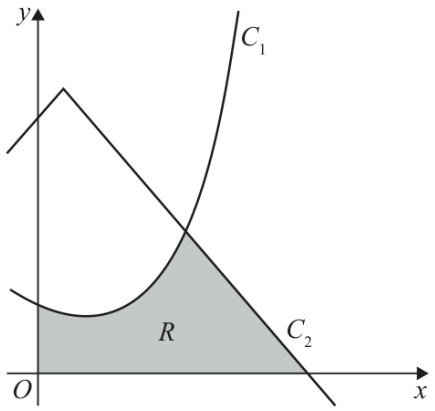


Figure 2

In this question you must show all stages of your working.
Solutions relying entirely on calculator technology are not acceptable.

(a) Find $\int \left(\frac{11}{2x+5} + 4e^{x-3} \right) dx$

(2)

Figure 2 shows a sketch of part of the curve C_1 and part of the graph C_2
The curve C_1 has equation

$$y = \frac{11}{2x+5} + 4e^{x-3}$$

The graph C_2 has equation

$$y = a - |bx - 1|$$

where a and b are positive integers.

(b) State, in terms of a and/or b, the coordinates of the maximum point of C_2

(2)

Given that

- C_1 intersects C_2 at $x = 3$
- C_2 crosses the y-axis at $(0, 9)$

(c) find the value of a and the value of b

(3)

The region R, shown shaded in Figure 2, is bounded by C_1, C_2, the x-axis and the y-axis.

(d) Find the area of R, giving the answer to 3 significant figures.

(4)

1 Part (a)

$$\int\left(\frac{11}{2x+5}+4e^{x-3}\right)dx=\boxed{\frac{11}{2}\ln|2x+5|+4e^{x-3}+C}.$$

The absolute value allows both intervals on which the integrand is defined.

2 Part (b)

The largest value of $a-|bx-1|$ occurs when $bx-1=0$. The maximum point is therefore $\boxed{(1/b,a)}$.

3 Part (c)

At $x=0$, $9=a-1$, so $\boxed{a=10}$. At the intersection $x=3$, the first curve has height $11/11+4=5$. Thus $10-|3b-1|=5$, giving $3b-1=5$ or -5. These give $b=2$ or $-4/3$; only the positive integer is allowed, so $\boxed{b=2}$.

4 Part (d)

For $x\geq 3$, the required branch of C_2 is $y=10-(2x-1)=11-2x$. Its x-intercept is $x=11/2$. The shaded area is the area under C_1 from 0 to 3 plus the triangle from 3 to $11/2$:

$$\begin{aligned}A&=\int_0^3\left(\frac{11}{2x+5}+4e^{x-3}\right)dx+\frac{1}{2}\left(\frac{11}{2}-3\right)5\\&=\left[\frac{11}{2}\ln(2x+5)+4e^{x-3}\right]_0^3+\frac{25}{4}\\&=\frac{11}{2}\ln\frac{11}{5}+4(1-e^{-3})+\frac{25}{4}=14.3873672\ldots.\end{aligned}$$

Hence $\boxed{A=14.4\text{ square units}}$ to 3 significant figures.

FINAL ANSWER

(a) $\frac{11}{2}\ln|2x+5|+4e^{x-3}+C$ (b) $(1/b,a)$ (c) $a=10$, $b=2$ (d) $A=14.4$ square units

TOPIC

Graph Transformations

Every past-paper question on this topic, January 2020 to June 2026.

Question 7

10 marks | Graph Transformations

7.

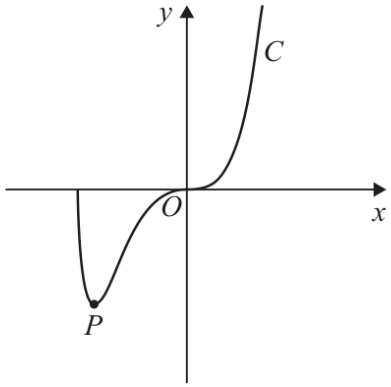


Figure 3

The curve C has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x^3\sqrt{4x+7} \qquad x \geqslant -\frac{7}{4}$$

(a) Show that

$$\mathrm{f}'(x) = \frac{kx^2(2x+3)}{\sqrt{4x+7}}$$

where k is a constant to be found.

(4)

The point P, shown in Figure 3, is the minimum turning point on C.

(b) Find the coordinates of P.

(2)

(c) Hence find the range of the function g defined by

$$\mathrm{g}(x) = -4\,\mathrm{f}(x) \qquad x \geqslant -\frac{7}{4}$$

(2)

The point Q with coordinates $\left(\frac{1}{2}, \frac{3}{8}\right)$ lies on C.

(d) Find the coordinates of the point to which Q is mapped when C is transformed to the curve with equation

$$y = 40\,\mathrm{f}\left(x - \frac{3}{2}\right) - 8$$

(2)

1 Part (a)

For $x > -7/4$, apply the product and chain rules:

$$f'(x) = 3x^2\sqrt{4x+7} + x^3\frac{2}{\sqrt{4x+7}} = \frac{3x^2(4x+7)+2x^3}{\sqrt{4x+7}} = \frac{7x^2(2x+3)}{\sqrt{4x+7}}.$$

$$\boxed{k = 7}$$

2 Part (b)

The interior stationary values satisfy $x^2(2x+3)=0$, so $x=0$ or $x=-3/2$. Near $x=-3/2$, the derivative changes from negative to positive; near $x=0$ it stays positive on both sides. Thus the minimum turning point is at $x=-3/2$.

$$f(-3/2) = (-3/2)^3\sqrt{-6+7} = -27/8.$$

$$\boxed{P = \left(-\frac{3}{2}, -\frac{27}{8}\right)}$$

3 Part (c)

The endpoint has $f(-7/4)=0$, the global minimum is $-27/8$ and $f(x)$ tends to infinity as x increases. Hence the range of f is $[-27/8, \infty)$. Multiplication by -4 reverses the inequality and makes the upper endpoint $27/2$.

$$\boxed{g(x) \le \frac{27}{2}}$$

4 Part (d)

The transformation is a shift right by $3/2$, vertical stretch by 40, then shift down by 8:

$$(x, y) \mapsto (x + 3/2,\, 40y - 8).$$

Therefore $Q = (1/2, 3/8)$ maps to $(1/2 + 3/2,\ 40\cdot 3/8 - 8)$.

$$\boxed{Q \mapsto (2, 7)}$$

FINAL ANSWER

(a) $k = 7$ (b) $P = \left(-\frac{3}{2}, -\frac{27}{8}\right)$ (c) $g(x) \le \frac{27}{2}$ (d) $Q \mapsto (2, 7)$

Question 6

9 marks | Graph Transformations | Main topic: Modulus

6.

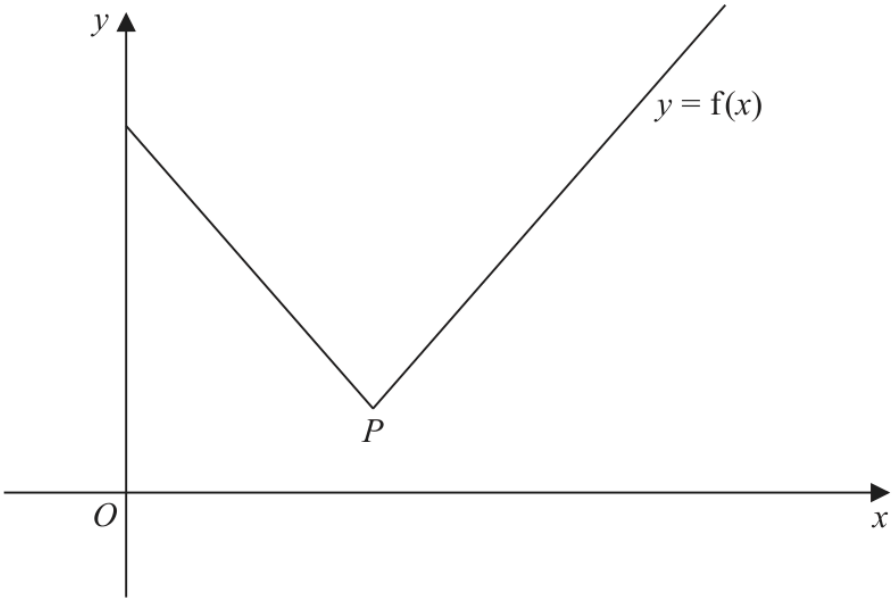


Figure 2

Figure 2 shows part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 2|2x - 5| + 3 \qquad x \geqslant 0$$

The vertex of the graph is at point P as shown.

(a) State the coordinates of P.

(2)

(b) Solve the equation $\mathrm{f}(x) = 3x - 2$

(4)

Given that the equation

$$\mathrm{f}(x) = kx + 2$$

where k is a constant, has exactly two roots,

(c) find the range of values of k.

(3)

1 Part (a)

At the vertex, the expression inside the modulus is zero:

$$2x - 5 = 0 \quad \Longrightarrow \quad x = \frac{5}{2}, \qquad f\left(\frac{5}{2}\right) = 3.$$

Hence $\boxed{P = \left(\frac{5}{2}, 3\right)}$.

2 Part (b)

Write the function in its two branches:

$$f(x) = \begin{cases} 13 - 4x, & 0 \le x \le \frac{5}{2}, \\ 4x - 7, & x \ge \frac{5}{2}. \end{cases}$$

On the left branch,

$$13 - 4x = 3x - 2 \quad \Longrightarrow \quad 7x = 15 \quad \Longrightarrow \quad x = \frac{15}{7}.$$

This lies in $0 \le x \le \frac{5}{2}$, so it is valid. On the right branch,

$$4x - 7 = 3x - 2 \quad \Longrightarrow \quad x = 5.$$

This satisfies $x \ge \frac{5}{2}$, so it is also valid. Thus

$$\boxed{x = \frac{15}{7} \text{ or } x = 5}.$$

3 Part (c)

Each straight branch can have at most one intersection with $y = kx + 2$. Two distinct roots therefore require one intersection strictly to each side of the vertex.

For the left branch,

$$13 - 4x = kx + 2 \quad \Longrightarrow \quad x_L = \frac{11}{k+4}.$$

The condition $0 \le x_L < \frac{5}{2}$ first requires $k + 4 > 0$, then gives

$$\frac{11}{k+4} < \frac{5}{2} \quad \Longrightarrow \quad 22 < 5(k+4) \quad \Longrightarrow \quad k > \frac{2}{5}.$$

For the right branch,

$$4x - 7 = kx + 2 \quad \Longrightarrow \quad x_R = \frac{9}{4-k}.$$

The condition $x_R > \frac{5}{2}$ requires $k < 4$, and then

$$\frac{9}{4-k} > \frac{5}{2} \quad \Longrightarrow \quad 18 > 5(4-k) \quad \Longrightarrow \quad k > \frac{2}{5}.$$

At $k = \frac{2}{5}$, both intersections coincide at the vertex, giving only one root. At $k = 4$, the line is parallel to the right branch and has no intersection with that branch, again leaving only one root. Hence

$$\boxed{\frac{2}{5} < k < 4}.$$

FINAL ANSWER

(a) $P = \left(\frac{5}{2}, 3\right)$ (b) $x = \frac{15}{7}$ or $x = 5$ (c) $\frac{2}{5} < k < 4$

Question 9

7 marks | Graph Transformations | Main topic: Trig Equations & R-Form

9. $\mathrm{f}(\theta) = 5\cos\theta - 4\sin\theta \qquad \theta \in \mathbb{R}$

(a) Express $\mathrm{f}(\theta)$ in the form $R\cos(\theta + \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

The curve with equation $y = \cos\theta$ is transformed onto the curve with equation $y = \mathrm{f}(\theta)$ by a sequence of two transformations.

Given that the first transformation is a stretch and the second a translation,

(b) (i) describe fully the transformation that is a stretch,

(ii) describe fully the transformation that is a translation.

(2)

Given

$$\mathrm{g}(\theta) = \frac{90}{4 + (\mathrm{f}(\theta))^2} \qquad \theta \in \mathbb{R}$$

(c) find the range of g.

(2)

1 Part (a)

Expanding the required form gives

$$R\cos(\theta+\alpha) = R\cos\alpha\cos\theta - R\sin\alpha\sin\theta.$$

Comparing coefficients with $5\cos\theta - 4\sin\theta$,

$$R\cos\alpha = 5, \qquad R\sin\alpha = 4.$$

Squaring and adding, using $R > 0$, gives

$$R^2 = 5^2 + 4^2 = 41 \quad \Longrightarrow \quad \boxed{R = \sqrt{41}}.$$

Also, since $0 < \alpha < \frac{\pi}{2}$,

$$\tan\alpha = \frac{4}{5}, \qquad \alpha = \arctan\left(\frac{4}{5}\right) = 0.6747409\ldots \quad \Longrightarrow \quad \boxed{\alpha = 0.675 \text{ radians (3 d.p.)}}.$$

Thus the exact form is

$$\boxed{f(\theta) = \sqrt{41}\cos\left(\theta + \arctan\frac{4}{5}\right)}.$$

2 Part (b)(i)

A stretch parallel to the y-axis with scale factor $\boxed{\sqrt{41}}$, taking $y = \cos\theta$ to $y = \sqrt{41}\cos\theta$.

3 Part (b)(ii)

A translation by the vector

$$\boxed{\begin{pmatrix} -\alpha \\ 0 \end{pmatrix}} = \begin{pmatrix} -\arctan(4/5) \\ 0 \end{pmatrix} \approx \begin{pmatrix} -0.675 \\ 0 \end{pmatrix}.$$

This moves the curve α units to the left, giving $y = \sqrt{41}\cos(\theta+\alpha)$.

4

Part (c)

Since θ can be any real number,

$$-\sqrt{41} \le f(\theta) \le \sqrt{41} \quad \Longrightarrow \quad 0 \le (f(\theta))^2 \le 41.$$

Hence $4 \le 4 + (f(\theta))^2 \le 45$. Taking reciprocals of these positive bounds and multiplying by 90 gives

$$\boxed{2 \le g(\theta) \le \frac{45}{2}}.$$

Both endpoints are attained: the minimum occurs when $f(\theta) = \pm\sqrt{41}$, and the maximum when $f(\theta) = 0$.

FINAL ANSWER

(a) $R = \sqrt{41}$, $\alpha = 0.675$ radians (3 d.p.), $f(\theta) = \sqrt{41}\cos\left(\theta + \arctan\frac{4}{5}\right)$ (b)(i) $\sqrt{41}$ (b)(ii) $\begin{pmatrix} -\alpha \\ 0 \end{pmatrix}$ (c) $2 \le g(\theta) \le \frac{45}{2}$

Question 6

9 marks | Graph Transformations | Main topic: Numerical Methods

6.

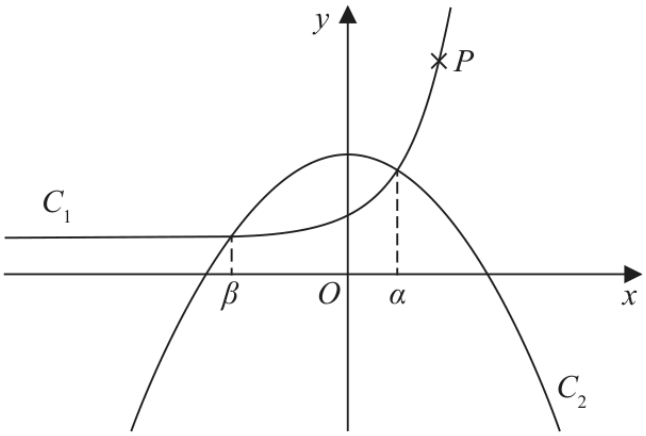


Figure 3

Figure 3 shows a sketch of curve C_1 with equation $y = 5e^{x-1} + 3$

and curve C_2 with equation $y = 10 - x^2$

The point P lies on C_1 and has y coordinate 18

(a) Find the x coordinate of P, writing your answer in the form $\ln k$, where k is a constant to be found.

(3)

The curve C_1 meets the curve C_2 at $x = \alpha$ and at $x = \beta$, as shown in Figure 3.

(b) Using a suitable interval and a suitable function that should be stated, show that to 3 decimal places $\alpha = 1.134$

(3)

The iterative equation

$$x_{n+1} = -\sqrt{7 - 5e^{x_n - 1}}$$

is used to find an approximation to β.

Using this iterative formula with $x_1 = -3$

(c) find the value of x_2 and the value of β, giving each answer to 6 decimal places.

(3)

1 Part (a)

At P, $18 = 5e^{x-1} + 3$, so $e^{x-1} = 3$. Taking natural logarithms gives $x - 1 = \ln 3$, hence $x = 1 + \ln 3 = \ln e + \ln 3 = \ln(3e)$.

$$\boxed{x = \ln(3e), \qquad k = 3e}$$

2 Part (b)

Define $F(x) = 5e^{x-1} + x^2 - 7$; its zeros are the intersection abscissae, and F is continuous. At the endpoints of the three-decimal rounding interval,

$$F(1.1335) = -0.0010714193\ldots < 0, \qquad F(1.1345) = 0.0069135450\ldots > 0.$$

Continuity guarantees a root between them. Moreover $F'(x) = 5e^{x-1} + 2x > 0$ on this positive interval, so that root is unique there. It is the positive intersection α, and rounds to 1.134 to three decimal places.

$$\boxed{\alpha = 1.134 \text{ to 3 d.p.}}$$

3 Part (c)

Start at $x_1 = -3$ and keep full precision in $x_{n+1} = -\sqrt{7 - 5e^{x_n - 1}}$:

n	x_n
1	-3
2	-2.628387681746
3	-2.620535277934
4	-2.620335527988
5	-2.620330426051
6	-2.620330295726
7	-2.620330292397

The later iterates agree to six decimal places. The negative root corresponds to β in the figure.

$$\boxed{x_2 = -2.628388, \qquad \beta = -2.620330}$$

FINAL ANSWER

(a) $x = \ln(3e), \qquad k = 3e$ (b) $\alpha = 1.134$ to 3 d.p. (c) $x_2 = -2.628388, \qquad \beta = -2.620330$

WMA13/01 OCTOBER 2020

Question 7

9 marks | Graph Transformations | Main topic: Trig Equations & R-Form

7. (a) Express $\cos x + 4\sin x$ in the form $R\cos(x - \alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

A scientist is studying the behaviour of seabirds in a colony.

She models the height above sea level, H metres, of one of the birds in the colony by the equation

$$H = \frac{24}{3 + \cos\left(\frac{1}{2}t\right) + 4\sin\left(\frac{1}{2}t\right)} \qquad 0 \leqslant t \leqslant 6.5$$

where t seconds is the time after it leaves the nest.

Find, according to the model,

(b) the minimum height of the seabird above sea level, giving your answer to the nearest cm,

(2)

(c) the value of t, to 2 decimal places, when $H = 10$

(4)

WORKED SOLUTION

Question 7 · WMA13/01 October 2020

1 **Part (a)**

Expand $R\cos(x-\alpha) = R\cos\alpha\cos x + R\sin\alpha\sin x$. Matching coefficients gives $R\cos\alpha = 1$, $R\sin\alpha = 4$. Thus $R = \sqrt{1^2 + 4^2} = \sqrt{17}$ and $\tan\alpha = 4$. Since $0 < \alpha < \pi/2$, $\alpha = \arctan 4 = 1.32581766\ldots$ radians.

$$\boxed{R = \sqrt{17}, \qquad \alpha = 1.326 \text{ radians}}$$

2 Part (b)

The model becomes $H = 24/[3 + \sqrt{17}\cos(t/2 - \alpha)]$. For $0 \le t \le 6.5$, the phase lies between $-\alpha$ and $3.25 - \alpha$; throughout this interval the denominator remains positive. Its maximum $3 + \sqrt{17}$ is attained at $t = 2\alpha = 2.651635\ldots$, inside the time interval. The height is then smallest:

$$H_{\min} = \frac{24}{3 + \sqrt{17}} = 3.369316876\ldots \text{ m}.$$

To the nearest centimetre this is 3.37 m (337 cm).

$$\boxed{H_{\min} = 3.37 \text{ m}}$$

3 Part (c)

For $H = 10$, $3 + \sqrt{17}\cos(t/2 - \alpha) = 24/10$, so

$$\cos(t/2 - \alpha) = -\frac{3}{5\sqrt{17}}.$$

Let $\gamma = \arccos[-3/(5\sqrt{17})]$. In the permitted phase interval $[-\alpha, 3.25 - \alpha]$, only $+\gamma$ occurs: $-\gamma < -\alpha$, and shifts by 2π are outside it. Using the unrounded $\alpha = \arctan 4$,

$$t = 2(\alpha + \gamma) = 6.085307854\ldots \text{ s}.$$

$$\boxed{t = 6.09 \text{ s}}$$

FINAL ANSWER

(a) $R = \sqrt{17}$, $\alpha = 1.326$ radians (b) $H_{\min} = 3.37$ m (c) $t = 6.09$ s

Question 6

8 marks | Graph Transformations | Main topic: Modulus

6. Given that k is a positive constant,

(a) on separate diagrams, sketch the graph with equation

(i) $y = k - 2|x|$

(ii) $y = \left|2x - \frac{k}{3}\right|$

Show on each sketch the coordinates, in terms of k, of each point where the graph meets or cuts the axes.

(4)

(b) Hence find, in terms of k, the values of x for which

$$\left|2x - \frac{k}{3}\right| = k - 2|x|$$

giving your answers in simplest form.

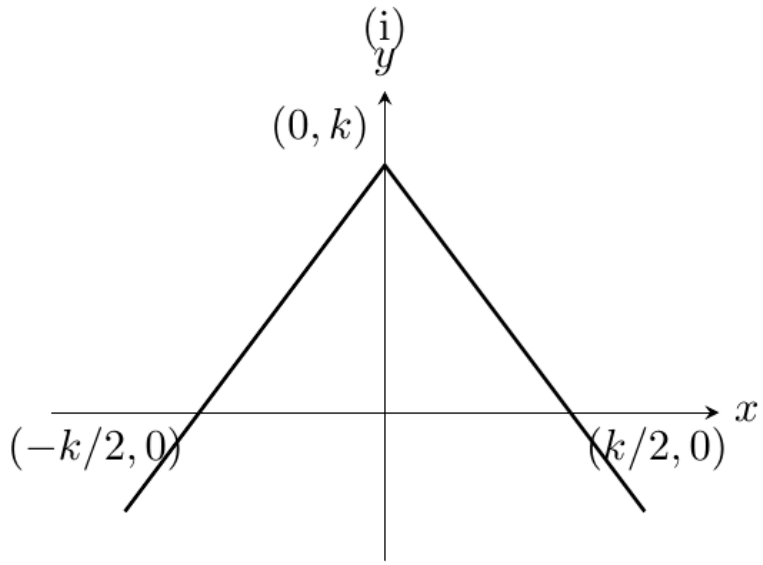
(4)

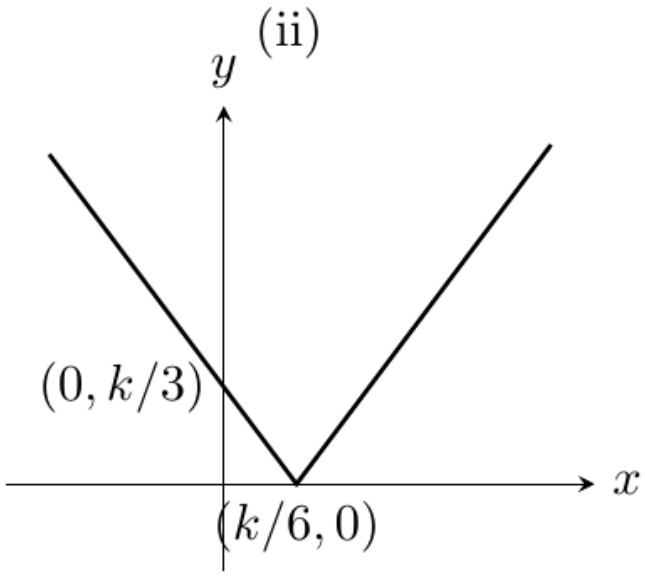
1 **Part (a)**

(i) $y = k - 2|x|$ is an inverted V with vertex $(0, k)$ and roots $x = \pm k/2$. (ii) $y = |2x - k/3|$ is an upright V with vertex $(k/6, 0)$ and y-intercept $(0, k/3)$. Because $k > 0$, both sketches have the ordering shown. Axes below use coordinates normalized by k, while labels give the original coordinates.

\qquad

$$\boxed{\text{(i) } (0, k), (\pm k/2, 0); \quad \text{(ii) } (0, k/3), (k/6, 0)}$$





2 **Part (b)**

Consider the breakpoints 0 and $k/6$. For $x < 0$, the equation is $k/3 - 2x = k + 2x$, giving $x = -k/6$, valid in this branch. For $0 \le x < k/6$, it is $k/3 - 2x = k - 2x$, impossible because $k > 0$. For $x \ge k/6$, it is $2x - k/3 = k - 2x$, giving $x = k/3$, valid in this branch. Both have nonnegative right side and satisfy the original equation.

$$\boxed{x = -\frac{k}{6}, \qquad x = \frac{k}{3}}$$

✓ **FINAL ANSWER**

(a) $\textbf{(i)}\ (0, k), (\pm k/2, 0); \quad \textbf{(ii)}\ (0, k/3), (k/6, 0)$ (b) $x = -\frac{k}{6}, \qquad x = \frac{k}{3}$

WMA13/01 MAY/JUNE 2021

Question 9

8 marks | Graph Transformations | Main topic: Trig Equations & R-Form

9. (a) Express $12\sin x - 5\cos x$ in the form $R\sin(x - \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the exact value of R and give the value of α in radians, to 3 decimal places.

(3)

The function g is defined by

$$g(\theta) = 10 + 12\sin\left(2\theta - \frac{\pi}{6}\right) - 5\cos\left(2\theta - \frac{\pi}{6}\right) \qquad \theta > 0$$

Find

(b) (i) the minimum value of $g(\theta)$

(ii) the smallest value of θ at which the minimum value occurs.

(3)

The function h is defined by

$$h(\beta) = 10 - (12\sin\beta - 5\cos\beta)^2$$

(c) Find the range of h.

(2)

WORKED SOLUTION

Question 9 · WMA13/01 May/June 2021

1 **Part (a)**

Expand $R\sin(x-\alpha) = R\cos\alpha\sin x - R\sin\alpha\cos x$. Therefore $R\cos\alpha = 12$, $R\sin\alpha = 5$. Thus $R = \sqrt{12^2 + 5^2} = 13$ and $\alpha = \arctan(5/12) = 0.3947911197\ldots$ radians, in the required first quadrant.

$$\boxed{13\sin(x - 0.395) \quad (R = 13,\ \alpha = 0.395 \text{ rad})}$$

2 Part (b)

Write $g(\theta) = 10 + 13\sin(2\theta - \pi/6 - \alpha)$. (i) The minimum is $10 - 13 = -3$. (ii) For sine to equal -1, $2\theta - \pi/6 - \alpha = -\pi/2 + 2\pi n$. Hence $\theta = -\pi/6 + \alpha/2 + \pi n$. For $n = 0$ this is negative, so the smallest permitted value uses $n = 1$:

$$\theta = \frac{5\pi}{6} + \frac{1}{2}\arctan(5/12) = 2.815389438\ldots \text{ radians}.$$

Use the unrounded phase angle for the calculation.

$$\boxed{\min g = -3; \qquad \theta_{\min} = \frac{5\pi}{6} + \frac{1}{2}\arctan(5/12)}$$

3 Part (c)

As β varies over the reals, $12\sin\beta - 5\cos\beta$ covers $[-13, 13]$. Its square therefore covers $[0, 169]$, not $[169, 169]$. Subtracting from 10 reverses the bounds, giving $[-159, 10]$; both endpoints are attained.

$$\boxed{-159 \le h(\beta) \le 10}$$

FINAL ANSWER

(a) $13\sin(x - 0.395) \quad (R = 13,\ \alpha = 0.395 \text{ rad})$ (b) $\min g = -3; \qquad \theta_{\min} = \frac{5\pi}{6} + \frac{1}{2}\arctan(5/12)$ (c) $-159 \le h(\beta) \le 10$

Question 8

7 marks | Graph Transformations

Main topic: Differentiation Techniques

8. A curve C has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \arcsin\left(\frac{1}{2}x\right) \qquad -2 \leqslant x \leqslant 2 \qquad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

(a) Sketch C.

(1)

(b) Given $x = 2\sin y$, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{A - x^2}}$$

where A is a constant to be found.

(3)

The point P lies on C and has y coordinate $\frac{\pi}{4}$

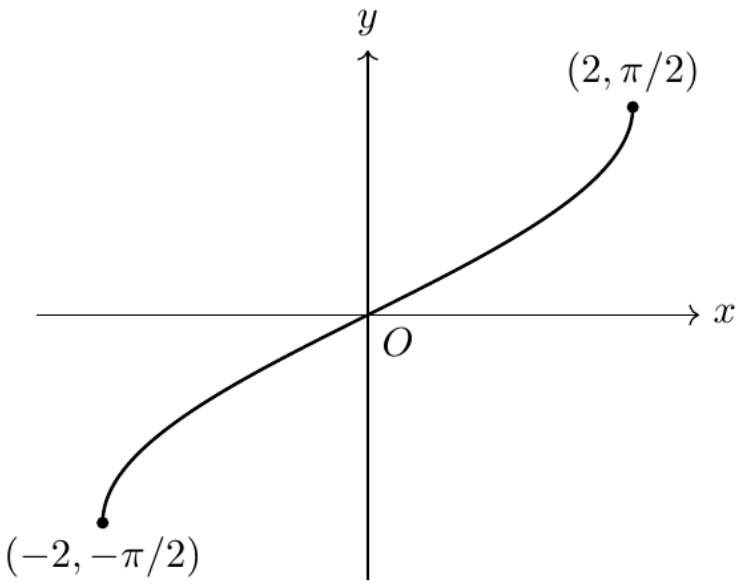
(c) Find the equation of the tangent to C at P. Write your answer in the form $y = mx + c$, where m and c are constants to be found.

(3)

1 Part (a)

The principal arcsine curve increases from $(-2, -\pi/2)$ through $(0, 0)$ to $(2, \pi/2)$. It is concave down on the negative-x half, concave up on the positive-x half, and vertical at both ends.

$$\boxed{y = \arcsin(x/2), \quad -2 \le x \le 2}$$



2 Part (b)

Differentiate the supplied inverse relation: $dx/dy = 2\cos y$. For $-\pi/2 < y < \pi/2$, cosine is positive, so $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2/4}$. Taking the reciprocal gives

$$\frac{dy}{dx} = \frac{1}{2\cos y} = \frac{1}{\sqrt{4 - x^2}}, \qquad -2 < x < 2.$$

Thus $A = 4$; the endpoints have vertical tangents, not finite derivatives.

$$\boxed{A = 4, \qquad \frac{dy}{dx} = \frac{1}{\sqrt{4 - x^2}}}$$

3 Part (c)

At $y = \pi/4$, $x = 2\sin(\pi/4) = \sqrt{2}$. The gradient is $1/\sqrt{4 - 2} = 1/\sqrt{2}$. Hence $y - \pi/4 = (x - \sqrt{2})/\sqrt{2}$, so $y = x/\sqrt{2} + \pi/4 - 1$.

$$\boxed{y = \frac{x}{\sqrt{2}} + \frac{\pi}{4} - 1}$$

FINAL ANSWER

(a) $y = \arcsin(x/2), \quad -2 \le x \le 2$ (b) $A = 4, \quad \frac{dy}{dx} = \frac{1}{\sqrt{4-x^2}}$ (c) $y = \frac{x}{\sqrt{2}} + \frac{\pi}{4} - 1$

Question 7

10 marks | Graph Transformations | Main topic: Modulus

7.

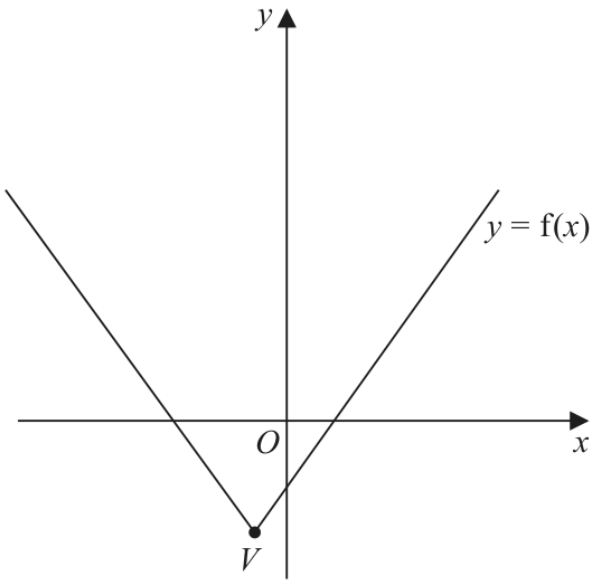


Figure 2

Figure 2 shows a sketch of part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \frac{1}{2}|2x + 7| - 10$$

(a) State the coordinates of the vertex, V, of the graph.

(2)

(b) Solve, using algebra,

$$\frac{1}{2}|2x + 7| - 10 \geqslant \frac{1}{3}x + 1$$

(4)

(c) Sketch the graph with equation

$$y = |\mathrm{f}(x)|$$

stating the coordinates of the local maximum point and each local minimum point.

(4)

1 Part (a)

At the vertex $2x + 7 = 0$, hence $x = -7/2$ and $f(x) = -10$.

$$\boxed{V = (-7/2, -10)}$$

2 Part (b)

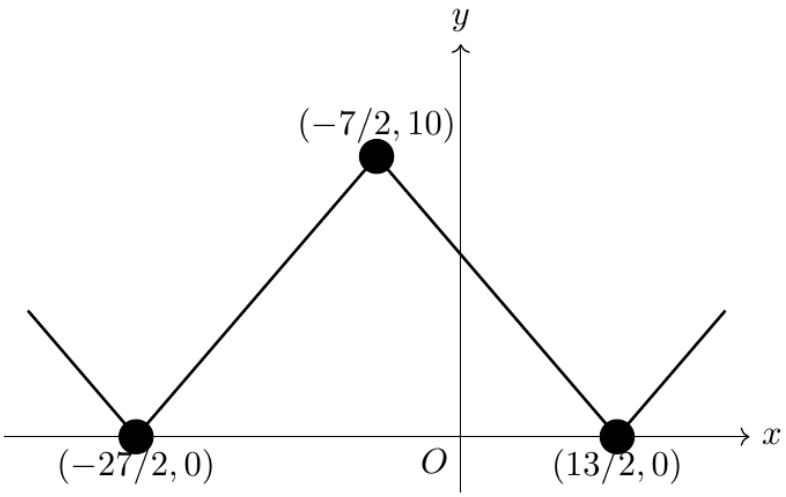
For $x \geq -7/2$, $f(x) = x - 13/2$. The inequality becomes $x - 13/2 \geq x/3 + 1$, giving $x \geq 45/4$, which satisfies this branch. For $x < -7/2$, $f(x) = -x - 27/2$. Then $-x - 27/2 \geq x/3 + 1$, giving $x \leq -87/8$, which satisfies that branch. Join the two permitted outer intervals, including the equality endpoints.

$$\boxed{x \leq -\frac{87}{8} \quad \text{or} \quad x \geq \frac{45}{4}}$$

3 Part (c)

The zeros of f satisfy $|2x + 7| = 20$, giving $x = -27/2$ and $x = 13/2$. Reflect the part below the x-axis upward, leaving the positive branches unchanged. This creates minima at $(-27/2, 0)$ and $(13/2, 0)$ and a local maximum at $(-7/2, 10)$.

$$\boxed{\text{Maximum } (-7/2, 10); \quad \text{minima } (-27/2, 0), (13/2, 0)}$$



FINAL ANSWER

(a) $V = (-7/2, -10)$ (b) $x \leq -\frac{87}{8}$ or $x \geq \frac{45}{4}$ (c)
Maximum $(-7/2, 10)$; minima $(-27/2, 0), (13/2, 0)$

Question 6

10 marks | Graph Transformations | Main topic: Applications of Differentiation

6.

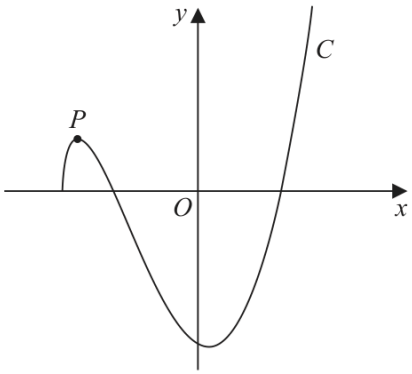


Figure 3

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

The function f is defined by

$$f(x) = 5(x^2 - 2)(4x + 9)^{\frac{1}{2}} \qquad x \geqslant -\frac{9}{4}$$

(a) Show that

$$f'(x) = \frac{k(5x^2 + 9x - 2)}{(4x + 9)^{\frac{1}{2}}}$$

where k is an integer to be found.

(4)

(b) Hence, find the values of x for which $f'(x) = 0$

(1)

Figure 3 shows a sketch of the curve C with equation $y = f(x)$.

The curve has a local maximum at the point P

(c) Find the exact coordinates of P

(2)

The function g is defined by

$$g(x) = 2f(x) + 4 \qquad -\frac{9}{4} \leqslant x \leqslant 0$$

(d) Find the range of g

(3)

1 Part (a)

Use product and chain rules:

$$f'(x) = 10x(4x+9)^{1/2} + 10(x^2-2)(4x+9)^{-1/2}.$$

With a common denominator,

$$f'(x) = \frac{10x(4x+9) + 10(x^2-2)}{\sqrt{4x+9}} = \frac{10(5x^2+9x-2)}{\sqrt{4x+9}}.$$

This derivative formula applies for $x > -9/4$; the endpoint derivative is not finite.

$$\boxed{k = 10}$$

2 Part (b)

The denominator is positive in the derivative domain. Thus $5x^2+9x-2 = (5x-1)(x+2) = 0$.

$$\boxed{x = -2, \quad \frac{1}{5}}$$

3 Part (c)

The derivative changes from positive to negative at $x = -2$, so this is the local maximum P. Substitute into f: $f(-2) = 5(4-2)\sqrt{-8+9} = 10$.

$$\boxed{P = (-2, 10)}$$

4 Part (d)

On $[-9/4, 0]$, f rises from $f(-9/4) = 0$ to $f(-2) = 10$, then decreases to $f(0) = -30$. The other stationary point $1/5$ is outside this restricted interval. Thus the minimum and maximum of f on it are -30 and 10. Applying the positive vertical transformation $g = 2f + 4$ gives endpoints $2(-30)+4 = -56$ and $2(10)+4 = 24$, both attained.

$$\boxed{-56 \le g(x) \le 24}$$

FINAL ANSWER

(a) $k = 10$ (b) $x = -2, \quad \frac{1}{5}$ (c) $P = (-2, 10)$ (d) $-56 \le g(x) \le 24$

Question 7

12 marks | Graph Transformations | Main topic: Modulus

7.

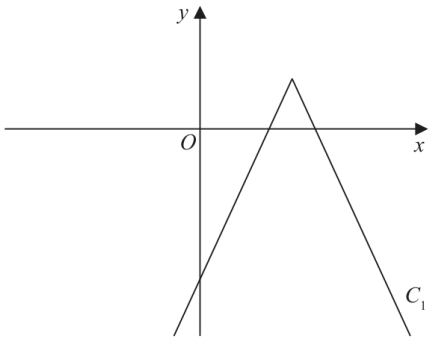


Figure 3

Figure 3 shows a sketch of the graph of C_1 with equation

$$y = 5 - |3x - 22|$$

(a) Write down the coordinates of

(i) the vertex of C_1

(ii) the intersection of C_1 with the y-axis.

(2)

(b) Find the x coordinates of the intersections of C_1 with the x-axis.

(2)

Diagram 1, shown on page 21, is a copy of Figure 3.

(c) On Diagram 1, sketch the curve C_2 with equation

$$y = \frac{1}{9}x^2 - 9$$

Identify clearly the coordinates of any points of intersection of C_2 with the coordinate axes.

(3)

(d) Find the coordinates of the points of intersection of C_1 and C_2
(*Solutions relying entirely on calculator technology are not acceptable.*)

(5)

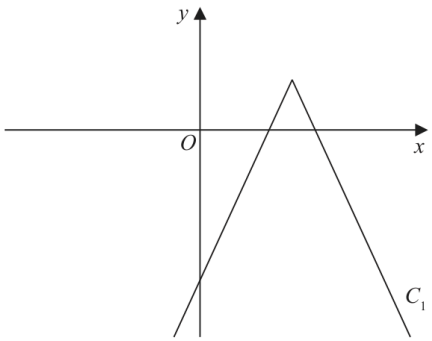


Diagram 1

1 Part (a)

(i) The modulus is zero at $3x - 22 = 0$, giving the vertex $(22/3, 5)$. (ii) At $x = 0$, $y = 5 - 22 = -17$, so the y-axis intersection is $(0, -17)$.

$$\boxed{(i)\ (22/3, 5), \qquad (ii)\ (0, -17)}$$

2 Part (b)

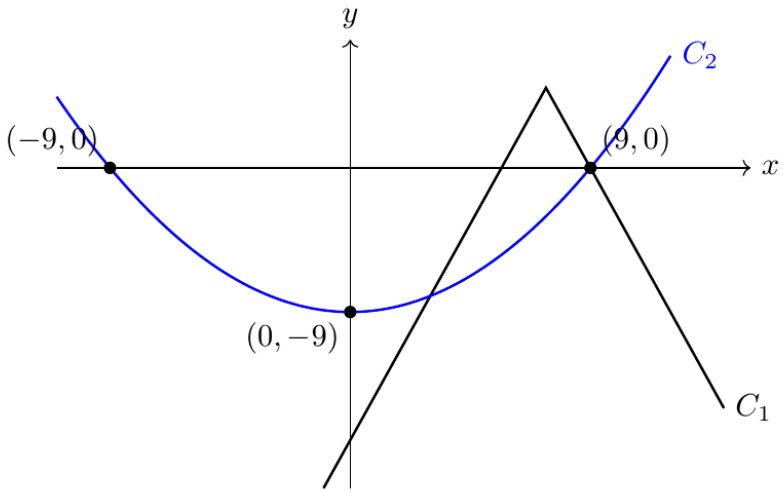
Set $y = 0$: $|3x - 22| = 5$. Thus $3x - 22 = 5$ or -5, giving $x = 9$ or $17/3$.

$$\boxed{x = \frac{17}{3}, \quad 9}$$

3 Part (c)

The curve $C_2 : y = x^2/9 - 9$ is an upward-opening parabola, symmetric about the y-axis. Its vertex and y-intercept are $(0, -9)$; setting y=0 gives $x^2 = 81$, hence x-intercepts $(-9, 0)$ and $(9, 0)$. The sketch below includes C_1 for context and labels all three axis intersections of C_2.

$$\boxed{(-9, 0), \quad (9, 0), \quad (0, -9)}$$



4 Part (d)

For $x \le 22/3$, C_1 is $y = 3x - 17$. Equating to C_2 gives

$$x^2/9 - 9 = 3x - 17 \implies x^2 - 27x + 72 = (x - 3)(x - 24) = 0.$$

Only $x = 3$ belongs to this branch; $y = -8$. For $x \ge 22/3$, C_1 is $y = 27 - 3x$. Then

$$x^2/9 - 9 = 27 - 3x \implies x^2 + 27x - 324 = (x - 9)(x + 36) = 0.$$

Only $x = 9$ belongs to this branch; $y = 0$. The rejected candidates 24 and -36 lie outside their assumed branch regions.

$$\boxed{(3, -8), \qquad (9, 0)}$$

FINAL ANSWER

(a) $(i)\ (22/3, 5), \quad (ii)\ (0, -17)$ (b) $x = \frac{17}{3}, \quad 9$ (c) $(-9, 0), \quad (9, 0), \quad (0, -9)$ (d) $(3, -8), \quad (9, 0)$

WMA13/01 OCTOBER 2022

Question 8

9 marks | Graph Transformations | Main topic: Trig Equations & R-Form

8. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Express $8\sin x - 15\cos x$ in the form $R\sin(x-\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R, and give the value of α, in radians, to 4 significant figures.

(3)

$$f(x) = \frac{15}{41 + 16\sin x - 30\cos x} \qquad x > 0$$

(b) Find

(i) the minimum value of $f(x)$

(ii) the smallest value of x at which this minimum value occurs.

(4)

(c) State the y coordinate of the minimum points on the curve with equation

$$y = 2f(x) - 5 \qquad x > 0$$

(1)

(d) State the smallest value of x at which a maximum point occurs for the curve with equation

$$y = -f(2x) \qquad x > 0$$

(1)

WORKED SOLUTION

Question 8 · WMA13/01 October 2022

1 **Part (a)**

Expand $R\sin(x-\alpha) = R\cos\alpha\sin x - R\sin\alpha\cos x$. Thus $R\cos\alpha = 8$, $R\sin\alpha = 15$, so $R = \sqrt{64+225} = 17$ and $\alpha = \arctan(15/8) = 1.0808390005\ldots$ radians, in the required quadrant.

$$\boxed{R = 17, \qquad \alpha = 1.081 \text{ radians}}$$

2 Part (b)

Using the exact phase from (a),

$$f(x) = \frac{15}{41 + 34\sin(x - \alpha)}.$$

The denominator lies between 7 and 75, so it is always positive. (i) The smallest fraction occurs at its largest denominator: $f_{\min} = 15/75 = 1/5$. (ii) This occurs when $x - \alpha = \pi/2 + 2n\pi$. The smallest positive value is $x = \alpha + \pi/2 = 2.6516353273\ldots$, or 2.65 to three significant figures.

$$\boxed{f_{\min} = \frac{1}{5}, \qquad x = 2.65}$$

---

3 Part (c)

The transformation $2f(x) - 5$ preserves the locations of minima and maps their y coordinate to $2(1/5) - 5 = -23/5$.

$$\boxed{y = -\frac{23}{5}}$$

---

4 Part (d)

Reflection in the x-axis changes minima of f into maxima of $-f$. The argument $2x$ halves their x coordinates, so the first positive maximum is at

$$x = \frac{\alpha + \pi/2}{2} = 1.3258176637\ldots \approx 1.33.$$

$$\boxed{x = 1.33}$$

✓ FINAL ANSWER

(a) $R = 17, \qquad \alpha = 1.081$ radians (b) $f_{\min} = \frac{1}{5}, \qquad x = 2.65$ (c) $y = -\frac{23}{5}$ (d) $x = 1.33$

Question 6

8 marks | Graph Transformations | Main topic: Modulus

6.

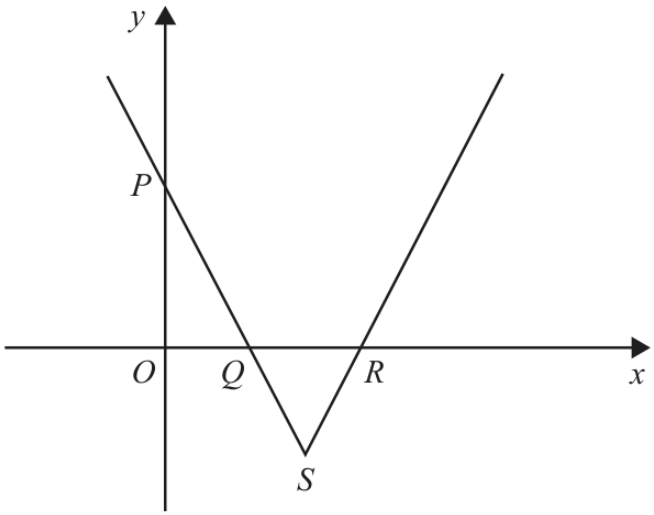


Figure 2

Figure 2 shows a sketch of the graph with equation

$$y = |3x - 5a| - 2a$$

where a is a positive constant.

The graph

- cuts the y-axis at the point P
- cuts the x-axis at the points Q and R
- has a minimum point at S

(a) Find, in simplest form in terms of a, the coordinates of

(i) point P

(ii) points Q and R

(iii) point S

(4)

(b) Find, in simplest form in terms of a, the values of x for which

$$|3x - 5a| - 2a = |x - 2a|$$

(4)

1 Part (a)

For $y = |3x - 5a| - 2a$ with $a > 0$: (i) At $x = 0$, $y = 5a - 2a = 3a$, so $P = (0, 3a)$. (ii) At $y = 0$, $|3x - 5a| = 2a$ gives $3x - 5a = \pm 2a$. Thus $x = a$ or $7a/3$; since $a < 7a/3$, $Q = (a, 0)$ and $R = (7a/3, 0)$. (iii) The modulus vanishes at $x = 5a/3$, giving the minimum $S = (5a/3, -2a)$.

$$\boxed{P = (0, 3a), \quad Q = (a, 0), \quad R = (7a/3, 0), \quad S = (5a/3, -2a)}$$

2 Part (b)

Split $|3x - 5a| - 2a = |x - 2a|$ at $x = 5a/3$ and $x = 2a$; their order uses $a > 0$. For $x < 5a/3$: $3a - 3x = 2a - x$, so $x = a/2$, which is in this region. For $5a/3 \le x < 2a$: $3x - 7a = 2a - x$, so $x = 9a/4$; this is outside the assumed region and is rejected. For $x \ge 2a$: $3x - 7a = x - 2a$, so $x = 5a/2$, which is valid. These three regions cover the whole real line, so there are exactly two solutions.

$$\boxed{x = \frac{a}{2}, \qquad x = \frac{5a}{2}}$$

FINAL ANSWER

(a) $P = (0, 3a), \quad Q = (a, 0), \quad R = (7a/3, 0), \quad S = (5a/3, -2a)$ (b) $x = \frac{a}{2}, \qquad x = \frac{5a}{2}$

Question 6

9 marks | Graph Transformations | Main topic: Modulus

6.

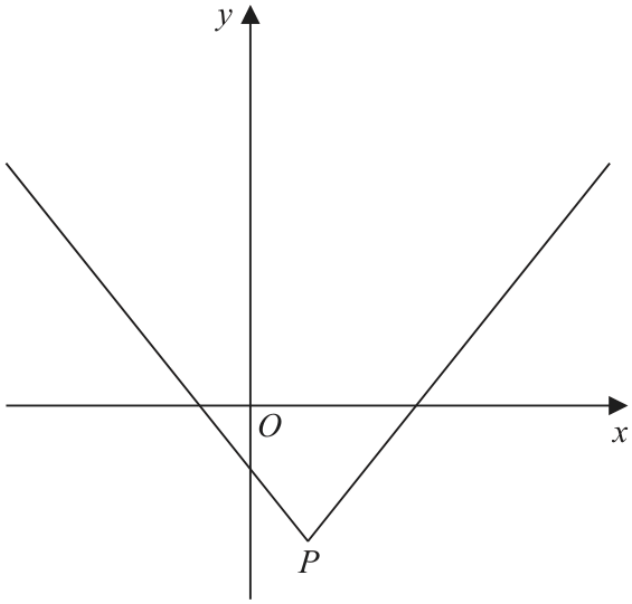


Figure 2

Figure 2 shows a sketch of the graph $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 3|x - 2| - 10$$

The vertex of the graph is at point P, shown in Figure 2.

(a) Find the coordinates of P

(2)

(b) Find $\mathrm{ff}(0)$

(2)

(c) Solve the inequality

$$3|x - 2| - 10 < 5x + 10$$

(2)

(d) Solve the equation

$$\mathrm{f}(|x|) = 0$$

(3)

1 Part (a)

The modulus is smallest when $x - 2 = 0$, giving $x = 2$ and $f(2) = -10$. The two straight-line branches meet there, so this is the vertex of the V-shaped graph.

$$\boxed{P = (2, -10)}$$

2 Part (b)

First $f(0) = 3|0 - 2| - 10 = -4$. Then $ff(0) = f(-4) = 3|-4 - 2| - 10 = 18 - 10 = 8$. In particular $ff(0)$ means $f(f(0))$, not the square of $f(0)$.

$$\boxed{ff(0) = 8}$$

3 Part (c)

For $x < 2$, the inequality becomes $3(2 - x) - 10 < 5x + 10$, so $-4 - 3x < 5x + 10$, giving $x > -7/4$. This yields $-7/4 < x < 2$. For $x \geq 2$, it becomes $3x - 16 < 5x + 10$, giving $x > -13$; every $x \geq 2$ satisfies this. The union is $x > -7/4$.

$$\boxed{x > -\frac{7}{4}}$$

4 Part (d)

Let $u = |x| \geq 0$. Then $f(|x|) = 0$ gives

$$3|u - 2| - 10 = 0 \quad \Longrightarrow \quad u - 2 = \pm\frac{10}{3}.$$

Thus $u = 16/3$ or $u = -4/3$. Reject $u = -4/3$ because $u \geq 0$; $|x| = 16/3$ has the two solutions below.

$$\boxed{x = -\frac{16}{3}, \quad \frac{16}{3}}$$

FINAL ANSWER

(a) $P = (2, -10)$ (b) $ff(0) = 8$ (c) $x > -\frac{7}{4}$ (d) $x = -\frac{16}{3}, \quad \frac{16}{3}$

Question 8

9 marks | Graph Transformations

Main topic: Applications of Differentiation

8.

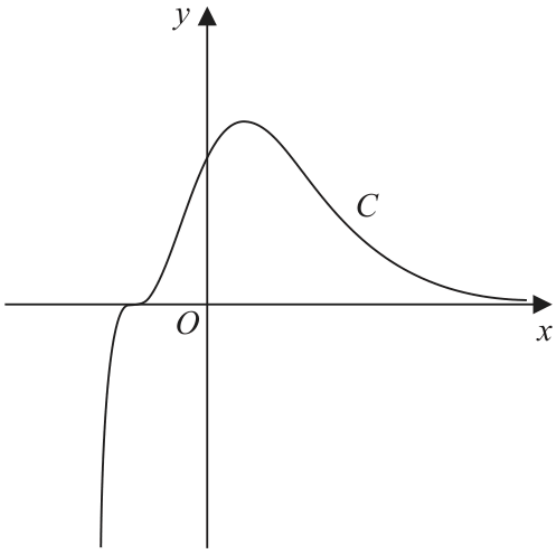


Figure 3

Figure 3 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = (2x + 1)^3 \mathrm{e}^{-4x}$$

(a) Show that

$$\mathrm{f}'(x) = A(2x + 1)^2 (1 - 4x)\, \mathrm{e}^{-4x}$$

where A is a constant to be found.

(4)

(b) Hence find the exact coordinates of the two stationary points on C.

(3)

The function g is defined by

$$\mathrm{g}(x) = 8\,\mathrm{f}(x - 2)$$

(c) Find the coordinates of the maximum stationary point on the curve with equation $y = \mathrm{g}(x)$.

(2)

1 Part (a)

The product rule gives

$$f'(x) = 6(2x+1)^2e^{-4x} - 4(2x+1)^3e^{-4x} = (2x+1)^2e^{-4x}[6-4(2x+1)] = 2(2x+1)^2(1-4x)e^{-4x}.$$

Comparison with the stated form gives $A = 2$.

$$\boxed{A = 2}$$

2 Part (b)

Since $e^{-4x} > 0$, $f'(x) = 0$ means $2x+1=0$ or $1-4x=0$. Thus $x = -1/2$ or $x = 1/4$.
Substitution gives $f(-1/2) = 0$ and $f(1/4) = (3/2)^3e^{-1} = 27/(8e)$.

$$\boxed{\left(-\frac{1}{2}, 0\right), \quad \left(\frac{1}{4}, \frac{27}{8e}\right)}$$

3 Part (c)

The derivative changes from positive to negative at $x = 1/4$, so this is the maximum; at $x = -1/2$ it does not change sign. The transformation $g(x) = 8f(x-2)$ sends (x, y) to $(x+2, 8y)$. Hence the maximum stationary point is $(1/4 + 2, 8 \cdot 27/(8e)) = (9/4, 27/e)$.

$$\boxed{\left(\frac{9}{4}, \frac{27}{e}\right)}$$

FINAL ANSWER

(a) $A = 2$ (b) $\left(-\frac{1}{2}, 0\right), \quad \left(\frac{1}{4}, \frac{27}{8e}\right)$ (c) $\left(\frac{9}{4}, \frac{27}{e}\right)$

Question 9

9 marks | Graph Transformations | Main topic: Exponentials & Logarithms

9. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

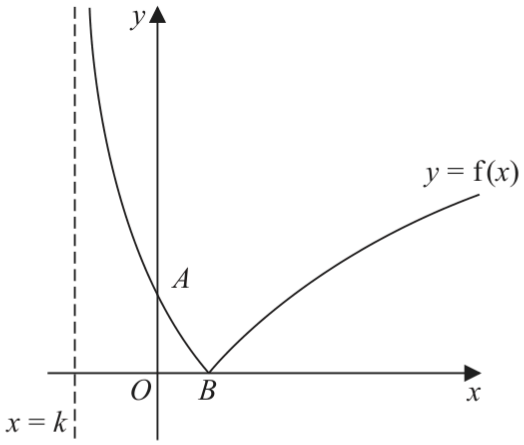


Figure 2

Figure 2 shows a sketch of the curve with equation

$$y = |2 - 4\ln(x + 1)| \qquad x > k$$

where k is a constant.

Given that the curve

- has an asymptote at $x = k$
- cuts the y-axis at point A
- meets the x-axis at point B

as shown in Figure 2,

(a) state the value of k

(1)

(b) (i) find the y coordinate of A

(ii) find the exact x coordinate of B

(3)

(c) Using algebra and showing your working, find the set of values of x such that

$$|2 - 4\ln(x + 1)| > 3$$

(5)

1 Part (a)

The logarithm requires $x + 1 > 0$, so the domain is $x > -1$. As $x \to -1^+$, $\ln(x+1) \to -\infty$ and the ordinate tends to infinity.

$$\boxed{k = -1}$$

2 Part (b)

(i) At the y-axis, $x = 0$, giving $y = |2 - 4\ln 1| = 2$.

(ii) At the x-axis, $|2 - 4\ln(x+1)| = 0$, so

$$\ln(x+1) = \frac{1}{2} \quad \Longrightarrow \quad x = e^{1/2} - 1.$$

Thus $A = (0, 2)$ and $B = (e^{1/2} - 1, 0)$.

$$\boxed{y_A = 2, \qquad x_B = e^{1/2} - 1}$$

3 Part (c)

On $x > -1$, split the absolute-value inequality into two alternatives:

$$2 - 4\ln(x+1) > 3 \quad \text{or} \quad 2 - 4\ln(x+1) < -3.$$

Dividing by -4 reverses the inequality in each branch:

$$\ln(x+1) < -\frac{1}{4} \quad \text{or} \quad \ln(x+1) > \frac{5}{4}.$$

Since the exponential is increasing, this becomes $x + 1 < e^{-1/4}$ or $x + 1 > e^{5/4}$. Intersect with the logarithmic domain; neither boundary is included because the source inequality is strict.

$$\boxed{-1 < x < e^{-1/4} - 1 \quad \text{or} \quad x > e^{5/4} - 1}$$

FINAL ANSWER

(a) $k = -1$ (b) $y_A = 2, \qquad x_B = e^{1/2} - 1$ (c) $-1 < x < e^{-1/4} - 1 \quad \text{or} \quad x > e^{5/4} - 1$

Question 8

11 marks | Graph Transformations | Main topic: Modulus

8.

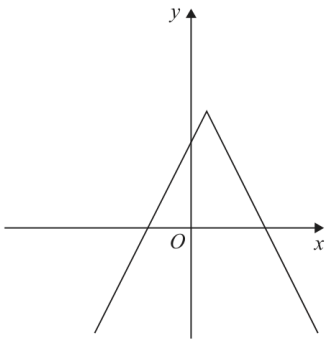


Figure 2

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

The graph shown in Figure 2 has equation

$$y = a - |2x - b|$$

where a and b are positive constants, $a > b$

(a) Find, giving your answer in terms of a and b,

(i) the coordinates of the maximum point of the graph,

(ii) the coordinates of the point of intersection of the graph with the y-axis,

(iii) the coordinates of the points of intersection of the graph with the x-axis.

(5)

On page 24 there is a copy of Figure 2 called Diagram 1.

(b) On Diagram 1, sketch the graph with equation

$$y = |x| - 1$$

(2)

Given that the graphs $y = |x| - 1$ and $y = a - |2x - b|$ intersect at $x = -3$ and $x = 5$

(c) find the value of a and the value of b

(4)

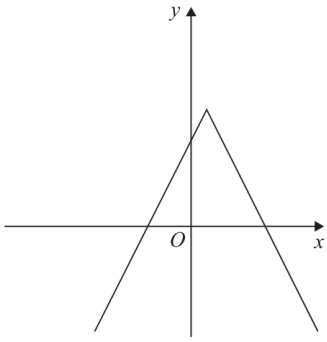


Diagram 1

1 Part (a)

(i) Since $|2x-b| \geq 0$, the maximum occurs at $2x-b=0$: $x=b/2$ and $y=a$. (ii) At $x=0$, $y=a-|-b|=a-b$ because $b>0$. (iii) At an x-axis crossing, $|2x-b|=a$, hence $2x-b=\pm a$. Since $a>b>0$, $(b-a)/2<0$ and $(b+a)/2>0$, matching the sketch.

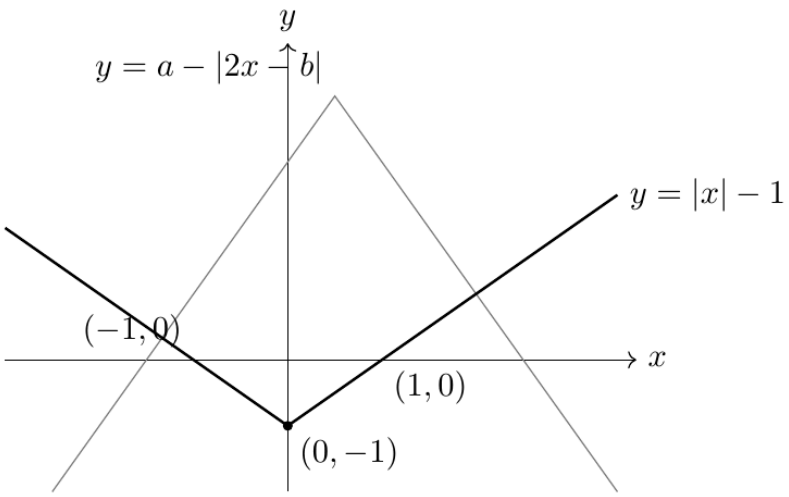
$$\boxed{\left(\frac{b}{2}, a\right); \quad (0, a-b); \quad \left(\frac{b-a}{2}, 0\right), \ \left(\frac{b+a}{2}, 0\right)}$$

2 Part (b)

The graph $y=|x|-1$ is a V with vertex $(0,-1)$ and intercepts $(-1,0)$ and $(1,0)$. Its left branch is $y=-x-1$ for $x<0$ and its right branch is $y=x-1$ for $x \geq 0$. Add this V to the supplied axes; the original inverted V remains unchanged.

The grey original curve is schematic; the three labelled points belong to the added graph.

$$\boxed{\text{Vertex } (0,-1); \quad \text{zeros } x=\pm 1}$$



3 Part (c)

At $x=-3$, the common ordinate is $|-3|-1=2$. Since $b>0$, $|-6-b|=6+b$, so $a-(6+b)=2$, giving $a=b+8$. At $x=5$, the common ordinate is $|5|-1=4$, so $a-|10-b|=4$. Substitute $a=b+8$ to obtain $|10-b|=b+4$. If $b \leq 10$, then $10-b=b+4$, giving $b=3$, $a=11$. If $b>10$, then $b-10=b+4$, impossible. Both accepted values satisfy $a>b>0$.

$$\boxed{a=11, \quad b=3}$$

FINAL ANSWER

(a) $\left(\frac{b}{2}, a\right); \quad (0, a-b); \quad \left(\frac{b-a}{2}, 0\right), \ \left(\frac{b+a}{2}, 0\right)$ (b) $\text{Vertex } (0,-1); \quad \text{zeros } x=\pm 1$ (c) $a=11, \quad b=3$

WMA13/01 JANUARY 2025

Question 6

8 marks | Graph Transformations

Main topic: Functions, Composition & Inverses

6. The function f is defined by

$$\mathrm{f}(x) = \frac{4x + 3}{x - 2} \qquad x \neq 2$$

(a) Find f^{-1}

(3)

(b) Show that

$$\mathrm{ff}(x) = \frac{ax + b}{cx + d}$$

where a, b, c and d are integers to be found.

(3)

The point $P(3, 15)$ lies on the curve with equation $y = \mathrm{f}(x)$.

(c) Find the point to which P is mapped when $y = \mathrm{f}(x)$ is transformed to the curve with equation $y = 2\mathrm{f}(3x) + 8$

(2)

WORKED SOLUTION

Question 6 · WMA13/01 January 2025

1 **Part (a)**

Let $y = (4x + 3)/(x - 2)$. Then $yx - 2y = 4x + 3$, so

$$x(y - 4) = 2y + 3 \quad \Longrightarrow \quad x = \frac{2y + 3}{y - 4}.$$

Swap the input and output names. Since $f(x) = 4 + 11/(x - 2)$ can never equal 4, the inverse domain excludes 4.

$$\boxed{f^{-1}(x) = \frac{2x + 3}{x - 4}, \quad x \neq 4}$$

2 Part (b)

Substitute $f(x)$ into both occurrences of the argument:

$$ff(x) = \frac{4\left(\frac{4x+3}{x-2}\right)+3}{\left(\frac{4x+3}{x-2}\right)-2} = \frac{4(4x+3)+3(x-2)}{4x+3-2(x-2)} = \frac{19x+6}{2x+7}.$$

Thus $a=19$, $b=6$, $c=2$, $d=7$. The composite retains $x \neq 2$ from the inner function and also requires $f(x) \neq 2$, that is $x \neq -7/2$.

$$\boxed{ff(x) = \frac{19x+6}{2x+7}, \quad (a,b,c,d) = (19,6,2,7)}$$

---

3 Part (c)

For $y = 2f(3x)+8$, the point mapping is $(x,y) \mapsto (x/3, 2y+8)$. Hence $(3,15) \mapsto (3/3, 2(15)+8) = (1,38)$.

$$\boxed{(1,38)}$$

FINAL ANSWER

(a) $f^{-1}(x) = \frac{2x+3}{x-4}, \quad x \neq 4$ (b) $ff(x) = \frac{19x+6}{2x+7}, \quad (a,b,c,d) = (19,6,2,7)$ (c) $(1,38)$

WMA13/01 JANUARY 2025

Question 7

8 marks | Graph Transformations | Main topic: Modulus

7. Given that a and b are positive constants with $a > b$,

(a) sketch, on **separate** diagrams, the graph with equation

(i) $y = |3x - a|$

(ii) $y = |3x - a| - b$

Show on each sketch

- the coordinates of the minimum point on the graph
- the coordinates of the point at which the graph crosses the y-axis

(6)

(b) Solve the equation

$$|3x - a| - b = 5x$$

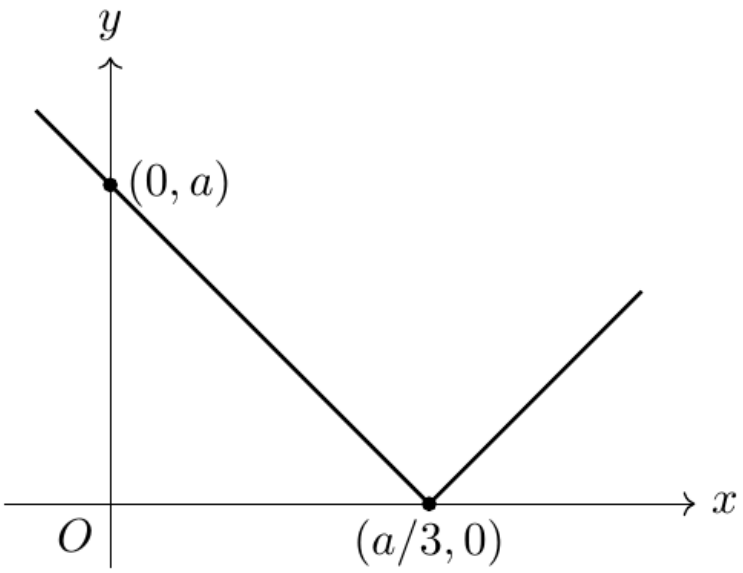
giving any solution for x in terms of a and b.

(2)

1 Part (a)(i)

For $x \leq a/3$, $y = a - 3x$; for $x \geq a/3$, $y = 3x - a$. These two straight branches form a V with minimum $(a/3, 0)$ and y-intercept $(0, a)$.

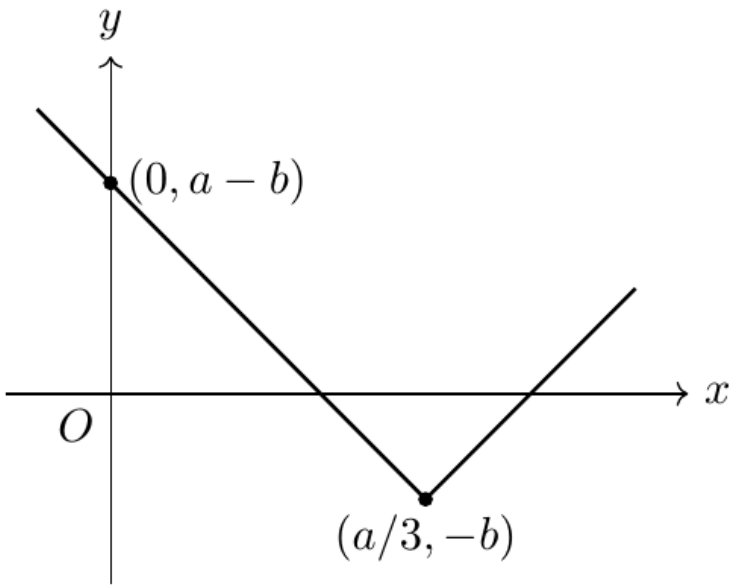
$$\boxed{\text{minimum } (a/3, 0), \quad y\text{-intercept } (0, a)}$$



2 Part (a)(ii)

Subtracting b translates the previous graph vertically down by b. The minimum becomes $(a/3, -b)$ and the y-intercept becomes $(0, a - b)$, which is above the x-axis because $a > b > 0$.

$$\boxed{\text{minimum } (a/3, -b), \quad y\text{-intercept } (0, a - b)}$$



3

Part (b)

For $x \leq a/3$, the equation becomes $a - 3x - b = 5x$, giving $x = (a-b)/8$. This is positive and less than $a/8 < a/3$, so it satisfies this branch condition. For $x \geq a/3$, the equation becomes $3x - a - b = 5x$, giving $x = -(a+b)/2 < 0$. This contradicts $x \geq a/3 > 0$ and must be rejected. Therefore only one solution remains.

$$\boxed{x = \frac{a-b}{8}}$$

FINAL ANSWER

(a)(i) $\textbf{minimum } (a/3, 0), \quad y\textbf{-intercept } (0, a)$ (a)(ii) $\textbf{minimum } (a/3, -b), \quad y\textbf{-intercept } (0, a-b)$ (b) $x = \frac{a-b}{8}$

Question 9

8 marks | Graph Transformations

Main topic: Applications of Differentiation

9.

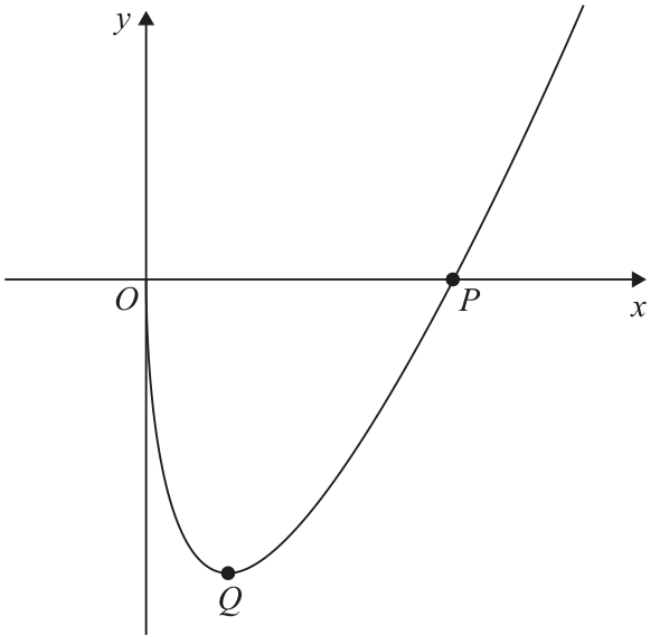


Figure 1

Figure 1 shows a sketch of part of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 6\sqrt{x}\ln(4x) \qquad x > 0$$

The curve cuts the x-axis at point P

(a) State the x coordinate of P

(1)

The point Q, shown in Figure 1, is the stationary point on C

(b) Use calculus to find the exact coordinates of Q

(5)

(c) Hence find the range of the function $\mathrm{g}(x)$ where

$$\mathrm{g}(x) = -2\,\mathrm{f}(x)$$

(2)

1 Part (a)

Since $x > 0$, $6\sqrt{x}$ is positive. Therefore $f(x) = 0$ requires $\ln(4x) = 0$, so $4x = 1$ and $x = 1/4$.

$$\boxed{x_P = \frac{1}{4}}$$

2 Part (b)

Apply the product and chain rules:

$$f'(x) = \frac{3}{\sqrt{x}}\ln(4x) + 6\sqrt{x}\frac{1}{x} = \frac{3}{\sqrt{x}}[\ln(4x) + 2].$$

At Q, set $f'(x) = 0$. Since $3/\sqrt{x} > 0$,

$$\ln(4x) + 2 = 0 \quad \Longrightarrow \quad 4x = e^{-2} \quad \Longrightarrow \quad x = \frac{1}{4e^2}.$$

Then $\sqrt{x} = 1/(2e)$ and $\ln(4x) = -2$, giving $f(x) = 6(1/(2e))(-2) = -6/e$.

$$\boxed{Q = \left(\frac{1}{4e^2}, -\frac{6}{e}\right)}$$

3 Part (c)

The factor $3/\sqrt{x}$ is positive and $\ln(4x) + 2$ increases through zero at Q, so f' changes from negative to positive: Q is the global minimum. As $x \to \infty$, $f(x) \to \infty$, hence the range of f is $[-6/e, \infty)$. Multiplication by -2 reverses the inequality and includes the maximum $12/e$.

$$\boxed{g(x) \leq \frac{12}{e}}$$

FINAL ANSWER

(a) $x_P = \frac{1}{4}$ (b) $Q = \left(\frac{1}{4e^2}, -\frac{6}{e}\right)$ (c) $g(x) \leq \frac{12}{e}$

Question 7

10 marks | Graph Transformations | Main topic: Modulus

7.

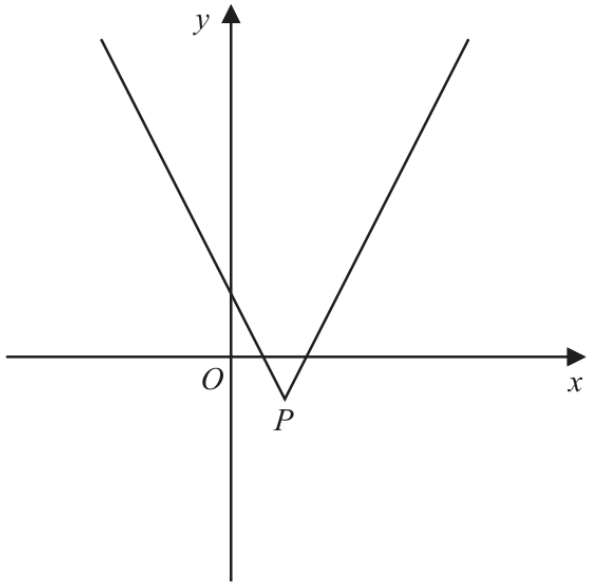


Figure 1

Figure 1 shows a sketch of part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = |3x - 11| - 4$$

The vertex of the graph is at the point P, as shown in Figure 1.

(a) State the coordinates of P.

(2)

(b) Solve the equation $\mathrm{f}(x) = 8$

(3)

The line l has equation $y = mx$, where m is a constant.

Given that l intersects the graph of $y = \mathrm{f}(x)$ at exactly one point,

(c) find the possible values of m.

(3)

The graph with equation $y = \mathrm{f}(x)$ is transformed onto the graph with equation $y = a\,\mathrm{f}(x - b)$, where a and b are constants.

Given that the vertex of the graph with equation $y = a\,\mathrm{f}(x - b)$ is $(5, 16)$

(d) find the value of a and the value of b.

(2)

1 Part (a)

At the vertex, $3x - 11 = 0$, so $P = (\frac{11}{3}, -4)$.

2 Part (b)

$$|3x - 11| - 4 = 8 \quad \Longrightarrow \quad |3x - 11| = 12.$$

Thus $3x - 11 = 12$ or $3x - 11 = -12$, giving $x = \frac{23}{3}$ or $x = -\frac{1}{3}$.

3 Part (c)

Write the two branches explicitly:

$$f(x) = \begin{cases} 7 - 3x, & x \le \frac{11}{3}, \\ 3x - 15, & x \ge \frac{11}{3}. \end{cases}$$

On the left branch, $mx = 7 - 3x$ gives $x = 7/(m + 3)$ for $m \neq -3$. This lies on the left branch precisely when $m < -3$ or $m \geq -12/11$. On the right branch, $mx = 3x - 15$ gives $x = 15/(3 - m)$ for $m \neq 3$. This lies on the right branch precisely when $-12/11 \leq m < 3$. At $m = -12/11$, the two branch intersections coincide at the vertex, so there is only one distinct point. For $m < -3$ or $m \geq 3$, only the left intersection exists. At $m = -3$ there is no intersection; at $m = 3$ there is one. Consequently the complete answer is

$$m < -3 \quad \text{or} \quad m = -\frac{12}{11} \quad \text{or} \quad m \geq 3.$$

4 Part (d)

Under $y = af(x - b)$, the vertex becomes $(\frac{11}{3} + b, -4a)$. Equating this with $(5, 16)$ gives

$$b = 5 - \frac{11}{3} = \frac{4}{3}, \qquad a = \frac{16}{-4} = -4.$$

✓ FINAL ANSWER

(a) $P = (\frac{11}{3}, -4)$ (b) $x = \frac{23}{3}$ or $-\frac{1}{3}$ (c) $m < -3 \quad \text{or} \quad m = -\frac{12}{11} \quad \text{or} \quad m \geq 3$ (d) $b = 5 - \frac{11}{3} = \frac{4}{3}, \qquad a = \frac{16}{-4} = -4$

Question 9

9 marks | Graph Transformations | Main topic: Trig Equations & R-Form

9. $$\mathrm{h}(x) = 8 + 3\sin x(\cos x - 2\sin x) \quad 0 \leqslant x \leqslant 2\pi$$

(a) Write $\mathrm{h}(x)$ in the form

$$P + R\sin(2x + \alpha)$$

where P, R and α are positive constants, $0 < \alpha < \frac{\pi}{2}$

Give the exact values of P and R and give the value of α, in radians, to 3 decimal places.

(6)

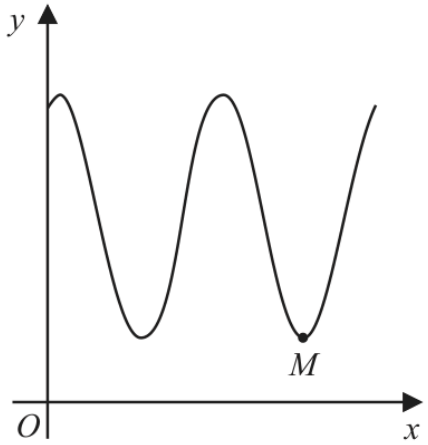


Figure 2

Figure 2 is a sketch of the curve with equation $y = \mathrm{h}(x)$

(b) State the range of h.

(1)

The point M, shown in Figure 2, is the minimum point on the curve with the largest x coordinate.

(c) Find the x coordinate of M.

(Solutions based entirely on calculator technology are not acceptable.)

(2)

1 Part (a)

Use $2\sin x\cos x = \sin 2x$ and $2\sin^2 x = 1 - \cos 2x$:

$$h(x) = 8 + 3\sin x\cos x - 6\sin^2 x = 5 + \frac{3}{2}\sin 2x + 3\cos 2x.$$

Expanding $R\sin(2x+\alpha)$ and comparing coefficients gives

$$R\cos\alpha = \frac{3}{2}, \qquad R\sin\alpha = 3.$$

Therefore

$$P = 5, \qquad R = \sqrt{\frac{9}{4} + 9} = \frac{3\sqrt{5}}{2}, \qquad \alpha = \arctan(2) = 1.107 \text{ radians (3 d.p.)}.$$

Thus $h(x) = 5 + \frac{3\sqrt{5}}{2}\sin(2x+\alpha)$ with the exact angle $\alpha = \arctan(2)$.

2 Part (b)

The given interval includes two complete periods. The range is

$$\left[5 - \frac{3\sqrt{5}}{2},\ 5 + \frac{3\sqrt{5}}{2}\right].$$

3 Part (c)

At a minimum, $\sin(2x+\alpha) = -1$, so

$$2x + \alpha = \frac{3\pi}{2} + 2\pi n, \qquad x = \frac{3\pi}{4} - \frac{\alpha}{2} + \pi n.$$

On $0 \le x \le 2\pi$, $n = 0$ and $n = 1$ give the two minima. The larger coordinate is

$$x_M = \frac{7\pi}{4} - \frac{1}{2}\arctan(2) = 4.944212785\ldots.$$

$$x_M = 4.94 \quad (3 \text{ significant figures}).$$

FINAL ANSWER

(a) $P = 5,\ R = \frac{3\sqrt{5}}{2},\ \alpha = \arctan 2$ (b) $\left[5 - \frac{3\sqrt{5}}{2},\ 5 + \frac{3\sqrt{5}}{2}\right]$ (c) $x_M = 4.94$

TOPIC

Modulus

Every past-paper question on this topic, January 2020 to June 2026.

Question 6

9 marks | Modulus

6.

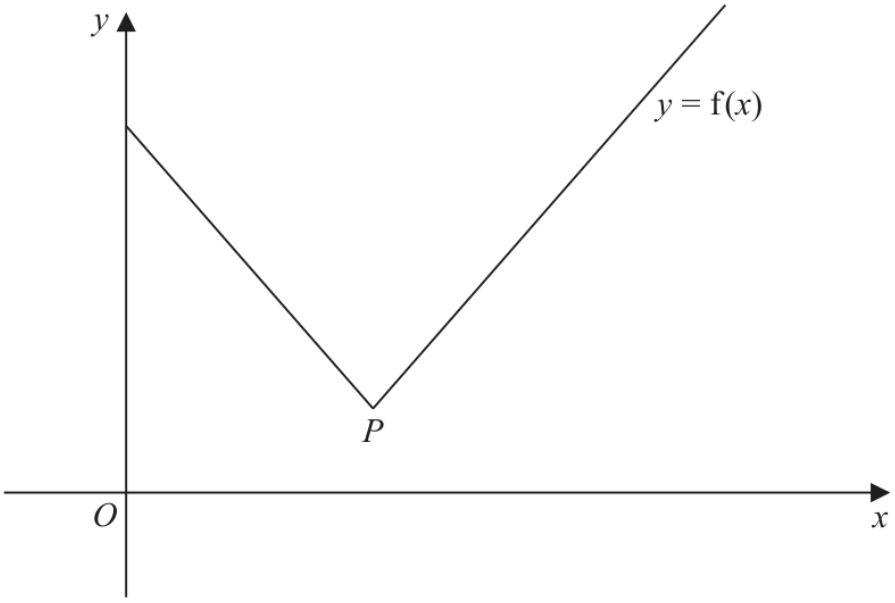


Figure 2

Figure 2 shows part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 2|2x - 5| + 3 \qquad x \geqslant 0$$

The vertex of the graph is at point P as shown.

(a) State the coordinates of P.

(2)

(b) Solve the equation $\mathrm{f}(x) = 3x - 2$

(4)

Given that the equation

$$\mathrm{f}(x) = kx + 2$$

where k is a constant, has exactly two roots,

(c) find the range of values of k.

(3)

1 Part (a)

At the vertex, the expression inside the modulus is zero:

$$2x - 5 = 0 \quad \Longrightarrow \quad x = \frac{5}{2}, \qquad f\left(\frac{5}{2}\right) = 3.$$

Hence $\boxed{P = \left(\frac{5}{2}, 3\right)}$.

2 Part (b)

Write the function in its two branches:

$$f(x) = \begin{cases} 13 - 4x, & 0 \le x \le \frac{5}{2}, \\ 4x - 7, & x \ge \frac{5}{2}. \end{cases}$$

On the left branch,

$$13 - 4x = 3x - 2 \quad \Longrightarrow \quad 7x = 15 \quad \Longrightarrow \quad x = \frac{15}{7}.$$

This lies in $0 \le x \le \frac{5}{2}$, so it is valid. On the right branch,

$$4x - 7 = 3x - 2 \quad \Longrightarrow \quad x = 5.$$

This satisfies $x \ge \frac{5}{2}$, so it is also valid. Thus

$$\boxed{x = \frac{15}{7} \text{ or } x = 5}.$$

3 **Part (c)**

Each straight branch can have at most one intersection with $y = kx + 2$. Two distinct roots therefore require one intersection strictly to each side of the vertex.

For the left branch,

$$13 - 4x = kx + 2 \quad \Longrightarrow \quad x_L = \frac{11}{k+4}.$$

The condition $0 \le x_L < \frac{5}{2}$ first requires $k + 4 > 0$, then gives

$$\frac{11}{k+4} < \frac{5}{2} \quad \Longrightarrow \quad 22 < 5(k+4) \quad \Longrightarrow \quad k > \frac{2}{5}.$$

For the right branch,

$$4x - 7 = kx + 2 \quad \Longrightarrow \quad x_R = \frac{9}{4-k}.$$

The condition $x_R > \frac{5}{2}$ requires $k < 4$, and then

$$\frac{9}{4-k} > \frac{5}{2} \quad \Longrightarrow \quad 18 > 5(4-k) \quad \Longrightarrow \quad k > \frac{2}{5}.$$

At $k = \frac{2}{5}$, both intersections coincide at the vertex, giving only one root. At $k = 4$, the line is parallel to the right branch and has no intersection with that branch, again leaving only one root. Hence

$$\boxed{\frac{2}{5} < k < 4}.$$

✓ **FINAL ANSWER**

(a) $P = \left(\frac{5}{2}, 3\right)$ (b) $x = \frac{15}{7}$ or $x = 5$ (c) $\frac{2}{5} < k < 4$

Question 6

8 marks | Modulus

6. Given that k is a positive constant,

(a) on separate diagrams, sketch the graph with equation

(i) $y = k - 2|x|$

(ii) $y = \left|2x - \frac{k}{3}\right|$

Show on each sketch the coordinates, in terms of k, of each point where the graph meets or cuts the axes.

(4)

(b) Hence find, in terms of k, the values of x for which

$$\left|2x - \frac{k}{3}\right| = k - 2|x|$$

giving your answers in simplest form.

(4)

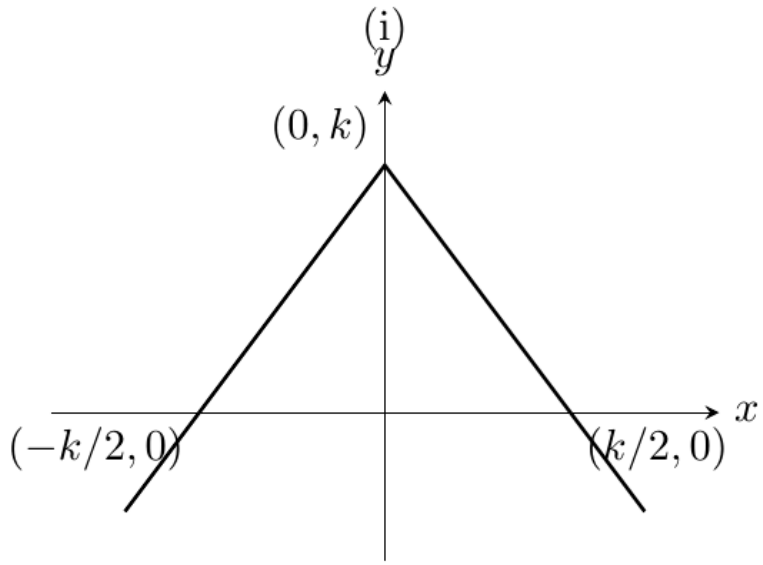
1

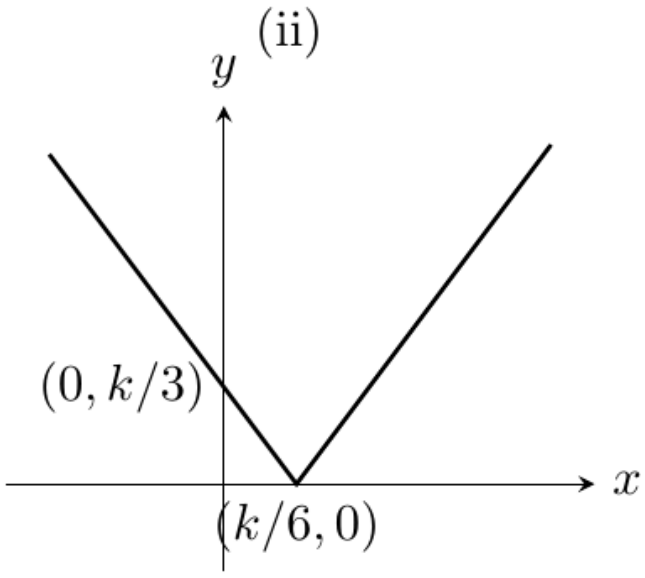
Part (a)

(i) $y = k - 2|x|$ is an inverted V with vertex $(0, k)$ and roots $x = \pm k/2$. (ii) $y = |2x - k/3|$ is an upright V with vertex $(k/6, 0)$ and y-intercept $(0, k/3)$. Because $k > 0$, both sketches have the ordering shown. Axes below use coordinates normalized by k, while labels give the original coordinates.

\qquad

$$\boxed{\text{(i) } (0,k),(\pm k/2,0); \quad \text{(ii) } (0,k/3),(k/6,0)}$$





2 **Part (b)**

Consider the breakpoints 0 and $k/6$. For $x < 0$, the equation is $k/3 - 2x = k + 2x$, giving $x = -k/6$, valid in this branch. For $0 \le x < k/6$, it is $k/3 - 2x = k - 2x$, impossible because $k > 0$. For $x \ge k/6$, it is $2x - k/3 = k - 2x$, giving $x = k/3$, valid in this branch. Both have nonnegative right side and satisfy the original equation.

$$\boxed{x = -\frac{k}{6}, \qquad x = \frac{k}{3}}$$

✓ **FINAL ANSWER**

(a) $\textbf{(i)}\ (0, k), (\pm k/2, 0); \quad \textbf{(ii)}\ (0, k/3), (k/6, 0)$ (b) $x = -\frac{k}{6}, \qquad x = \frac{k}{3}$

Question 7

10 marks | Modulus

7.

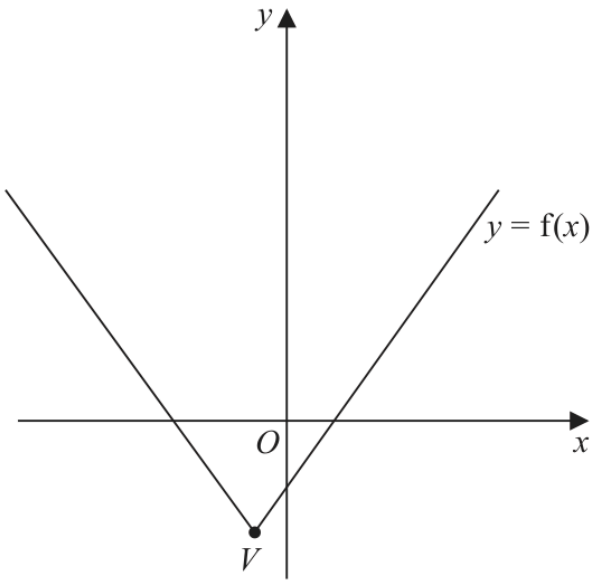


Figure 2

Figure 2 shows a sketch of part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \frac{1}{2}|2x + 7| - 10$$

(a) State the coordinates of the vertex, V, of the graph.

(2)

(b) Solve, using algebra,

$$\frac{1}{2}|2x + 7| - 10 \geqslant \frac{1}{3}x + 1$$

(4)

(c) Sketch the graph with equation

$$y = |\mathrm{f}(x)|$$

stating the coordinates of the local maximum point and each local minimum point.

(4)

1 Part (a)

At the vertex $2x + 7 = 0$, hence $x = -7/2$ and $f(x) = -10$.

$$\boxed{V = (-7/2, -10)}$$

2 Part (b)

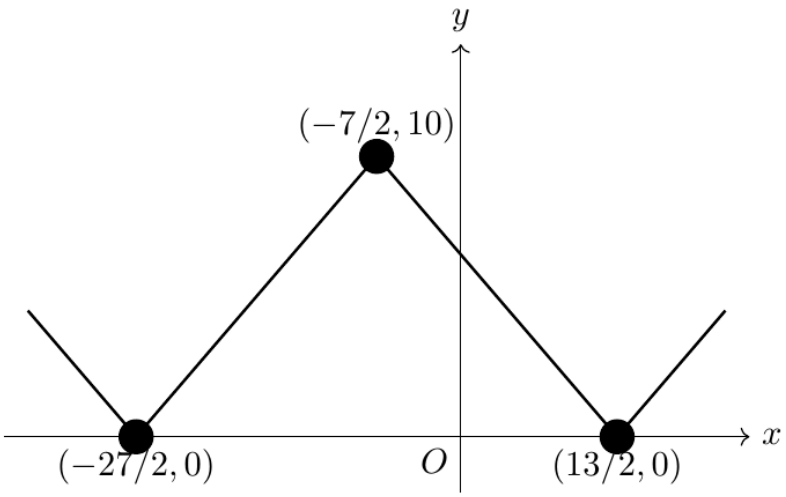
For $x \geq -7/2$, $f(x) = x - 13/2$. The inequality becomes $x - 13/2 \geq x/3 + 1$, giving $x \geq 45/4$, which satisfies this branch. For $x < -7/2$, $f(x) = -x - 27/2$. Then $-x - 27/2 \geq x/3 + 1$, giving $x \leq -87/8$, which satisfies that branch. Join the two permitted outer intervals, including the equality endpoints.

$$\boxed{x \leq -\frac{87}{8} \quad \text{or} \quad x \geq \frac{45}{4}}$$

3 Part (c)

The zeros of f satisfy $|2x + 7| = 20$, giving $x = -27/2$ and $x = 13/2$. Reflect the part below the x-axis upward, leaving the positive branches unchanged. This creates minima at $(-27/2, 0)$ and $(13/2, 0)$ and a local maximum at $(-7/2, 10)$.

$$\boxed{\text{Maximum } (-7/2, 10); \quad \text{minima } (-27/2, 0), (13/2, 0)}$$



FINAL ANSWER

(a) $V = (-7/2, -10)$ (b) $x \leq -\frac{87}{8}$ or $x \geq \frac{45}{4}$ (c)
Maximum $(-7/2, 10)$; minima $(-27/2, 0), (13/2, 0)$

Question 7

12 marks | Modulus

7.

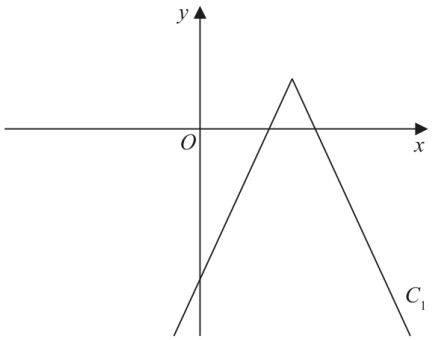


Figure 3

Figure 3 shows a sketch of the graph of C_1 with equation

$$y = 5 - |3x - 22|$$

(a) Write down the coordinates of

(i) the vertex of C_1

(ii) the intersection of C_1 with the y-axis.

(2)

(b) Find the x coordinates of the intersections of C_1 with the x-axis.

(2)

Diagram 1, shown on page 21, is a copy of Figure 3.

(c) On Diagram 1, sketch the curve C_2 with equation

$$y = \frac{1}{9}x^2 - 9$$

Identify clearly the coordinates of any points of intersection of C_2 with the coordinate axes.

(3)

(d) Find the coordinates of the points of intersection of C_1 and C_2
(*Solutions relying entirely on calculator technology are not acceptable.*)

(5)

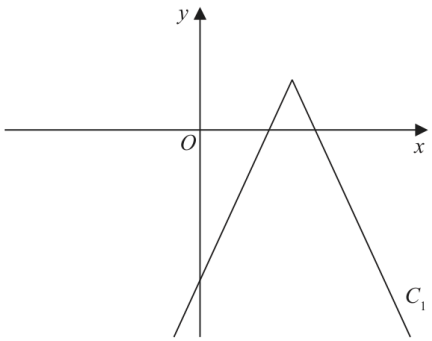


Diagram 1

1 Part (a)

(i) The modulus is zero at $3x - 22 = 0$, giving the vertex $(22/3, 5)$. (ii) At $x = 0$, $y = 5 - 22 = -17$, so the y-axis intersection is $(0, -17)$.

$$\boxed{(i)\ (22/3, 5), \qquad (ii)\ (0, -17)}$$

2 Part (b)

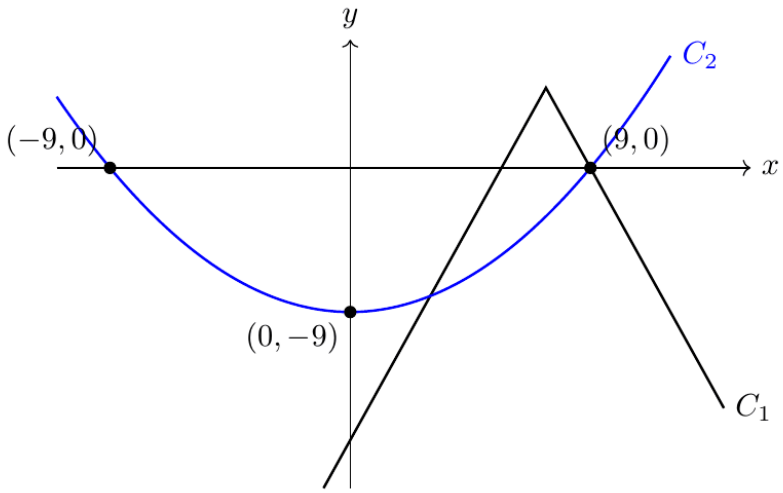
Set $y = 0$: $|3x - 22| = 5$. Thus $3x - 22 = 5$ or -5, giving $x = 9$ or $17/3$.

$$\boxed{x = \frac{17}{3}, \quad 9}$$

3 Part (c)

The curve $C_2 : y = x^2/9 - 9$ is an upward-opening parabola, symmetric about the y-axis. Its vertex and y-intercept are $(0, -9)$; setting y=0 gives $x^2 = 81$, hence x-intercepts $(-9, 0)$ and $(9, 0)$. The sketch below includes C_1 for context and labels all three axis intersections of C_2.

$$\boxed{(-9, 0), \quad (9, 0), \quad (0, -9)}$$



4 Part (d)

For $x \le 22/3$, C_1 is $y = 3x - 17$. Equating to C_2 gives

$$x^2/9 - 9 = 3x - 17 \implies x^2 - 27x + 72 = (x - 3)(x - 24) = 0.$$

Only $x = 3$ belongs to this branch; $y = -8$. For $x \ge 22/3$, C_1 is $y = 27 - 3x$. Then

$$x^2/9 - 9 = 27 - 3x \implies x^2 + 27x - 324 = (x - 9)(x + 36) = 0.$$

Only $x = 9$ belongs to this branch; $y = 0$. The rejected candidates 24 and -36 lie outside their assumed branch regions.

$$\boxed{(3, -8), \qquad (9, 0)}$$

FINAL ANSWER

(a) $(i)\ (22/3, 5), \quad (ii)\ (0, -17)$ (b) $x = \frac{17}{3}, \quad 9$ (c) $(-9, 0), \quad (9, 0), \quad (0, -9)$ (d) $(3, -8), \quad (9, 0)$

Question 6

8 marks | Modulus

6.

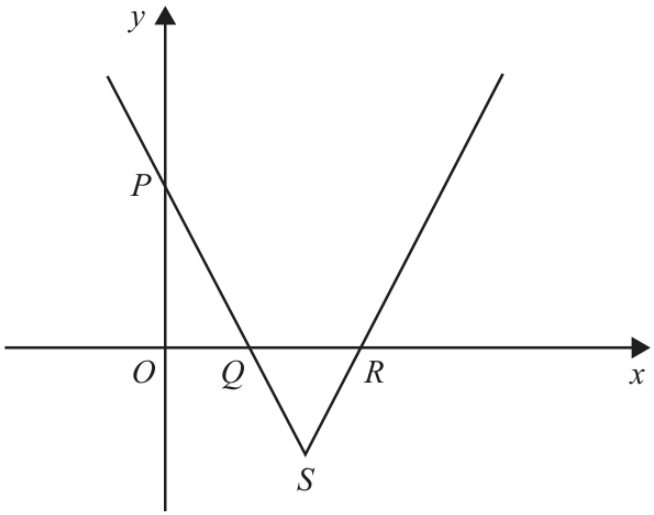


Figure 2

Figure 2 shows a sketch of the graph with equation

$$y = |3x - 5a| - 2a$$

where a is a positive constant.

The graph

- cuts the y-axis at the point P
- cuts the x-axis at the points Q and R
- has a minimum point at S

(a) Find, in simplest form in terms of a, the coordinates of

(i) point P

(ii) points Q and R

(iii) point S

(4)

(b) Find, in simplest form in terms of a, the values of x for which

$$|3x - 5a| - 2a = |x - 2a|$$

(4)

1 Part (a)

For $y = |3x - 5a| - 2a$ with $a > 0$: (i) At $x = 0$, $y = 5a - 2a = 3a$, so $P = (0, 3a)$. (ii) At $y = 0$, $|3x - 5a| = 2a$ gives $3x - 5a = \pm 2a$. Thus $x = a$ or $7a/3$; since $a < 7a/3$, $Q = (a, 0)$ and $R = (7a/3, 0)$. (iii) The modulus vanishes at $x = 5a/3$, giving the minimum $S = (5a/3, -2a)$.

$$\boxed{P = (0, 3a), \quad Q = (a, 0), \quad R = (7a/3, 0), \quad S = (5a/3, -2a)}$$

2 Part (b)

Split $|3x - 5a| - 2a = |x - 2a|$ at $x = 5a/3$ and $x = 2a$; their order uses $a > 0$. For $x < 5a/3$: $3a - 3x = 2a - x$, so $x = a/2$, which is in this region. For $5a/3 \le x < 2a$: $3x - 7a = 2a - x$, so $x = 9a/4$; this is outside the assumed region and is rejected. For $x \ge 2a$: $3x - 7a = x - 2a$, so $x = 5a/2$, which is valid. These three regions cover the whole real line, so there are exactly two solutions.

$$\boxed{x = \frac{a}{2}, \qquad x = \frac{5a}{2}}$$

FINAL ANSWER

(a) $P = (0, 3a), \quad Q = (a, 0), \quad R = (7a/3, 0), \quad S = (5a/3, -2a)$ (b) $x = \frac{a}{2}, \qquad x = \frac{5a}{2}$

Question 6

9 marks | Modulus

6.

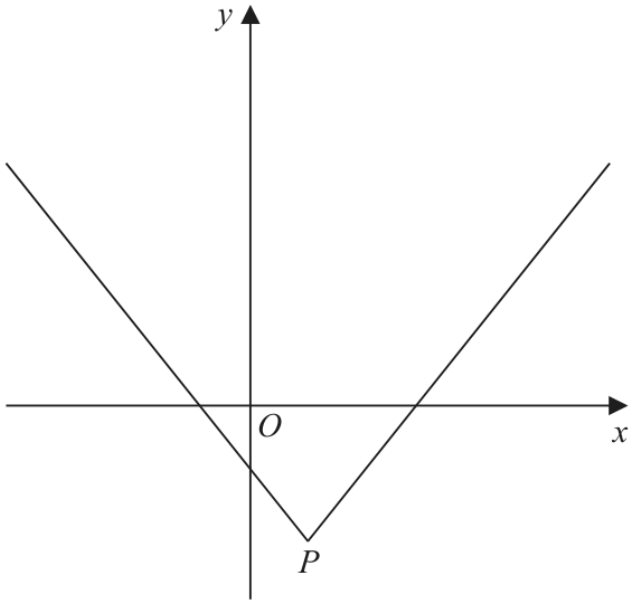


Figure 2

Figure 2 shows a sketch of the graph $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 3|x - 2| - 10$$

The vertex of the graph is at point P, shown in Figure 2.

(a) Find the coordinates of P

(2)

(b) Find $\mathrm{ff}(0)$

(2)

(c) Solve the inequality

$$3|x - 2| - 10 < 5x + 10$$

(2)

(d) Solve the equation

$$\mathrm{f}(|x|) = 0$$

(3)

1 Part (a)

The modulus is smallest when $x - 2 = 0$, giving $x = 2$ and $f(2) = -10$. The two straight-line branches meet there, so this is the vertex of the V-shaped graph.

$$\boxed{P = (2, -10)}$$

2 Part (b)

First $f(0) = 3|0 - 2| - 10 = -4$. Then $ff(0) = f(-4) = 3|-4 - 2| - 10 = 18 - 10 = 8$. In particular $ff(0)$ means $f(f(0))$, not the square of $f(0)$.

$$\boxed{ff(0) = 8}$$

3 Part (c)

For $x < 2$, the inequality becomes $3(2 - x) - 10 < 5x + 10$, so $-4 - 3x < 5x + 10$, giving $x > -7/4$. This yields $-7/4 < x < 2$. For $x \geq 2$, it becomes $3x - 16 < 5x + 10$, giving $x > -13$; every $x \geq 2$ satisfies this. The union is $x > -7/4$.

$$\boxed{x > -\frac{7}{4}}$$

4 Part (d)

Let $u = |x| \geq 0$. Then $f(|x|) = 0$ gives

$$3|u - 2| - 10 = 0 \quad \Longrightarrow \quad u - 2 = \pm\frac{10}{3}.$$

Thus $u = 16/3$ or $u = -4/3$. Reject $u = -4/3$ because $u \geq 0$; $|x| = 16/3$ has the two solutions below.

$$\boxed{x = -\frac{16}{3}, \quad \frac{16}{3}}$$

FINAL ANSWER

(a) $P = (2, -10)$ (b) $ff(0) = 8$ (c) $x > -\frac{7}{4}$ (d) $x = -\frac{16}{3}, \quad \frac{16}{3}$

Question 8

11 marks | Modulus

8.

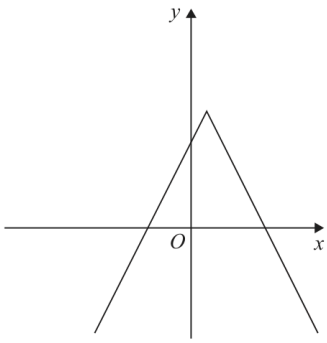


Figure 2

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

The graph shown in Figure 2 has equation

$$y = a - |2x - b|$$

where a and b are positive constants, $a > b$

(a) Find, giving your answer in terms of a and b,

(i) the coordinates of the maximum point of the graph,

(ii) the coordinates of the point of intersection of the graph with the y-axis,

(iii) the coordinates of the points of intersection of the graph with the x-axis.

(5)

On page 24 there is a copy of Figure 2 called Diagram 1.

(b) On Diagram 1, sketch the graph with equation

$$y = |x| - 1$$

(2)

Given that the graphs $y = |x| - 1$ and $y = a - |2x - b|$ intersect at $x = -3$ and $x = 5$

(c) find the value of a and the value of b

(4)

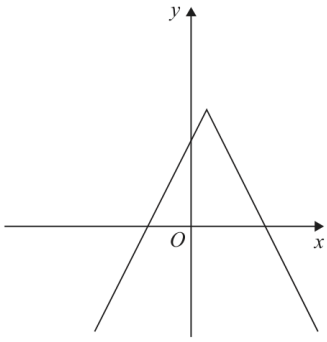


Diagram 1

1 Part (a)

(i) Since $|2x-b| \geq 0$, the maximum occurs at $2x-b=0$: $x=b/2$ and $y=a$. (ii) At $x=0$, $y=a-|-b|=a-b$ because $b>0$. (iii) At an x-axis crossing, $|2x-b|=a$, hence $2x-b=\pm a$. Since $a>b>0$, $(b-a)/2<0$ and $(b+a)/2>0$, matching the sketch.

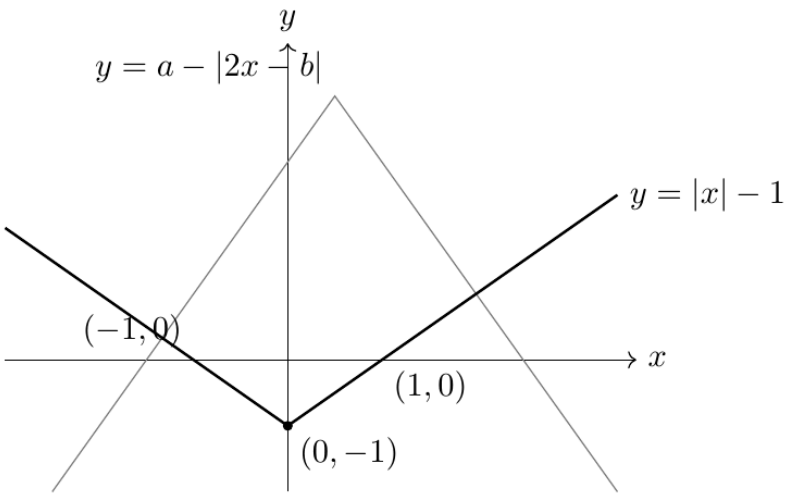
$$\boxed{\left(\frac{b}{2}, a\right); \quad (0, a-b); \quad \left(\frac{b-a}{2}, 0\right), \ \left(\frac{b+a}{2}, 0\right)}$$

2 Part (b)

The graph $y=|x|-1$ is a V with vertex $(0,-1)$ and intercepts $(-1,0)$ and $(1,0)$. Its left branch is $y=-x-1$ for $x<0$ and its right branch is $y=x-1$ for $x \geq 0$. Add this V to the supplied axes; the original inverted V remains unchanged.

The grey original curve is schematic; the three labelled points belong to the added graph.

$$\boxed{\text{Vertex } (0,-1); \quad \text{zeros } x=\pm 1}$$



3 Part (c)

At $x=-3$, the common ordinate is $|-3|-1=2$. Since $b>0$, $|-6-b|=6+b$, so $a-(6+b)=2$, giving $a=b+8$. At $x=5$, the common ordinate is $|5|-1=4$, so $a-|10-b|=4$. Substitute $a=b+8$ to obtain $|10-b|=b+4$. If $b \leq 10$, then $10-b=b+4$, giving $b=3$, $a=11$. If $b>10$, then $b-10=b+4$, impossible. Both accepted values satisfy $a>b>0$.

$$\boxed{a=11, \quad b=3}$$

FINAL ANSWER

(a) $\left(\frac{b}{2}, a\right)$; $(0, a-b)$; $\left(\frac{b-a}{2}, 0\right)$, $\left(\frac{b+a}{2}, 0\right)$ (b) $\text{Vertex } (0,-1)$; $\text{zeros } x=\pm 1$ (c) $a=11, \quad b=3$

Question 7

8 marks | Modulus

7. Given that a and b are positive constants with $a > b$,

(a) sketch, on **separate** diagrams, the graph with equation

(i) $y = |3x - a|$

(ii) $y = |3x - a| - b$

Show on each sketch

- the coordinates of the minimum point on the graph
- the coordinates of the point at which the graph crosses the y-axis

(6)

(b) Solve the equation

$$|3x - a| - b = 5x$$

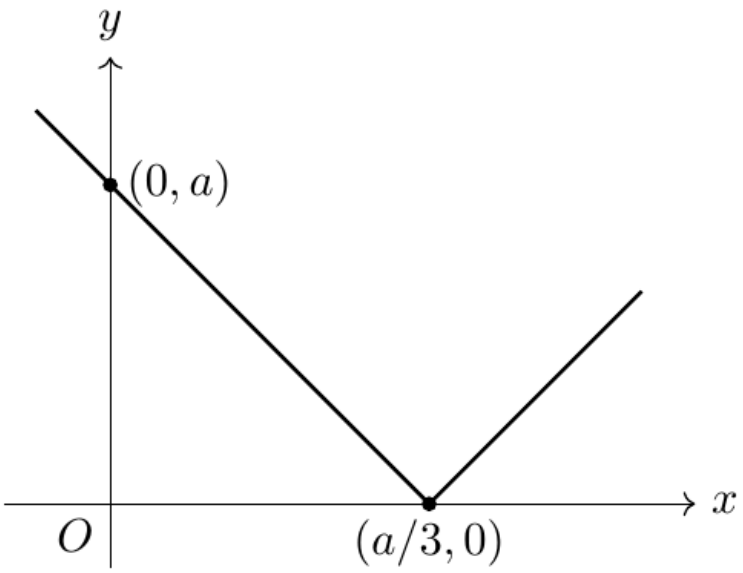
giving any solution for x in terms of a and b.

(2)

1 Part (a)(i)

For $x \leq a/3$, $y = a - 3x$; for $x \geq a/3$, $y = 3x - a$. These two straight branches form a V with minimum $(a/3, 0)$ and y-intercept $(0, a)$.

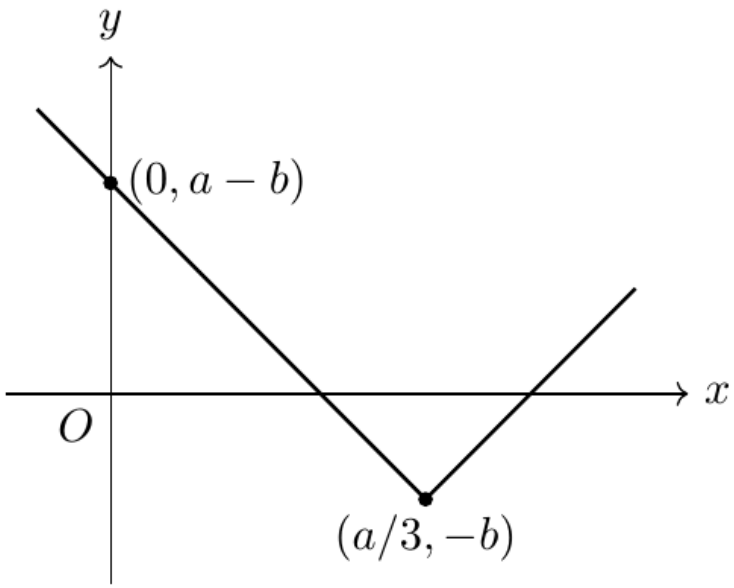
$$\boxed{\text{minimum } (a/3, 0), \quad y\text{-intercept } (0, a)}$$



2 Part (a)(ii)

Subtracting b translates the previous graph vertically down by b. The minimum becomes $(a/3, -b)$ and the y-intercept becomes $(0, a - b)$, which is above the x-axis because $a > b > 0$.

$$\boxed{\text{minimum } (a/3, -b), \quad y\text{-intercept } (0, a - b)}$$



3

Part (b)

For $x \leq a/3$, the equation becomes $a - 3x - b = 5x$, giving $x = (a-b)/8$. This is positive and less than $a/8 < a/3$, so it satisfies this branch condition. For $x \geq a/3$, the equation becomes $3x - a - b = 5x$, giving $x = -(a+b)/2 < 0$. This contradicts $x \geq a/3 > 0$ and must be rejected. Therefore only one solution remains.

$$\boxed{x = \frac{a-b}{8}}$$

✓ **FINAL ANSWER**

(a)(i) $\textbf{minimum } (a/3, 0), \quad y\textbf{-intercept } (0, a)$ (a)(ii) $\textbf{minimum } (a/3, -b), \quad y\textbf{-intercept } (0, a-b)$ (b) $x = \frac{a-b}{8}$

Question 10

10 marks | Modulus

10:

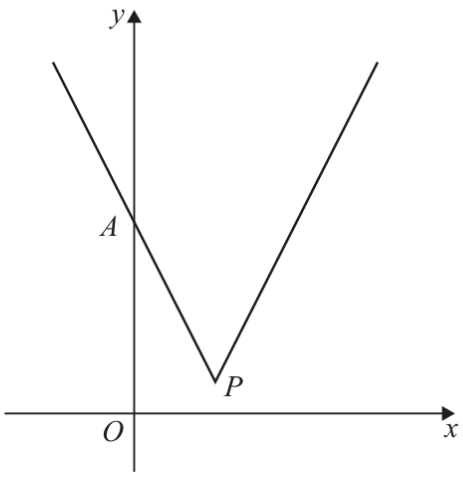


Figure 3

Figure 3 shows a sketch of part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = |kx - 10| + k \qquad x \in \mathbb{R}$$

and k is a positive constant.

The graph

- cuts the y-axis at the point A
- has a vertex at the point P

(a) Find, in simplest form in terms of k,

(i) the y coordinate of A

(ii) the coordinates of P

(3)

(b) Find, in terms of k, the range of values of x which satisfy

$$|kx - 10| + k \geqslant 2k$$

(3)

Given that the line with equation $y = 3x + 1$ intersects the graph of $y = \mathrm{f}(x)$ at 2 distinct points,

(c) find the range of values of k.

(4)

1 **Part (i)**

At $x = 0$, $f(0) = |-10| + k$, so $\boxed{y_A = 10 + k}$.

2 **Part (ii)**

At the vertex the expression inside the modulus vanishes:

$$kx - 10 = 0 \quad \Longrightarrow \quad \boxed{P = \left(\frac{10}{k}, k\right)}.$$

3 **Part (b)**

$$|kx - 10| + k \geq 2k \quad \Longleftrightarrow \quad |kx - 10| \geq k.$$

Since $k > 0$, the two branches give

$$kx - 10 \leq -k \quad \text{or} \quad kx - 10 \geq k,$$

$$\boxed{x \leq \frac{10}{k} - 1 \quad \text{or} \quad x \geq \frac{10}{k} + 1}.$$

4 **Part (c)**

Consider the vertical difference $d(x) = f(x) - (3x + 1)$:

$$d(x) = \begin{cases} 9 + k - (k+3)x, & x \leq 10/k, \\ (k-3)x + k - 11, & x \geq 10/k. \end{cases}$$

The left branch is strictly decreasing. If $0 < k < 3$, the right branch also decreases, so there cannot be two roots. If $k = 3$, the right branch is the nonzero constant -8, again giving no second root. For $k > 3$ the right branch increases and both outer ends tend to $+\infty$. There are two distinct roots precisely when the vertex lies below zero:

$$d(10/k) = k - 1 - \frac{30}{k} < 0.$$

Multiplying by $k > 0$ gives

$$k^2 - k - 30 = (k-6)(k+5) < 0 \quad \Longrightarrow \quad 0 < k < 6.$$

Combining with $k > 3$ yields $\boxed{3 < k < 6}$. At $k = 6$ the two roots merge at the vertex, so the endpoint is excluded.

FINAL ANSWER

(i) $y_A = 10 + k$ (ii) $P = \left(\frac{10}{k}, k\right)$ (b) $x \leq \frac{10}{k} - 1 \quad \text{or} \quad x \geq \frac{10}{k} + 1$ (c) $3 < k < 6$ (a)(i) $y_A = 10 + k$ (a)(ii) $P = \left(\frac{10}{k}, k\right)$

Question 8

13 marks | Modulus

8.

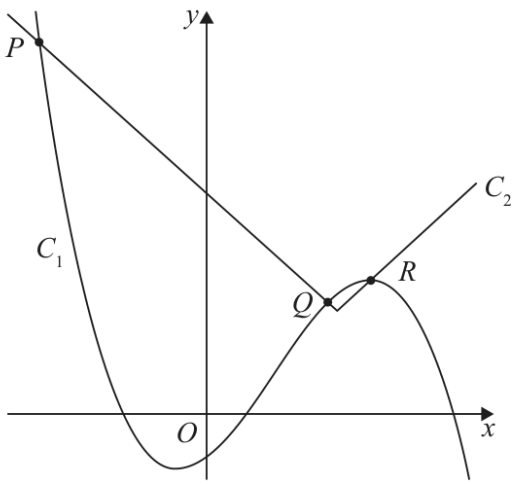


Figure 2

Figure 2 shows a sketch of the graph C_1 with equation

$$y = -2x^3 + 5x^2 + 4x - 3$$

and a sketch of the graph C_2 with equation

$$y = a + |5x + b|$$

where a and b are constants.

Graphs C_1 and C_2 intersect at point P, point Q and point R, as shown in Figure 2.

Given that P has coordinates $(-2, 25)$

(a) show that

$$a = 15 + b$$

(2)

Given also that R has coordinates $(2, 9)$

(b) find the value of a and the value of b

(3)

Using the answer to part (b),

(c) state the coordinates of the vertex of C_2

(2)

(d) Find, using algebra, the coordinates of Q. Show each stage of your working.

(*Solutions relying on calculator technology are not acceptable.*)

(6)

1 Part (a)

Point P lies on the descending branch of C_2, where $5x + b < 0$. At $P = (-2, 25)$,

$$25 = a - (-10 + b) = a + 10 - b,$$

hence $a = 15 + b$.

2 Part (b)

Point $R = (2, 9)$ lies on the ascending branch, so

$$9 = a + (10 + b) \Longrightarrow a + b = -1.$$

Combining with $a - b = 15$ gives $2a = 14$, hence $a = 7$ and $b = -8$.

3 Part (c)

The vertex occurs at $5x - 8 = 0$, giving $(x, y) = (8/5, 7)$.

4 Part (d)

The diagram places Q on the descending branch $y = 7 - (5x - 8) = 15 - 5x$, with $x < 8/5$. Equate this with the cubic:

$$-2x^3 + 5x^2 + 4x - 3 = 15 - 5x \Longrightarrow 2x^3 - 5x^2 - 9x + 18 = 0.$$

Since $x = -2$ is the known intersection P, divide by $x + 2$:

$$2x^3 - 5x^2 - 9x + 18 = (x + 2)(2x^2 - 9x + 9) = (x + 2)(2x - 3)(x - 3).$$

The candidates are $-2, 3/2, 3$. Reject $x = 3$ because it is outside the descending branch, and $x = -2$ is P, not Q. Thus $x_Q = 3/2$ and $y_Q = 15 - 5(3/2) = 15/2$, giving $Q = (3/2, 15/2)$.

FINAL ANSWER

(a) Shown, as required. (b) 7; -8 (c) $(8/5, 7)$ (d) $(3/2, 15/2)$

Question 7

10 marks | Modulus

7.

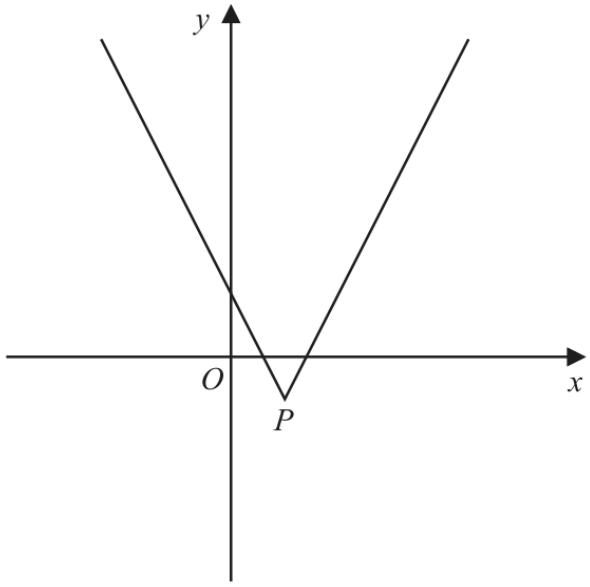


Figure 1

Figure 1 shows a sketch of part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = |3x - 11| - 4$$

The vertex of the graph is at the point P, as shown in Figure 1.

(a) State the coordinates of P.

(2)

(b) Solve the equation $\mathrm{f}(x) = 8$

(3)

The line l has equation $y = mx$, where m is a constant.

Given that l intersects the graph of $y = \mathrm{f}(x)$ at exactly one point,

(c) find the possible values of m.

(3)

The graph with equation $y = \mathrm{f}(x)$ is transformed onto the graph with equation $y = a\,\mathrm{f}(x - b)$, where a and b are constants.

Given that the vertex of the graph with equation $y = a\,\mathrm{f}(x - b)$ is $(5, 16)$

(d) find the value of a and the value of b.

(2)

1 Part (a)

At the vertex, $3x - 11 = 0$, so $P = (\frac{11}{3}, -4)$.

2 Part (b)

$$|3x - 11| - 4 = 8 \quad \Longrightarrow \quad |3x - 11| = 12.$$

Thus $3x - 11 = 12$ or $3x - 11 = -12$, giving $x = \frac{23}{3}$ or $x = -\frac{1}{3}$.

3 Part (c)

Write the two branches explicitly:

$$f(x) = \begin{cases} 7 - 3x, & x \le \frac{11}{3}, \\ 3x - 15, & x \ge \frac{11}{3}. \end{cases}$$

On the left branch, $mx = 7 - 3x$ gives $x = 7/(m + 3)$ for $m \neq -3$. This lies on the left branch precisely when $m < -3$ or $m \ge -12/11$. On the right branch, $mx = 3x - 15$ gives $x = 15/(3 - m)$ for $m \neq 3$. This lies on the right branch precisely when $-12/11 \le m < 3$. At $m = -12/11$, the two branch intersections coincide at the vertex, so there is only one distinct point. For $m < -3$ or $m \ge 3$, only the left intersection exists. At $m = -3$ there is no intersection; at $m = 3$ there is one. Consequently the complete answer is

$$m < -3 \quad \text{or} \quad m = -\frac{12}{11} \quad \text{or} \quad m \ge 3.$$

4 Part (d)

Under $y = af(x - b)$, the vertex becomes $(\frac{11}{3} + b, -4a)$. Equating this with $(5, 16)$ gives

$$b = 5 - \frac{11}{3} = \frac{4}{3}, \qquad a = \frac{16}{-4} = -4.$$

FINAL ANSWER

(a) $P = (\frac{11}{3}, -4)$ (b) $x = \frac{23}{3}$ or $-\frac{1}{3}$ (c) $m < -3 \quad \text{or} \quad m = -\frac{12}{11} \quad \text{or} \quad m \ge 3$ (d) $b = 5 - \frac{11}{3} = \frac{4}{3}, \qquad a = \frac{16}{-4} = -4$

Question 9

11 marks | Modulus

9.

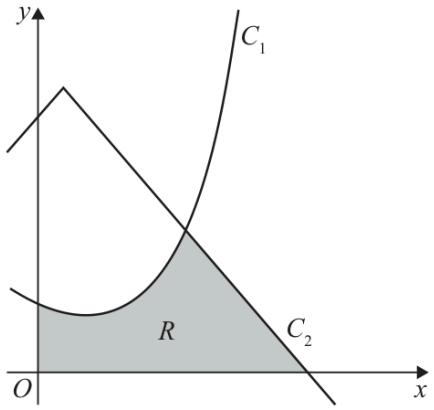


Figure 2

In this question you must show all stages of your working.
Solutions relying entirely on calculator technology are not acceptable.

(a) Find $\int \left(\frac{11}{2x+5} + 4e^{x-3} \right) dx$

(2)

Figure 2 shows a sketch of part of the curve C_1 and part of the graph C_2
The curve C_1 has equation

$$y = \frac{11}{2x+5} + 4e^{x-3}$$

The graph C_2 has equation

$$y = a - |bx - 1|$$

where a and b are positive integers.

(b) State, in terms of a and/or b, the coordinates of the maximum point of C_2

(2)

Given that

- C_1 intersects C_2 at $x = 3$
- C_2 crosses the y-axis at $(0, 9)$

(c) find the value of a and the value of b

(3)

The region R, shown shaded in Figure 2, is bounded by C_1, C_2, the x-axis and the y-axis.

(d) Find the area of R, giving the answer to 3 significant figures.

(4)

1 Part (a)

$$\int\left(\frac{11}{2x+5}+4e^{x-3}\right)dx=\boxed{\frac{11}{2}\ln|2x+5|+4e^{x-3}+C}.$$

The absolute value allows both intervals on which the integrand is defined.

2 Part (b)

The largest value of $a-|bx-1|$ occurs when $bx-1=0$. The maximum point is therefore $\boxed{(1/b,a)}$.

3 Part (c)

At $x=0$, $9=a-1$, so $\boxed{a=10}$. At the intersection $x=3$, the first curve has height $11/11+4=5$. Thus $10-|3b-1|=5$, giving $3b-1=5$ or -5. These give $b=2$ or $-4/3$; only the positive integer is allowed, so $\boxed{b=2}$.

4 Part (d)

For $x\geq 3$, the required branch of C_2 is $y=10-(2x-1)=11-2x$. Its x-intercept is $x=11/2$. The shaded area is the area under C_1 from 0 to 3 plus the triangle from 3 to $11/2$:

$$\begin{aligned}A&=\int_0^3\left(\frac{11}{2x+5}+4e^{x-3}\right)dx+\frac{1}{2}\left(\frac{11}{2}-3\right)5\\&=\left[\frac{11}{2}\ln(2x+5)+4e^{x-3}\right]_0^3+\frac{25}{4}\\&=\frac{11}{2}\ln\frac{11}{5}+4(1-e^{-3})+\frac{25}{4}=14.3873672\ldots.\end{aligned}$$

Hence $\boxed{A=14.4\text{ square units}}$ to 3 significant figures.

FINAL ANSWER

(a) $\frac{11}{2}\ln|2x+5|+4e^{x-3}+C$ (b) $(1/b,a)$ (c) $a=10$, $b=2$ (d) $A=14.4$ square units

Question 9

9 marks | Modulus | Main topic: Exponentials & Logarithms

9. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

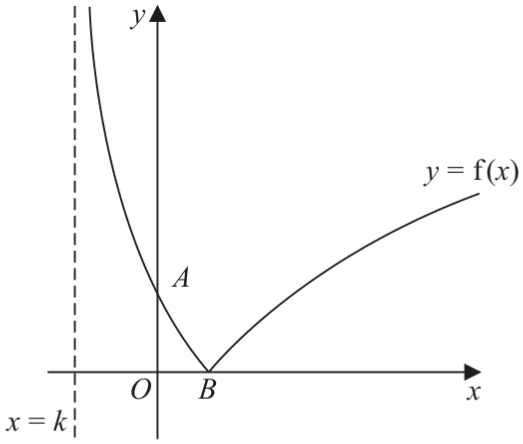


Figure 2

Figure 2 shows a sketch of the curve with equation

$$y = |2 - 4\ln(x + 1)| \qquad x > k$$

where k is a constant.

Given that the curve

- has an asymptote at $x = k$
- cuts the y-axis at point A
- meets the x-axis at point B

as shown in Figure 2,

(a) state the value of k

(1)

(b) (i) find the y coordinate of A

(ii) find the exact x coordinate of B

(3)

(c) Using algebra and showing your working, find the set of values of x such that

$$|2 - 4\ln(x + 1)| > 3$$

(5)

1 Part (a)

The logarithm requires $x+1>0$, so the domain is $x>-1$. As $x\to -1^{+}$, $\ln(x+1)\to -\infty$ and the ordinate tends to infinity.

$$\boxed{k=-1}$$

2 Part (b)

(i) At the y-axis, $x=0$, giving $y=|2-4\ln 1|=2$.

(ii) At the x-axis, $|2-4\ln(x+1)|=0$, so

$$\ln(x+1)=\frac{1}{2}\quad\Longrightarrow\quad x=e^{1/2}-1.$$

Thus $A=(0,2)$ and $B=(e^{1/2}-1,0)$.

$$\boxed{y_A=2,\qquad x_B=e^{1/2}-1}$$

3 Part (c)

On $x>-1$, split the absolute-value inequality into two alternatives:

$$2-4\ln(x+1)>3\quad\text{or}\quad 2-4\ln(x+1)<-3.$$

Dividing by -4 reverses the inequality in each branch:

$$\ln(x+1)<-\frac{1}{4}\quad\text{or}\quad\ln(x+1)>\frac{5}{4}.$$

Since the exponential is increasing, this becomes $x+1<e^{-1/4}$ or $x+1>e^{5/4}$. Intersect with the logarithmic domain; neither boundary is included because the source inequality is strict.

$$\boxed{-1<x<e^{-1/4}-1\quad\text{or}\quad x>e^{5/4}-1}$$

FINAL ANSWER

(a) $k=-1$ (b) $y_A=2,\qquad x_B=e^{1/2}-1$ (c) $-1<x<e^{-1/4}-1\quad\text{or}\quad x>e^{5/4}-1$

WMA13/01 OCTOBER 2025

Question 7

7 marks | Modulus

Main topic: Applications of Differentiation

7. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

A curve C has equation

$$y = 2x\mathrm{e}^{x^2+(3k-2)x}$$

where k is a constant.

Given that C has two distinct turning points, find the range of possible values of k.

(7)

WORKED SOLUTION

Question 7 · WMA13/01 October 2025

1 Solution

\paragraph{Solution} Put $u = x^2 + (3k-2)x$, so $u' = 2x + 3k - 2$. By the product and chain rules,

$$\frac{dy}{dx} = 2e^u + 2xe^u(2x + 3k - 2) = 2e^u\big(2x^2 + (3k-2)x + 1\big).$$

As $2e^u > 0$, stationary points satisfy

$$2x^2 + (3k-2)x + 1 = 0.$$

For two distinct real roots, the discriminant must be positive:

$$(3k-2)^2 - 8 > 0 \iff |3k-2| > 2\sqrt{2}.$$

Consequently

$$k < \frac{2-2\sqrt{2}}{3} \quad \text{or} \quad k > \frac{2+2\sqrt{2}}{3}.$$

For these values the quadratic changes sign at both simple roots, so both stationary points are genuine turning points. Equality gives a repeated root with no derivative sign change and is excluded.

FINAL ANSWER

(main) $(-infinity, (2 - 2 * sqrt(2))/3) union ((2 + 2 * sqrt(2))/3, infinity)$

TOPIC

Trigonometric Functions & Graphs

Every past-paper question on this topic, January 2020 to June 2026.

Question 9

7 marks | Trigonometric Functions & Graphs | Main topic: Trig Equations & R-Form

9. $\mathrm{f}(\theta) = 5\cos\theta - 4\sin\theta \qquad \theta \in \mathbb{R}$

(a) Express $\mathrm{f}(\theta)$ in the form $R\cos(\theta + \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

The curve with equation $y = \cos\theta$ is transformed onto the curve with equation $y = \mathrm{f}(\theta)$ by a sequence of two transformations.

Given that the first transformation is a stretch and the second a translation,

(b) (i) describe fully the transformation that is a stretch,

(ii) describe fully the transformation that is a translation.

(2)

Given

$$\mathrm{g}(\theta) = \frac{90}{4 + (\mathrm{f}(\theta))^2} \qquad \theta \in \mathbb{R}$$

(c) find the range of g.

(2)

1 Part (a)

Expanding the required form gives

$$R\cos(\theta+\alpha) = R\cos\alpha\cos\theta - R\sin\alpha\sin\theta.$$

Comparing coefficients with $5\cos\theta - 4\sin\theta$,

$$R\cos\alpha = 5, \qquad R\sin\alpha = 4.$$

Squaring and adding, using $R > 0$, gives

$$R^2 = 5^2 + 4^2 = 41 \quad \Longrightarrow \quad \boxed{R = \sqrt{41}}.$$

Also, since $0 < \alpha < \frac{\pi}{2}$,

$$\tan\alpha = \frac{4}{5}, \qquad \alpha = \arctan\left(\frac{4}{5}\right) = 0.6747409\ldots \quad \Longrightarrow \quad \boxed{\alpha = 0.675 \text{ radians (3 d.p.)}}.$$

Thus the exact form is

$$\boxed{f(\theta) = \sqrt{41}\cos\left(\theta + \arctan\frac{4}{5}\right)}.$$

2 Part (b)(i)

A stretch parallel to the y-axis with scale factor $\boxed{\sqrt{41}}$, taking $y = \cos\theta$ to $y = \sqrt{41}\cos\theta$.

3 Part (b)(ii)

A translation by the vector

$$\boxed{\begin{pmatrix} -\alpha \\ 0 \end{pmatrix}} = \begin{pmatrix} -\arctan(4/5) \\ 0 \end{pmatrix} \approx \begin{pmatrix} -0.675 \\ 0 \end{pmatrix}.$$

This moves the curve α units to the left, giving $y = \sqrt{41}\cos(\theta+\alpha)$.

4

Part (c)

Since θ can be any real number,

$$-\sqrt{41} \le f(\theta) \le \sqrt{41} \quad \Longrightarrow \quad 0 \le (f(\theta))^2 \le 41.$$

Hence $4 \le 4 + (f(\theta))^2 \le 45$. Taking reciprocals of these positive bounds and multiplying by 90 gives

$$\boxed{2 \le g(\theta) \le \frac{45}{2}}.$$

Both endpoints are attained: the minimum occurs when $f(\theta) = \pm\sqrt{41}$, and the maximum when $f(\theta) = 0$.

FINAL ANSWER

(a) $R = \sqrt{41}$, $\alpha = 0.675$ radians (3 d.p.), $f(\theta) = \sqrt{41}\cos\left(\theta + \arctan\frac{4}{5}\right)$ (b)(i) $\sqrt{41}$ (b)(ii) $\begin{pmatrix} -\alpha \\ 0 \end{pmatrix}$ (c) $2 \le g(\theta) \le \frac{45}{2}$

WMA13/01 OCTOBER 2020

Question 7

9 marks | Trigonometric Functions & Graphs | Main topic: Trig Equations & R-Form

7. (a) Express $\cos x + 4\sin x$ in the form $R\cos(x - \alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

A scientist is studying the behaviour of seabirds in a colony.

She models the height above sea level, H metres, of one of the birds in the colony by the equation

$$H = \frac{24}{3 + \cos\left(\frac{1}{2}t\right) + 4\sin\left(\frac{1}{2}t\right)} \qquad 0 \leqslant t \leqslant 6.5$$

where t seconds is the time after it leaves the nest.

Find, according to the model,

(b) the minimum height of the seabird above sea level, giving your answer to the nearest cm,

(2)

(c) the value of t, to 2 decimal places, when $H = 10$

(4)

WORKED SOLUTION

Question 7 · WMA13/01 October 2020

1 **Part (a)**

Expand $R\cos(x - \alpha) = R\cos\alpha\cos x + R\sin\alpha\sin x$. Matching coefficients gives $R\cos\alpha = 1$, $R\sin\alpha = 4$. Thus $R = \sqrt{1^2 + 4^2} = \sqrt{17}$ and $\tan\alpha = 4$. Since $0 < \alpha < \pi/2$, $\alpha = \arctan 4 = 1.32581766\ldots$ radians.

$$\boxed{R = \sqrt{17}, \qquad \alpha = 1.326 \text{ radians}}$$

2 Part (b)

The model becomes $H = 24/[3 + \sqrt{17}\cos(t/2 - \alpha)]$. For $0 \le t \le 6.5$, the phase lies between $-\alpha$ and $3.25 - \alpha$; throughout this interval the denominator remains positive. Its maximum $3 + \sqrt{17}$ is attained at $t = 2\alpha = 2.651635\ldots$, inside the time interval. The height is then smallest:

$$H_{\min} = \frac{24}{3 + \sqrt{17}} = 3.369316876\ldots \text{ m}.$$

To the nearest centimetre this is 3.37 m (337 cm).

$$\boxed{H_{\min} = 3.37 \text{ m}}$$

---

3 Part (c)

For $H = 10$, $3 + \sqrt{17}\cos(t/2 - \alpha) = 24/10$, so

$$\cos(t/2 - \alpha) = -\frac{3}{5\sqrt{17}}.$$

Let $\gamma = \arccos[-3/(5\sqrt{17})]$. In the permitted phase interval $[-\alpha, 3.25 - \alpha]$, only $+\gamma$ occurs: $-\gamma < -\alpha$, and shifts by 2π are outside it. Using the unrounded $\alpha = \arctan 4$,

$$t = 2(\alpha + \gamma) = 6.085307854\ldots \text{ s}.$$

$$\boxed{t = 6.09 \text{ s}}$$

FINAL ANSWER

(a) $R = \sqrt{17}, \qquad \alpha = 1.326$ radians (b) $H_{\min} = 3.37$ m (c) $t = 6.09$ s

WMA13/01 MAY/JUNE 2021

Question 9

8 marks | Trigonometric Functions & Graphs | Main topic: Trig Equations & R-Form

9. (a) Express $12\sin x - 5\cos x$ in the form $R\sin(x - \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the exact value of R and give the value of α in radians, to 3 decimal places.

(3)

The function g is defined by

$$g(\theta) = 10 + 12\sin\left(2\theta - \frac{\pi}{6}\right) - 5\cos\left(2\theta - \frac{\pi}{6}\right) \qquad \theta > 0$$

Find

(b) (i) the minimum value of $g(\theta)$

(ii) the smallest value of θ at which the minimum value occurs.

(3)

The function h is defined by

$$h(\beta) = 10 - (12\sin\beta - 5\cos\beta)^2$$

(c) Find the range of h.

(2)

WORKED SOLUTION

Question 9 · WMA13/01 May/June 2021

1 **Part (a)**

Expand $R\sin(x-\alpha) = R\cos\alpha\sin x - R\sin\alpha\cos x$. Therefore $R\cos\alpha = 12$, $R\sin\alpha = 5$. Thus $R = \sqrt{12^2 + 5^2} = 13$ and $\alpha = \arctan(5/12) = 0.3947911197\ldots$ radians, in the required first quadrant.

$$\boxed{13\sin(x - 0.395) \quad (R = 13,\ \alpha = 0.395 \text{ rad})}$$

2 Part (b)

Write $g(\theta) = 10 + 13\sin(2\theta - \pi/6 - \alpha)$. (i) The minimum is $10 - 13 = -3$. (ii) For sine to equal -1, $2\theta - \pi/6 - \alpha = -\pi/2 + 2\pi n$. Hence $\theta = -\pi/6 + \alpha/2 + \pi n$. For $n = 0$ this is negative, so the smallest permitted value uses $n = 1$:

$$\theta = \frac{5\pi}{6} + \frac{1}{2}\arctan(5/12) = 2.815389438\ldots \text{ radians}.$$

Use the unrounded phase angle for the calculation.

$$\boxed{\min g = -3; \qquad \theta_{\min} = \frac{5\pi}{6} + \frac{1}{2}\arctan(5/12)}$$

3 Part (c)

As β varies over the reals, $12\sin\beta - 5\cos\beta$ covers $[-13, 13]$. Its square therefore covers $[0, 169]$, not $[169, 169]$. Subtracting from 10 reverses the bounds, giving $[-159, 10]$; both endpoints are attained.

$$\boxed{-159 \le h(\beta) \le 10}$$

FINAL ANSWER

(a) $13\sin(x - 0.395) \quad (R = 13,\ \alpha = 0.395 \text{ rad})$ (b) $\min g = -3; \qquad \theta_{\min} = \frac{5\pi}{6} + \frac{1}{2}\arctan(5/12)$ (c) $-159 \le h(\beta) \le 10$

WMA13/01 OCTOBER 2021

Question 6

8 marks | Trigonometric Functions & Graphs | Main topic: Differentiation Techniques

6. (i) The curve C_1 has equation

$$y = 3\ln(x^2 - 5) - 4x^2 + 15 \qquad x > \sqrt{5}$$

Show that C_1 has a stationary point at $x = \dfrac{\sqrt{p}}{2}$ where p is a constant to be found.

(4)

(ii) A different curve C_2 has equation

$$y = 4x - 12\sin^2 x$$

(a) Show that, for this curve,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = A + B\sin 2x$$

where A and B are constants to be found.

(b) Hence, state the maximum gradient of this curve.

(4)

WORKED SOLUTION

Question 6 · WMA13/01 October 2021

1 **Part (i)**

Differentiate to obtain $y' = 6x/(x^2 - 5) - 8x$. At a stationary point in $x > \sqrt{5}$, x is nonzero, so $6/(x^2 - 5) = 8$, giving $6 = 8x^2 - 40$ and $x^2 = 23/4$. The original positive domain selects $x = \sqrt{23}/2$, which exceeds $\sqrt{5}$. Hence $p = 23$.

$$\boxed{p = 23, \qquad x = \frac{\sqrt{23}}{2}}$$

2 **Part (ii)**

(a) Differentiate the squared sine with the chain rule: $y' = 4 - 24\sin x\cos x = 4 - 12\sin 2x$. Thus $A = 4$, $B = -12$. (b) Since $-1 \le \sin 2x \le 1$, the largest derivative is attained when $\sin 2x = -1$: $4 - 12(-1) = 16$.

$$\boxed{A = 4, \quad B = -12; \qquad \text{maximum gradient} = 16}$$

FINAL ANSWER

(i) $p = 23, \quad x = \frac{\sqrt{23}}{2}$ (ii) $A = 4, \quad B = -12; \quad$ maximum gradient $= 16$

Question 8

7 marks | Trigonometric Functions & Graphs | Main topic: Differentiation Techniques

8. A curve C has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \arcsin\left(\frac{1}{2}x\right) \qquad -2 \leqslant x \leqslant 2 \qquad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

(a) Sketch C.

(1)

(b) Given $x = 2\sin y$, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{A - x^2}}$$

where A is a constant to be found.

(3)

The point P lies on C and has y coordinate $\frac{\pi}{4}$

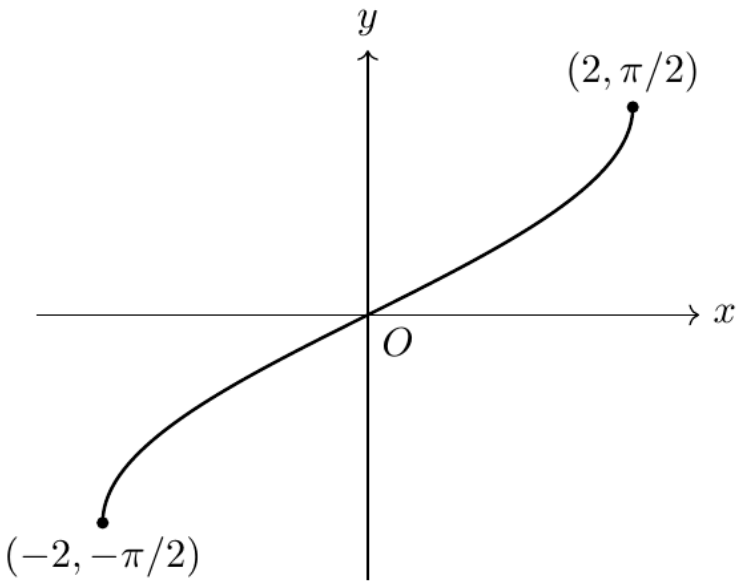
(c) Find the equation of the tangent to C at P. Write your answer in the form $y = mx + c$, where m and c are constants to be found.

(3)

1 Part (a)

The principal arcsine curve increases from $(-2, -\pi/2)$ through $(0, 0)$ to $(2, \pi/2)$. It is concave down on the negative-x half, concave up on the positive-x half, and vertical at both ends.

$$\boxed{y = \arcsin(x/2), \quad -2 \le x \le 2}$$



2 Part (b)

Differentiate the supplied inverse relation: $dx/dy = 2\cos y$. For $-\pi/2 < y < \pi/2$, cosine is positive, so $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2/4}$. Taking the reciprocal gives

$$\frac{dy}{dx} = \frac{1}{2\cos y} = \frac{1}{\sqrt{4 - x^2}}, \qquad -2 < x < 2.$$

Thus $A = 4$; the endpoints have vertical tangents, not finite derivatives.

$$\boxed{A = 4, \qquad \frac{dy}{dx} = \frac{1}{\sqrt{4 - x^2}}}$$

3 Part (c)

At $y = \pi/4$, $x = 2\sin(\pi/4) = \sqrt{2}$. The gradient is $1/\sqrt{4 - 2} = 1/\sqrt{2}$. Hence $y - \pi/4 = (x - \sqrt{2})/\sqrt{2}$, so $y = x/\sqrt{2} + \pi/4 - 1$.

$$\boxed{y = \frac{x}{\sqrt{2}} + \frac{\pi}{4} - 1}$$

FINAL ANSWER

(a) $y = \arcsin(x/2), \quad -2 \le x \le 2$ (b) $A = 4, \quad \frac{dy}{dx} = \frac{1}{\sqrt{4-x^2}}$ (c) $y = \frac{x}{\sqrt{2}} + \frac{\pi}{4} - 1$

WMA13/01 OCTOBER 2022

Question 8

9 marks | Trigonometric Functions & Graphs | Main topic: Trig Equations & R-Form

8. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Express $8\sin x - 15\cos x$ in the form $R\sin(x-\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R, and give the value of α, in radians, to 4 significant figures.

(3)

$$f(x) = \frac{15}{41 + 16\sin x - 30\cos x} \qquad x > 0$$

(b) Find

(i) the minimum value of $f(x)$

(ii) the smallest value of x at which this minimum value occurs.

(4)

(c) State the y coordinate of the minimum points on the curve with equation

$$y = 2f(x) - 5 \qquad x > 0$$

(1)

(d) State the smallest value of x at which a maximum point occurs for the curve with equation

$$y = -f(2x) \qquad x > 0$$

(1)

WORKED SOLUTION

Question 8 · WMA13/01 October 2022

1 **Part (a)**

Expand $R\sin(x-\alpha) = R\cos\alpha\sin x - R\sin\alpha\cos x$. Thus $R\cos\alpha = 8$, $R\sin\alpha = 15$, so $R = \sqrt{64+225} = 17$ and $\alpha = \arctan(15/8) = 1.0808390005\ldots$ radians, in the required quadrant.

$$\boxed{R = 17, \qquad \alpha = 1.081 \text{ radians}}$$

2 Part (b)

Using the exact phase from (a),

$$f(x) = \frac{15}{41 + 34\sin(x - \alpha)}.$$

The denominator lies between 7 and 75, so it is always positive. (i) The smallest fraction occurs at its largest denominator: $f_{\min} = 15/75 = 1/5$. (ii) This occurs when $x - \alpha = \pi/2 + 2n\pi$. The smallest positive value is $x = \alpha + \pi/2 = 2.6516353273\ldots$, or 2.65 to three significant figures.

$$\boxed{f_{\min} = \frac{1}{5}, \qquad x = 2.65}$$

3 Part (c)

The transformation $2f(x) - 5$ preserves the locations of minima and maps their y coordinate to $2(1/5) - 5 = -23/5$.

$$\boxed{y = -\frac{23}{5}}$$

4 Part (d)

Reflection in the x-axis changes minima of f into maxima of $-f$. The argument $2x$ halves their x coordinates, so the first positive maximum is at

$$x = \frac{\alpha + \pi/2}{2} = 1.3258176637\ldots \approx 1.33.$$

$$\boxed{x = 1.33}$$

FINAL ANSWER

(a) $R = 17, \qquad \alpha = 1.081$ radians (b) $f_{\min} = \frac{1}{5}, \qquad x = 2.65$ (c) $y = -\frac{23}{5}$ (d) $x = 1.33$

Question 9

10 marks | Trigonometric Functions & Graphs | Main topic: Applications of Differentiation

9.

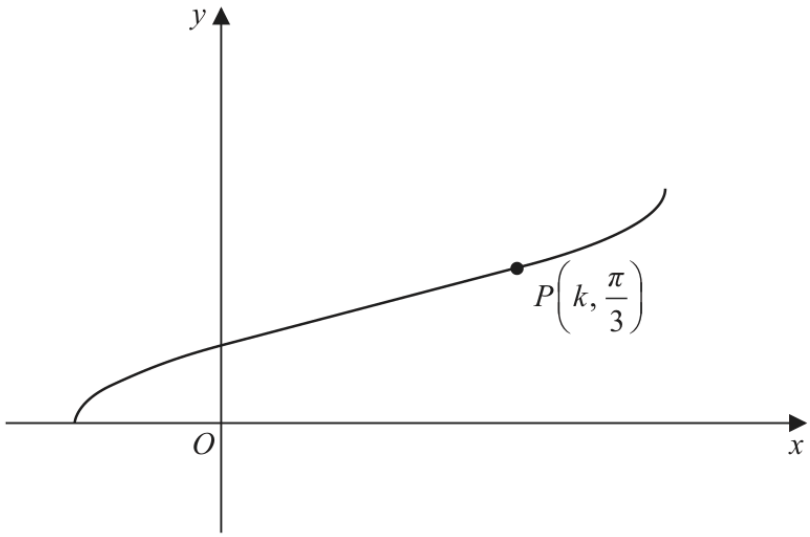


Figure 5

The curve shown in Figure 5 has equation

$$x = 4\sin^2 y - 1 \qquad 0 \leqslant y \leqslant \frac{\pi}{2}$$

The point $P\left(k, \frac{\pi}{3}\right)$ lies on the curve.

(a) Verify that $k = 2$

(1)

(b) (i) Find $\frac{\mathrm{d}x}{\mathrm{d}y}$ in terms of y

(ii) Hence show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\sqrt{x+1}\sqrt{3-x}}$

(6)

The normal to the curve at P cuts the x-axis at the point N.

(c) Find the exact area of triangle OPN, where O is the origin.

Give your answer in the form $a\pi + b\pi^2$ where a and b are constants.

(3)

1 Part (a)

At $y = \pi/3$, $x = 4(\sqrt{3}/2)^2 - 1 = 3 - 1 = 2$.

$$\boxed{k = 2}$$

2 Part (b)

(i) Differentiating with respect to y gives

$$\frac{dx}{dy} = 8\sin y \cos y.$$

(ii) For $0 < y < \pi/2$, both sine and cosine are positive, and $-1 < x < 3$. From $x + 1 = 4\sin^2 y$,

$$\sin y = \frac{\sqrt{x+1}}{2}, \qquad \cos y = \sqrt{1 - \frac{x+1}{4}} = \frac{\sqrt{3-x}}{2}.$$

Consequently $dx/dy = 2\sqrt{x+1}\sqrt{3-x}$. Taking the reciprocal proves the required formula on $-1 < x < 3$. At $x = -1$ and $x = 3$ the denominator is zero: these are endpoint vertical tangents, not finite derivatives.

$$\boxed{\frac{dx}{dy} = 8\sin y\cos y, \qquad \frac{dy}{dx} = \frac{1}{2\sqrt{x+1}\sqrt{3-x}}}$$

3 Part (c)

At $P = (2, \pi/3)$ the tangent gradient is $1/(2\sqrt{3})$, so the normal gradient is $-2\sqrt{3}$. Thus the normal is $y - \pi/3 = -2\sqrt{3}(x - 2)$. Setting $y = 0$ gives $N = (2 + \pi/(6\sqrt{3}), 0)$. Triangle OPN has horizontal base $ON = 2 + \pi/(6\sqrt{3})$ and perpendicular height $\pi/3$:

$$\text{Area}(OPN) = \frac{1}{2}\left(2 + \frac{\pi}{6\sqrt{3}}\right)\frac{\pi}{3} = \frac{\pi}{3} + \frac{\pi^2}{36\sqrt{3}}.$$

$$\boxed{\text{Area}(OPN) = \frac{1}{3}\pi + \frac{\sqrt{3}}{108}\pi^2}$$

FINAL ANSWER

(a) $k = 2$ (b) $\frac{dx}{dy} = 8\sin y\cos y$, $\frac{dy}{dx} = \frac{1}{2\sqrt{x+1}\sqrt{3-x}}$ (c) $\text{Area}(OPN) = \frac{1}{3}\pi + \frac{\sqrt{3}}{108}\pi^2$

Question 9

9 marks | Trigonometric Functions & Graphs | Main topic: Trig Equations & R-Form

9. $$\mathrm{h}(x) = 8 + 3\sin x(\cos x - 2\sin x) \quad 0 \leqslant x \leqslant 2\pi$$

(a) Write $\mathrm{h}(x)$ in the form

$$P + R\sin(2x + \alpha)$$

where P, R and α are positive constants, $0 < \alpha < \frac{\pi}{2}$

Give the exact values of P and R and give the value of α, in radians, to 3 decimal places.

(6)

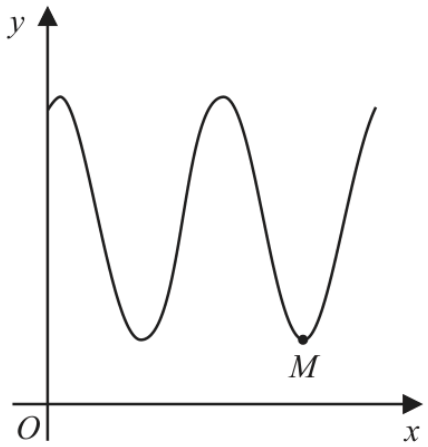


Figure 2

Figure 2 is a sketch of the curve with equation $y = \mathrm{h}(x)$

(b) State the range of h.

(1)

The point M, shown in Figure 2, is the minimum point on the curve with the largest x coordinate.

(c) Find the x coordinate of M.

(Solutions based entirely on calculator technology are not acceptable.)

(2)

1 Part (a)

Use $2\sin x\cos x = \sin 2x$ and $2\sin^2 x = 1 - \cos 2x$:

$$h(x) = 8 + 3\sin x\cos x - 6\sin^2 x = 5 + \frac{3}{2}\sin 2x + 3\cos 2x.$$

Expanding $R\sin(2x+\alpha)$ and comparing coefficients gives

$$R\cos\alpha = \frac{3}{2}, \qquad R\sin\alpha = 3.$$

Therefore

$$P = 5, \qquad R = \sqrt{\frac{9}{4} + 9} = \frac{3\sqrt{5}}{2}, \qquad \alpha = \arctan(2) = 1.107 \text{ radians (3 d.p.)}.$$

Thus $h(x) = 5 + \frac{3\sqrt{5}}{2}\sin(2x+\alpha)$ with the exact angle $\alpha = \arctan(2)$.

2 Part (b)

The given interval includes two complete periods. The range is

$$\left[5 - \frac{3\sqrt{5}}{2},\ 5 + \frac{3\sqrt{5}}{2}\right].$$

3 Part (c)

At a minimum, $\sin(2x+\alpha) = -1$, so

$$2x + \alpha = \frac{3\pi}{2} + 2\pi n, \qquad x = \frac{3\pi}{4} - \frac{\alpha}{2} + \pi n.$$

On $0 \le x \le 2\pi$, $n = 0$ and $n = 1$ give the two minima. The larger coordinate is

$$x_M = \frac{7\pi}{4} - \frac{1}{2}\arctan(2) = 4.944212785\ldots.$$

$$x_M = 4.94 \quad (3 \text{ significant figures}).$$

FINAL ANSWER

(a) $P = 5,\ R = \frac{3\sqrt{5}}{2},\ \alpha = \arctan 2$ (b) $\left[5 - \frac{3\sqrt{5}}{2},\ 5 + \frac{3\sqrt{5}}{2}\right]$ (c) $x_M = 4.94$

TOPIC

Trig Identities & Exact Values

Every past-paper question on this topic, January 2020 to June 2026.

WMA13/01 JANUARY 2021

Question 7

9 marks | Trig Identities & Exact Values

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

7. (a) Prove that

$$\frac{\sin 2x}{\cos x} + \frac{\cos 2x}{\sin x} \equiv \operatorname{cosec} x \qquad x \neq \frac{n\pi}{2} \quad n \in \mathbb{Z}$$

(3)

(b) Hence solve, for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$7 + \frac{\sin 4\theta}{\cos 2\theta} + \frac{\cos 4\theta}{\sin 2\theta} = 3\cot^2 2\theta$$

giving your answers in radians to 3 significant figures where appropriate.

(6)

WORKED SOLUTION

Question 7 · WMA13/01 January 2021

1 Part (a)

Apply the double-angle identities to the left side:

$$\frac{2\sin x\cos x}{\cos x} + \frac{1 - 2\sin^2 x}{\sin x} = 2\sin x + \frac{1}{\sin x} - 2\sin x = \csc x.$$

All divisions and cancellations are valid under the given exclusion $x \neq n\pi/2$.

$$\boxed{\frac{\sin 2x}{\cos x} + \frac{\cos 2x}{\sin x} = \csc x}$$

2 Part (b)

Using (a) with $x = 2\theta$ gives $7 + \csc 2\theta = 3\cot^2 2\theta$. Set $u = \csc 2\theta$ and use $\cot^2 2\theta = u^2 - 1$: $3u^2 - u - 10 = (3u + 5)(u - 2) = 0$. Thus $\sin 2\theta = 1/2$ or $\sin 2\theta = -3/5$. As $-\pi < 2\theta < \pi$, the first gives $2\theta = \pi/6, 5\pi/6$. Write $\gamma = \arcsin(3/5)$; the second gives $2\theta = -\gamma, -\pi + \gamma$. Therefore the four values are $\pi/12, 5\pi/12, -0.3217505544\ldots, -1.249045772\ldots$. None makes an original sine or cosine denominator zero.

$$\boxed{\theta = \frac{\pi}{12},\ \frac{5\pi}{12},\ -0.322,\ -1.25}$$

FINAL ANSWER

(a) $\frac{\sin 2x}{\cos x} + \frac{\cos 2x}{\sin x} = \csc x$ (b) $\theta = \frac{\pi}{12},\ \frac{5\pi}{12},\ -0.322,\ -1.25$

WMA13/01 JANUARY 2022

Question 9

8 marks | Trig Identities & Exact Values

9. **In this question you must show detailed reasoning.**

Solutions relying entirely on calculator technology are not acceptable.

(i) Solve, for $0 < x \leqslant \pi$, the equation

$$2\sec^2 x - 3\tan x = 2$$

giving the answers, as appropriate, to 3 significant figures.

(4)

(ii) Prove that

$$\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} \equiv 2$$

(4)

WORKED SOLUTION

Question 9 · WMA13/01 January 2022

1 **Part (i)**

Using $\sec^2 x = 1 + \tan^2 x$, the equation becomes $2\tan^2 x - 3\tan x = 0$, or $\tan x(2\tan x - 3) = 0$. If $\tan x = 0$, the given interval $0 < x \leq \pi$ contains only $x = \pi$. If $\tan x = 3/2$, the sole root in this interval is $x = \arctan(3/2) = 0.982793723\ldots$ radians. The excluded pole $x = \pi/2$ supplies no solution.

$$\boxed{x = 0.983 \text{ radians}, \quad \pi}$$

2 **Part (ii)**

Use angle addition to expand:

$$\sin 3\theta = \sin 2\theta\cos\theta + \cos 2\theta\sin\theta = 2\sin\theta\cos^2\theta + (1 - 2\sin^2\theta)\sin\theta = 3\sin\theta - 4\sin^3\theta,$$

$$\cos 3\theta = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta = (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta = 4\cos^3\theta - 3\cos\theta.$$

Where $\sin\theta\cos\theta \neq 0$, divide by the stated denominators:

$$\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = 3 - 4\sin^2\theta - (4\cos^2\theta - 3) = 6 - 4(\sin^2\theta + \cos^2\theta) = 2.$$

$$\boxed{\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = 2}$$

FINAL ANSWER

(i) $x = 0.983 \text{ radians}, \quad \pi$ (ii) $\frac{\sin 3\theta}{\sin \theta} - \frac{\cos 3\theta}{\cos \theta} = 2$

WMA13/01 MAY/JUNE 2022

Question 7

8 marks | Trig Identities & Exact Values

7. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that the equation

$$2\sin\theta\,(3\cot^2 2\theta - 7) = 13\sec\theta$$

can be written as

$$3\operatorname{cosec}^2 2\theta - 13\operatorname{cosec} 2\theta - 10 = 0$$

(4)

(b) Hence solve, for $0 < \theta < \frac{\pi}{2}$, the equation

$$2\sin\theta\,(3\cot^2 2\theta - 7) = 13\sec\theta$$

giving your answers to 3 significant figures.

(4)

WORKED SOLUTION

Question 7 · WMA13/01 May/June 2022

1 **Part (a)**

On the domain of the original equation, multiply by $\cos\theta$:

$$2\sin\theta\cos\theta(3\cot^2 2\theta - 7) = 13 \implies \sin 2\theta(3\cot^2 2\theta - 7) = 13.$$

Divide by the nonzero $\sin 2\theta$ and use $\cot^2 u = \csc^2 u - 1$:

$$3(\csc^2 2\theta - 1) - 7 = 13\csc 2\theta.$$

Collect the terms to obtain the displayed equation.

$$\boxed{3\csc^2 2\theta - 13\csc 2\theta - 10 = 0}$$

2

Part (b)

Put $u = \csc 2\theta$. Then $(3u + 2)(u - 5) = 0$, so $u = 5$ or $-2/3$. The second would give $\sin 2\theta = -3/2$, which is impossible. For $0 < \theta < \pi/2$, we have $0 < 2\theta < \pi$. Thus

$$2\theta = \arcsin(1/5) \quad \text{or} \quad \pi - \arcsin(1/5).$$

Hence $\theta = 0.1006789604\ldots$ or $1.4701173664\ldots$ radians. Both satisfy the original denominator restrictions.

$$\boxed{\theta = 0.101, \quad 1.47 \text{ radians}}$$

FINAL ANSWER

(a) $3\csc^2 2\theta - 13\csc 2\theta - 10 = 0$ (b) $\theta = 0.101, \quad 1.47$ radians

WMA13/01 MAY/JUNE 2023

Question 9

10 marks | Trig Identities & Exact Values

9. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that

$$\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} \equiv \operatorname{cosec} x \qquad x \neq \frac{n\pi}{2} \qquad n \in \mathbb{Z}$$

(3)

(b) Hence solve, for $0 < \theta < \frac{\pi}{2}$

$$\left(\frac{\cos 2\theta}{\sin \theta} + \frac{\sin 2\theta}{\cos \theta}\right)^2 = 6\cot\theta - 4$$

giving your answers to 3 significant figures as appropriate.

(5)

(c) Using the result from part (a), or otherwise, find the exact value of

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x}\right) \cot x \, dx$$

(2)

WORKED SOLUTION

Question 9 · WMA13/01 May/June 2023

1 Part (a)

Using a common denominator,

$$\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} = \frac{\cos 2x \cos x + \sin 2x \sin x}{\sin x \cos x} = \frac{\cos(2x - x)}{\sin x \cos x} = \frac{1}{\sin x} = \csc x.$$

The stated exclusion $x \neq n\pi/2$ ensures both denominators are nonzero.

$$\boxed{\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} = \csc x}$$

2 Part (b)

By part (a), the squared bracket is $\csc^2\theta$. Using $\csc^2\theta = 1 + \cot^2\theta$ and setting $u = \cot\theta$ gives

$$1 + u^2 = 6u - 4 \quad \Longrightarrow \quad u^2 - 6u + 5 = (u-1)(u-5) = 0.$$

On $0 < \theta < \pi/2$, both positive cotangents are permitted: $u = 1$ gives $\theta = \pi/4$, while $u = 5$ gives $\tan\theta = 1/5$, $\theta = 0.1973955598\ldots$ radians.

$$\boxed{\theta = \frac{\pi}{4}, \quad 0.197 \text{ radians}}$$

---

3 Part (c)

The bracket in part (a) equals $\csc x$. Therefore

$$\int_{\pi/6}^{\pi/4} \csc x \cot x \, dx = \left[-\csc x\right]_{\pi/6}^{\pi/4} = -\sqrt{2} - (-2) = 2 - \sqrt{2}.$$

$$\boxed{2 - \sqrt{2}}$$

✓ FINAL ANSWER

(a) $\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} = \csc x$ (b) $\theta = \frac{\pi}{4}, \quad 0.197 \text{ radians}$ (c) $2 - \sqrt{2}$

WMA13/01 OCTOBER 2023

Question 8

8 marks | Trig Identities & Exact Values

8. (a) Prove that

$$2\operatorname{cosec}^2 2\theta(1-\cos 2\theta) \equiv 1+\tan^2\theta$$

(4)

(b) Hence solve for $0 < x < 360°$, where $x \neq (90n)°$, $n \in \mathbb{N}$, the equation

$$2\operatorname{cosec}^2 2x(1-\cos 2x) = 4+3\sec x$$

giving your answers to one decimal place.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(4)

WORKED SOLUTION

Question 8 · WMA13/01 October 2023

1 Part (a)

Starting from the left side and using double-angle identities,

$$2\csc^2 2\theta(1-\cos 2\theta) = \frac{2(2\sin^2\theta)}{(2\sin\theta\cos\theta)^2} = \frac{1}{\cos^2\theta} = \sec^2\theta = 1+\tan^2\theta.$$

These steps hold wherever the original expression is defined, so $\sin 2\theta \neq 0$; no claim is made at the excluded angles.

$$\boxed{2\csc^2 2\theta(1-\cos 2\theta) = 1+\tan^2\theta}$$

2 Part (b)

By part (a), the equation becomes

$$\sec^2 x = 4+3\sec x \quad\Longrightarrow\quad (\sec x-4)(\sec x+1) = 0.$$

The branch $\sec x = 4$ gives $\cos x = 1/4$ and hence $x = 75.5225\ldots°$ or $284.4775\ldots°$. The other branch gives $\cos x = -1$, hence $x = 180°$ in the stated interval. Reject $180°$: it is a multiple of $90°$ and the original $\csc 2x$ is undefined there. The two remaining angles satisfy the exclusions.

$$\boxed{x = 75.5°, \qquad 284.5°}$$

FINAL ANSWER

(a) $2\csc^2 2\theta(1-\cos 2\theta) = 1+\tan^2\theta$ (b) $x = 75.5°, \qquad 284.5°$

WMA13/01 MAY/JUNE 2024

Question 7

8 marks | Trig Identities & Exact Values

7. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Given that

$$\sqrt{2}\sin(x+45°) = \cos(x-60°)$$

show that

$$\tan x = -2-\sqrt{3}$$

(4)

(b) Hence or otherwise, solve, for $0 \leqslant \theta < 180°$

$$\sqrt{2}\sin(2\theta) = \cos(2\theta-105°)$$

(4)

WORKED SOLUTION

Question 7 · WMA13/01 May/June 2024

1 **Part (a)**

Expand both compound angles:

$$\sin x + \cos x = \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x.$$

Thus $(2-\sqrt{3})\sin x = -\cos x$. Here $\cos x \neq 0$: otherwise this equation would require $\sin x = 0$ as well, which is impossible. Dividing by $(2-\sqrt{3})\cos x$ gives

$$\tan x = -\frac{1}{2-\sqrt{3}} = -\frac{2+\sqrt{3}}{(2-\sqrt{3})(2+\sqrt{3})}.$$

$$\boxed{\tan x = -2-\sqrt{3}}$$

2 **Part (b)**

Set $x = 2\theta - 45^\circ$. Then $x + 45^\circ = 2\theta$ and $x - 60^\circ = 2\theta - 105^\circ$, so part (a) gives $\tan(2\theta - 45^\circ) = -2 - \sqrt{3}$. Since $\tan 75^\circ = 2 + \sqrt{3}$,

$$2\theta - 45^\circ = -75^\circ + 180^\circ k, \qquad \theta = -15^\circ + 90^\circ k.$$

In $0 \le \theta < 180^\circ$, only $k = 1, 2$ are allowed. Substitution in the original equation confirms both values.

$$\boxed{\theta = 75^\circ,\ 165^\circ}$$

FINAL ANSWER

(a) $\tan x = -2 - \sqrt{3}$ (b) $\theta = 75^\circ,\ 165^\circ$

WMA13/01 MAY/JUNE 2025

Question 8

8 marks | Trig Identities & Exact Values

8: **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(a) Prove that

$$\tan 3x \equiv \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x} \qquad x \neq (2n+1)\frac{\pi}{6} \qquad n \in \mathbb{Z}$$

(3)

(b) Hence solve, for $0 < \theta < \frac{\pi}{2}$

$$\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = 2\sec^2 3\theta - 8$$

giving your answers to 2 decimal places.

(5)

WORKED SOLUTION

Question 8 · WMA13/01 May/June 2025

1 **Part (a)**

Put $s = \sin x$ and $c = \cos x$. Using the addition and double-angle formulae,

$$\sin 3x = \sin 2x \cos x + \cos 2x \sin x = 3sc^2 - s^3,$$

$$\cos 3x = \cos 2x \cos x - \sin 2x \sin x = c^3 - 3s^2c.$$

Where the stated expressions are defined, $c \neq 0$ and $\cos 3x \neq 0$. Dividing numerator and denominator by c^3 gives

$$\tan 3x = \frac{3sc^2 - s^3}{c^3 - 3s^2c} = \boxed{\frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}}.$$

2 Part (b)

By part (a), the equation is $\tan 3\theta = 2\sec^2 3\theta - 8$. Set $u = \tan 3\theta$ and use $\sec^2 3\theta = 1 + u^2$:

$$u = 2(1 + u^2) - 8 \implies 2u^2 - u - 6 = 0 \implies (2u + 3)(u - 2) = 0.$$

Thus $\tan 3\theta = 2$ or $\tan 3\theta = -3/2$. The transformed interval is $0 < 3\theta < 3\pi/2$. Its three allowed angles are

$$3\theta = \arctan 2, \quad \pi - \arctan\frac{3}{2}, \quad \pi + \arctan 2.$$

Consequently

$$\theta = 0.36904957\ldots, \quad 0.71959964\ldots, \quad 1.41624712\ldots,$$

$$\boxed{\theta = 0.37,\ 0.72,\ 1.42 \text{ radians}}.$$

None lies at an excluded pole of either side.

FINAL ANSWER

(a) $\frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$ (b) $\theta = 0.37,\ 0.72,\ 1.42$ radians

WMA13/01 OCTOBER 2025

Question 9

9 marks | Trig Identities & Exact Values

9. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(a) Express

$$6\sin^2\theta\cot 2\theta + 4\sin\theta\cos\theta$$

in terms of $\sin 2\theta$ and $\cos 2\theta$ only.

(3)

(b) Hence show that the equation

$$3\cot 2\theta - 14 = 6\sin^2\theta\cot 2\theta + 4\sin\theta\cos\theta$$

can be written in the form

$$5\sin^2 2\theta + 14\sin 2\theta - 3 = 0$$

(3)

(c) Hence solve, for $0 < x < 90°$, the equation

$$3\cot 2x - 14 = 6\sin^2 x\cot 2x + 4\sin x\cos x$$

giving your answers to one decimal place.

(3)

WORKED SOLUTION

Question 9 · WMA13/01 October 2025

1 **Part (a)**

Use $2\sin^2\theta = 1 - \cos 2\theta$, $\cot 2\theta = \cos 2\theta / \sin 2\theta$, and $2\sin\theta\cos\theta = \sin 2\theta$:

$$6\sin^2\theta\cot 2\theta + 4\sin\theta\cos\theta = \frac{3(1-\cos 2\theta)\cos 2\theta}{\sin 2\theta} + 2\sin 2\theta.$$

This is in terms of $\sin 2\theta$ and $\cos 2\theta$ only, on the original domain $\sin 2\theta \neq 0$.

2 **Part (b)**

Substitute the preceding expression and multiply by $\sin 2\theta$:

$$3\cos 2\theta - 14\sin 2\theta = 3\cos 2\theta - 3\cos^2 2\theta + 2\sin^2 2\theta.$$

Cancel $3\cos 2\theta$ and use $\cos^2 2\theta = 1 - \sin^2 2\theta$:

$$-14\sin 2\theta = -3 + 5\sin^2 2\theta.$$

Hence $5\sin^2 2\theta + 14\sin 2\theta - 3 = 0$.

---

3 **Part (c)**

Let $s = \sin 2x$. Then $(5s - 1)(s + 3) = 0$, so $s = 1/5$ or $s = -3$. The latter is impossible. As $0 < 2x < 180°$,

$$2x = \arcsin(1/5) = 11.53695903\ldots° \quad \text{or} \quad 180° - 11.53695903\ldots°.$$

Therefore $x = 5.8°$ or $84.2°$ to one decimal place. Both have nonzero $\sin 2x$, so the original cotangent is defined.

✓ **FINAL ANSWER**

(a) $3 * (1 - cos(2 * theta)) * cos(2 * theta)/sin(2 * theta) + 2 * sin(2 * theta)$ (b) Shown, as required. (c) $5.8 degrees$; $84.2 degrees$

WMA13/01 JANUARY 2026

Question 6

7 marks | Trig Identities & Exact Values

6. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(a) Show that the equation

$$2\cos(\theta - 60°) = 3\sin\theta$$

can be written in the form

$$\tan\theta = \frac{1}{6}\left(3+\sqrt{3}\right)$$

(4)

(b) Hence solve, for $0 \leqslant x < 180°$, the equation

$$2\cos(2x - 70°) = 3\sin(2x - 10°)$$

giving your solutions to one decimal place.

(3)

WORKED SOLUTION

Question 6 · WMA13/01 January 2026

1 Part (a)

Expanding the compound angle,

$$2\left(\cos\theta\cos 60° + \sin\theta\sin 60°\right) = 3\sin\theta,$$

so $\cos\theta + \sqrt{3}\sin\theta = 3\sin\theta$ and $\cos\theta = (3-\sqrt{3})\sin\theta$. If $\cos\theta = 0$, this equation would require $\sin\theta = 0$, which is impossible. We may therefore divide by $\cos\theta$:

$$\tan\theta = \frac{1}{3-\sqrt{3}} = \frac{3+\sqrt{3}}{9-3} = \frac{3+\sqrt{3}}{6}.$$

2 Part (b)

Set $\theta = 2x - 10°$, so $2x - 70° = \theta - 60°$ and $-10° \leq \theta < 350°$.

$$\theta = \arctan\left(\frac{3+\sqrt{3}}{6}\right) + 180°n = 38.2619662\ldots° + 180°n.$$

The admissible values are $38.2619662\ldots°$ and $218.2619662\ldots°$. Hence $x = (\theta + 10°)/2$, giving $x = 24.1°$ or $114.1°$ (1 decimal place).

FINAL ANSWER

(a) $\frac{3+\sqrt{3}}{6}$ (b) $x = 24.1° \text{ or } 114.1°$

WMA13/01 MAY/JUNE 2026

Question 8

9 marks | Trig Identities & Exact Values

8. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(a) Show that

$$\frac{\sec\theta\,\operatorname{cosec}\theta}{4\operatorname{cosec}4\theta} \equiv \cos 2\theta$$

(2)

(b) Hence show that the equation

$$4(5\cos^2\theta - 9)\operatorname{cosec}4\theta = 3(1+\tan^2 2\theta)\sec\theta\operatorname{cosec}\theta$$

can be written as

$$a\cos^2 2\theta + b\cos 2\theta + c = 0$$

where a, b and c are integers to be found.

(3)

(c) Hence, solve for $0 < x < 180°$, $x \neq (45n)°$, $n \in \mathbb{N}$, the equation

$$4(5\cos^2 x - 9)\operatorname{cosec}4x = 3(1+\tan^2 2x)\sec x\operatorname{cosec}x$$

giving the answers to one decimal place.

(4)

WORKED SOLUTION

Question 8 · WMA13/01 May/June 2026

1 **Part (a)**

For angles where the original expression is defined,

$$\frac{\sec\theta\ \operatorname{cosec}\theta}{4\operatorname{cosec}4\theta} = \frac{\sin 4\theta}{4\sin\theta\cos\theta} = \frac{2\sin 2\theta\cos 2\theta}{2\sin 2\theta} = \boxed{\cos 2\theta}.$$

2 Part (b)

Divide the given equation by $4\operatorname{cosec} 4\theta$ and use (a):

$$5\cos^2\theta - 9 = 3(1+\tan^2 2\theta)\cos 2\theta = \frac{3}{\cos 2\theta}.$$

Use $\cos^2\theta = (1+\cos 2\theta)/2$ and multiply by $2\cos 2\theta$:

$$(5\cos 2\theta - 13)\cos 2\theta = 6.$$

Therefore

$$\boxed{5\cos^2 2\theta - 13\cos 2\theta - 6 = 0}, \qquad a = 5,\ b = -13,\ c = -6.$$

The divisions are valid on the domain of the original equation.

3 Part (c)

Set $u = \cos 2x$:

$$5u^2 - 13u - 6 = (5u+2)(u-3) = 0.$$

Reject $u = 3$ because cosine lies between -1 and 1. Thus $\cos 2x = -2/5$. Since $0 < 2x < 360°$,

$$2x = 113.5781785\ldots° \quad \text{or} \quad 246.4218215\ldots°.$$

Hence $\boxed{x = 56.8° \text{ or } 123.2°}$ to one decimal place. Neither is an excluded multiple of $45°$; these are all the solutions in the interval.

FINAL ANSWER

(a) $\cos 2\theta$ (b) $5\cos^2 2\theta - 13\cos 2\theta - 6 = 0$ (c) $x = 56.8° \text{ or } 123.2°$

Question 9

7 marks | Trig Identities & Exact Values | Main topic: Trig Equations & R-Form

9. $\mathrm{f}(\theta) = 5\cos\theta - 4\sin\theta \qquad \theta \in \mathbb{R}$

(a) Express $\mathrm{f}(\theta)$ in the form $R\cos(\theta + \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

The curve with equation $y = \cos\theta$ is transformed onto the curve with equation $y = \mathrm{f}(\theta)$ by a sequence of two transformations.

Given that the first transformation is a stretch and the second a translation,

(b) (i) describe fully the transformation that is a stretch,

(ii) describe fully the transformation that is a translation.

(2)

Given

$$\mathrm{g}(\theta) = \frac{90}{4 + (\mathrm{f}(\theta))^2} \qquad \theta \in \mathbb{R}$$

(c) find the range of g.

(2)

1 Part (a)

Expanding the required form gives

$$R\cos(\theta+\alpha) = R\cos\alpha\cos\theta - R\sin\alpha\sin\theta.$$

Comparing coefficients with $5\cos\theta - 4\sin\theta$,

$$R\cos\alpha = 5, \qquad R\sin\alpha = 4.$$

Squaring and adding, using $R > 0$, gives

$$R^2 = 5^2 + 4^2 = 41 \quad \Longrightarrow \quad \boxed{R = \sqrt{41}}.$$

Also, since $0 < \alpha < \frac{\pi}{2}$,

$$\tan\alpha = \frac{4}{5}, \qquad \alpha = \arctan\left(\frac{4}{5}\right) = 0.6747409\ldots \quad \Longrightarrow \quad \boxed{\alpha = 0.675 \text{ radians (3 d.p.)}}.$$

Thus the exact form is

$$\boxed{f(\theta) = \sqrt{41}\cos\left(\theta + \arctan\frac{4}{5}\right)}.$$

2 Part (b)(i)

A stretch parallel to the y-axis with scale factor $\boxed{\sqrt{41}}$, taking $y = \cos\theta$ to $y = \sqrt{41}\cos\theta$.

3 Part (b)(ii)

A translation by the vector

$$\boxed{\begin{pmatrix} -\alpha \\ 0 \end{pmatrix}} = \begin{pmatrix} -\arctan(4/5) \\ 0 \end{pmatrix} \approx \begin{pmatrix} -0.675 \\ 0 \end{pmatrix}.$$

This moves the curve α units to the left, giving $y = \sqrt{41}\cos(\theta+\alpha)$.

4

Part (c)

Since θ can be any real number,

$$-\sqrt{41} \le f(\theta) \le \sqrt{41} \quad \Longrightarrow \quad 0 \le (f(\theta))^2 \le 41.$$

Hence $4 \le 4 + (f(\theta))^2 \le 45$. Taking reciprocals of these positive bounds and multiplying by 90 gives

$$\boxed{2 \le g(\theta) \le \frac{45}{2}}.$$

Both endpoints are attained: the minimum occurs when $f(\theta) = \pm\sqrt{41}$, and the maximum when $f(\theta) = 0$.

FINAL ANSWER

(a) $R = \sqrt{41}$, $\alpha = 0.675$ radians (3 d.p.), $f(\theta) = \sqrt{41}\cos\left(\theta + \arctan\frac{4}{5}\right)$ (b)(i) $\sqrt{41}$ (b)(ii) $\begin{pmatrix} -\alpha \\ 0 \end{pmatrix}$ (c) $2 \le g(\theta) \le \frac{45}{2}$

WMA13/01 OCTOBER 2020

Question 7

9 marks | Trig Identities & Exact Values | Main topic: Trig Equations & R-Form

7. (a) Express $\cos x + 4\sin x$ in the form $R\cos(x - \alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

A scientist is studying the behaviour of seabirds in a colony.

She models the height above sea level, H metres, of one of the birds in the colony by the equation

$$H = \frac{24}{3 + \cos\left(\frac{1}{2}t\right) + 4\sin\left(\frac{1}{2}t\right)} \qquad 0 \leqslant t \leqslant 6.5$$

where t seconds is the time after it leaves the nest.

Find, according to the model,

(b) the minimum height of the seabird above sea level, giving your answer to the nearest cm,

(2)

(c) the value of t, to 2 decimal places, when $H = 10$

(4)

WORKED SOLUTION

Question 7 · WMA13/01 October 2020

1 **Part (a)**

Expand $R\cos(x-\alpha) = R\cos\alpha\cos x + R\sin\alpha\sin x$. Matching coefficients gives $R\cos\alpha = 1$, $R\sin\alpha = 4$. Thus $R = \sqrt{1^2 + 4^2} = \sqrt{17}$ and $\tan\alpha = 4$. Since $0 < \alpha < \pi/2$, $\alpha = \arctan 4 = 1.32581766\ldots$ radians.

$$\boxed{R = \sqrt{17}, \qquad \alpha = 1.326 \text{ radians}}$$

2 Part (b)

The model becomes $H = 24/[3 + \sqrt{17}\cos(t/2 - \alpha)]$. For $0 \le t \le 6.5$, the phase lies between $-\alpha$ and $3.25 - \alpha$; throughout this interval the denominator remains positive. Its maximum $3 + \sqrt{17}$ is attained at $t = 2\alpha = 2.651635\ldots$, inside the time interval. The height is then smallest:

$$H_{\min} = \frac{24}{3 + \sqrt{17}} = 3.369316876\ldots \text{ m}.$$

To the nearest centimetre this is 3.37 m (337 cm).

$$\boxed{H_{\min} = 3.37 \text{ m}}$$

---

3 Part (c)

For $H = 10$, $3 + \sqrt{17}\cos(t/2 - \alpha) = 24/10$, so

$$\cos(t/2 - \alpha) = -\frac{3}{5\sqrt{17}}.$$

Let $\gamma = \arccos[-3/(5\sqrt{17})]$. In the permitted phase interval $[-\alpha, 3.25 - \alpha]$, only $+\gamma$ occurs: $-\gamma < -\alpha$, and shifts by 2π are outside it. Using the unrounded $\alpha = \arctan 4$,

$$t = 2(\alpha + \gamma) = 6.085307854\ldots \text{ s}.$$

$$\boxed{t = 6.09 \text{ s}}$$

FINAL ANSWER

(a) $R = \sqrt{17}, \qquad \alpha = 1.326$ radians (b) $H_{\min} = 3.37$ m (c) $t = 6.09$ s

WMA13/01 OCTOBER 2020

Question 8

9 marks | Trig Identities & Exact Values | Main topic: Differentiation Techniques

8. (i) The curve C has equation $y = \mathrm{g}(x)$ where

$$\mathrm{g}(x) = \mathrm{e}^{3x} \sec 2x \qquad -\frac{\pi}{4} < x < \frac{\pi}{4}$$

(a) Find $\mathrm{g}'(x)$

(2)

(b) Hence find the x coordinate of the stationary point of C.

(3)

(ii) A different curve has equation

$$x = \ln(\sin y) \qquad 0 < y < \frac{\pi}{2}$$

Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^x}{\mathrm{f}(x)}$$

where $\mathrm{f}(x)$ is a function of e^x that should be found.

(4)

WORKED SOLUTION

Question 8 · WMA13/01 October 2020

1 Part (i)(a)

By the product and chain rules,

$$g'(x) = 3e^{3x}\sec 2x + 2e^{3x}\sec 2x \tan 2x = e^{3x}\sec 2x(3 + 2\tan 2x).$$

$$\boxed{g'(x) = e^{3x}\sec 2x(3 + 2\tan 2x)}$$

2 Part (i)(b)

For $-\pi/4 < x < \pi/4$, $e^{3x}\sec 2x$ is positive and nonzero. Hence $g'(x) = 0$ precisely when $3 + 2\tan 2x = 0$, or $\tan 2x = -3/2$. Since $-\pi/2 < 2x < \pi/2$, tangent is one-to-one there:

$$x = -\frac{1}{2}\arctan\frac{3}{2} = -0.4913968616\ldots.$$

$$\boxed{x = -\frac{1}{2}\arctan\frac{3}{2} \approx -0.491}$$

---

3 Part (ii)

Differentiate $x = \ln(\sin y)$ with respect to y:

$$\frac{dx}{dy} = \frac{\cos y}{\sin y}, \qquad \frac{dy}{dx} = \frac{\sin y}{\cos y}.$$

The original equation gives $\sin y = e^{x}$. Since $0 < y < \pi/2$, cosine is positive, so $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - e^{2x}}$. Therefore

$$\frac{dy}{dx} = \frac{e^{x}}{\sqrt{1 - e^{2x}}}, \qquad f(x) = \sqrt{1 - e^{2x}}.$$

Here $x < 0$ from the original interval, so the square root is real and nonzero.

$$\boxed{f(x) = \sqrt{1 - e^{2x}}, \qquad x < 0}$$

FINAL ANSWER

(i)(a) $g'(x) = e^{3x}\sec 2x(3 + 2\tan 2x)$ (i)(b) $x = -\frac{1}{2}\arctan\frac{3}{2} \approx -0.491$ (ii) $f(x) = \sqrt{1 - e^{2x}}, \qquad x < 0$

WMA13/01 JANUARY 2021

Question 6

8 marks | Trig Identities & Exact Values | Main topic: Numerical Methods

6.

$$f(x) = x\cos\left(\frac{x}{3}\right) \qquad x > 0$$

(a) Find $f'(x)$

(2)

(b) Show that the equation $f'(x) = 0$ can be written as

$$x = k\arctan\left(\frac{k}{x}\right)$$

where k is an integer to be found.

(2)

(c) Starting with $x_1 = 2.5$ use the iteration formula

$$x_{n+1} = k\arctan\left(\frac{k}{x_n}\right)$$

with the value of k found in part (b), to calculate the values of x_2 and x_6 giving your answers to 3 decimal places.

(2)

(d) Using a suitable interval and a suitable function that should be stated, show that a root of $f'(x) = 0$ is 2.581 correct to 3 decimal places.

(2)

WORKED SOLUTION

Question 6 · WMA13/01 January 2021

1 **Part (a)**

By the product and chain rules, $f'(x) = \cos(x/3) - (x/3)\sin(x/3)$.

$$\boxed{f'(x) = \cos(x/3) - \frac{x}{3}\sin(x/3)}$$

2 Part (b)

Set $f'(x) = 0$: $\cos(x/3) = (x/3)\sin(x/3)$. At a solution, cosine cannot vanish (that would leave a nonzero right side), so $\tan(x/3) = 3/x$. On the principal positive branch $0 < x/3 < \pi/2$, this gives

$$x = 3\arctan(3/x), \qquad k = 3.$$

This is the branch used by the supplied iteration for the root near 2.581. The original domain $x > 0$ also admits later tangent branches: in general $x = 3\arctan(3/x) + 3n\pi$. The principal-branch rearrangement must not be interpreted as listing every positive stationary point.

$$\boxed{k = 3; \quad x = 3\arctan(3/x) \text{ on the selected branch}}$$

3 Part (c)

Using radians and $x_1 = 2.5$, the successive values are

n	1	2	3	4	5	6
x_n	2.5	2.628174152	2.554102174	2.596526871	2.572101407	2.586122337

Keep full precision between steps and round the requested terms only.

$$\boxed{x_2 = 2.628, \qquad x_6 = 2.586}$$

4 Part (d)

Define $F(x) = \cos(x/3) - (x/3)\sin(x/3)$, which is continuous near 2.581. The rounding interval endpoints give

$$F(2.5805) = 0.0003467157\ldots > 0, \qquad F(2.5815) = -0.0003456897\ldots < 0.$$

By continuity a root lies strictly between these endpoints; every number in this open interval rounds to 2.581 to three decimal places.

$$\boxed{x = 2.581 \text{ correct to 3 d.p.}}$$

✓ FINAL ANSWER

(a) $f'(x) = \cos(x/3) - \frac{x}{3}\sin(x/3)$ (b) $k = 3; \quad x = 3\arctan(3/x)$ on the selected branch (c) $x_2 = 2.628, \qquad x_6 = 2.586$ (d) $x = 2.581$ correct to 3 d.p.

WMA13/01 JANUARY 2021

Question 10

10 marks | Trig Identities & Exact Values | Main topic: Differentiation Techniques

10. The curve C has equation

$$x = 3\sec^2 2y \qquad x > 3 \qquad 0 < y < \frac{\pi}{4}$$

(a) Find $\dfrac{\mathrm{d}x}{\mathrm{d}y}$ in terms of y.

(2)

(b) Hence show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{p}{qx\sqrt{x-3}}$$

where p is irrational and q is an integer, stating the values of p and q.

(3)

(c) Find the equation of the normal to C at the point where $y = \dfrac{\pi}{12}$, giving your answer in the form $y = mx + c$, giving m and c as exact irrational numbers.

(5)

WORKED SOLUTION

Question 10 · WMA13/01 January 2021

1 **Part (a)**

Apply the chain rule twice:

$$\frac{dx}{dy} = 3 \cdot 2\sec 2y\,(2\sec 2y \tan 2y) = 12\sec^2 2y \tan 2y.$$

$$\boxed{\frac{dx}{dy} = 12\sec^2 2y \tan 2y}$$

2 **Part (b)**

From $x = 3\sec^2 2y$, $\sec^2 2y = x/3$ and $\tan^2 2y = x/3 - 1$. Since $0 < 2y < \pi/2$, tangent is positive, so $\tan 2y = \sqrt{(x-3)/3}$. Thus $dx/dy = 4x\sqrt{(x-3)/3} > 0$. Taking the reciprocal gives

$$\frac{dy}{dx} = \frac{\sqrt{3}}{4x\sqrt{x-3}}, \qquad p = \sqrt{3}, \quad q = 4.$$

$$\boxed{p = \sqrt{3}, \quad q = 4}$$

3 Part (c)

At $y = \pi/12$, $2y = \pi/6$ and $x = 3(4/3) = 4$. The tangent gradient is $\sqrt{3}/(4 \cdot 4\sqrt{1}) = \sqrt{3}/16$. Therefore the normal gradient is $-16/\sqrt{3} = -16\sqrt{3}/3$. Using the point $(4, \pi/12)$,

$$y - \frac{\pi}{12} = -\frac{16\sqrt{3}}{3}(x - 4) \Longrightarrow y = -\frac{16\sqrt{3}}{3}x + \frac{64\sqrt{3}}{3} + \frac{\pi}{12}.$$

$$\boxed{y = -\frac{16\sqrt{3}}{3}x + \frac{64\sqrt{3}}{3} + \frac{\pi}{12}}$$

FINAL ANSWER

(a) $\frac{dx}{dy} = 12\sec^2 2y \tan 2y$ (b) $p = \sqrt{3}, \quad q = 4$ (c) $y = -\frac{16\sqrt{3}}{3}x + \frac{64\sqrt{3}}{3} + \frac{\pi}{12}$

WMA13/01 MAY/JUNE 2021

Question 7

5 marks | Trig Identities & Exact Values | Main topic: Differentiation Techniques

7. Given that

$$x = 6\sin^2 2y \qquad 0 < y < \frac{\pi}{4}$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{A\sqrt{(Bx - x^2)}}$$

where A and B are integers to be found.

(5)

WORKED SOLUTION

Question 7 · WMA13/01 May/June 2021

1 Part (main)

Differentiate x with respect to y:

$$\frac{dx}{dy} = 6 \cdot 2\sin 2y \cos 2y \cdot 2 = 24 \sin 2y \cos 2y.$$

Since $0 < 2y < \pi/2$, both sine and cosine are positive. Using $\sin^2 2y = x/6$ and $\cos^2 2y = 1 - x/6$,

$$\frac{dx}{dy} = 24\sqrt{x/6}\sqrt{1 - x/6} = 4\sqrt{6x - x^2}.$$

On $0 < x < 6$ this is nonzero, so take its reciprocal.

$$\boxed{\frac{dy}{dx} = \frac{1}{4\sqrt{6x - x^2}}, \qquad A = 4,\ B = 6}$$

FINAL ANSWER

(main) $\frac{dy}{dx} = \frac{1}{4\sqrt{6x-x^2}}, \qquad A = 4,\ B = 6$

WMA13/01 MAY/JUNE 2021

Question 9

8 marks | Trig Identities & Exact Values | Main topic: Trig Equations & R-Form

9. (a) Express $12\sin x - 5\cos x$ in the form $R\sin(x - \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the exact value of R and give the value of α in radians, to 3 decimal places.

(3)

The function g is defined by

$$g(\theta) = 10 + 12\sin\left(2\theta - \frac{\pi}{6}\right) - 5\cos\left(2\theta - \frac{\pi}{6}\right) \qquad \theta > 0$$

Find

(b) (i) the minimum value of $g(\theta)$

(ii) the smallest value of θ at which the minimum value occurs.

(3)

The function h is defined by

$$h(\beta) = 10 - (12\sin\beta - 5\cos\beta)^2$$

(c) Find the range of h.

(2)

WORKED SOLUTION

Question 9 · WMA13/01 May/June 2021

1 **Part (a)**

Expand $R\sin(x-\alpha) = R\cos\alpha\sin x - R\sin\alpha\cos x$. Therefore $R\cos\alpha = 12$, $R\sin\alpha = 5$. Thus $R = \sqrt{12^2 + 5^2} = 13$ and $\alpha = \arctan(5/12) = 0.3947911197\ldots$ radians, in the required first quadrant.

$$\boxed{13\sin(x - 0.395) \quad (R = 13,\ \alpha = 0.395 \text{ rad})}$$

2 Part (b)

Write $g(\theta) = 10 + 13\sin(2\theta - \pi/6 - \alpha)$. (i) The minimum is $10 - 13 = -3$. (ii) For sine to equal -1, $2\theta - \pi/6 - \alpha = -\pi/2 + 2\pi n$. Hence $\theta = -\pi/6 + \alpha/2 + \pi n$. For $n = 0$ this is negative, so the smallest permitted value uses $n = 1$:

$$\theta = \frac{5\pi}{6} + \frac{1}{2}\arctan(5/12) = 2.815389438\ldots \text{ radians}.$$

Use the unrounded phase angle for the calculation.

$$\boxed{\min g = -3; \qquad \theta_{\min} = \frac{5\pi}{6} + \frac{1}{2}\arctan(5/12)}$$

3 Part (c)

As β varies over the reals, $12\sin\beta - 5\cos\beta$ covers $[-13, 13]$. Its square therefore covers $[0, 169]$, not $[169, 169]$. Subtracting from 10 reverses the bounds, giving $[-159, 10]$; both endpoints are attained.

$$\boxed{-159 \le h(\beta) \le 10}$$

FINAL ANSWER

(a) $13\sin(x - 0.395) \quad (R = 13,\ \alpha = 0.395 \text{ rad})$ (b) $\min g = -3; \qquad \theta_{\min} = \frac{5\pi}{6} + \frac{1}{2}\arctan(5/12)$ (c) $-159 \le h(\beta) \le 10$

WMA13/01 OCTOBER 2021

Question 6

8 marks | Trig Identities & Exact Values | Main topic: Differentiation Techniques

6. (i) The curve C_1 has equation

$$y = 3\ln(x^2 - 5) - 4x^2 + 15 \qquad x > \sqrt{5}$$

Show that C_1 has a stationary point at $x = \dfrac{\sqrt{p}}{2}$ where p is a constant to be found.

(4)

(ii) A different curve C_2 has equation

$$y = 4x - 12\sin^2 x$$

(a) Show that, for this curve,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = A + B\sin 2x$$

where A and B are constants to be found.

(b) Hence, state the maximum gradient of this curve.

(4)

WORKED SOLUTION

Question 6 · WMA13/01 October 2021

1 **Part (i)**

Differentiate to obtain $y' = 6x/(x^2 - 5) - 8x$. At a stationary point in $x > \sqrt{5}$, x is nonzero, so $6/(x^2 - 5) = 8$, giving $6 = 8x^2 - 40$ and $x^2 = 23/4$. The original positive domain selects $x = \sqrt{23}/2$, which exceeds $\sqrt{5}$. Hence $p = 23$.

$$\boxed{p = 23, \qquad x = \frac{\sqrt{23}}{2}}$$

2 **Part (ii)**

(a) Differentiate the squared sine with the chain rule: $y' = 4 - 24\sin x\cos x = 4 - 12\sin 2x$. Thus $A = 4$, $B = -12$. (b) Since $-1 \le \sin 2x \le 1$, the largest derivative is attained when $\sin 2x = -1$: $4 - 12(-1) = 16$.

$$\boxed{A = 4, \quad B = -12; \qquad \text{maximum gradient} = 16}$$

FINAL ANSWER

(i) $p = 23, \quad x = \frac{\sqrt{23}}{2}$ (ii) $A = 4, \quad B = -12; \quad$ maximum gradient $= 16$

Question 8

7 marks | Trig Identities & Exact Values | Main topic: Differentiation Techniques

8. A curve C has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \arcsin\left(\frac{1}{2}x\right) \qquad -2 \leqslant x \leqslant 2 \qquad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

(a) Sketch C.

(1)

(b) Given $x = 2\sin y$, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{A - x^2}}$$

where A is a constant to be found.

(3)

The point P lies on C and has y coordinate $\frac{\pi}{4}$

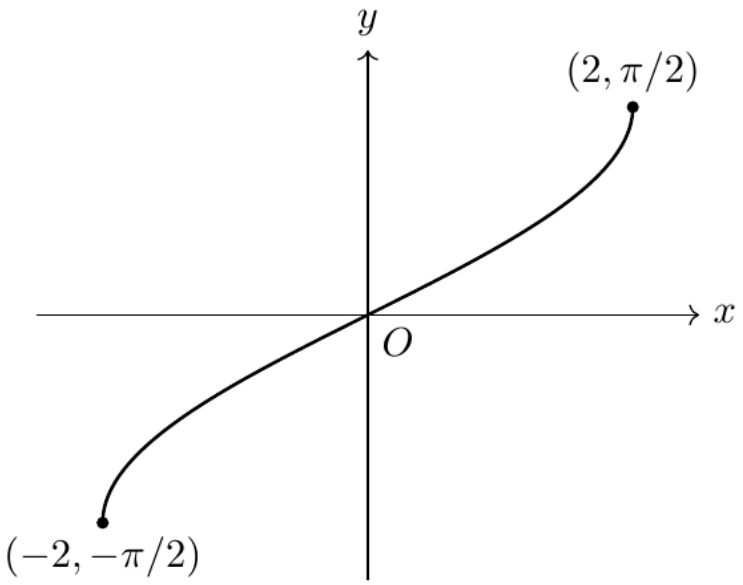
(c) Find the equation of the tangent to C at P. Write your answer in the form $y = mx + c$, where m and c are constants to be found.

(3)

1 Part (a)

The principal arcsine curve increases from $(-2, -\pi/2)$ through $(0, 0)$ to $(2, \pi/2)$. It is concave down on the negative-x half, concave up on the positive-x half, and vertical at both ends.

$$\boxed{y = \arcsin(x/2), \quad -2 \le x \le 2}$$



2 Part (b)

Differentiate the supplied inverse relation: $dx/dy = 2\cos y$. For $-\pi/2 < y < \pi/2$, cosine is positive, so $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2/4}$. Taking the reciprocal gives

$$\frac{dy}{dx} = \frac{1}{2\cos y} = \frac{1}{\sqrt{4 - x^2}}, \qquad -2 < x < 2.$$

Thus $A = 4$; the endpoints have vertical tangents, not finite derivatives.

$$\boxed{A = 4, \qquad \frac{dy}{dx} = \frac{1}{\sqrt{4 - x^2}}}$$

3 Part (c)

At $y = \pi/4$, $x = 2\sin(\pi/4) = \sqrt{2}$. The gradient is $1/\sqrt{4 - 2} = 1/\sqrt{2}$. Hence $y - \pi/4 = (x - \sqrt{2})/\sqrt{2}$, so $y = x/\sqrt{2} + \pi/4 - 1$.

$$\boxed{y = \frac{x}{\sqrt{2}} + \frac{\pi}{4} - 1}$$

FINAL ANSWER

(a) $y = \arcsin(x/2), \quad -2 \le x \le 2$ (b) $A = 4, \quad \frac{dy}{dx} = \frac{1}{\sqrt{4-x^2}}$ (c) $y = \frac{x}{\sqrt{2}} + \frac{\pi}{4} - 1$

WMA13/01 OCTOBER 2021

Question 10

7 marks | Trig Identities & Exact Values | Main topic: Integration

10.

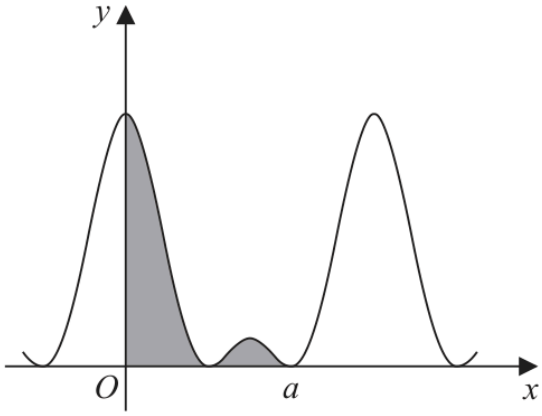


Figure 4

Figure 4 shows a sketch of part of the curve with equation

$$y = (1 + 2\cos 2x)^2$$

(a) Show that

$$(1 + 2\cos 2x)^2 \equiv p + q\cos 2x + r\cos 4x$$

where p, q and r are constants to be found.

(2)

The curve touches the positive x-axis for the second time when $x = a$, as shown in Figure 4.

The regions bounded by the curve, the y-axis and the x-axis up to $x = a$ are shown shaded in Figure 4.

(b) Find, using algebraic integration and making your method clear, the exact total area of the shaded regions. Write your answer in simplest form.

(5)

WORKED SOLUTION

Question 10 · WMA13/01 October 2021

1 **Part (a)**

Expand and use $\cos^2 2x = (1 + \cos 4x)/2$:

$$(1 + 2\cos 2x)^2 = 1 + 4\cos 2x + 4\cos^2 2x = 3 + 4\cos 2x + 2\cos 4x.$$

$$\boxed{p = 3, \quad q = 4, \quad r = 2}$$

2 **Part (b)**

The zeros satisfy $\cos 2x = -1/2$. The first two positive values are $x = \pi/3$ and $x = 2\pi/3$, so $a = 2\pi/3$. The squared curve is nonnegative, so integrate directly for the total shaded area:

$$\mathcal{A} = \int_0^{2\pi/3} (3 + 4\cos 2x + 2\cos 4x)\,dx = \left[3x + 2\sin 2x + \tfrac{1}{2}\sin 4x\right]_0^{2\pi/3}.$$

Use $\sin(4\pi/3) = -\sqrt{3}/2$ and $\sin(8\pi/3) = \sqrt{3}/2$ to obtain $\mathcal{A} = 2\pi - \sqrt{3} + \sqrt{3}/4 = 2\pi - 3\sqrt{3}/4$.

$$\boxed{2\pi - \frac{3\sqrt{3}}{4}}$$

FINAL ANSWER

(a) $p = 3, \quad q = 4, \quad r = 2$ (b) $2\pi - \frac{3\sqrt{3}}{4}$

Question 9

9 marks | Trig Identities & Exact Values | Main topic: Trig Equations & R-Form

9.

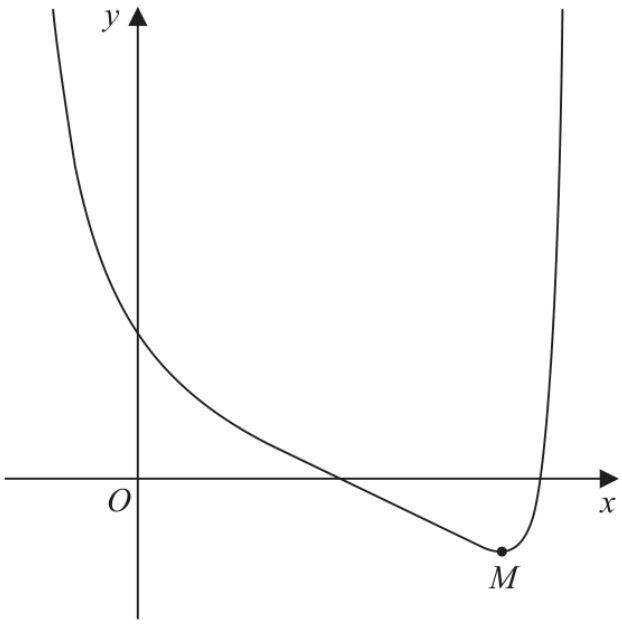


Figure 5

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 5 shows the curve with equation

$$y = \frac{1 + 2\cos x}{1 + \sin x} \qquad -\frac{\pi}{2} < x < \frac{3\pi}{2}$$

The point M, shown in Figure 5, is the minimum point on the curve.

(a) Show that the x coordinate of M is a solution of the equation

$$2\sin x + \cos x = -2$$

(4)

(b) Hence find, to 3 significant figures, the x coordinate of M.

(5)

1 Part (a)

Differentiate by the quotient rule:

$$y' = \frac{-2\sin x(1+\sin x) - (1+2\cos x)\cos x}{(1+\sin x)^2} = \frac{-2\sin x - \cos x - 2(\sin^2 x + \cos^2 x)}{(1+\sin x)^2} = \frac{-2\sin x - \cos x - 2}{(1+\sin x)^2}.$$

Inside the stated interval, $1+\sin x > 0$. At the interior minimum M, $y' = 0$, so its numerator is zero, giving the required equation.

$$\boxed{2\sin x + \cos x = -2}$$

2 Part (b)

Write $2\sin x + \cos x = \sqrt{5}\sin(x+\beta)$, where $\cos\beta = 2/\sqrt{5}$, $\sin\beta = 1/\sqrt{5}$ and $\beta = \arctan(1/2)$. The equation becomes $\sin(x+\beta) = -2/\sqrt{5}$. Let $\gamma = \arcsin(2/\sqrt{5})$, so $\gamma + \beta = \pi/2$. The solutions in the closed boundary range include $x = -\pi/2$, $3\pi/2$, and

$$x = \pi + \gamma - \beta = 3.7850937624\ldots.$$

Both boundary values are outside the original open domain (and make its denominator zero). The sole interior stationary point is M; the derivative changes from negative to positive there.

$$\boxed{x_M = 3.79}$$

FINAL ANSWER

(a) $2\sin x + \cos x = -2$ (b) $x_M = 3.79$

WMA13/01 OCTOBER 2022

Question 6

6 marks | Trig Identities & Exact Values | Main topic: Differentiation Techniques

6.

$$y = \frac{2 + 3\sin x}{\cos x + \sin x}$$

Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a\tan x + b\sec x + c}{\sec x + 2\sin x}$$

where a, b and c are integers to be found.

(6)

WORKED SOLUTION

Question 6 · WMA13/01 October 2022

1 Part (main)

With $y = (2 + 3\sin x)/(\cos x + \sin x)$, the quotient rule gives

$$\frac{dy}{dx} = \frac{3\cos x(\cos x + \sin x) - (2 + 3\sin x)(\cos x - \sin x)}{(\cos x + \sin x)^2}.$$

Expand the numerator and cancel the opposite mixed terms:

$$3\cos^2 x + 3\sin^2 x - 2\cos x + 2\sin x = 3 - 2\cos x + 2\sin x.$$

The denominator is $1 + 2\sin x\cos x$. Divide numerator and denominator by $\cos x$ to obtain

$$\frac{dy}{dx} = \frac{2\tan x + 3\sec x - 2}{\sec x + 2\sin x}.$$

This displayed form is used where its trigonometric terms are defined; the preceding quotient form also covers points with $\cos x = 0$ when the original denominator is nonzero.

$$\boxed{a = 2, \quad b = 3, \quad c = -2}$$

FINAL ANSWER

(main) $a = 2, \quad b = 3, \quad c = -2$

WMA13/01 OCTOBER 2022

Question 8

9 marks | Trig Identities & Exact Values | Main topic: Trig Equations & R-Form

8. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Express $8\sin x - 15\cos x$ in the form $R\sin(x-\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R, and give the value of α, in radians, to 4 significant figures.

(3)

$$f(x) = \frac{15}{41 + 16\sin x - 30\cos x} \qquad x > 0$$

(b) Find

(i) the minimum value of $f(x)$

(ii) the smallest value of x at which this minimum value occurs.

(4)

(c) State the y coordinate of the minimum points on the curve with equation

$$y = 2f(x) - 5 \qquad x > 0$$

(1)

(d) State the smallest value of x at which a maximum point occurs for the curve with equation

$$y = -f(2x) \qquad x > 0$$

(1)

WORKED SOLUTION

Question 8 · WMA13/01 October 2022

1 **Part (a)**

Expand $R\sin(x-\alpha) = R\cos\alpha\sin x - R\sin\alpha\cos x$. Thus $R\cos\alpha = 8$, $R\sin\alpha = 15$, so $R = \sqrt{64+225} = 17$ and $\alpha = \arctan(15/8) = 1.0808390005\ldots$ radians, in the required quadrant.

$$\boxed{R = 17, \qquad \alpha = 1.081 \text{ radians}}$$

2 Part (b)

Using the exact phase from (a),

$$f(x) = \frac{15}{41 + 34\sin(x - \alpha)}.$$

The denominator lies between 7 and 75, so it is always positive. (i) The smallest fraction occurs at its largest denominator: $f_{\min} = 15/75 = 1/5$. (ii) This occurs when $x - \alpha = \pi/2 + 2n\pi$. The smallest positive value is $x = \alpha + \pi/2 = 2.6516353273\ldots$, or 2.65 to three significant figures.

$$\boxed{f_{\min} = \frac{1}{5}, \qquad x = 2.65}$$

---

3 Part (c)

The transformation $2f(x) - 5$ preserves the locations of minima and maps their y coordinate to $2(1/5) - 5 = -23/5$.

$$\boxed{y = -\frac{23}{5}}$$

---

4 Part (d)

Reflection in the x-axis changes minima of f into maxima of $-f$. The argument $2x$ halves their x coordinates, so the first positive maximum is at

$$x = \frac{\alpha + \pi/2}{2} = 1.3258176637\ldots \approx 1.33.$$

$$\boxed{x = 1.33}$$

FINAL ANSWER

(a) $R = 17, \qquad \alpha = 1.081$ radians (b) $f_{\min} = \frac{1}{5}, \qquad x = 2.65$ (c) $y = -\frac{23}{5}$ (d) $x = 1.33$

WMA13/01 OCTOBER 2022

Question 9

9 marks | Trig Identities & Exact Values | Main topic: Trig Equations & R-Form

9. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

Given that $\cos 2\theta - \sin 3\theta \neq 0$

(a) prove that

$$\frac{\cos^2\theta}{\cos 2\theta - \sin 3\theta} \equiv \frac{1+\sin\theta}{1-2\sin\theta-4\sin^2\theta}$$

(4)

(b) Hence solve, for $0 < \theta \leqslant 360°$

$$\frac{\cos^2\theta}{\cos 2\theta - \sin 3\theta} = 2\operatorname{cosec}\theta$$

Give your answers to one decimal place.

(5)

WORKED SOLUTION

Question 9 · WMA13/01 October 2022

Part (a)

Write $s = \sin\theta$. The compound-angle formula gives

$$\sin 3\theta = \sin(2\theta+\theta) = 2\sin\theta\cos^2\theta + (1-2\sin^2\theta)\sin\theta = 3s - 4s^3.$$

Hence

$$\cos 2\theta - \sin 3\theta = 1 - 2s^2 - 3s + 4s^3 = (1-s)(1-2s-4s^2), \quad \cos^2\theta = (1-s)(1+s).$$

The given nonzero denominator ensures that $1 - s \neq 0$ and the other factor is nonzero. Cancel the common factor to obtain the required identity.

$$\boxed{\frac{\cos^2\theta}{\cos 2\theta - \sin 3\theta} = \frac{1+\sin\theta}{1-2\sin\theta-4\sin^2\theta}}$$

2 Part (b)

Use (a) and $\csc\theta = 1/\sin\theta$. With $s = \sin\theta \neq 0$,

$$\frac{1+s}{1-2s-4s^2} = \frac{2}{s} \implies s + s^2 = 2 - 4s - 8s^2 \implies 9s^2 + 5s - 2 = 0.$$

Thus $s = (-5 \pm \sqrt{97})/18$, approximately 0.269380989 and -0.824936545. Both lie in $[-1, 1]$, are nonzero, and satisfy the original nonzero-denominator condition. The positive value gives $15.6274\ldots$° and $164.3726\ldots$°; the negative value gives $235.5821\ldots$° and $304.4179\ldots$°. All four angles lie in $0 < \theta \leq 360$°.

$$\boxed{\theta = 15.6^\circ,\ 164.4^\circ,\ 235.6^\circ,\ 304.4^\circ}$$

FINAL ANSWER

(a) $\frac{\cos^2\theta}{\cos 2\theta - \sin 3\theta} = \frac{1+\sin\theta}{1-2\sin\theta-4\sin^2\theta}$ (b) $\theta = 15.6^\circ,\ 164.4^\circ,\ 235.6^\circ,\ 304.4^\circ$

WMA13/01 JANUARY 2023

Question 7

9 marks | Trig Identities & Exact Values | Main topic: Applications of Differentiation

7. The curve C has equation

$$x = 3\tan\left(y - \frac{\pi}{6}\right) \qquad x \in \mathbb{R} \quad -\frac{\pi}{3} < y < \frac{2\pi}{3}$$

(a) Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a}{x^2 + b}$$

where a and b are integers to be found.

(4)

The point P with y coordinate $\frac{\pi}{3}$ lies on C.

Given that the tangent to C at P crosses the x-axis at the point Q.

(b) find, in simplest form, the exact x coordinate of Q.

(5)

WORKED SOLUTION

Question 7 · WMA13/01 January 2023

1 **Part (a)**

Put $u = y - \pi/6$. Since $x = 3\tan u$,

$$\frac{dx}{dy} = 3\sec^2 u = 3(1 + \tan^2 u) = 3\left(1 + \frac{x^2}{9}\right) = \frac{x^2 + 9}{3}.$$

This is positive for all real x, so we may take the reciprocal: $dy/dx = 3/(x^2 + 9)$. Thus the required integers are $a = 3$, $b = 9$.

$$\boxed{\frac{dy}{dx} = \frac{3}{x^2 + 9}, \qquad a = 3, \quad b = 9}$$

2 **Part (b)**

At $y = \pi/3$, $x = 3\tan(\pi/3 - \pi/6) = 3\tan(\pi/6) = \sqrt{3}$. The tangent gradient is $3/((\sqrt{3})^2 + 9) = 1/4$. Its point-gradient equation is

$$y - \frac{\pi}{3} = \frac{1}{4}(x - \sqrt{3}).$$

At its x-axis crossing $y = 0$, $-4\pi/3 = x - \sqrt{3}$.

$$\boxed{x_Q = \sqrt{3} - \frac{4\pi}{3}}$$

✓ **FINAL ANSWER**

(a) $\frac{dy}{dx} = \frac{3}{x^2+9}$, $a = 3$, $b = 9$ (b) $x_Q = \sqrt{3} - \frac{4\pi}{3}$

WMA13/01 JANUARY 2023

Question 8

5 marks | Trig Identities & Exact Values | Main topic: Integration

8. Find, in simplest form,

$$\int (2\cos x - \sin x)^2 \, \mathrm{d}x$$

(5)

WORKED SOLUTION

Question 8 · WMA13/01 January 2023

1 Part (main)

Expand the square first:

$$(2\cos x - \sin x)^2 = 4\cos^2 x - 4\sin x \cos x + \sin^2 x.$$

Use $\cos^2 x = (1+\cos 2x)/2$, $\sin^2 x = (1-\cos 2x)/2$, and $2\sin x\cos x = \sin 2x$. The integrand becomes

$$\frac{5}{2} + \frac{3}{2}\cos 2x - 2\sin 2x.$$

Integrating with the angle-2 chain factors gives $5x/2 + 3\sin 2x/4 + \cos 2x + K$.

$$\boxed{\int (2\cos x - \sin x)^2 \, dx = \frac{5}{2}x + \frac{3}{4}\sin 2x + \cos 2x + K}$$

FINAL ANSWER

(main) $\int (2\cos x - \sin x)^2 \, dx = \frac{5}{2}x + \frac{3}{4}\sin 2x + \cos 2x + K$

Question 10

9 marks | Trig Identities & Exact Values | Main topic: Differentiation Techniques

10. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

A curve C has equation

$$x = \sin^2 4y \qquad 0 \leqslant y \leqslant \frac{\pi}{8} \qquad 0 \leqslant x \leqslant 1$$

The point P with x coordinate $\frac{1}{4}$ lies on C

(a) Find the exact y coordinate of P

(2)

(b) Find $\frac{\mathrm{d}x}{\mathrm{d}y}$

(2)

(c) Hence show that $\frac{\mathrm{d}y}{\mathrm{d}x}$ can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{q + r(x + s)^2}}$$

where q, r and s are constants to be found.

(3)

Using the answer to part (c),

(d) (i) state the x coordinate of the point where the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ is a minimum,

(ii) state the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at this point.

(2)

1 Part (a)

At $x = 1/4$, $\sin^2 4y = 1/4$. Since $0 \le 4y \le \pi/2$,

$$\sin 4y = \frac{1}{2} \quad \Longrightarrow \quad 4y = \frac{\pi}{6}.$$

Only this branch lies in the given range of y.

$$\boxed{y = \frac{\pi}{24}}$$

2 Part (b)

Differentiate $x = (\sin 4y)^2$ with respect to y:

$$\frac{dx}{dy} = 2\sin 4y\,(4\cos 4y).$$

$$\boxed{\frac{dx}{dy} = 8\sin 4y\cos 4y}$$

3 Part (c)

On the interior of the given interval, $\sin 4y = \sqrt{x}$ and $\cos 4y = \sqrt{1-x}$ are positive. Taking the reciprocal,

$$\frac{dy}{dx} = \frac{1}{8\sqrt{x}\sqrt{1-x}} = \frac{1}{\sqrt{64x - 64x^2}}.$$

Complete the square:

$$64x - 64x^2 = 16 - 64\left(x - \frac{1}{2}\right)^2.$$

Thus $dy/dx = (q + r(x+s)^2)^{-1/2}$ has the constants below, with $0 < x < 1$ for a finite derivative; the endpoints have vertical tangents.

$$\boxed{q = 16, \qquad r = -64, \qquad s = -\frac{1}{2}}$$

4 Part (d)

The derivative is positive. Its minimum occurs when the square root in the denominator is largest. Since $(x - 1/2)^2 \ge 0$, the largest radicand is 16, attained at $x = 1/2$. Hence the smallest gradient is $1/\sqrt{16}$.

$$\boxed{\text{(i) } x = \frac{1}{2}, \qquad \text{(ii) } \min \frac{dy}{dx} = \frac{1}{4}}$$

FINAL ANSWER

(a) $y = \frac{\pi}{24}$ (b) $\frac{dx}{dy} = 8\sin 4y \cos 4y$ (c) $q = 16, \qquad r = -64, \qquad s = -\frac{1}{2}$ (d)
(i) $x = \frac{1}{2}, \qquad$ (ii) $\min \frac{dy}{dx} = \frac{1}{4}$

Question 6

7 marks | Trig Identities & Exact Values

Main topic: Applications of Differentiation

6.

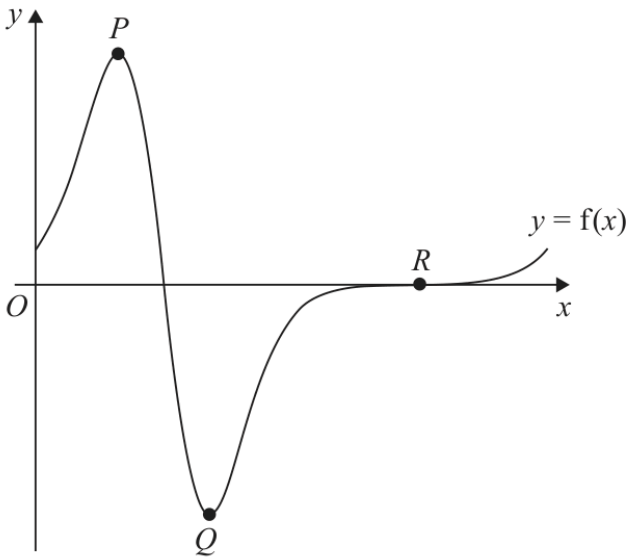


Figure 1

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 1 shows a sketch of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 2\mathrm{e}^{3\sin x}\cos x \qquad 0 \leqslant x \leqslant 2\pi$$

The curve intersects the x-axis at point R, as shown in Figure 1.

(a) State the coordinates of R

(1)

The curve has two turning points, at point P and point Q, also shown in Figure 1.

(b) Show that, at points P and Q,

$$a\sin^2 x + b\sin x + c = 0$$

where a, b and c are integers to be found.

(4)

(c) Hence find the x coordinate of point Q, giving your answer to 3 decimal places.

(2)

1 Part (a)

Since $2e^{3\sin x} > 0$, zeros occur when $\cos x = 0$. The labelled point R is the right-hand crossing, at $x = 3\pi/2$.

$$\boxed{R = \left(\frac{3\pi}{2}, 0\right)}$$

2 Part (b)

For $f(x) = 2e^{3\sin x}\cos x$, the product and chain rules give

$$f'(x) = 2\left(3e^{3\sin x}\cos^2 x - e^{3\sin x}\sin x\right) = 2e^{3\sin x}(3\cos^2 x - \sin x).$$

At a turning point the derivative is zero. The exponential factor is nonzero, so $3\cos^2 x - \sin x = 0$. Substitute $\cos^2 x = 1 - \sin^2 x$ and rearrange: $3 - 3\sin^2 x - \sin x = 0$.

$$\boxed{3\sin^2 x + \sin x - 3 = 0, \quad a = 3,\ b = 1,\ c = -3}$$

3 Part (c)

Solving $3s^2 + s - 3 = 0$ for $s = \sin x$ gives

$$s = \frac{-1 \pm \sqrt{37}}{6}.$$

The negative root is less than -1, so it is impossible for a real sine. Let $\beta = \arcsin\left((\sqrt{37} - 1)/6\right)$ in the first quadrant. The labelled minimum Q lies in the second quadrant, hence $x_Q = \pi - \beta = 2.131037293\ldots$.

$$\boxed{x_Q = 2.131}$$

FINAL ANSWER

(a) $R = \left(\frac{3\pi}{2}, 0\right)$ (b) $3\sin^2 x + \sin x - 3 = 0, \quad a = 3,\ b = 1,\ c = -3$ (c) $x_Q = 2.131$

WMA13/01 JANUARY 2024

Question 9

8 marks | Trig Identities & Exact Values | Main topic: Trig Equations & R-Form

9. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that the equation

$$\frac{3\sin\theta\cos\theta}{\cos\theta+\sin\theta} = (2+\sec 2\theta)(\cos\theta-\sin\theta)$$

can be written in the form

$$3\sin 2\theta - 4\cos 2\theta = 2$$

(3)

(b) Hence solve for $\pi < x < \frac{3\pi}{2}$

$$\frac{3\sin x\cos x}{\cos x+\sin x} = (2+\sec 2x)(\cos x-\sin x)$$

giving the answer to 3 significant figures.

(5)

WORKED SOLUTION

Question 9 · WMA13/01 January 2024

Part (a)

The original equation is defined only when $\cos\theta+\sin\theta\neq 0$ and $\cos 2\theta\neq 0$. Multiplying by $\cos\theta+\sin\theta$ gives

$$3\sin\theta\cos\theta = (2+\sec 2\theta)(\cos^2\theta-\sin^2\theta) = (2+\sec 2\theta)\cos 2\theta = 2\cos 2\theta+1.$$

Using $2\sin\theta\cos\theta=\sin 2\theta$, multiply through by 2: $3\sin 2\theta = 4\cos 2\theta+2$.

$$\boxed{3\sin 2\theta - 4\cos 2\theta = 2}$$

2 **Part (b)**

Write $3\sin 2x - 4\cos 2x = 5\sin(2x - \phi)$, where $\cos\phi = 3/5$, $\sin\phi = 4/5$ and $\phi = \arctan(4/3)$. Then $\sin(2x - \phi) = 2/5$. Because $\pi < x < 3\pi/2$, $2\pi - \phi < 2x - \phi < 3\pi - \phi$, approximately $(5.356, 8.497)$. With $\gamma = \arcsin(2/5)$, the only angle in this interval is $2\pi + \gamma$; $3\pi - \gamma \approx 9.013$ is outside it. Thus

$$x = \frac{2\pi + \arcsin(2/5) + \arctan(4/3)}{2} = 3.810998686\ldots.$$

At this value $\cos x + \sin x \neq 0$ and $\cos 2x \neq 0$; direct substitution satisfies the original equation.

$$\boxed{x = 3.81}$$

FINAL ANSWER

(a) $3\sin 2\theta - 4\cos 2\theta = 2$ (b) $x = 3.81$

Question 9

10 marks | Trig Identities & Exact Values | Main topic: Applications of Differentiation

9.

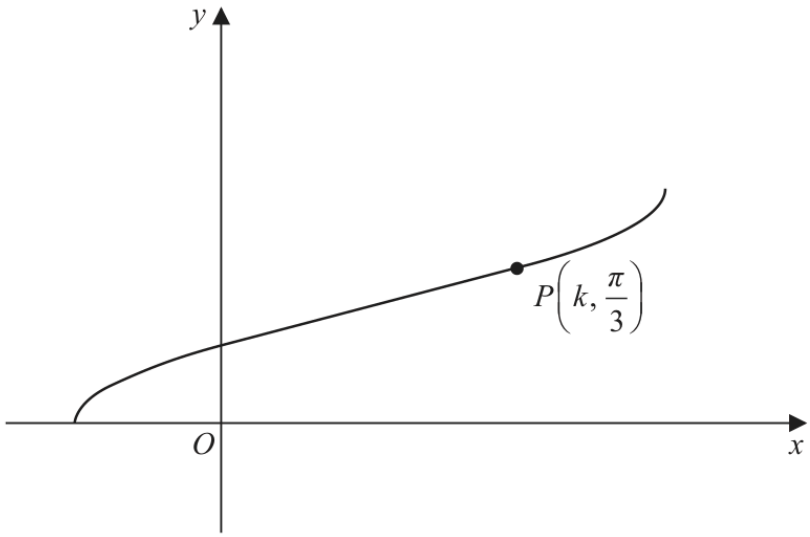


Figure 5

The curve shown in Figure 5 has equation

$$x = 4\sin^2 y - 1 \qquad 0 \leqslant y \leqslant \frac{\pi}{2}$$

The point $P\left(k, \frac{\pi}{3}\right)$ lies on the curve.

(a) Verify that $k = 2$

(1)

(b) (i) Find $\frac{\mathrm{d}x}{\mathrm{d}y}$ in terms of y

(ii) Hence show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\sqrt{x+1}\sqrt{3-x}}$

(6)

The normal to the curve at P cuts the x-axis at the point N.

(c) Find the exact area of triangle OPN, where O is the origin.

Give your answer in the form $a\pi + b\pi^2$ where a and b are constants.

(3)

1 Part (a)

At $y = \pi/3$, $x = 4(\sqrt{3}/2)^2 - 1 = 3 - 1 = 2$.

$$\boxed{k = 2}$$

2 Part (b)

(i) Differentiating with respect to y gives

$$\frac{dx}{dy} = 8\sin y \cos y.$$

(ii) For $0 < y < \pi/2$, both sine and cosine are positive, and $-1 < x < 3$. From $x + 1 = 4\sin^2 y$,

$$\sin y = \frac{\sqrt{x+1}}{2}, \qquad \cos y = \sqrt{1 - \frac{x+1}{4}} = \frac{\sqrt{3-x}}{2}.$$

Consequently $dx/dy = 2\sqrt{x+1}\sqrt{3-x}$. Taking the reciprocal proves the required formula on $-1 < x < 3$. At $x = -1$ and $x = 3$ the denominator is zero: these are endpoint vertical tangents, not finite derivatives.

$$\boxed{\frac{dx}{dy} = 8\sin y\cos y, \qquad \frac{dy}{dx} = \frac{1}{2\sqrt{x+1}\sqrt{3-x}}}$$

3 Part (c)

At $P = (2, \pi/3)$ the tangent gradient is $1/(2\sqrt{3})$, so the normal gradient is $-2\sqrt{3}$. Thus the normal is $y - \pi/3 = -2\sqrt{3}(x - 2)$. Setting $y = 0$ gives $N = (2 + \pi/(6\sqrt{3}), 0)$. Triangle OPN has horizontal base $ON = 2 + \pi/(6\sqrt{3})$ and perpendicular height $\pi/3$:

$$\text{Area}(OPN) = \frac{1}{2}\left(2 + \frac{\pi}{6\sqrt{3}}\right)\frac{\pi}{3} = \frac{\pi}{3} + \frac{\pi^2}{36\sqrt{3}}.$$

$$\boxed{\text{Area}(OPN) = \frac{1}{3}\pi + \frac{\sqrt{3}}{108}\pi^2}$$

FINAL ANSWER

(a) $k = 2$ (b) $\frac{dx}{dy} = 8\sin y\cos y$, $\frac{dy}{dx} = \frac{1}{2\sqrt{x+1}\sqrt{3-x}}$ (c) $\text{Area}(OPN) = \frac{1}{3}\pi + \frac{\sqrt{3}}{108}\pi^2$

WMA13/01 JANUARY 2025

Question 8

9 marks | Trig Identities & Exact Values | Main topic: Trig Equations & R-Form

8. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(i) Solve, for $0 < \theta < \pi$

$$3\operatorname{cosec}\theta = 8\cos\theta$$

giving your answers, in radians, to 3 significant figures.

(5)

(ii) Solve, for $0 < x < 180°$

$$\frac{\tan 2x - \tan 70°}{1 + \tan 2x \tan 70°} = -\frac{3}{8}$$

giving your answers, in degrees, to one decimal place.

(4)

WORKED SOLUTION

Question 8 · WMA13/01 January 2025

1 **Part (i)**

Use $\csc\theta = 1/\sin\theta$. Since $0 < \theta < \pi$, $\sin\theta \neq 0$ and

$$\frac{3}{\sin\theta} = 8\cos\theta \quad \Longrightarrow \quad 3 = 8\sin\theta\cos\theta = 4\sin 2\theta.$$

Thus $\sin 2\theta = 3/4$. Since $0 < 2\theta < 2\pi$, the two possibilities are

$$2\theta = \arcsin(3/4) \quad \text{or} \quad 2\theta = \pi - \arcsin(3/4).$$

They give $\theta = 0.4240310395\ldots$ and $1.1467652873\ldots$ radians.

$$\boxed{\theta = 0.424,\ 1.15 \text{ radians}}$$

2 Part (ii)

Apply the tangent subtraction identity:

$$\frac{\tan 2x - \tan 70^\circ}{1 + \tan 2x \tan 70^\circ} = \tan(2x - 70^\circ) = -\frac{3}{8}.$$

Set $u = 2x - 70^\circ$. The interval becomes $-70^\circ < u < 290^\circ$. With $\arctan(-3/8) = -20.5560452196\ldots^\circ$, the permitted values are $u = -20.5560452196\ldots^\circ$ and $159.4439547804\ldots^\circ$. Adding 70° and halving gives $x = 24.7219773902\ldots^\circ$ and $114.7219773902\ldots^\circ$. Both keep the original tangents and denominator defined.

$$\boxed{x = 24.7^\circ,\ 114.7^\circ}$$

FINAL ANSWER

(i) $\theta = 0.424,\ 1.15$ radians (ii) $x = 24.7^\circ,\ 114.7^\circ$

Question 10

9 marks | Trig Identities & Exact Values | Main topic: Differentiation Techniques

10.

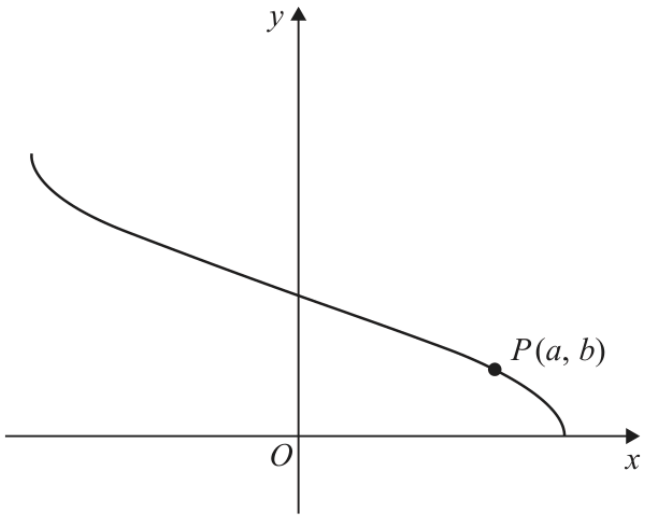


Figure 2

Figure 2 shows a sketch of the curve with equation

$$x = 3\cos 2y \qquad -3 \leqslant x \leqslant 3 \qquad 0 \leqslant y \leqslant \frac{\pi}{2}$$

(a) Find $\frac{dx}{dy}$ in terms of y.

(2)

(b) Hence show that

$$\frac{dy}{dx} = \frac{k}{\sqrt{9 - x^2}}$$

where k is a constant to be found.

(3)

The point $P(a, b)$ lies on the curve and is shown in Figure 2.

Given that

- the gradient of the curve at P is $-\frac{1}{4}$
- both a and b are positive

(c) find the exact values of a and b.

(4)

1 **Part (a)**

Differentiate $x = 3\cos 2y$ with respect to y using the chain rule:

$$\frac{dx}{dy} = 3(-\sin 2y)\cdot 2 = -6\sin 2y.$$

$$\boxed{\frac{dx}{dy} = -6\sin 2y}$$

2 **Part (b)**

For $0 < y < \pi/2$, $\sin 2y > 0$ and

$$\frac{dy}{dx} = \frac{1}{dx/dy} = -\frac{1}{6\sin 2y}.$$

Since $\cos 2y = x/3$ and the sine is positive on this interval,

$$\sin 2y = \sqrt{1-\cos^2 2y} = \sqrt{1 - x^2/9} = \frac{\sqrt{9-x^2}}{3}.$$

Consequently $dy/dx = -1/(2\sqrt{9-x^2})$, valid for $-3 < x < 3$. The endpoint tangents are vertical; there is no finite derivative there.

$$\boxed{\frac{dy}{dx} = -\frac{1}{2\sqrt{9-x^2}}, \qquad k = -\frac{1}{2}}$$

3 **Part (c)**

Set the gradient equal to $-1/4$:

$$-\frac{1}{2\sqrt{9-a^2}} = -\frac{1}{4} \quad\Longrightarrow\quad \sqrt{9-a^2} = 2 \quad\Longrightarrow\quad a^2 = 5.$$

Because $a > 0$, take $a = \sqrt{5}$. Substitute into $a = 3\cos 2b$:

$$\cos 2b = \frac{\sqrt{5}}{3} \quad\Longrightarrow\quad b = \frac{1}{2}\arccos\left(\frac{\sqrt{5}}{3}\right).$$

Here $0 < 2b < \pi/2$ since both coordinates are positive, so the principal arccos value is the required one.

$$\boxed{a = \sqrt{5}, \qquad b = \frac{1}{2}\arccos\left(\frac{\sqrt{5}}{3}\right)}$$

FINAL ANSWER

(a) $\frac{dx}{dy} = -6\sin 2y$ (b) $\frac{dy}{dx} = -\frac{1}{2\sqrt{9-x^2}}$, $\quad k = -\frac{1}{2}$ (c) $a = \sqrt{5}$, $\quad b = \frac{1}{2}\arccos\left(\frac{\sqrt{5}}{3}\right)$

Question 7

9 marks | Trig Identities & Exact Values | Main topic: Numerical Methods

7: A continuous curve has equation

$$y = e^{-x^2} \sin 3x \qquad 0 \leqslant x \leqslant \frac{\pi}{3}$$

The curve has a stationary point at the point P.

(a) Show, using calculus, that the x coordinate of P is a solution of the equation

$$x = \frac{1}{3}\arctan\left(\frac{3}{2x}\right)$$

(4)

Using the iteration formula

$$x_{n+1} = \frac{1}{3}\arctan\left(\frac{3}{2x_n}\right) \qquad x_1 = 0.4$$

(b) find the value of

(i) x_2

(ii) x_4

giving your answers to 4 decimal places.

(3)

(c) Using a suitable interval and a suitable function which should be stated, show that the x coordinate of P is 0.430 correct to 3 decimal places.

(2)

1 Part (a)

Apply the product and chain rules:

$$\frac{dy}{dx} = e^{-x^2}\left(3\cos 3x - 2x\sin 3x\right).$$

Since $e^{-x^2} > 0$, a stationary point satisfies

$$3\cos 3x = 2x\sin 3x.$$

Neither endpoint $x = 0, \pi/3$ is stationary. In the interior $\sin 3x > 0$ and $x > 0$, so $\cos 3x > 0$ at a stationary point and $0 < 3x < \pi/2$. Therefore division and the principal inverse tangent are valid:

$$\tan 3x = \frac{3}{2x} \quad\Longrightarrow\quad \boxed{x = \frac{1}{3}\arctan\left(\frac{3}{2x}\right)}.$$

2 Part (b)

Work in radians and retain calculator precision between iterations:

$$x_2 = \frac{1}{3}\arctan\frac{3}{0.8} = 0.4367313117\ldots,$$

$$x_3 = 0.4291581882\ldots, \qquad x_4 = 0.4307116849\ldots.$$

Thus (i) $\boxed{x_2 = 0.4367}$ and (ii) $\boxed{x_4 = 0.4307}$.

3 Part (c)

Let $h(x) = 3\cos 3x - 2x\sin 3x$, which is continuous. At the rounding boundaries,

$$h(0.4295) = 0.0106861\ldots > 0, \qquad h(0.4305) = -0.000598027\ldots < 0.$$

The sign change places a root strictly between 0.4295 and 0.4305. On $0 < x < \pi/6$, $h'(x) = -11\sin 3x - 6x\cos 3x < 0$, so the stationary root is unique; for $\pi/6 \le x \le \pi/3$, $h(x) < 0$. Hence the x coordinate of P is $\boxed{0.430 \text{ to } 3 \text{ decimal places}}$.

FINAL ANSWER

(a) $x = \frac{1}{3}\arctan\left(\frac{3}{2x}\right)$ (b) $x_2 = 0.4367$, $x_4 = 0.4307$ (c) 0.430 to 3 decimal places

Question 9

11 marks | Trig Identities & Exact Values | Main topic: Differentiation Techniques

9:

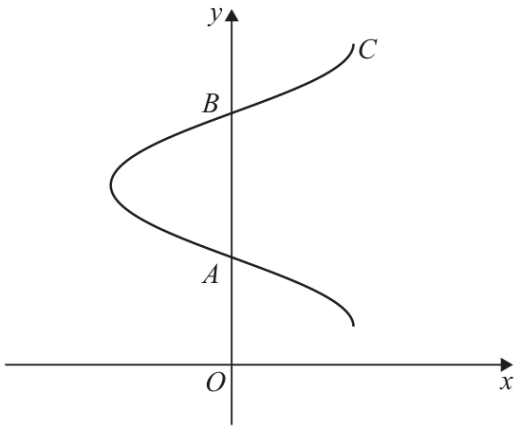


Figure 2

Figure 2 shows a sketch of the curve C with equation

$$x = \frac{2}{3}\sin\left(3y + \frac{\pi}{4}\right) \qquad \frac{\pi}{12} < y < \frac{3\pi}{4}$$

The curve intersects the y-axis at the points A and B as shown.

(a) Find the exact value of the y coordinate of

- point A
- point B

(3)

(b) Show that

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{p - qx^2}$$

where p and q are integers to be found.

(4)

The **normal** to C at A and the **tangent** to C at B intersect at the point D.

Using

- the answer to part (b)
- the sketch of curve C in Figure 2

(c) find, in simplest form, the exact x coordinate of D.

(4)

1 Part (a)

On the y axis, $x = 0$, so $\sin(3y + \pi/4) = 0$. Since $\pi/12 < y < 3\pi/4$, we have $\pi/2 < 3y + \pi/4 < 5\pi/2$. Therefore the possible angles are π and 2π, giving

$$\boxed{y_A = \frac{\pi}{4}, \qquad y_B = \frac{7\pi}{12}}.$$

2 Part (b)

Differentiate x with respect to y, then take the reciprocal:

$$\frac{dx}{dy} = 2\cos(3y + \pi/4), \qquad \frac{dy}{dx} = \frac{1}{2\cos(3y + \pi/4)}.$$

Using $\sin(3y + \pi/4) = 3x/2$,

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4(1 - \sin^2(3y + \pi/4))} = \boxed{\frac{1}{4 - 9x^2}}.$$

Thus $\boxed{p = 4,\ q = 9}$. The derivative formula applies where the tangent is not vertical.

3 Part (c)

At $x = 0$, part (b) gives $|dy/dx| = 1/2$. The sketch shows a negative tangent gradient at A and a positive one at B. The normal at A therefore has gradient 2, while the tangent at B has gradient $1/2$:

$$y = 2x + \frac{\pi}{4}, \qquad y = \frac{x}{2} + \frac{7\pi}{12}.$$

At their intersection,

$$\frac{3}{2}x = \frac{7\pi}{12} - \frac{\pi}{4} = \frac{\pi}{3} \quad \Longrightarrow \quad \boxed{x_D = \frac{2\pi}{9}}.$$

FINAL ANSWER

(a) $y_A = \frac{\pi}{4}, \qquad y_B = \frac{7\pi}{12}$ (b) $\frac{1}{4-9x^2}$, $p = 4,\ q = 9$ (c) $x_D = \frac{2\pi}{9}$

WMA13/01 JANUARY 2026

Question 8

4 marks | Trig Identities & Exact Values | Main topic: Differentiation Techniques

8. A curve has equation

$$x = \mathrm{e}^{2\tan y} \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a}{x\left(b + (\ln x)^n\right)}$$

where a, b and n are integers to be found.

(4)

WORKED SOLUTION

Question 8 · WMA13/01 January 2026

1 **Solution**

\paragraph{Solution} Differentiate $x = e^{2\tan y}$ with respect to y:

$$\frac{dx}{dy} = 2\sec^2 y\, e^{2\tan y} = 2x\sec^2 y.$$

This is positive on the stated interval, so

$$\frac{dy}{dx} = \frac{1}{2x\sec^2 y}.$$

Taking logarithms of the original relation gives $\ln x = 2\tan y$. Using $\sec^2 y = 1 + \tan^2 y$,

$$\frac{dy}{dx} = \frac{1}{2x\left(1 + \frac{(\ln x)^2}{4}\right)} = \frac{2}{x\left(4 + (\ln x)^2\right)}.$$

Hence $a = 2$, $b = 4$ and $n = 2$.

FINAL ANSWER

(main) $a = 2,\ b = 4,\ n = 2$

Question 9

9 marks | Trig Identities & Exact Values | Main topic: Trig Equations & R-Form

9. $$\mathrm{h}(x) = 8 + 3\sin x(\cos x - 2\sin x) \quad 0 \leqslant x \leqslant 2\pi$$

(a) Write $\mathrm{h}(x)$ in the form

$$P + R\sin(2x + \alpha)$$

where P, R and α are positive constants, $0 < \alpha < \frac{\pi}{2}$

Give the exact values of P and R and give the value of α, in radians, to 3 decimal places.

(6)

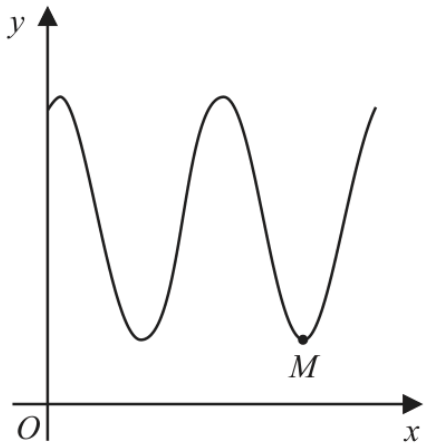


Figure 2

Figure 2 is a sketch of the curve with equation $y = \mathrm{h}(x)$

(b) State the range of h.

(1)

The point M, shown in Figure 2, is the minimum point on the curve with the largest x coordinate.

(c) Find the x coordinate of M.

(Solutions based entirely on calculator technology are not acceptable.)

(2)

1 Part (a)

Use $2\sin x\cos x = \sin 2x$ and $2\sin^2 x = 1 - \cos 2x$:

$$h(x) = 8 + 3\sin x\cos x - 6\sin^2 x = 5 + \frac{3}{2}\sin 2x + 3\cos 2x.$$

Expanding $R\sin(2x+\alpha)$ and comparing coefficients gives

$$R\cos\alpha = \frac{3}{2}, \qquad R\sin\alpha = 3.$$

Therefore

$$P = 5, \qquad R = \sqrt{\frac{9}{4} + 9} = \frac{3\sqrt{5}}{2}, \qquad \alpha = \arctan(2) = 1.107 \text{ radians (3 d.p.)}.$$

Thus $h(x) = 5 + \frac{3\sqrt{5}}{2}\sin(2x+\alpha)$ with the exact angle $\alpha = \arctan(2)$.

2 Part (b)

The given interval includes two complete periods. The range is

$$\left[5 - \frac{3\sqrt{5}}{2},\ 5 + \frac{3\sqrt{5}}{2}\right].$$

3 Part (c)

At a minimum, $\sin(2x+\alpha) = -1$, so

$$2x + \alpha = \frac{3\pi}{2} + 2\pi n, \qquad x = \frac{3\pi}{4} - \frac{\alpha}{2} + \pi n.$$

On $0 \le x \le 2\pi$, $n = 0$ and $n = 1$ give the two minima. The larger coordinate is

$$x_M = \frac{7\pi}{4} - \frac{1}{2}\arctan(2) = 4.944212785\ldots.$$

$$x_M = 4.94 \quad (3 \text{ significant figures}).$$

FINAL ANSWER

(a) $P = 5,\ R = \frac{3\sqrt{5}}{2},\ \alpha = \arctan 2$ (b) $\left[5 - \frac{3\sqrt{5}}{2},\ 5 + \frac{3\sqrt{5}}{2}\right]$ (c) $x_M = 4.94$

TOPIC

Trig Equations & R-Form

Every past-paper question on this topic, January 2020 to June 2026.

Question 9

7 marks | Trig Equations & R-Form

9. $$f(\theta) = 5\cos\theta - 4\sin\theta \qquad \theta \in \mathbb{R}$$

(a) Express $f(\theta)$ in the form $R\cos(\theta + \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

The curve with equation $y = \cos\theta$ is transformed onto the curve with equation $y = f(\theta)$ by a sequence of two transformations.

Given that the first transformation is a stretch and the second a translation,

(b) (i) describe fully the transformation that is a stretch,

(ii) describe fully the transformation that is a translation.

(2)

Given

$$g(\theta) = \frac{90}{4 + (f(\theta))^2} \qquad \theta \in \mathbb{R}$$

(c) find the range of g.

(2)

1 Part (a)

Expanding the required form gives

$$R\cos(\theta+\alpha) = R\cos\alpha\cos\theta - R\sin\alpha\sin\theta.$$

Comparing coefficients with $5\cos\theta - 4\sin\theta$,

$$R\cos\alpha = 5, \qquad R\sin\alpha = 4.$$

Squaring and adding, using $R > 0$, gives

$$R^2 = 5^2 + 4^2 = 41 \quad \Longrightarrow \quad \boxed{R = \sqrt{41}}.$$

Also, since $0 < \alpha < \frac{\pi}{2}$,

$$\tan\alpha = \frac{4}{5}, \qquad \alpha = \arctan\left(\frac{4}{5}\right) = 0.6747409\ldots \quad \Longrightarrow \quad \boxed{\alpha = 0.675 \text{ radians (3 d.p.)}}.$$

Thus the exact form is

$$\boxed{f(\theta) = \sqrt{41}\cos\left(\theta + \arctan\frac{4}{5}\right)}.$$

2 Part (b)(i)

A stretch parallel to the y-axis with scale factor $\boxed{\sqrt{41}}$, taking $y = \cos\theta$ to $y = \sqrt{41}\cos\theta$.

3 Part (b)(ii)

A translation by the vector

$$\boxed{\begin{pmatrix} -\alpha \\ 0 \end{pmatrix}} = \begin{pmatrix} -\arctan(4/5) \\ 0 \end{pmatrix} \approx \begin{pmatrix} -0.675 \\ 0 \end{pmatrix}.$$

This moves the curve α units to the left, giving $y = \sqrt{41}\cos(\theta+\alpha)$.

4

Part (c)

Since θ can be any real number,

$$-\sqrt{41} \le f(\theta) \le \sqrt{41} \quad \Longrightarrow \quad 0 \le (f(\theta))^2 \le 41.$$

Hence $4 \le 4 + (f(\theta))^2 \le 45$. Taking reciprocals of these positive bounds and multiplying by 90 gives

$$\boxed{2 \le g(\theta) \le \frac{45}{2}}.$$

Both endpoints are attained: the minimum occurs when $f(\theta) = \pm\sqrt{41}$, and the maximum when $f(\theta) = 0$.

FINAL ANSWER

(a) $R = \sqrt{41}$, $\alpha = 0.675$ radians (3 d.p.), $f(\theta) = \sqrt{41}\cos\left(\theta + \arctan\frac{4}{5}\right)$ (b)(i) $\sqrt{41}$ (b)(ii) $\begin{pmatrix} -\alpha \\ 0 \end{pmatrix}$ (c) $2 \le g(\theta) \le \frac{45}{2}$

WMA13/01 OCTOBER 2020

Question 7

9 marks | Trig Equations & R-Form

7. (a) Express $\cos x + 4\sin x$ in the form $R\cos(x - \alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

A scientist is studying the behaviour of seabirds in a colony.

She models the height above sea level, H metres, of one of the birds in the colony by the equation

$$H = \frac{24}{3 + \cos\left(\frac{1}{2}t\right) + 4\sin\left(\frac{1}{2}t\right)} \qquad 0 \leqslant t \leqslant 6.5$$

where t seconds is the time after it leaves the nest.

Find, according to the model,

(b) the minimum height of the seabird above sea level, giving your answer to the nearest cm,

(2)

(c) the value of t, to 2 decimal places, when $H = 10$

(4)

WORKED SOLUTION

Question 7 · WMA13/01 October 2020

1 **Part (a)**

Expand $R\cos(x-\alpha) = R\cos\alpha\cos x + R\sin\alpha\sin x$. Matching coefficients gives $R\cos\alpha = 1$, $R\sin\alpha = 4$. Thus $R = \sqrt{1^2 + 4^2} = \sqrt{17}$ and $\tan\alpha = 4$. Since $0 < \alpha < \pi/2$, $\alpha = \arctan 4 = 1.32581766\ldots$ radians.

$$\boxed{R = \sqrt{17}, \qquad \alpha = 1.326 \text{ radians}}$$

2 Part (b)

The model becomes $H = 24/[3 + \sqrt{17}\cos(t/2 - \alpha)]$. For $0 \le t \le 6.5$, the phase lies between $-\alpha$ and $3.25 - \alpha$; throughout this interval the denominator remains positive. Its maximum $3 + \sqrt{17}$ is attained at $t = 2\alpha = 2.651635\ldots$, inside the time interval. The height is then smallest:

$$H_{\min} = \frac{24}{3 + \sqrt{17}} = 3.369316876\ldots \text{ m}.$$

To the nearest centimetre this is 3.37 m (337 cm).

$$\boxed{H_{\min} = 3.37 \text{ m}}$$

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3 Part (c)

For $H = 10$, $3 + \sqrt{17}\cos(t/2 - \alpha) = 24/10$, so

$$\cos(t/2 - \alpha) = -\frac{3}{5\sqrt{17}}.$$

Let $\gamma = \arccos[-3/(5\sqrt{17})]$. In the permitted phase interval $[-\alpha, 3.25 - \alpha]$, only $+\gamma$ occurs: $-\gamma < -\alpha$, and shifts by 2π are outside it. Using the unrounded $\alpha = \arctan 4$,

$$t = 2(\alpha + \gamma) = 6.085307854\ldots \text{ s}.$$

$$\boxed{t = 6.09 \text{ s}}$$

FINAL ANSWER

(a) $R = \sqrt{17}, \qquad \alpha = 1.326$ radians (b) $H_{\min} = 3.37$ m (c) $t = 6.09$ s

WMA13/01 MAY/JUNE 2021

Question 9

8 marks | Trig Equations & R-Form

9. (a) Express $12\sin x - 5\cos x$ in the form $R\sin(x - \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the exact value of R and give the value of α in radians, to 3 decimal places.

(3)

The function g is defined by

$$g(\theta) = 10 + 12\sin\left(2\theta - \frac{\pi}{6}\right) - 5\cos\left(2\theta - \frac{\pi}{6}\right) \qquad \theta > 0$$

Find

(b) (i) the minimum value of $g(\theta)$

(ii) the smallest value of θ at which the minimum value occurs.

(3)

The function h is defined by

$$h(\beta) = 10 - (12\sin\beta - 5\cos\beta)^2$$

(c) Find the range of h.

(2)

WORKED SOLUTION

Question 9 · WMA13/01 May/June 2021

1 **Part (a)**

Expand $R\sin(x-\alpha) = R\cos\alpha\sin x - R\sin\alpha\cos x$. Therefore $R\cos\alpha = 12$, $R\sin\alpha = 5$. Thus $R = \sqrt{12^2 + 5^2} = 13$ and $\alpha = \arctan(5/12) = 0.3947911197\ldots$ radians, in the required first quadrant.

$$\boxed{13\sin(x - 0.395) \quad (R = 13,\ \alpha = 0.395\ \text{rad})}$$

2 Part (b)

Write $g(\theta) = 10 + 13\sin(2\theta - \pi/6 - \alpha)$. (i) The minimum is $10 - 13 = -3$. (ii) For sine to equal -1, $2\theta - \pi/6 - \alpha = -\pi/2 + 2\pi n$. Hence $\theta = -\pi/6 + \alpha/2 + \pi n$. For $n = 0$ this is negative, so the smallest permitted value uses $n = 1$:

$$\theta = \frac{5\pi}{6} + \frac{1}{2}\arctan(5/12) = 2.815389438\ldots \text{ radians}.$$

Use the unrounded phase angle for the calculation.

$$\boxed{\min g = -3; \qquad \theta_{\min} = \frac{5\pi}{6} + \frac{1}{2}\arctan(5/12)}$$

3 Part (c)

As β varies over the reals, $12\sin\beta - 5\cos\beta$ covers $[-13, 13]$. Its square therefore covers $[0, 169]$, not $[169, 169]$. Subtracting from 10 reverses the bounds, giving $[-159, 10]$; both endpoints are attained.

$$\boxed{-159 \le h(\beta) \le 10}$$

FINAL ANSWER

(a) $13\sin(x - 0.395) \quad (R = 13,\ \alpha = 0.395\ \text{rad})$ (b) $\min g = -3; \qquad \theta_{\min} = \frac{5\pi}{6} + \frac{1}{2}\arctan(5/12)$ (c) $-159 \le h(\beta) \le 10$

Question 9

9 marks | Trig Equations & R-Form

9.

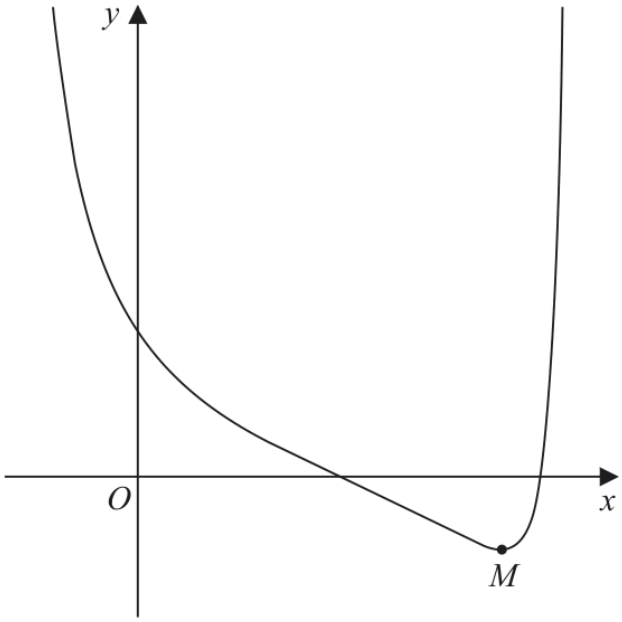


Figure 5

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 5 shows the curve with equation

$$y = \frac{1 + 2\cos x}{1 + \sin x} \qquad -\frac{\pi}{2} < x < \frac{3\pi}{2}$$

The point M, shown in Figure 5, is the minimum point on the curve.

(a) Show that the x coordinate of M is a solution of the equation

$$2\sin x + \cos x = -2$$

(4)

(b) Hence find, to 3 significant figures, the x coordinate of M.

(5)

1 Part (a)

Differentiate by the quotient rule:

$$y' = \frac{-2\sin x(1+\sin x) - (1+2\cos x)\cos x}{(1+\sin x)^2} = \frac{-2\sin x - \cos x - 2(\sin^2 x + \cos^2 x)}{(1+\sin x)^2} = \frac{-2\sin x - \cos x - 2}{(1+\sin x)^2}.$$

Inside the stated interval, $1+\sin x > 0$. At the interior minimum M, $y' = 0$, so its numerator is zero, giving the required equation.

$$\boxed{2\sin x + \cos x = -2}$$

2 Part (b)

Write $2\sin x + \cos x = \sqrt{5}\sin(x+\beta)$, where $\cos\beta = 2/\sqrt{5}$, $\sin\beta = 1/\sqrt{5}$ and $\beta = \arctan(1/2)$. The equation becomes $\sin(x+\beta) = -2/\sqrt{5}$. Let $\gamma = \arcsin(2/\sqrt{5})$, so $\gamma + \beta = \pi/2$. The solutions in the closed boundary range include $x = -\pi/2$, $3\pi/2$, and

$$x = \pi + \gamma - \beta = 3.7850937624\ldots.$$

Both boundary values are outside the original open domain (and make its denominator zero). The sole interior stationary point is M; the derivative changes from negative to positive there.

$$\boxed{x_M = 3.79}$$

FINAL ANSWER

(a) $2\sin x + \cos x = -2$ (b) $x_M = 3.79$

WMA13/01 OCTOBER 2022

Question 8

9 marks | Trig Equations & R-Form

8. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Express $8\sin x - 15\cos x$ in the form $R\sin(x-\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R, and give the value of α, in radians, to 4 significant figures.

(3)

$$f(x) = \frac{15}{41 + 16\sin x - 30\cos x} \qquad x > 0$$

(b) Find

(i) the minimum value of $f(x)$

(ii) the smallest value of x at which this minimum value occurs.

(4)

(c) State the y coordinate of the minimum points on the curve with equation

$$y = 2f(x) - 5 \qquad x > 0$$

(1)

(d) State the smallest value of x at which a maximum point occurs for the curve with equation

$$y = -f(2x) \qquad x > 0$$

(1)

WORKED SOLUTION

Question 8 · WMA13/01 October 2022

1 **Part (a)**

Expand $R\sin(x-\alpha) = R\cos\alpha\sin x - R\sin\alpha\cos x$. Thus $R\cos\alpha = 8$, $R\sin\alpha = 15$, so $R = \sqrt{64+225} = 17$ and $\alpha = \arctan(15/8) = 1.0808390005\ldots$ radians, in the required quadrant.

$$\boxed{R = 17, \qquad \alpha = 1.081 \text{ radians}}$$

2 Part (b)

Using the exact phase from (a),

$$f(x) = \frac{15}{41 + 34\sin(x - \alpha)}.$$

The denominator lies between 7 and 75, so it is always positive. (i) The smallest fraction occurs at its largest denominator: $f_{\min} = 15/75 = 1/5$. (ii) This occurs when $x - \alpha = \pi/2 + 2n\pi$. The smallest positive value is $x = \alpha + \pi/2 = 2.6516353273\ldots$, or 2.65 to three significant figures.

$$\boxed{f_{\min} = \frac{1}{5}, \qquad x = 2.65}$$

---

3 Part (c)

The transformation $2f(x) - 5$ preserves the locations of minima and maps their y coordinate to $2(1/5) - 5 = -23/5$.

$$\boxed{y = -\frac{23}{5}}$$

---

4 Part (d)

Reflection in the x-axis changes minima of f into maxima of $-f$. The argument $2x$ halves their x coordinates, so the first positive maximum is at

$$x = \frac{\alpha + \pi/2}{2} = 1.3258176637\ldots \approx 1.33.$$

$$\boxed{x = 1.33}$$

✓ FINAL ANSWER

(a) $R = 17, \qquad \alpha = 1.081$ **radians** (b) $f_{\min} = \frac{1}{5}, \qquad x = 2.65$ (c) $y = -\frac{23}{5}$ (d) $x = 1.33$

WMA13/01 OCTOBER 2022

Question 9

9 marks | Trig Equations & R-Form

9. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

Given that $\cos 2\theta - \sin 3\theta \neq 0$

(a) prove that

$$\frac{\cos^2\theta}{\cos 2\theta - \sin 3\theta} \equiv \frac{1+\sin\theta}{1-2\sin\theta-4\sin^2\theta}$$

(4)

(b) Hence solve, for $0 < \theta \leqslant 360°$

$$\frac{\cos^2\theta}{\cos 2\theta - \sin 3\theta} = 2\operatorname{cosec}\theta$$

Give your answers to one decimal place.

(5)

WORKED SOLUTION

Question 9 · WMA13/01 October 2022

1 Part (a)

Write $s = \sin\theta$. The compound-angle formula gives

$$\sin 3\theta = \sin(2\theta+\theta) = 2\sin\theta\cos^2\theta + (1-2\sin^2\theta)\sin\theta = 3s - 4s^3.$$

Hence

$$\cos 2\theta - \sin 3\theta = 1 - 2s^2 - 3s + 4s^3 = (1-s)(1-2s-4s^2), \quad \cos^2\theta = (1-s)(1+s).$$

The given nonzero denominator ensures that $1 - s \neq 0$ and the other factor is nonzero. Cancel the common factor to obtain the required identity.

$$\boxed{\frac{\cos^2\theta}{\cos 2\theta - \sin 3\theta} = \frac{1+\sin\theta}{1-2\sin\theta-4\sin^2\theta}}$$

2 Part (b)

Use (a) and $\csc\theta = 1/\sin\theta$. With $s = \sin\theta \neq 0$,

$$\frac{1+s}{1-2s-4s^2} = \frac{2}{s} \implies s + s^2 = 2 - 4s - 8s^2 \implies 9s^2 + 5s - 2 = 0.$$

Thus $s = (-5 \pm \sqrt{97})/18$, approximately 0.269380989 and -0.824936545. Both lie in $[-1, 1]$, are nonzero, and satisfy the original nonzero-denominator condition. The positive value gives $15.6274\ldots$° and $164.3726\ldots$°; the negative value gives $235.5821\ldots$° and $304.4179\ldots$°. All four angles lie in $0 < \theta \leq 360$°.

$$\boxed{\theta = 15.6^\circ,\ 164.4^\circ,\ 235.6^\circ,\ 304.4^\circ}$$

FINAL ANSWER

(a) $\frac{\cos^2\theta}{\cos 2\theta - \sin 3\theta} = \frac{1+\sin\theta}{1-2\sin\theta-4\sin^2\theta}$ (b) $\theta = 15.6^\circ,\ 164.4^\circ,\ 235.6^\circ,\ 304.4^\circ$

WMA13/01 JANUARY 2024

Question 9

8 marks | Trig Equations & R-Form

9. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that the equation

$$\frac{3\sin\theta\cos\theta}{\cos\theta+\sin\theta} = (2+\sec 2\theta)(\cos\theta-\sin\theta)$$

can be written in the form

$$3\sin 2\theta - 4\cos 2\theta = 2$$

(3)

(b) Hence solve for $\pi < x < \frac{3\pi}{2}$

$$\frac{3\sin x\cos x}{\cos x+\sin x} = (2+\sec 2x)(\cos x-\sin x)$$

giving the answer to 3 significant figures.

(5)

WORKED SOLUTION

Question 9 · WMA13/01 January 2024

1 Part (a)

The original equation is defined only when $\cos\theta+\sin\theta\neq 0$ and $\cos 2\theta\neq 0$. Multiplying by $\cos\theta+\sin\theta$ gives

$$3\sin\theta\cos\theta = (2+\sec 2\theta)(\cos^2\theta-\sin^2\theta) = (2+\sec 2\theta)\cos 2\theta = 2\cos 2\theta+1.$$

Using $2\sin\theta\cos\theta=\sin 2\theta$, multiply through by 2: $3\sin 2\theta = 4\cos 2\theta+2$.

$$\boxed{3\sin 2\theta - 4\cos 2\theta = 2}$$

2 **Part (b)**

Write $3\sin 2x - 4\cos 2x = 5\sin(2x - \phi)$, where $\cos\phi = 3/5$, $\sin\phi = 4/5$ and $\phi = \arctan(4/3)$. Then $\sin(2x - \phi) = 2/5$. Because $\pi < x < 3\pi/2$, $2\pi - \phi < 2x - \phi < 3\pi - \phi$, approximately $(5.356, 8.497)$. With $\gamma = \arcsin(2/5)$, the only angle in this interval is $2\pi + \gamma$; $3\pi - \gamma \approx 9.013$ is outside it. Thus

$$x = \frac{2\pi + \arcsin(2/5) + \arctan(4/3)}{2} = 3.810998686\ldots.$$

At this value $\cos x + \sin x \neq 0$ and $\cos 2x \neq 0$; direct substitution satisfies the original equation.

$$\boxed{x = 3.81}$$

FINAL ANSWER

(a) $3\sin 2\theta - 4\cos 2\theta = 2$ (b) $x = 3.81$

WMA13/01 JANUARY 2025

Question 8

9 marks | Trig Equations & R-Form

8. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(i) Solve, for $0 < \theta < \pi$

$$3\,\text{cosec}\,\theta = 8\cos\theta$$

giving your answers, in radians, to 3 significant figures.

(5)

(ii) Solve, for $0 < x < 180°$

$$\frac{\tan 2x - \tan 70°}{1 + \tan 2x \tan 70°} = -\frac{3}{8}$$

giving your answers, in degrees, to one decimal place.

(4)

WORKED SOLUTION

Question 8 · WMA13/01 January 2025

1 **Part (i)**

Use $\csc\theta = 1/\sin\theta$. Since $0 < \theta < \pi$, $\sin\theta \neq 0$ and

$$\frac{3}{\sin\theta} = 8\cos\theta \quad \Longrightarrow \quad 3 = 8\sin\theta\cos\theta = 4\sin 2\theta.$$

Thus $\sin 2\theta = 3/4$. Since $0 < 2\theta < 2\pi$, the two possibilities are

$$2\theta = \arcsin(3/4) \quad \text{or} \quad 2\theta = \pi - \arcsin(3/4).$$

They give $\theta = 0.4240310395\ldots$ and $1.1467652873\ldots$ radians.

$$\boxed{\theta = 0.424,\ 1.15 \text{ radians}}$$

2 Part (ii)

Apply the tangent subtraction identity:

$$\frac{\tan 2x - \tan 70^\circ}{1 + \tan 2x \tan 70^\circ} = \tan(2x - 70^\circ) = -\frac{3}{8}.$$

Set $u = 2x - 70^\circ$. The interval becomes $-70^\circ < u < 290^\circ$. With $\arctan(-3/8) = -20.5560452196\ldots^\circ$, the permitted values are $u = -20.5560452196\ldots^\circ$ and $159.4439547804\ldots^\circ$. Adding 70° and halving gives $x = 24.7219773902\ldots^\circ$ and $114.7219773902\ldots^\circ$. Both keep the original tangents and denominator defined.

$$\boxed{x = 24.7^\circ,\ 114.7^\circ}$$

FINAL ANSWER

(i) $\theta = 0.424,\ 1.15$ radians (ii) $x = 24.7^\circ,\ 114.7^\circ$

Question 9

9 marks | Trig Equations & R-Form

9. $$\mathrm{h}(x) = 8 + 3\sin x(\cos x - 2\sin x) \quad 0 \leqslant x \leqslant 2\pi$$

(a) Write $\mathrm{h}(x)$ in the form

$$P + R\sin(2x + \alpha)$$

where P, R and α are positive constants, $0 < \alpha < \frac{\pi}{2}$

Give the exact values of P and R and give the value of α, in radians, to 3 decimal places.

(6)

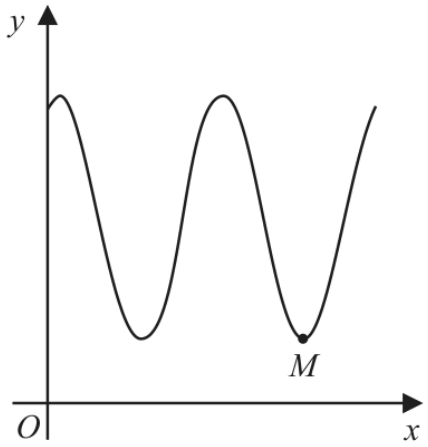


Figure 2

Figure 2 is a sketch of the curve with equation $y = \mathrm{h}(x)$

(b) State the range of h.

(1)

The point M, shown in Figure 2, is the minimum point on the curve with the largest x coordinate.

(c) Find the x coordinate of M.

(Solutions based entirely on calculator technology are not acceptable.)

(2)

1 Part (a)

Use $2\sin x\cos x = \sin 2x$ and $2\sin^2 x = 1 - \cos 2x$:

$$h(x) = 8 + 3\sin x\cos x - 6\sin^2 x = 5 + \frac{3}{2}\sin 2x + 3\cos 2x.$$

Expanding $R\sin(2x+\alpha)$ and comparing coefficients gives

$$R\cos\alpha = \frac{3}{2}, \qquad R\sin\alpha = 3.$$

Therefore

$$P = 5, \qquad R = \sqrt{\frac{9}{4} + 9} = \frac{3\sqrt{5}}{2}, \qquad \alpha = \arctan(2) = 1.107 \text{ radians (3 d.p.)}.$$

Thus $h(x) = 5 + \frac{3\sqrt{5}}{2}\sin(2x+\alpha)$ with the exact angle $\alpha = \arctan(2)$.

2 Part (b)

The given interval includes two complete periods. The range is

$$\left[5 - \frac{3\sqrt{5}}{2},\ 5 + \frac{3\sqrt{5}}{2}\right].$$

3 Part (c)

At a minimum, $\sin(2x+\alpha) = -1$, so

$$2x + \alpha = \frac{3\pi}{2} + 2\pi n, \qquad x = \frac{3\pi}{4} - \frac{\alpha}{2} + \pi n.$$

On $0 \le x \le 2\pi$, $n = 0$ and $n = 1$ give the two minima. The larger coordinate is

$$x_M = \frac{7\pi}{4} - \frac{1}{2}\arctan(2) = 4.944212785\ldots.$$

$$x_M = 4.94 \quad \text{(3 significant figures)}.$$

FINAL ANSWER

(a) $P = 5,\ R = \frac{3\sqrt{5}}{2},\ \alpha = \arctan 2$ (b) $\left[5 - \frac{3\sqrt{5}}{2},\ 5 + \frac{3\sqrt{5}}{2}\right]$ (c) $x_M = 4.94$

Question 7

11 marks | Trig Equations & R-Form

Main topic: Applications of Differentiation

7.

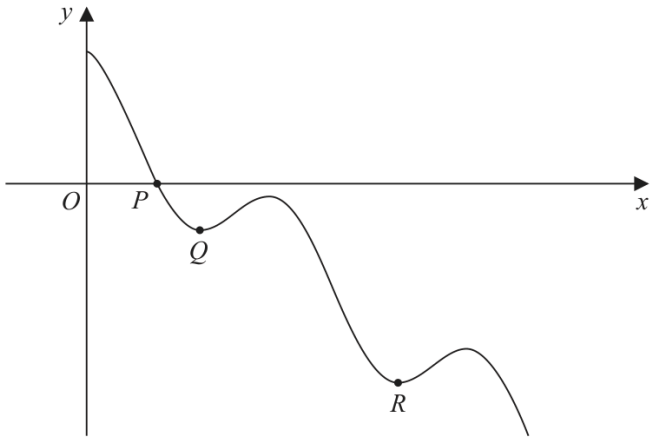


Figure 3

Figure 3 shows a sketch of part of the curve with equation

$$y = 2\cos 3x - 3x + 4 \qquad x > 0$$

where x is measured in radians.

The curve crosses the x-axis at the point P, as shown in Figure 3.

Given that the x coordinate of P is α,

(a) show that α lies between 0.8 and 0.9

(2)

The iteration formula

$$x_{n+1} = \frac{1}{3}\arccos(1.5x_n - 2)$$

can be used to find an approximate value for α.

(b) Using this iteration formula with $x_1 = 0.8$ find, to 4 decimal places, the value of

(i) x_2

(ii) x_5

(3)

The point Q and the point R are local minimum points on the curve, as shown in Figure 3.

Given that the x coordinates of Q and R are β and λ respectively, and that they are the two smallest values of x at which local minima occur,

(c) find, using calculus, the exact value of β and the exact value of λ.

(6)

1 Part (a)

Let $f(x) = 2\cos 3x - 3x + 4$, using radians. Then

$$f(0.8) = 2\cos 2.4 - 2.4 + 4 = 0.1252125\ldots > 0,$$
$$f(0.9) = 2\cos 2.7 - 2.7 + 4 = -0.5081443\ldots < 0.$$

The function is continuous on $[0.8, 0.9]$, so this change of sign shows that there is a root between the endpoints. Moreover, $f'(x) = -6\sin 3x - 3 < 0$ on this interval, so this root is unique there. Thus the crossing shown at P has $\boxed{0.8 < \alpha < 0.9}$.

2 Part (b)(i)

Using the inverse cosine in radians,

$$x_2 = \frac{1}{3}\arccos(1.5(0.8) - 2) = \frac{1}{3}\arccos(-0.8) = 0.8326971\ldots.$$

Hence $\boxed{x_2 = 0.8327}$ to 4 decimal places.

3 Part (b)(ii)

Continue the iteration, retaining the full calculator values at each step:

$$x_3 = \frac{1}{3}\arccos(1.5x_2 - 2) = 0.8067674\ldots,$$
$$x_4 = \frac{1}{3}\arccos(1.5x_3 - 2) = 0.8271196\ldots,$$
$$x_5 = \frac{1}{3}\arccos(1.5x_4 - 2) = 0.8110216\ldots.$$

Therefore $\boxed{x_5 = 0.8110}$ to 4 decimal places.

4 **Part (c)**

Differentiate the curve equation:

$$\frac{dy}{dx} = -6\sin 3x - 3, \qquad \frac{d^2y}{dx^2} = -18\cos 3x.$$

At a stationary point,

$$-6\sin 3x - 3 = 0 \quad \Longrightarrow \quad \sin 3x = -\frac{1}{2}.$$

For $x > 0$, the stationary points therefore belong to the two families

$$3x = \frac{7\pi}{6} + 2n\pi \quad \text{or} \quad 3x = \frac{11\pi}{6} + 2n\pi, \qquad n = 0, 1, 2, \ldots.$$

In the first family, $\cos 3x = -\frac{\sqrt{3}}{2}$, giving $y'' = 9\sqrt{3} > 0$, so these are local minima. In the second family, $\cos 3x = \frac{\sqrt{3}}{2}$ and $y'' = -9\sqrt{3} < 0$, so these are local maxima and are excluded. Thus all positive minimum coordinates are

$$x = \frac{7\pi}{18} + \frac{2n\pi}{3}, \qquad n = 0, 1, 2, \ldots.$$

The two smallest are obtained with $n = 0$ and $n = 1$:

$$\boxed{\beta = \frac{7\pi}{18}}, \qquad \boxed{\lambda = \frac{19\pi}{18}}.$$

FINAL ANSWER

(a) $0.8 < \alpha < 0.9$ (b)(i) $x_2 = 0.8327$ (b)(ii) $x_5 = 0.8110$ (c) $\beta = \frac{7\pi}{18}$, $\lambda = \frac{19\pi}{18}$

WMA13/01 OCTOBER 2020

Question 8

9 marks | Trig Equations & R-Form

Main topic: Differentiation Techniques

8. (i) The curve C has equation $y = \mathrm{g}(x)$ where

$$\mathrm{g}(x) = \mathrm{e}^{3x} \sec 2x \qquad -\frac{\pi}{4} < x < \frac{\pi}{4}$$

(a) Find $\mathrm{g}'(x)$

(2)

(b) Hence find the x coordinate of the stationary point of C.

(3)

(ii) A different curve has equation

$$x = \ln(\sin y) \qquad 0 < y < \frac{\pi}{2}$$

Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^x}{\mathrm{f}(x)}$$

where $\mathrm{f}(x)$ is a function of e^x that should be found.

(4)

WORKED SOLUTION

Question 8 · WMA13/01 October 2020

1 Part (i)(a)

By the product and chain rules,

$$g'(x) = 3e^{3x}\sec 2x + 2e^{3x}\sec 2x \tan 2x = e^{3x}\sec 2x(3 + 2\tan 2x).$$

$$\boxed{g'(x) = e^{3x}\sec 2x(3 + 2\tan 2x)}$$

2 Part (i)(b)

For $-\pi/4 < x < \pi/4$, $e^{3x}\sec 2x$ is positive and nonzero. Hence $g'(x) = 0$ precisely when $3 + 2\tan 2x = 0$, or $\tan 2x = -3/2$. Since $-\pi/2 < 2x < \pi/2$, tangent is one-to-one there:

$$x = -\frac{1}{2}\arctan\frac{3}{2} = -0.4913968616\ldots.$$

$$\boxed{x = -\frac{1}{2}\arctan\frac{3}{2} \approx -0.491}$$

---

3 Part (ii)

Differentiate $x = \ln(\sin y)$ with respect to y:

$$\frac{dx}{dy} = \frac{\cos y}{\sin y}, \qquad \frac{dy}{dx} = \frac{\sin y}{\cos y}.$$

The original equation gives $\sin y = e^{x}$. Since $0 < y < \pi/2$, cosine is positive, so $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - e^{2x}}$. Therefore

$$\frac{dy}{dx} = \frac{e^{x}}{\sqrt{1 - e^{2x}}}, \qquad f(x) = \sqrt{1 - e^{2x}}.$$

Here $x < 0$ from the original interval, so the square root is real and nonzero.

$$\boxed{f(x) = \sqrt{1 - e^{2x}}, \qquad x < 0}$$

FINAL ANSWER

(i)(a) $g'(x) = e^{3x}\sec 2x(3 + 2\tan 2x)$ (i)(b) $x = -\frac{1}{2}\arctan\frac{3}{2} \approx -0.491$ (ii) $f(x) = \sqrt{1 - e^{2x}}, \qquad x < 0$

WMA13/01 JANUARY 2021

Question 6

8 marks | Trig Equations & R-Form | Main topic: Numerical Methods

6.

$$f(x) = x\cos\left(\frac{x}{3}\right) \qquad x > 0$$

(a) Find $f'(x)$

(2)

(b) Show that the equation $f'(x) = 0$ can be written as

$$x = k\arctan\left(\frac{k}{x}\right)$$

where k is an integer to be found.

(2)

(c) Starting with $x_1 = 2.5$ use the iteration formula

$$x_{n+1} = k\arctan\left(\frac{k}{x_n}\right)$$

with the value of k found in part (b), to calculate the values of x_2 and x_6 giving your answers to 3 decimal places.

(2)

(d) Using a suitable interval and a suitable function that should be stated, show that a root of $f'(x) = 0$ is 2.581 correct to 3 decimal places.

(2)

WORKED SOLUTION

Question 6 · WMA13/01 January 2021

1 **Part (a)**

By the product and chain rules, $f'(x) = \cos(x/3) - (x/3)\sin(x/3)$.

$$\boxed{f'(x) = \cos(x/3) - \frac{x}{3}\sin(x/3)}$$

2 Part (b)

Set $f'(x) = 0$: $\cos(x/3) = (x/3)\sin(x/3)$. At a solution, cosine cannot vanish (that would leave a nonzero right side), so $\tan(x/3) = 3/x$. On the principal positive branch $0 < x/3 < \pi/2$, this gives

$$x = 3\arctan(3/x), \qquad k = 3.$$

This is the branch used by the supplied iteration for the root near 2.581. The original domain $x > 0$ also admits later tangent branches: in general $x = 3\arctan(3/x) + 3n\pi$. The principal-branch rearrangement must not be interpreted as listing every positive stationary point.

$$\boxed{k = 3; \quad x = 3\arctan(3/x) \text{ on the selected branch}}$$

---

3 Part (c)

Using radians and $x_1 = 2.5$, the successive values are

n	1	2	3	4	5	6
x_n	2.5	2.628174152	2.554102174	2.596526871	2.572101407	2.586122337

Keep full precision between steps and round the requested terms only.

$$\boxed{x_2 = 2.628, \qquad x_6 = 2.586}$$

---

4 Part (d)

Define $F(x) = \cos(x/3) - (x/3)\sin(x/3)$, which is continuous near 2.581. The rounding interval endpoints give

$$F(2.5805) = 0.0003467157\ldots > 0, \qquad F(2.5815) = -0.0003456897\ldots < 0.$$

By continuity a root lies strictly between these endpoints; every number in this open interval rounds to 2.581 to three decimal places.

$$\boxed{x = 2.581 \text{ correct to 3 d.p.}}$$

✓ FINAL ANSWER

(a) $f'(x) = \cos(x/3) - \frac{x}{3}\sin(x/3)$ (b) $k = 3; \quad x = 3\arctan(3/x)$ on the selected branch (c) $x_2 = 2.628, \qquad x_6 = 2.586$ (d) $x = 2.581$ correct to 3 d.p.

WMA13/01 JANUARY 2021

Question 7

9 marks | Trig Equations & R-Form | Main topic: Trig Identities & Exact Values

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

7. (a) Prove that

$$\frac{\sin 2x}{\cos x} + \frac{\cos 2x}{\sin x} \equiv \operatorname{cosec} x \qquad x \neq \frac{n\pi}{2} \quad n \in \mathbb{Z}$$

(3)

(b) Hence solve, for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$7 + \frac{\sin 4\theta}{\cos 2\theta} + \frac{\cos 4\theta}{\sin 2\theta} = 3\cot^2 2\theta$$

giving your answers in radians to 3 significant figures where appropriate.

(6)

WORKED SOLUTION

Question 7 · WMA13/01 January 2021

1 Part (a)

Apply the double-angle identities to the left side:

$$\frac{2\sin x\cos x}{\cos x} + \frac{1-2\sin^2 x}{\sin x} = 2\sin x + \frac{1}{\sin x} - 2\sin x = \csc x.$$

All divisions and cancellations are valid under the given exclusion $x \neq n\pi/2$.

$$\boxed{\frac{\sin 2x}{\cos x} + \frac{\cos 2x}{\sin x} = \csc x}$$

2 Part (b)

Using (a) with $x = 2\theta$ gives $7 + \csc 2\theta = 3\cot^2 2\theta$. Set $u = \csc 2\theta$ and use $\cot^2 2\theta = u^2 - 1$: $3u^2 - u - 10 = (3u+5)(u-2) = 0$. Thus $\sin 2\theta = 1/2$ or $\sin 2\theta = -3/5$. As $-\pi < 2\theta < \pi$, the first gives $2\theta = \pi/6, 5\pi/6$. Write $\gamma = \arcsin(3/5)$; the second gives $2\theta = -\gamma, -\pi+\gamma$. Therefore the four values are $\pi/12, 5\pi/12, -0.3217505544\ldots, -1.249045772\ldots$. None makes an original sine or cosine denominator zero.

$$\boxed{\theta = \frac{\pi}{12},\ \frac{5\pi}{12},\ -0.322,\ -1.25}$$

FINAL ANSWER

(a) $\frac{\sin 2x}{\cos x} + \frac{\cos 2x}{\sin x} = \csc x$ (b) $\theta = \frac{\pi}{12},\ \frac{5\pi}{12},\ -0.322,\ -1.25$

WMA13/01 OCTOBER 2021

Question 10

7 marks | Trig Equations & R-Form | Main topic: Integration

10.

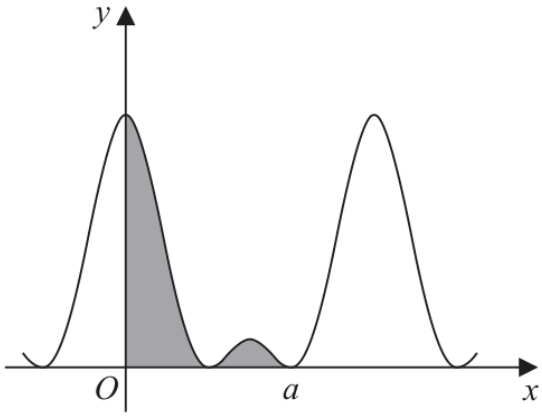


Figure 4

Figure 4 shows a sketch of part of the curve with equation

$$y = (1 + 2\cos 2x)^2$$

(a) Show that

$$(1 + 2\cos 2x)^2 \equiv p + q\cos 2x + r\cos 4x$$

where p, q and r are constants to be found.

(2)

The curve touches the positive x-axis for the second time when $x = a$, as shown in Figure 4.

The regions bounded by the curve, the y-axis and the x-axis up to $x = a$ are shown shaded in Figure 4.

(b) Find, using algebraic integration and making your method clear, the exact total area of the shaded regions. Write your answer in simplest form.

(5)

WORKED SOLUTION

Question 10 · WMA13/01 October 2021

1 **Part (a)**

Expand and use $\cos^2 2x = (1 + \cos 4x)/2$:

$$(1 + 2\cos 2x)^2 = 1 + 4\cos 2x + 4\cos^2 2x = 3 + 4\cos 2x + 2\cos 4x.$$

$$\boxed{p = 3, \quad q = 4, \quad r = 2}$$

2 **Part (b)**

The zeros satisfy $\cos 2x = -1/2$. The first two positive values are $x = \pi/3$ and $x = 2\pi/3$, so $a = 2\pi/3$. The squared curve is nonnegative, so integrate directly for the total shaded area:

$$\mathcal{A} = \int_0^{2\pi/3} (3 + 4\cos 2x + 2\cos 4x)\,dx = \left[3x + 2\sin 2x + \tfrac{1}{2}\sin 4x\right]_0^{2\pi/3}.$$

Use $\sin(4\pi/3) = -\sqrt{3}/2$ and $\sin(8\pi/3) = \sqrt{3}/2$ to obtain $\mathcal{A} = 2\pi - \sqrt{3} + \sqrt{3}/4 = 2\pi - 3\sqrt{3}/4$.

$$\boxed{2\pi - \frac{3\sqrt{3}}{4}}$$

FINAL ANSWER

(a) $p = 3, \quad q = 4, \quad r = 2$ (b) $2\pi - \frac{3\sqrt{3}}{4}$

WMA13/01 JANUARY 2022

Question 9

8 marks | Trig Equations & R-Form

Main topic: Trig Identities & Exact Values

9. **In this question you must show detailed reasoning.**

Solutions relying entirely on calculator technology are not acceptable.

(i) Solve, for $0 < x \leqslant \pi$, the equation

$$2\sec^2 x - 3\tan x = 2$$

giving the answers, as appropriate, to 3 significant figures.

(4)

(ii) Prove that

$$\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} \equiv 2$$

(4)

WORKED SOLUTION

Question 9 · WMA13/01 January 2022

1 Part (i)

Using $\sec^2 x = 1 + \tan^2 x$, the equation becomes $2\tan^2 x - 3\tan x = 0$, or $\tan x(2\tan x - 3) = 0$. If $\tan x = 0$, the given interval $0 < x \leq \pi$ contains only $x = \pi$. If $\tan x = 3/2$, the sole root in this interval is $x = \arctan(3/2) = 0.982793723\ldots$ radians. The excluded pole $x = \pi/2$ supplies no solution.

$$\boxed{x = 0.983 \text{ radians}, \quad \pi}$$

2 Part (ii)

Use angle addition to expand:

$$\sin 3\theta = \sin 2\theta\cos\theta + \cos 2\theta\sin\theta = 2\sin\theta\cos^2\theta + (1 - 2\sin^2\theta)\sin\theta = 3\sin\theta - 4\sin^3\theta,$$

$$\cos 3\theta = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta = (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta = 4\cos^3\theta - 3\cos\theta.$$

Where $\sin\theta\cos\theta \neq 0$, divide by the stated denominators:

$$\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = 3 - 4\sin^2\theta - (4\cos^2\theta - 3) = 6 - 4(\sin^2\theta + \cos^2\theta) = 2.$$

$$\boxed{\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = 2}$$

FINAL ANSWER

(i) $x = 0.983 \text{ radians}, \quad \pi$ (ii) $\frac{\sin 3\theta}{\sin \theta} - \frac{\cos 3\theta}{\cos \theta} = 2$

WMA13/01 MAY/JUNE 2022

Question 7

8 marks | Trig Equations & R-Form

Main topic: Trig Identities & Exact Values

7. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that the equation

$$2\sin\theta\,(3\cot^2 2\theta - 7) = 13\sec\theta$$

can be written as

$$3\operatorname{cosec}^2 2\theta - 13\operatorname{cosec} 2\theta - 10 = 0$$

(4)

(b) Hence solve, for $0 < \theta < \frac{\pi}{2}$, the equation

$$2\sin\theta\,(3\cot^2 2\theta - 7) = 13\sec\theta$$

giving your answers to 3 significant figures.

(4)

WORKED SOLUTION

Question 7 · WMA13/01 May/June 2022

1 **Part (a)**

On the domain of the original equation, multiply by $\cos\theta$:

$$2\sin\theta\cos\theta(3\cot^2 2\theta - 7) = 13 \implies \sin 2\theta(3\cot^2 2\theta - 7) = 13.$$

Divide by the nonzero $\sin 2\theta$ and use $\cot^2 u = \csc^2 u - 1$:

$$3(\csc^2 2\theta - 1) - 7 = 13\csc 2\theta.$$

Collect the terms to obtain the displayed equation.

$$\boxed{3\csc^2 2\theta - 13\csc 2\theta - 10 = 0}$$

2

Part (b)

Put $u = \csc 2\theta$. Then $(3u + 2)(u - 5) = 0$, so $u = 5$ or $-2/3$. The second would give $\sin 2\theta = -3/2$, which is impossible. For $0 < \theta < \pi/2$, we have $0 < 2\theta < \pi$. Thus

$$2\theta = \arcsin(1/5) \quad \text{or} \quad \pi - \arcsin(1/5).$$

Hence $\theta = 0.1006789604\ldots$ or $1.4701173664\ldots$ radians. Both satisfy the original denominator restrictions.

$$\boxed{\theta = 0.101, \quad 1.47 \text{ radians}}$$

FINAL ANSWER

(a) $3\csc^2 2\theta - 13\csc 2\theta - 10 = 0$ (b) $\theta = 0.101, \quad 1.47$ radians

WMA13/01 MAY/JUNE 2023

Question 9

10 marks | Trig Equations & R-Form

Main topic: Trig Identities & Exact Values

9. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that

$$\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} \equiv \operatorname{cosec} x \qquad x \neq \frac{n\pi}{2} \qquad n \in \mathbb{Z}$$

(3)

(b) Hence solve, for $0 < \theta < \frac{\pi}{2}$

$$\left(\frac{\cos 2\theta}{\sin \theta} + \frac{\sin 2\theta}{\cos \theta}\right)^2 = 6\cot\theta - 4$$

giving your answers to 3 significant figures as appropriate.

(5)

(c) Using the result from part (a), or otherwise, find the exact value of

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x}\right) \cot x \, dx$$

(2)

WORKED SOLUTION

Question 9 · WMA13/01 May/June 2023

1 **Part (a)**

Using a common denominator,

$$\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} = \frac{\cos 2x \cos x + \sin 2x \sin x}{\sin x \cos x} = \frac{\cos(2x - x)}{\sin x \cos x} = \frac{1}{\sin x} = \csc x.$$

The stated exclusion $x \neq n\pi/2$ ensures both denominators are nonzero.

$$\boxed{\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} = \csc x}$$

2 Part (b)

By part (a), the squared bracket is $\csc^2\theta$. Using $\csc^2\theta = 1 + \cot^2\theta$ and setting $u = \cot\theta$ gives

$$1 + u^2 = 6u - 4 \quad \Longrightarrow \quad u^2 - 6u + 5 = (u-1)(u-5) = 0.$$

On $0 < \theta < \pi/2$, both positive cotangents are permitted: $u = 1$ gives $\theta = \pi/4$, while $u = 5$ gives $\tan\theta = 1/5$, $\theta = 0.1973955598\ldots$ radians.

$$\boxed{\theta = \frac{\pi}{4}, \quad 0.197 \text{ radians}}$$

---

3 Part (c)

The bracket in part (a) equals $\csc x$. Therefore

$$\int_{\pi/6}^{\pi/4} \csc x \cot x \, dx = \left[-\csc x\right]_{\pi/6}^{\pi/4} = -\sqrt{2} - (-2) = 2 - \sqrt{2}.$$

$$\boxed{2 - \sqrt{2}}$$

✓ FINAL ANSWER

(a) $\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} = \csc x$ (b) $\theta = \frac{\pi}{4}, \quad 0.197 \text{ radians}$ (c) $2 - \sqrt{2}$

WMA13/01 OCTOBER 2023

Question 8

8 marks | Trig Equations & R-Form | Main topic: Trig Identities & Exact Values

8. (a) Prove that

$$2\operatorname{cosec}^2 2\theta(1-\cos 2\theta) \equiv 1+\tan^2\theta$$

(4)

(b) Hence solve for $0 < x < 360°$, where $x \neq (90n)°$, $n \in \mathbb{N}$, the equation

$$2\operatorname{cosec}^2 2x(1-\cos 2x) = 4+3\sec x$$

giving your answers to one decimal place.

(Solutions relying entirely on calculator technology are not acceptable.)

(4)

WORKED SOLUTION

Question 8 · WMA13/01 October 2023

1 Part (a)

Starting from the left side and using double-angle identities,

$$2\csc^2 2\theta(1-\cos 2\theta) = \frac{2(2\sin^2\theta)}{(2\sin\theta\cos\theta)^2} = \frac{1}{\cos^2\theta} = \sec^2\theta = 1+\tan^2\theta.$$

These steps hold wherever the original expression is defined, so $\sin 2\theta \neq 0$; no claim is made at the excluded angles.

$$\boxed{2\csc^2 2\theta(1-\cos 2\theta) = 1+\tan^2\theta}$$

2 Part (b)

By part (a), the equation becomes

$$\sec^2 x = 4+3\sec x \quad\Longrightarrow\quad (\sec x-4)(\sec x+1) = 0.$$

The branch $\sec x = 4$ gives $\cos x = 1/4$ and hence $x = 75.5225\ldots°$ or $284.4775\ldots°$. The other branch gives $\cos x = -1$, hence $x = 180°$ in the stated interval. Reject $180°$: it is a multiple of $90°$ and the original $\csc 2x$ is undefined there. The two remaining angles satisfy the exclusions.

$$\boxed{x = 75.5°, \qquad 284.5°}$$

FINAL ANSWER

(a) $2\csc^2 2\theta(1-\cos 2\theta) = 1+\tan^2\theta$ (b) $x = 75.5°, \qquad 284.5°$

Question 10

9 marks | Trig Equations & R-Form | Main topic: Differentiation Techniques

10. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

A curve C has equation

$$x = \sin^2 4y \qquad 0 \leqslant y \leqslant \frac{\pi}{8} \qquad 0 \leqslant x \leqslant 1$$

The point P with x coordinate $\frac{1}{4}$ lies on C

(a) Find the exact y coordinate of P

(2)

(b) Find $\frac{\mathrm{d}x}{\mathrm{d}y}$

(2)

(c) Hence show that $\frac{\mathrm{d}y}{\mathrm{d}x}$ can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{q + r(x + s)^2}}$$

where q, r and s are constants to be found.

(3)

Using the answer to part (c),

(d) (i) state the x coordinate of the point where the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ is a minimum,

(ii) state the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at this point.

(2)

1 Part (a)

At $x = 1/4$, $\sin^2 4y = 1/4$. Since $0 \le 4y \le \pi/2$,

$$\sin 4y = \frac{1}{2} \quad \Longrightarrow \quad 4y = \frac{\pi}{6}.$$

Only this branch lies in the given range of y.

$$\boxed{y = \frac{\pi}{24}}$$

2 Part (b)

Differentiate $x = (\sin 4y)^2$ with respect to y:

$$\frac{dx}{dy} = 2\sin 4y\,(4\cos 4y).$$

$$\boxed{\frac{dx}{dy} = 8\sin 4y\cos 4y}$$

3 Part (c)

On the interior of the given interval, $\sin 4y = \sqrt{x}$ and $\cos 4y = \sqrt{1-x}$ are positive. Taking the reciprocal,

$$\frac{dy}{dx} = \frac{1}{8\sqrt{x}\sqrt{1-x}} = \frac{1}{\sqrt{64x - 64x^2}}.$$

Complete the square:

$$64x - 64x^2 = 16 - 64\left(x - \frac{1}{2}\right)^2.$$

Thus $dy/dx = (q + r(x+s)^2)^{-1/2}$ has the constants below, with $0 < x < 1$ for a finite derivative; the endpoints have vertical tangents.

$$\boxed{q = 16, \qquad r = -64, \qquad s = -\frac{1}{2}}$$

4 Part (d)

The derivative is positive. Its minimum occurs when the square root in the denominator is largest. Since $(x - 1/2)^2 \ge 0$, the largest radicand is 16, attained at $x = 1/2$. Hence the smallest gradient is $1/\sqrt{16}$.

$$\boxed{\text{(i) } x = \frac{1}{2}, \qquad \text{(ii) } \min \frac{dy}{dx} = \frac{1}{4}}$$

FINAL ANSWER

(a) $y = \frac{\pi}{24}$ (b) $\frac{dx}{dy} = 8\sin 4y \cos 4y$ (c) $q = 16, \qquad r = -64, \qquad s = -\frac{1}{2}$ (d)
(i) $x = \frac{1}{2}$, (ii) $\min \frac{dy}{dx} = \frac{1}{4}$

Question 6

7 marks | Trig Equations & R-Form

Main topic: Applications of Differentiation

6.

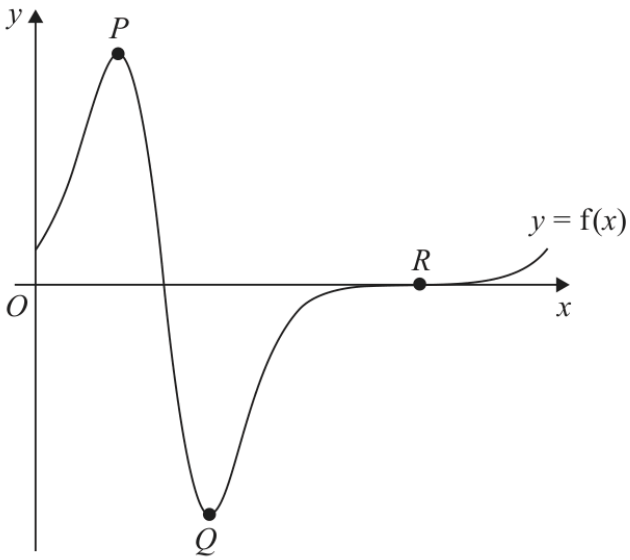


Figure 1

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 1 shows a sketch of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 2\mathrm{e}^{3\sin x}\cos x \qquad 0 \leqslant x \leqslant 2\pi$$

The curve intersects the x-axis at point R, as shown in Figure 1.

(a) State the coordinates of R

(1)

The curve has two turning points, at point P and point Q, also shown in Figure 1.

(b) Show that, at points P and Q,

$$a\sin^2 x + b\sin x + c = 0$$

where a, b and c are integers to be found.

(4)

(c) Hence find the x coordinate of point Q, giving your answer to 3 decimal places.

(2)

1 Part (a)

Since $2e^{3\sin x} > 0$, zeros occur when $\cos x = 0$. The labelled point R is the right-hand crossing, at $x = 3\pi/2$.

$$\boxed{R = \left(\frac{3\pi}{2}, 0\right)}$$

2 Part (b)

For $f(x) = 2e^{3\sin x}\cos x$, the product and chain rules give

$$f'(x) = 2\left(3e^{3\sin x}\cos^2 x - e^{3\sin x}\sin x\right) = 2e^{3\sin x}(3\cos^2 x - \sin x).$$

At a turning point the derivative is zero. The exponential factor is nonzero, so $3\cos^2 x - \sin x = 0$. Substitute $\cos^2 x = 1 - \sin^2 x$ and rearrange: $3 - 3\sin^2 x - \sin x = 0$.

$$\boxed{3\sin^2 x + \sin x - 3 = 0, \quad a = 3,\ b = 1,\ c = -3}$$

3 Part (c)

Solving $3s^2 + s - 3 = 0$ for $s = \sin x$ gives

$$s = \frac{-1 \pm \sqrt{37}}{6}.$$

The negative root is less than -1, so it is impossible for a real sine. Let $\beta = \arcsin\left((\sqrt{37} - 1)/6\right)$ in the first quadrant. The labelled minimum Q lies in the second quadrant, hence $x_Q = \pi - \beta = 2.131037293\ldots$.

$$\boxed{x_Q = 2.131}$$

FINAL ANSWER

(a) $R = \left(\frac{3\pi}{2}, 0\right)$ (b) $3\sin^2 x + \sin x - 3 = 0, \quad a = 3,\ b = 1,\ c = -3$ (c) $x_Q = 2.131$

WMA13/01 MAY/JUNE 2024

Question 7

8 marks | Trig Equations & R-Form

Main topic: Trig Identities & Exact Values

7. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Given that

$$\sqrt{2}\sin(x+45°) = \cos(x-60°)$$

show that

$$\tan x = -2-\sqrt{3}$$

(4)

(b) Hence or otherwise, solve, for $0 \leqslant \theta < 180°$

$$\sqrt{2}\sin(2\theta) = \cos(2\theta-105°)$$

(4)

WORKED SOLUTION

Question 7 · WMA13/01 May/June 2024

1 **Part (a)**

Expand both compound angles:

$$\sin x + \cos x = \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x.$$

Thus $(2-\sqrt{3})\sin x = -\cos x$. Here $\cos x \neq 0$: otherwise this equation would require $\sin x = 0$ as well, which is impossible. Dividing by $(2-\sqrt{3})\cos x$ gives

$$\tan x = -\frac{1}{2-\sqrt{3}} = -\frac{2+\sqrt{3}}{(2-\sqrt{3})(2+\sqrt{3})}.$$

$$\boxed{\tan x = -2-\sqrt{3}}$$

2 **Part (b)**

Set $x = 2\theta - 45°$. Then $x + 45° = 2\theta$ and $x - 60° = 2\theta - 105°$, so part (a) gives $\tan(2\theta - 45°) = -2 - \sqrt{3}$. Since $\tan 75° = 2 + \sqrt{3}$,

$$2\theta - 45° = -75° + 180°k, \qquad \theta = -15° + 90°k.$$

In $0 \le \theta < 180°$, only $k = 1, 2$ are allowed. Substitution in the original equation confirms both values.

$$\boxed{\theta = 75°,\ 165°}$$

FINAL ANSWER

(a) $\tan x = -2 - \sqrt{3}$ (b) $\theta = 75°,\ 165°$

Question 10

9 marks | Trig Equations & R-Form

Main topic: Differentiation Techniques

10.

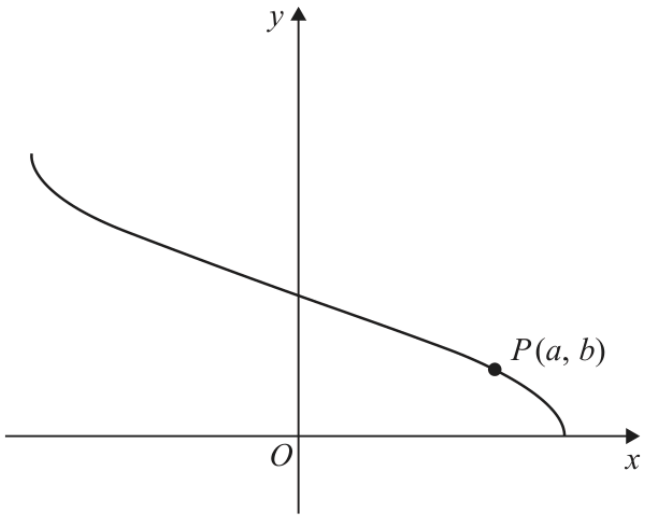


Figure 2

Figure 2 shows a sketch of the curve with equation

$$x = 3\cos 2y \qquad -3 \leqslant x \leqslant 3 \qquad 0 \leqslant y \leqslant \frac{\pi}{2}$$

(a) Find $\frac{dx}{dy}$ in terms of y.

(2)

(b) Hence show that

$$\frac{dy}{dx} = \frac{k}{\sqrt{9 - x^2}}$$

where k is a constant to be found.

(3)

The point $P(a, b)$ lies on the curve and is shown in Figure 2.

Given that

- the gradient of the curve at P is $-\frac{1}{4}$
- both a and b are positive

(c) find the exact values of a and b.

(4)

1 Part (a)

Differentiate $x = 3\cos 2y$ with respect to y using the chain rule:

$$\frac{dx}{dy} = 3(-\sin 2y)\cdot 2 = -6\sin 2y.$$

$$\boxed{\frac{dx}{dy} = -6\sin 2y}$$

2 Part (b)

For $0 < y < \pi/2$, $\sin 2y > 0$ and

$$\frac{dy}{dx} = \frac{1}{dx/dy} = -\frac{1}{6\sin 2y}.$$

Since $\cos 2y = x/3$ and the sine is positive on this interval,

$$\sin 2y = \sqrt{1-\cos^2 2y} = \sqrt{1 - x^2/9} = \frac{\sqrt{9-x^2}}{3}.$$

Consequently $dy/dx = -1/(2\sqrt{9-x^2})$, valid for $-3 < x < 3$. The endpoint tangents are vertical; there is no finite derivative there.

$$\boxed{\frac{dy}{dx} = -\frac{1}{2\sqrt{9-x^2}}, \qquad k = -\frac{1}{2}}$$

3 Part (c)

Set the gradient equal to $-1/4$:

$$-\frac{1}{2\sqrt{9-a^2}} = -\frac{1}{4} \quad\Longrightarrow\quad \sqrt{9-a^2} = 2 \quad\Longrightarrow\quad a^2 = 5.$$

Because $a > 0$, take $a = \sqrt{5}$. Substitute into $a = 3\cos 2b$:

$$\cos 2b = \frac{\sqrt{5}}{3} \quad\Longrightarrow\quad b = \frac{1}{2}\arccos\left(\frac{\sqrt{5}}{3}\right).$$

Here $0 < 2b < \pi/2$ since both coordinates are positive, so the principal arccos value is the required one.

$$\boxed{a = \sqrt{5}, \qquad b = \frac{1}{2}\arccos\left(\frac{\sqrt{5}}{3}\right)}$$

FINAL ANSWER

(a) $\frac{dx}{dy} = -6\sin 2y$ (b) $\frac{dy}{dx} = -\frac{1}{2\sqrt{9-x^2}}, \qquad k = -\frac{1}{2}$ (c) $a = \sqrt{5}, \qquad b = \frac{1}{2}\arccos\left(\frac{\sqrt{5}}{3}\right)$

Question 7

9 marks | Trig Equations & R-Form | Main topic: Numerical Methods

7: A continuous curve has equation

$$y = e^{-x^2}\sin 3x \qquad 0 \leqslant x \leqslant \frac{\pi}{3}$$

The curve has a stationary point at the point P.

(a) Show, using calculus, that the x coordinate of P is a solution of the equation

$$x = \frac{1}{3}\arctan\left(\frac{3}{2x}\right)$$

(4)

Using the iteration formula

$$x_{n+1} = \frac{1}{3}\arctan\left(\frac{3}{2x_n}\right) \qquad x_1 = 0.4$$

(b) find the value of

(i) x_2

(ii) x_4

giving your answers to 4 decimal places.

(3)

(c) Using a suitable interval and a suitable function which should be stated, show that the x coordinate of P is 0.430 correct to 3 decimal places.

(2)

1 Part (a)

Apply the product and chain rules:

$$\frac{dy}{dx} = e^{-x^2}\left(3\cos 3x - 2x\sin 3x\right).$$

Since $e^{-x^2} > 0$, a stationary point satisfies

$$3\cos 3x = 2x\sin 3x.$$

Neither endpoint $x = 0, \pi/3$ is stationary. In the interior $\sin 3x > 0$ and $x > 0$, so $\cos 3x > 0$ at a stationary point and $0 < 3x < \pi/2$. Therefore division and the principal inverse tangent are valid:

$$\tan 3x = \frac{3}{2x} \quad \Longrightarrow \quad \boxed{x = \frac{1}{3}\arctan\left(\frac{3}{2x}\right)}.$$

2 Part (b)

Work in radians and retain calculator precision between iterations:

$$x_2 = \frac{1}{3}\arctan\frac{3}{0.8} = 0.4367313117\ldots,$$

$$x_3 = 0.4291581882\ldots, \qquad x_4 = 0.4307116849\ldots.$$

Thus (i) $\boxed{x_2 = 0.4367}$ and (ii) $\boxed{x_4 = 0.4307}$.

3 Part (c)

Let $h(x) = 3\cos 3x - 2x\sin 3x$, which is continuous. At the rounding boundaries,

$$h(0.4295) = 0.0106861\ldots > 0, \qquad h(0.4305) = -0.000598027\ldots < 0.$$

The sign change places a root strictly between 0.4295 and 0.4305. On $0 < x < \pi/6$, $h'(x) = -11\sin 3x - 6x\cos 3x < 0$, so the stationary root is unique; for $\pi/6 \le x \le \pi/3$, $h(x) < 0$. Hence the x coordinate of P is $\boxed{0.430 \text{ to } 3 \text{ decimal places}}$.

FINAL ANSWER

(a) $x = \frac{1}{3}\arctan\left(\frac{3}{2x}\right)$ (b) $x_2 = 0.4367$, $x_4 = 0.4307$ (c) 0.430 to 3 decimal places

WMA13/01 MAY/JUNE 2025

Question 8

8 marks | Trig Equations & R-Form | Main topic: Trig Identities & Exact Values

8: **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(a) Prove that

$$\tan 3x \equiv \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x} \qquad x \neq (2n+1)\frac{\pi}{6} \qquad n \in \mathbb{Z}$$

(3)

(b) Hence solve, for $0 < \theta < \frac{\pi}{2}$

$$\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = 2\sec^2 3\theta - 8$$

giving your answers to 2 decimal places.

(5)

WORKED SOLUTION

Question 8 · WMA13/01 May/June 2025

Part (a)

Put $s = \sin x$ and $c = \cos x$. Using the addition and double-angle formulae,

$$\sin 3x = \sin 2x \cos x + \cos 2x \sin x = 3sc^2 - s^3,$$

$$\cos 3x = \cos 2x \cos x - \sin 2x \sin x = c^3 - 3s^2c.$$

Where the stated expressions are defined, $c \neq 0$ and $\cos 3x \neq 0$. Dividing numerator and denominator by c^3 gives

$$\tan 3x = \frac{3sc^2 - s^3}{c^3 - 3s^2c} = \boxed{\frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}}.$$

2 Part (b)

By part (a), the equation is $\tan 3\theta = 2\sec^2 3\theta - 8$. Set $u = \tan 3\theta$ and use $\sec^2 3\theta = 1 + u^2$:

$$u = 2(1 + u^2) - 8 \implies 2u^2 - u - 6 = 0 \implies (2u + 3)(u - 2) = 0.$$

Thus $\tan 3\theta = 2$ or $\tan 3\theta = -3/2$. The transformed interval is $0 < 3\theta < 3\pi/2$. Its three allowed angles are

$$3\theta = \arctan 2, \quad \pi - \arctan \frac{3}{2}, \quad \pi + \arctan 2.$$

Consequently

$$\theta = 0.36904957\ldots, \quad 0.71959964\ldots, \quad 1.41624712\ldots,$$

$$\boxed{\theta = 0.37,\ 0.72,\ 1.42 \text{ radians}}.$$

None lies at an excluded pole of either side.

FINAL ANSWER

(a) $\frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$ (b) $\theta = 0.37,\ 0.72,\ 1.42$ radians

Question 9

11 marks | Trig Equations & R-Form

Main topic: Differentiation Techniques

9:

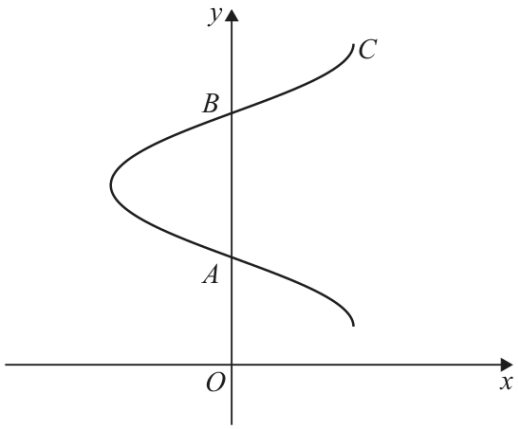


Figure 2

Figure 2 shows a sketch of the curve C with equation

$$x = \frac{2}{3}\sin\left(3y + \frac{\pi}{4}\right) \qquad \frac{\pi}{12} < y < \frac{3\pi}{4}$$

The curve intersects the y-axis at the points A and B as shown.

(a) Find the exact value of the y coordinate of

- point A
- point B

(3)

(b) Show that

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{p - qx^2}$$

where p and q are integers to be found.

(4)

The **normal** to C at A and the **tangent** to C at B intersect at the point D.

Using

- the answer to part (b)
- the sketch of curve C in Figure 2

(c) find, in simplest form, the exact x coordinate of D.

(4)

1 Part (a)

On the y axis, $x = 0$, so $\sin(3y + \pi/4) = 0$. Since $\pi/12 < y < 3\pi/4$, we have $\pi/2 < 3y + \pi/4 < 5\pi/2$. Therefore the possible angles are π and 2π, giving

$$\boxed{y_A = \frac{\pi}{4}, \qquad y_B = \frac{7\pi}{12}}.$$

2 Part (b)

Differentiate x with respect to y, then take the reciprocal:

$$\frac{dx}{dy} = 2\cos(3y + \pi/4), \qquad \frac{dy}{dx} = \frac{1}{2\cos(3y + \pi/4)}.$$

Using $\sin(3y + \pi/4) = 3x/2$,

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4(1 - \sin^2(3y + \pi/4))} = \boxed{\frac{1}{4 - 9x^2}}.$$

Thus $\boxed{p = 4,\ q = 9}$. The derivative formula applies where the tangent is not vertical.

3 Part (c)

At $x = 0$, part (b) gives $|dy/dx| = 1/2$. The sketch shows a negative tangent gradient at A and a positive one at B. The normal at A therefore has gradient 2, while the tangent at B has gradient $1/2$:

$$y = 2x + \frac{\pi}{4}, \qquad y = \frac{x}{2} + \frac{7\pi}{12}.$$

At their intersection,

$$\frac{3}{2}x = \frac{7\pi}{12} - \frac{\pi}{4} = \frac{\pi}{3} \quad \Longrightarrow \quad \boxed{x_D = \frac{2\pi}{9}}.$$

FINAL ANSWER

(a) $y_A = \frac{\pi}{4}, \quad y_B = \frac{7\pi}{12}$ (b) $\frac{1}{4-9x^2}$, $p = 4,\ q = 9$ (c) $x_D = \frac{2\pi}{9}$

WMA13/01 OCTOBER 2025

Question 9

9 marks | Trig Equations & R-Form | Main topic: Trig Identities & Exact Values

9. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(a) Express

$$6\sin^2\theta\cot 2\theta + 4\sin\theta\cos\theta$$

in terms of $\sin 2\theta$ and $\cos 2\theta$ only.

(3)

(b) Hence show that the equation

$$3\cot 2\theta - 14 = 6\sin^2\theta\cot 2\theta + 4\sin\theta\cos\theta$$

can be written in the form

$$5\sin^2 2\theta + 14\sin 2\theta - 3 = 0$$

(3)

(c) Hence solve, for $0 < x < 90°$, the equation

$$3\cot 2x - 14 = 6\sin^2 x\cot 2x + 4\sin x\cos x$$

giving your answers to one decimal place.

(3)

WORKED SOLUTION

Question 9 · WMA13/01 October 2025

1 **Part (a)**

Use $2\sin^2\theta = 1 - \cos 2\theta$, $\cot 2\theta = \cos 2\theta / \sin 2\theta$, and $2\sin\theta\cos\theta = \sin 2\theta$:

$$6\sin^2\theta\cot 2\theta + 4\sin\theta\cos\theta = \frac{3(1-\cos 2\theta)\cos 2\theta}{\sin 2\theta} + 2\sin 2\theta.$$

This is in terms of $\sin 2\theta$ and $\cos 2\theta$ only, on the original domain $\sin 2\theta \neq 0$.

2

Part (b)

Substitute the preceding expression and multiply by $\sin 2\theta$:

$$3\cos 2\theta - 14\sin 2\theta = 3\cos 2\theta - 3\cos^2 2\theta + 2\sin^2 2\theta.$$

Cancel $3\cos 2\theta$ and use $\cos^2 2\theta = 1 - \sin^2 2\theta$:

$$-14\sin 2\theta = -3 + 5\sin^2 2\theta.$$

Hence $5\sin^2 2\theta + 14\sin 2\theta - 3 = 0$.

3

Part (c)

Let $s = \sin 2x$. Then $(5s - 1)(s + 3) = 0$, so $s = 1/5$ or $s = -3$. The latter is impossible. As $0 < 2x < 180°$,

$$2x = \arcsin(1/5) = 11.53695903\ldots° \quad \text{or} \quad 180° - 11.53695903\ldots°.$$

Therefore $x = 5.8°$ or $84.2°$ to one decimal place. Both have nonzero $\sin 2x$, so the original cotangent is defined.

✓ **FINAL ANSWER**

(a) $3 * (1 - cos(2 * theta)) * cos(2 * theta)/sin(2 * theta) + 2 * sin(2 * theta)$ (b) Shown, as required. (c) $5.8 degrees$; $84.2 degrees$

WMA13/01 JANUARY 2026

Question 6

7 marks | Trig Equations & R-Form | Main topic: Trig Identities & Exact Values

6. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(a) Show that the equation

$$2\cos(\theta - 60°) = 3\sin\theta$$

can be written in the form

$$\tan\theta = \frac{1}{6}\left(3 + \sqrt{3}\right)$$

(4)

(b) Hence solve, for $0 \leqslant x < 180°$, the equation

$$2\cos(2x - 70°) = 3\sin(2x - 10°)$$

giving your solutions to one decimal place.

(3)

WORKED SOLUTION

Question 6 · WMA13/01 January 2026

1 Part (a)

Expanding the compound angle,

$$2\left(\cos\theta\cos 60° + \sin\theta\sin 60°\right) = 3\sin\theta,$$

so $\cos\theta + \sqrt{3}\sin\theta = 3\sin\theta$ and $\cos\theta = (3 - \sqrt{3})\sin\theta$. If $\cos\theta = 0$, this equation would require $\sin\theta = 0$, which is impossible. We may therefore divide by $\cos\theta$:

$$\tan\theta = \frac{1}{3 - \sqrt{3}} = \frac{3 + \sqrt{3}}{9 - 3} = \frac{3 + \sqrt{3}}{6}.$$

2 Part (b)

Set $\theta = 2x - 10°$, so $2x - 70° = \theta - 60°$ and $-10° \leq \theta < 350°$.

$$\theta = \arctan\left(\frac{3 + \sqrt{3}}{6}\right) + 180°n = 38.2619662\ldots° + 180°n.$$

The admissible values are $38.2619662\ldots°$ and $218.2619662\ldots°$. Hence $x = (\theta + 10°)/2$, giving $x = 24.1°$ or $114.1°$ (1 decimal place).

FINAL ANSWER

(a) $\frac{3+\sqrt{3}}{6}$ (b) $x = 24.1° \text{ or } 114.1°$

WMA13/01 MAY/JUNE 2026

Question 8

9 marks | Trig Equations & R-Form

Main topic: Trig Identities & Exact Values

8. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(a) Show that

$$\frac{\sec\theta\,\operatorname{cosec}\theta}{4\operatorname{cosec}4\theta} \equiv \cos 2\theta$$

(2)

(b) Hence show that the equation

$$4(5\cos^2\theta - 9)\operatorname{cosec}4\theta = 3(1+\tan^2 2\theta)\sec\theta\operatorname{cosec}\theta$$

can be written as

$$a\cos^2 2\theta + b\cos 2\theta + c = 0$$

where a, b and c are integers to be found.

(3)

(c) Hence, solve for $0 < x < 180°$, $x \neq (45n)°$, $n \in \mathbb{N}$, the equation

$$4(5\cos^2 x - 9)\operatorname{cosec}4x = 3(1+\tan^2 2x)\sec x\operatorname{cosec}x$$

giving the answers to one decimal place.

(4)

WORKED SOLUTION

Question 8 · WMA13/01 May/June 2026

1 **Part (a)**

For angles where the original expression is defined,

$$\frac{\sec\theta\ \operatorname{cosec}\theta}{4\operatorname{cosec}4\theta} = \frac{\sin 4\theta}{4\sin\theta\cos\theta} = \frac{2\sin 2\theta\cos 2\theta}{2\sin 2\theta} = \boxed{\cos 2\theta}.$$

2 Part (b)

Divide the given equation by $4\operatorname{cosec} 4\theta$ and use (a):

$$5\cos^2\theta - 9 = 3(1+\tan^2 2\theta)\cos 2\theta = \frac{3}{\cos 2\theta}.$$

Use $\cos^2\theta = (1+\cos 2\theta)/2$ and multiply by $2\cos 2\theta$:

$$(5\cos 2\theta - 13)\cos 2\theta = 6.$$

Therefore

$$\boxed{5\cos^2 2\theta - 13\cos 2\theta - 6 = 0}, \qquad a = 5,\ b = -13,\ c = -6.$$

The divisions are valid on the domain of the original equation.

3 Part (c)

Set $u = \cos 2x$:

$$5u^2 - 13u - 6 = (5u+2)(u-3) = 0.$$

Reject $u = 3$ because cosine lies between -1 and 1. Thus $\cos 2x = -2/5$. Since $0 < 2x < 360°$,

$$2x = 113.5781785\ldots° \quad \text{or} \quad 246.4218215\ldots°.$$

Hence $\boxed{x = 56.8° \text{ or } 123.2°}$ to one decimal place. Neither is an excluded multiple of $45°$; these are all the solutions in the interval.

FINAL ANSWER

(a) $\cos 2\theta$ (b) $5\cos^2 2\theta - 13\cos 2\theta - 6 = 0$ (c) $x = 56.8° \text{ or } 123.2°$

TOPIC

Exponentials & Logarithms

Every past-paper question on this topic, January 2020 to June 2026.

Question 9

9 marks | Exponentials & Logarithms

9. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

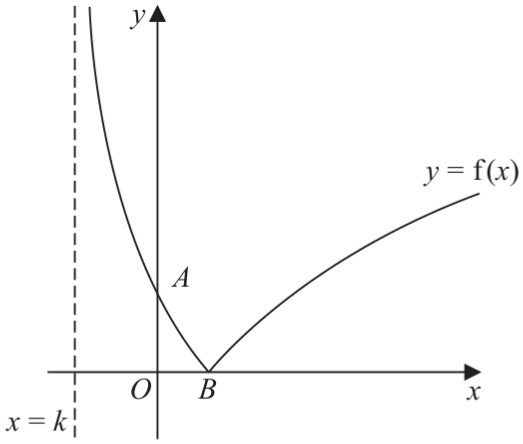


Figure 2

Figure 2 shows a sketch of the curve with equation

$$y = |2 - 4\ln(x + 1)| \qquad x > k$$

where k is a constant.

Given that the curve

- has an asymptote at $x = k$
- cuts the y-axis at point A
- meets the x-axis at point B

as shown in Figure 2,

(a) state the value of k

(1)

(b) (i) find the y coordinate of A

(ii) find the exact x coordinate of B

(3)

(c) Using algebra and showing your working, find the set of values of x such that

$$|2 - 4\ln(x + 1)| > 3$$

(5)

1 Part (a)

The logarithm requires $x+1>0$, so the domain is $x>-1$. As $x\to -1^{+}$, $\ln(x+1)\to -\infty$ and the ordinate tends to infinity.

$$\boxed{k=-1}$$

2 Part (b)

(i) At the y-axis, $x=0$, giving $y=|2-4\ln 1|=2$.
(ii) At the x-axis, $|2-4\ln(x+1)|=0$, so

$$\ln(x+1)=\frac{1}{2}\quad\Longrightarrow\quad x=e^{1/2}-1.$$

Thus $A=(0,2)$ and $B=(e^{1/2}-1,0)$.

$$\boxed{y_A=2,\qquad x_B=e^{1/2}-1}$$

3 Part (c)

On $x>-1$, split the absolute-value inequality into two alternatives:

$$2-4\ln(x+1)>3\quad\text{or}\quad 2-4\ln(x+1)<-3.$$

Dividing by -4 reverses the inequality in each branch:

$$\ln(x+1)<-\frac{1}{4}\quad\text{or}\quad\ln(x+1)>\frac{5}{4}.$$

Since the exponential is increasing, this becomes $x+1<e^{-1/4}$ or $x+1>e^{5/4}$. Intersect with the logarithmic domain; neither boundary is included because the source inequality is strict.

$$\boxed{-1<x<e^{-1/4}-1\quad\text{or}\quad x>e^{5/4}-1}$$

FINAL ANSWER

(a) $k=-1$ (b) $y_A=2,\qquad x_B=e^{1/2}-1$ (c) $-1<x<e^{-1/4}-1\quad\text{or}\quad x>e^{5/4}-1$

WMA13/01 JANUARY 2020

Question 8

10 marks | Exponentials & Logarithms | Main topic: Integration

8. (i) Find, using algebraic integration, the exact value of

$$\int_3^{42} \frac{2}{3x-1}\,\mathrm{d}x$$

giving your answer in simplest form.

(4)

(ii) $$\mathrm{h}(x) = \frac{2x^3 - 7x^2 + 8x + 1}{(x-1)^2} \qquad x > 1$$

Given $\mathrm{h}(x) = Ax + B + \frac{C}{(x-1)^2}$ where A, B and C are constants to be found, find

$$\int \mathrm{h}(x)\,\mathrm{d}x$$

(6)

WORKED SOLUTION

Question 8 · WMA13/01 January 2020

1 **Part (i)**

Since $3x - 1 > 0$ throughout $3 \le x \le 42$, an antiderivative is

$$\int \frac{2}{3x-1}\,dx = \frac{2}{3}\ln(3x-1) + K.$$

The factor $1/3$ compensates for the derivative of $3x - 1$. Thus

$$\begin{aligned}\int_3^{42} \frac{2}{3x-1}\,dx &= \left[\frac{2}{3}\ln(3x-1)\right]_3^{42} \\ &= \frac{2}{3}\left(\ln 125 - \ln 8\right) \\ &= \frac{2}{3}\ln\left(\frac{125}{8}\right) = \frac{2}{3}\ln\left(\left(\frac{5}{2}\right)^3\right).\end{aligned}$$

Using $\ln(a^3) = 3\ln a$ for $a > 0$, the exact value in simplest form is

$$\boxed{2\ln\left(\frac{5}{2}\right).}$$

2 **Part (ii)**

Multiply the given decomposition by $(x-1)^2$:

$$\begin{aligned} 2x^3 - 7x^2 + 8x + 1 &= (Ax + B)(x-1)^2 + C \\ &= Ax^3 + (B - 2A)x^2 + (A - 2B)x + (B + C). \end{aligned}$$

Comparing coefficients gives

$$A = 2, \qquad B - 2A = -7 \implies B = -3, \qquad B + C = 1 \implies C = 4.$$

The coefficient of x also agrees, since $A - 2B = 2 - 2(-3) = 8$. Hence

$$\boxed{A = 2, \qquad B = -3, \qquad C = 4,} \qquad h(x) = 2x - 3 + 4(x-1)^{-2}.$$

Integrating each term,

$$\begin{aligned} \int h(x)\,dx &= \int \left(2x - 3 + 4(x-1)^{-2}\right) dx \\ &= x^2 - 3x - 4(x-1)^{-1} + K. \end{aligned}$$

Therefore, for $x > 1$,

$$\boxed{\int h(x)\,dx = x^2 - 3x - \frac{4}{x-1} + K,}$$

where K is an arbitrary constant of integration, distinct from the coefficient $C = 4$.

FINAL ANSWER

(i) $2\ln\left(\frac{5}{2}\right)$ (ii) $A = 2, \qquad B = -3, \qquad C = 4, \int h(x)\,dx = x^2 - 3x - \frac{4}{x-1} + K$

Question 6

9 marks | Exponentials & Logarithms | Main topic: Numerical Methods

6.

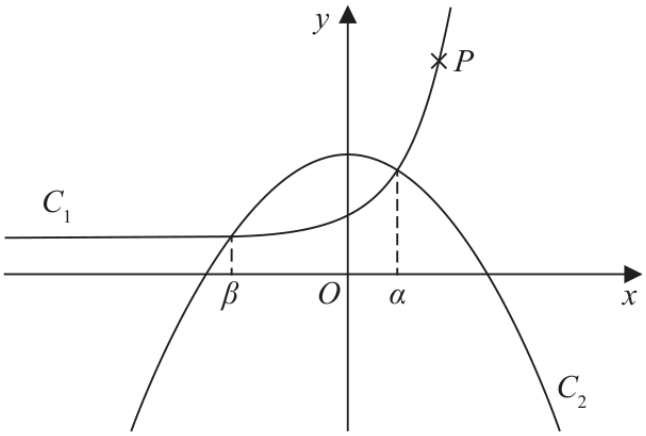


Figure 3

Figure 3 shows a sketch of curve C_1 with equation $y = 5e^{x-1} + 3$

and curve C_2 with equation $y = 10 - x^2$

The point P lies on C_1 and has y coordinate 18

(a) Find the x coordinate of P, writing your answer in the form $\ln k$, where k is a constant to be found.

(3)

The curve C_1 meets the curve C_2 at $x = \alpha$ and at $x = \beta$, as shown in Figure 3.

(b) Using a suitable interval and a suitable function that should be stated, show that to 3 decimal places $\alpha = 1.134$

(3)

The iterative equation

$$x_{n+1} = -\sqrt{7 - 5e^{x_n - 1}}$$

is used to find an approximation to β.

Using this iterative formula with $x_1 = -3$

(c) find the value of x_2 and the value of β, giving each answer to 6 decimal places.

(3)

1 Part (a)

At P, $18 = 5e^{x-1} + 3$, so $e^{x-1} = 3$. Taking natural logarithms gives $x - 1 = \ln 3$, hence $x = 1 + \ln 3 = \ln e + \ln 3 = \ln(3e)$.

$$\boxed{x = \ln(3e), \qquad k = 3e}$$

2 Part (b)

Define $F(x) = 5e^{x-1} + x^2 - 7$; its zeros are the intersection abscissae, and F is continuous. At the endpoints of the three-decimal rounding interval,

$$F(1.1335) = -0.0010714193\ldots < 0, \qquad F(1.1345) = 0.0069135450\ldots > 0.$$

Continuity guarantees a root between them. Moreover $F'(x) = 5e^{x-1} + 2x > 0$ on this positive interval, so that root is unique there. It is the positive intersection α, and rounds to 1.134 to three decimal places.

$$\boxed{\alpha = 1.134 \text{ to 3 d.p.}}$$

3 Part (c)

Start at $x_1 = -3$ and keep full precision in $x_{n+1} = -\sqrt{7 - 5e^{x_n - 1}}$:

n	x_n
1	-3
2	-2.628387681746
3	-2.620535277934
4	-2.620335527988
5	-2.620330426051
6	-2.620330295726
7	-2.620330292397

The later iterates agree to six decimal places. The negative root corresponds to β in the figure.

$$\boxed{x_2 = -2.628388, \qquad \beta = -2.620330}$$

FINAL ANSWER

(a) $x = \ln(3e), \qquad k = 3e$ (b) $\alpha = 1.134$ to 3 d.p. (c) $x_2 = -2.628388, \qquad \beta = -2.620330$

WMA13/01 OCTOBER 2020

Question 8

9 marks | Exponentials & Logarithms | Main topic: Differentiation Techniques

8. (i) The curve C has equation $y = \mathrm{g}(x)$ where

$$\mathrm{g}(x) = \mathrm{e}^{3x} \sec 2x \qquad -\frac{\pi}{4} < x < \frac{\pi}{4}$$

(a) Find $\mathrm{g}'(x)$

(2)

(b) Hence find the x coordinate of the stationary point of C.

(3)

(ii) A different curve has equation

$$x = \ln(\sin y) \qquad 0 < y < \frac{\pi}{2}$$

Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^{x}}{\mathrm{f}(x)}$$

where $\mathrm{f}(x)$ is a function of e^{x} that should be found.

(4)

WORKED SOLUTION

Question 8 · WMA13/01 October 2020

1 Part (i)(a)

By the product and chain rules,

$$g'(x) = 3e^{3x}\sec 2x + 2e^{3x}\sec 2x \tan 2x = e^{3x}\sec 2x(3 + 2\tan 2x).$$

$$\boxed{g'(x) = e^{3x}\sec 2x(3 + 2\tan 2x)}$$

2 Part (i)(b)

For $-\pi/4 < x < \pi/4$, $e^{3x}\sec 2x$ is positive and nonzero. Hence $g'(x) = 0$ precisely when $3 + 2\tan 2x = 0$, or $\tan 2x = -3/2$. Since $-\pi/2 < 2x < \pi/2$, tangent is one-to-one there:

$$x = -\frac{1}{2}\arctan\frac{3}{2} = -0.4913968616\ldots.$$

$$\boxed{x = -\frac{1}{2}\arctan\frac{3}{2} \approx -0.491}$$

---

3 Part (ii)

Differentiate $x = \ln(\sin y)$ with respect to y:

$$\frac{dx}{dy} = \frac{\cos y}{\sin y}, \qquad \frac{dy}{dx} = \frac{\sin y}{\cos y}.$$

The original equation gives $\sin y = e^{x}$. Since $0 < y < \pi/2$, cosine is positive, so $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - e^{2x}}$. Therefore

$$\frac{dy}{dx} = \frac{e^{x}}{\sqrt{1 - e^{2x}}}, \qquad f(x) = \sqrt{1 - e^{2x}}.$$

Here $x < 0$ from the original interval, so the square root is real and nonzero.

$$\boxed{f(x) = \sqrt{1 - e^{2x}}, \qquad x < 0}$$

FINAL ANSWER

(i)(a) $g'(x) = e^{3x}\sec 2x(3 + 2\tan 2x)$ (i)(b) $x = -\frac{1}{2}\arctan\frac{3}{2} \approx -0.491$ (ii) $f(x) = \sqrt{1 - e^{2x}}, \qquad x < 0$

Question 9

14 marks | Exponentials & Logarithms | Main topic: Applications of Differentiation

9. (a) Given that

$$\frac{x^4 - x^3 - 10x^2 + 3x - 9}{x^2 - x - 12} \equiv x^2 + P + \frac{Q}{x - 4} \qquad x > -3$$

find the value of the constant P and show that $Q = 5$

(4)

The curve C has equation $y = g(x)$, where

$$g(x) = \frac{x^4 - x^3 - 10x^2 + 3x - 9}{x^2 - x - 12} \qquad -3 < x < 3.5 \qquad x \in \mathbb{R}$$

(b) Find the equation of the tangent to C at the point where $x = 2$
Give your answer in the form $y = mx + c$, where m and c are constants to be found.

(5)

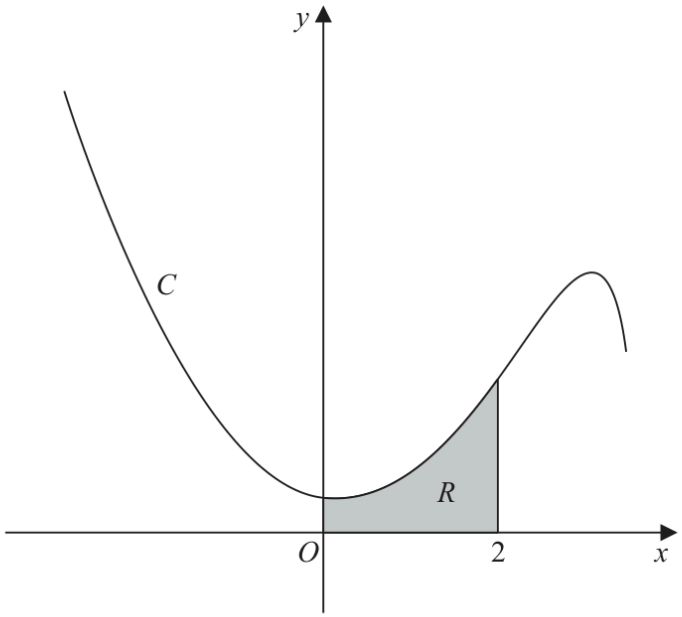


Figure 4

Figure 4 shows a sketch of the curve C.
The region R, shown shaded in Figure 4, is bounded by C, the y-axis, the x-axis and the line with equation $x = 2$

(c) Find the exact area of R, writing your answer in the form $a + b\ln 2$, where a and b are constants to be found.

(5)

1 Part (a)

Factor the denominator as $(x-4)(x+3)$ and multiply the proposed identity through by it:

$$x^4 - x^3 - 10x^2 + 3x - 9 = (x^2 + P)(x^2 - x - 12) + Q(x+3).$$

The right side expands to $x^4 - x^3 + (P-12)x^2 + (Q-P)x + 3Q - 12P$. Comparing coefficients gives $P - 12 = -10$, so $P = 2$, and $Q - P = 3$, so $Q = 5$. The constant check is $3(5) - 12(2) = -9$, as required. The identity holds where the original fraction is defined: the stated $x > -3$ still excludes $x = 4$.

$$\boxed{P = 2, \qquad Q = 5}$$

2 Part (b)

From (a), $g(x) = x^2 + 2 + 5/(x-4)$ on the curve domain. Thus $g'(x) = 2x - 5/(x-4)^2$. At $x = 2$, $g(2) = 4 + 2 - 5/2 = 7/2$ and $g'(2) = 4 - 5/4 = 11/4$. The tangent through $(2, 7/2)$ is

$$y - \frac{7}{2} = \frac{11}{4}(x-2) \quad \Longrightarrow \quad y = \frac{11}{4}x - 2.$$

$$\boxed{y = \frac{11}{4}x - 2}$$

3 Part (c)

The source shading is above the x-axis from $x = 0$ to $x = 2$, so integrate g over those bounds.

$$\text{Area}(R) = \int_0^2 \left(x^2 + 2 + \frac{5}{x-4}\right) dx = \left[\frac{x^3}{3} + 2x + 5\ln|x-4|\right]_0^2.$$

The absolute value is essential because $x - 4 < 0$ throughout this interval. Evaluating gives

$$\frac{8}{3} + 4 + 5\ln 2 - 5\ln 4 = \frac{20}{3} - 5\ln 2.$$

Thus $a = 20/3$ and $b = -5$ in the requested form.

$$\boxed{\text{Area}(R) = \frac{20}{3} - 5\ln 2}$$

FINAL ANSWER

(a) $P = 2, \quad Q = 5$ (b) $y = \frac{11}{4}x - 2$ (c) $\text{Area}(R) = \frac{20}{3} - 5\ln 2$

WMA13/01 JANUARY 2021

Question 8

7 marks | Exponentials & Logarithms | Main topic: Log Graphs & Exponential Models

8. The percentage, P, of the population of a small country who have access to the internet, is modelled by the equation

$$P = ab^t$$

where a and b are constants and t is the number of years after the start of 2005

Using the data for the years between the start of 2005 and the start of 2010, a graph is plotted of $\log_{10} P$ against t.

The points are found to lie approximately on a straight line with gradient 0.09 and intercept 0.68 on the $\log_{10} P$ axis.

(a) Find, according to the model, the value of a and the value of b, giving your answers to 2 decimal places.

(4)

(b) In the context of the model, give a practical interpretation of the constant a.

(1)

(c) Use the model to estimate the percentage of the population who had access to the internet at the start of 2015

(2)

WORKED SOLUTION

Question 8 · WMA13/01 January 2021

1 Part (a)

Taking logarithms gives $\log_{10} P = \log_{10} a + t \log_{10} b$. Match the intercept and gradient: $\log_{10} a = 0.68$, $\log_{10} b = 0.09$. Thus $a = 10^{0.68} = 4.786300923\ldots$ and $b = 10^{0.09} = 1.230268771\ldots$.

$$\boxed{a = 4.79, \qquad b = 1.23}$$

2 Part (b)

At $t = 0$, $P = a$. Thus a is the percentage of the country's population with internet access at the start of 2005, approximately 4.79%.

$$\boxed{\text{Initial percentage with internet access at the start of 2005}}$$

3 Part (c)

The start of 2015 is $t = 10$ years after the start of 2005. Using the model parameters, $P = 4.79(1.23)^{10} = 37.9652818\ldots\%$, giving 38.0% to three significant figures. Using unrounded parameters gives $10^{0.68+0.09(10)} = 38.0189396\ldots\%$, the same rounded answer.

$$\boxed{38.0\%}$$

FINAL ANSWER

(a) $a = 4.79, \quad b = 1.23$ (b) $\textbf{Initial percentage with internet access at the start of 2005}$ (c) 38.0%

WMA13/01 MAY/JUNE 2021

Question 8

13 marks | Exponentials & Logarithms | Main topic: Log Graphs & Exponential Models

8. A scientist is studying a population of fish in a lake. The number of fish, N, in the population, t years after the start of the study, is modelled by the equation

$$N = \frac{600e^{0.3t}}{2 + e^{0.3t}} \qquad t \geqslant 0$$

Use the equation of the model to answer parts (a), (b), (c), (d) and (e).

(a) Find the number of fish in the lake at the start of the study.

(1)

(b) Find the upper limit to the number of fish in the lake.

(1)

(c) Find the time, after the start of the study, when there are predicted to be 500 fish in the lake. Give your answer in years and months to the nearest month.

(4)

(d) Show that

$$\frac{dN}{dt} = \frac{Ae^{0.3t}}{(2 + e^{0.3t})^2}$$

where A is a constant to be found.

(3)

Given that when $t = T$, $\frac{dN}{dt} = 8$

(e) find the value of T to one decimal place.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(4)

WORKED SOLUTION

Question 8 · WMA13/01 May/June 2021

Part (a)

At the start, $t = 0$ and $e^{0.3t} = 1$. Thus $N(0) = 600/(2+1) = 200$.

$$\boxed{200 \text{ fish}}$$

2 Part (b)

Divide numerator and denominator by $e^{0.3t}$ to write $N = 600/(1 + 2e^{-0.3t})$. As $t \to \infty$, the denominator tends to 1, so the upper limit is 600 fish.

$$\boxed{600 \text{ fish}}$$

3 Part (c)

Set $N = 500$ and multiply through: $1000 + 500e^{0.3t} = 600e^{0.3t}$, so $e^{0.3t} = 10$. Hence $t = \ln 10/0.3 = 7.675283643\ldots$ years. The fractional year is $12(0.675283643\ldots) = 8.10340372\ldots$ months. Round only the months to the nearest whole month.

$$\boxed{7 \text{ years } 8 \text{ months}}$$

4 Part (d)

The quotient rule gives

$$\frac{dN}{dt} = \frac{180e^{0.3t}(2 + e^{0.3t}) - 600e^{0.3t}(0.3e^{0.3t})}{(2 + e^{0.3t})^2} = \frac{360e^{0.3t}}{(2 + e^{0.3t})^2}.$$

The two $e^{0.6t}$ terms cancel exactly.

$$\boxed{A = 360}$$

5 Part (e)

Let $z = e^{0.3T}$. Since $T \geq 0$, require $z \geq 1$. Then $360z/(2 + z)^2 = 8$ gives $45z = (2 + z)^2$, hence

$$z^2 - 41z + 4 = 0, \qquad z = \frac{41 \pm \sqrt{1665}}{2}.$$

The smaller root lies between 0 and 1 and would give $T < 0$, so reject it. For the larger root,

$$T = \frac{1}{0.3}\ln\left(\frac{41 + \sqrt{1665}}{2}\right) = 12.37061331\ldots.$$

$$\boxed{T = 12.4 \text{ years}}$$

FINAL ANSWER

(a) 200 fish (b) 600 fish (c) 7 years 8 months (d) $A = 360$ (e) $T = 12.4$ years

WMA13/01 OCTOBER 2021

Question 7

6 marks | Exponentials & Logarithms | Main topic: Log Graphs & Exponential Models

7 The mass, M kg, of a species of tree can be modelled by the equation

$$\log_{10} M = 1.93 \log_{10} r + 0.684$$

where r cm is the base radius of the tree.

The base radius of a particular tree of this species is 45 cm.

According to the model,

(a) find the mass of this tree, giving your answer to 2 significant figures.

(2)

(b) Show that the equation of the model can be written in the form

$$M = pr^q$$

giving the values of the constants p and q to 3 significant figures.

(3)

(c) With reference to the model, interpret the value of the constant p.

(1)

WORKED SOLUTION

Question 7 · WMA13/01 October 2021

1 **Part (a)**

At $r = 45$, $\log_{10} M = 1.93 \log_{10} 45 + 0.684$. Exponentiate to obtain $M = 10^{0.684} 45^{1.93} = 7493.766412\ldots$ kg.

$$\boxed{M = 7.5 \times 10^3 \text{ kg}}$$

2 **Part (b)**

Exponentiate the full equation:

$$M = 10^{1.93 \log_{10} r + 0.684} = 10^{0.684} r^{1.93}.$$

Therefore $p = 10^{0.684} = 4.83058802\ldots$ and $q = 1.93$.

$$\boxed{p = 4.83, \qquad q = 1.93}$$

3

Part (c)

If the base radius is $r = 1$ cm, the model gives $M = p \cdot 1^q = p$ kg. Thus p is the predicted mass in kilograms of a tree of base radius 1 cm.

$$\boxed{\text{Predicted mass (kg) when base radius is 1 cm}}$$

✓ **FINAL ANSWER**

(a) $M = 7.5 \times 10^3$ kg (b) $p = 4.83, \qquad q = 1.93$ (c)

$\text{Predicted mass (kg) when base radius is 1 cm}$

WMA13/01 JANUARY 2022

Question 8

8 marks | Exponentials & Logarithms | Main topic: Log Graphs & Exponential Models

8. A dose of antibiotics is given to a patient.

The amount of the antibiotic, x milligrams, in the patient's bloodstream t hours after the dose was given, is found to satisfy the equation

$$\log_{10} x = 2.74 - 0.079t$$

(a) Show that this equation can be written in the form

$$x = pq^{-t}$$

where p and q are constants to be found. Give the value of p to the nearest whole number and the value of q to 2 significant figures.

(4)

(b) With reference to the equation in part (a), interpret the value of the constant p.

(1)

When a different dose of the antibiotic is given to another patient, the values of x and t satisfy the equation

$$x = 400 \times 1.4^{-t}$$

(c) Use calculus to find, to 2 significant figures, the value of $\dfrac{dx}{dt}$ when $t = 5$

(3)

WORKED SOLUTION

Question 8 · WMA13/01 January 2022

1 Part (a)

Exponentiate base 10:

$$x = 10^{2.74-0.079t} = 10^{2.74}(10^{0.079})^{-t}.$$

Thus $p = 10^{2.74} = 549.5408739\ldots$ and $q = 10^{0.079} = 1.199499303\ldots$. Round p to the nearest integer and q to two significant figures.

$$\boxed{p = 550, \qquad q = 1.2}$$

2 Part (b)

At $t = 0$, $x = pq^0 = p$. Thus p is the initial amount of antibiotic in the patient's bloodstream, approximately 550 milligrams immediately after the dose in this model.

$$\boxed{p \text{ represents the initial amount, approximately } 550 \text{ mg}}$$

3 Part (c)

For the second patient, write $x = 400e^{-t\ln 1.4}$. Differentiate to obtain $dx/dt = -400\ln(1.4)\,1.4^{-t}$. At $t = 5$, the derivative is $-25.02470969\ldots \text{ mg}\,\text{h}^{-1}$. The negative sign indicates the amount is decreasing.

$$\boxed{\left.\frac{dx}{dt}\right|_{t=5} = -25 \text{ mg}\,\text{h}^{-1}}$$

FINAL ANSWER

(a) $p = 550, \quad q = 1.2$ (b) p represents the initial amount, approximately 550 mg (c) $\left.\frac{dx}{dt}\right|_{t=5} = -25 \text{ mg}\,\text{h}^{-1}$

Question 8

12 marks | Exponentials & Logarithms | Main topic: Numerical Methods

8.

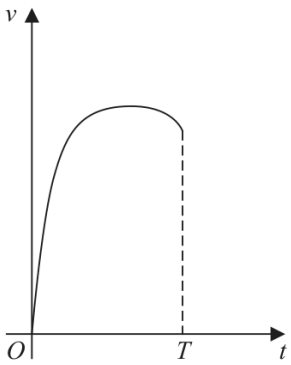


Figure 4

Figure 4 is a graph showing the velocity of a sprinter during a 100 m race.

The sprinter's velocity during the race, $v\,\text{m s}^{-1}$, is modelled by the equation

$$v = 12 - \text{e}^{t-10} - 12\text{e}^{-0.75t} \qquad t \geqslant 0$$

where t seconds is the time after the sprinter begins to run.

According to the model,

(a) find, using calculus, the sprinter's maximum velocity during the race.

(5)

Given that the sprinter runs 100 m in T seconds, such that

$$\int_0^T v\,\text{d}t = 100$$

(b) show that T is a solution of the equation

$$T = \frac{1}{12}\left(116 - 16\text{e}^{-0.75T} + \text{e}^{T-10} - \text{e}^{-10}\right)$$

(4)

The iteration formula

$$T_{n+1} = \frac{1}{12}\left(116 - 16\text{e}^{-0.75T_n} + \text{e}^{T_n-10} - \text{e}^{-10}\right)$$

is used to find an approximate value for T

Using this iteration formula with $T_1 = 10$

(c) find, to 4 decimal places,

(i) the value of T_2

(ii) the time taken by the sprinter to run the race, according to the model.

(3)

1 Part (a)

Differentiate the velocity:

$$v'(t) = -e^{t-10} + 9e^{-0.75t}.$$

At a stationary velocity, $e^{t-10} = 9e^{-0.75t}$, so $e^{1.75t} = 9e^{10}$ and $t = (10 + \ln 9)/1.75 = 6.9698426\ldots$ seconds. The second derivative $-e^{t-10} - 6.75e^{-0.75t}$ is negative, so this is a maximum. Substitute this unrounded time into v to get $11.88728125\ldots$ metres per second.

$$\boxed{v_{\max} = 11.9\ \mathrm{m\,s^{-1}}}$$

2 Part (b)

Integrating the velocity gives $12t - e^{t-10} + 16e^{-0.75t}$. Evaluate between 0 and T:

$$100 = 12T - e^{T-10} + 16e^{-0.75T} - (-e^{-10} + 16).$$

Rearrange to $12T = 116 - 16e^{-0.75T} + e^{T-10} - e^{-10}$ and divide by 12.

$$\boxed{T = \frac{1}{12}(116 - 16e^{-0.75T} + e^{T-10} - e^{-10})}$$

3 Part (c)

With $T_1 = 10$, substitution gives $T_2 = 9.7492587708\ldots$. Continuing with unrounded values gives $T_3 = 9.7306248365\ldots$, $T_4 = 9.7294150526\ldots$, $T_5 = 9.7293372715\ldots$, $T_6 = 9.7293322738\ldots$.

Further iteration settles at $9.7293319307\ldots$ seconds. Thus (i) $T_2 = 9.7493$ and (ii) the race time is 9.7293 seconds, to four decimal places.

$$\boxed{T_2 = 9.7493, \qquad T = 9.7293\ \mathrm{s}}$$

FINAL ANSWER

(a) $v_{\max} = 11.9\ \mathrm{m\,s^{-1}}$ (b) $T = \frac{1}{12}(116 - 16e^{-0.75T} + e^{T-10} - e^{-10})$ (c) $T_2 = 9.7493, \qquad T = 9.7293\ \mathrm{s}$

WMA13/01 JANUARY 2023

Question 10

9 marks | Exponentials & Logarithms

Main topic: Log Graphs & Exponential Models

10. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

A population of fruit flies is being studied.

The number of fruit flies, F, in the population, t days after the start of the study, is modelled by the equation

$$F = \frac{350e^{kt}}{9 + e^{kt}}$$

where k is a constant.

Use the equation of the model to answer parts (a), (b) and (c).

(a) Find the number of fruit flies in the population at the start of the study.

(1)

Given that there are 200 fruit flies in the population 15 days after the start of the study,

(b) show that $k = \frac{1}{15}\ln 12$

(3)

Given also that, when $t = T$, the number of fruit flies in the population is increasing at a rate of 10 per day,

(c) find the possible values of T, giving your answers to one decimal place.

(5)

WORKED SOLUTION

Question 10 · WMA13/01 January 2023

1 **Part (a)**

At the start $t = 0$, so $e^{kt} = 1$ and the model gives $F(0) = 350/(9 + 1) = 35$ fruit flies.

$$\boxed{F(0) = 35}$$

2 Part (b)

Using $F(15) = 200$,

$$200 = \frac{350e^{15k}}{9 + e^{15k}} \quad \Longrightarrow \quad 1800 + 200e^{15k} = 350e^{15k} \quad \Longrightarrow \quad e^{15k} = 12.$$

Take natural logs to obtain $15k = \ln 12$, as required.

$$\boxed{k = \frac{\ln 12}{15}}$$

3 Part (c)

By the quotient rule,

$$F'(t) = \frac{350ke^{kt}(9 + e^{kt}) - 350ke^{2kt}}{(9 + e^{kt})^2} = \frac{3150ke^{kt}}{(9 + e^{kt})^2}.$$

Set $F'(T) = 10$ and let $u = e^{kT} > 0$. Then

$$315ku = (9 + u)^2 \quad \Longrightarrow \quad u^2 + (18 - 315k)u + 81 = 0.$$

With the exact $k = \ln 12/15$, the quadratic formula gives

$$u = \frac{315k - 18 \pm \sqrt{(315k - 18)^2 - 324}}{2} = 2.5615490777\ldots \text{ or } 31.6214905678\ldots.$$

Both values are positive and at least 1, hence both correspond to $T \geq 0$. Finally $T = \ln u/k$ gives $5.6779528361\ldots$ or $20.8488936858\ldots$ days. To one decimal place the two times are as follows.

$$\boxed{T = 5.7 \text{ days}, \qquad T = 20.8 \text{ days}}$$

FINAL ANSWER

(a) $F(0) = 35$ (b) $k = \frac{\ln 12}{15}$ (c) $T = 5.7$ days, $\qquad T = 20.8$ days

WMA13/01 MAY/JUNE 2023

Question 7

8 marks | Exponentials & Logarithms | Main topic: Log Graphs & Exponential Models

7. A scientist is studying two different populations of bacteria.

The number of bacteria N in the first population is modelled by the equation

$$N = Ae^{kt} \qquad t \geqslant 0$$

where A and k are positive constants and t is the time in hours from the start of the study.

Given that
- there were 2500 bacteria in this population at the start of the study
- there were 10 000 bacteria 8 hours later

(a) find the exact value of A and the value of k to 4 significant figures.

(3)

The number of bacteria N in the second population is modelled by the equation

$$N = 60\,000e^{-0.6t} \qquad t \geqslant 0$$

where t is the time in hours from the start of the study.

(b) Find the rate of decrease of bacteria in this population exactly 5 hours from the start of the study. Give your answer to 3 significant figures.

(2)

When $t = T$, the number of bacteria in the two different populations was the same.

(c) Find the value of T, giving your answer to 3 significant figures.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(3)

WORKED SOLUTION

Question 7 · WMA13/01 May/June 2023

1 **Part (a)**

At $t = 0$, $2500 = Ae^0$, so $A = 2500$ exactly. At $t = 8$, $10000 = 2500e^{8k}$; hence $e^{8k} = 4$ and $k = \ln 4/8 = 0.1732867951\ldots$. To four significant figures:

$$\boxed{A = 2500, \qquad k = 0.1733}$$

2 Part (b)

For the second population,

$$\frac{dN}{dt} = -36000e^{-0.6t}, \qquad \left.\frac{dN}{dt}\right|_{t=5} = -36000e^{-3} = -1792.334461\ldots.$$

The requested rate of decrease is the positive magnitude, to three significant figures.

$$\boxed{1.79 \times 10^3 \text{ bacteria per hour}}$$

3 Part (c)

Retain the exact value $k = \ln 4/8$. Equating populations gives

$$2500e^{kT} = 60000e^{-0.6T} \quad \Longrightarrow \quad e^{(k+0.6)T} = 24.$$

Take natural logs and solve:

$$T = \frac{\ln 24}{0.6 + \ln 4/8} = 4.109799689\ldots \text{ hours}.$$

$$\boxed{T = 4.11 \text{ hours}}$$

FINAL ANSWER

(a) $A = 2500, \qquad k = 0.1733$ (b) 1.79×10^3 bacteria per hour (c) $T = 4.11$ hours

WMA13/01 OCTOBER 2023

Question 6

6 marks | Exponentials & Logarithms | Main topic: Log Graphs & Exponential Models

6. An area of sea floor is being monitored.

The area of the sea floor, $S\,\text{km}^2$, covered by coral reefs is modelled by the equation

$$S = pq^t$$

where p and q are constants and t is the number of years after monitoring began.

Given that

$$\log_{10} S = 4.5 - 0.006t$$

(a) find, according to the model, the area of sea floor covered by coral reefs when $t = 2$

(2)

(b) find a complete equation for the model in the form

$$S = pq^t$$

giving the value of p and the value of q each to 3 significant figures.

(3)

(c) With reference to the model, interpret the value of the constant q

(1)

WORKED SOLUTION

Question 6 · WMA13/01 October 2023

1 **Part (a)**

Substitute $t = 2$ into the logarithmic model:

$$\log_{10} S = 4.5 - 0.006(2) = 4.488 \quad \Longrightarrow \quad S = 10^{4.488}.$$

Thus $S = 30760.968\ldots\ \text{km}^2$. To 3 significant figures, the area is $3.08 \times 10^4\ \text{km}^2$.

$$\boxed{S = 3.08 \times 10^4\ \text{km}^2}$$

2 Part (b)

Exponentiating and using index laws,

$$S = 10^{4.5-0.006t} = 10^{4.5}(10^{-0.006})^t.$$

Thus $p = 31622.7766\ldots$ and $q = 0.9862795\ldots$. To 3 significant figures, $p = 3.16 \times 10^4$ and $q = 0.986$. Hence the complete model, with both constants rounded as requested, is $S = 31600(0.986)^t$, where S is in km^2 and t is in years.

$$\boxed{S = 31600(0.986)^t}$$

3 Part (c)

The factor q is the yearly multiplier for the sea-floor area covered by coral reefs. Each year, about 98.6% of the previous year's coral-covered sea-floor area remains. This is a yearly decrease of about 1.4% (or $1.372\ldots\%$ using the unrounded value of q).

$$\boxed{\text{About } 98.6\% \text{ of the coral-covered sea-floor area remains each year.}}$$

FINAL ANSWER

(a) $S = 3.08 \times 10^4\ \text{km}^2$ (b) $S = 31600(0.986)^t$ (c) About 98.6% of the coral-covered sea-floor area remains each year.

Question 7

12 marks | Exponentials & Logarithms | Main topic: Applications of Differentiation

7. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

The curve C has equation

$$y = \frac{16}{9(3x - k)} \qquad x \neq \frac{k}{3}$$

where k is a positive constant not equal to 3

(a) Find $\frac{dy}{dx}$ giving your answer in simplest form in terms of k.

(2)

The point P with x coordinate 1 lies on C.

Given that the gradient of the curve at P is -12

(b) find the two possible values of k

(3)

Given also that $k < 3$

(c) find the equation of the normal to C at P, writing your answer in the form $ax + by + c = 0$, where a, b and c are integers to be found.

(3)

(d) show, using algebraic integration that,

$$\int_1^3 \frac{16}{9(3x - k)} dx = \lambda \ln 10$$

where λ is a constant to be found.

(4)

1 Part (a)

Write $y = (16/9)(3x-k)^{-1}$. The chain rule gives

$$\frac{dy}{dx} = \frac{16}{9}(-1)(3x-k)^{-2}\cdot 3.$$

$$\boxed{\frac{dy}{dx} = -\frac{16}{3(3x-k)^2}}$$

2 Part (b)

At $x = 1$, the given gradient yields

$$-\frac{16}{3(3-k)^2} = -12 \quad\Longrightarrow\quad (3-k)^2 = \frac{4}{9}.$$

Hence $3-k = \pm 2/3$. Both resulting k-values are positive and different from 3.

$$\boxed{k = \frac{7}{3} \text{ or } \frac{11}{3}}$$

3 Part (c)

Since $k < 3$, use $k = 7/3$. At $x = 1$, $y = 16/[9(3-7/3)] = 8/3$. The tangent gradient is -12, so the normal gradient is $1/12$:

$$y - \frac{8}{3} = \frac{1}{12}(x-1), \qquad 12y - 32 = x - 1.$$

$$\boxed{x - 12y + 31 = 0}$$

4 Part (d)

With $k = 7/3$, the denominator does not vanish on $[1,3]$. Using the reciprocal-linear antiderivative,

$$\int_1^3 \frac{16}{9(3x-7/3)}\,dx = \left[\frac{16}{27}\ln|3x-7/3|\right]_1^3 = \frac{16}{27}\left(\ln\frac{20}{3} - \ln\frac{2}{3}\right) = \frac{16}{27}\ln\left(\frac{20/3}{2/3}\right).$$

$$\boxed{\frac{16}{27}\ln 10, \quad \lambda = \frac{16}{27}}$$

FINAL ANSWER

(a) $\frac{dy}{dx} = -\frac{16}{3(3x-k)^2}$ (b) $k = \frac{7}{3}$ or $\frac{11}{3}$ (c) $x - 12y + 31 = 0$ (d) $\frac{16}{27}\ln 10, \quad \lambda = \frac{16}{27}$

Question 8

10 marks | Exponentials & Logarithms

Main topic: Log Graphs & Exponential Models

8.

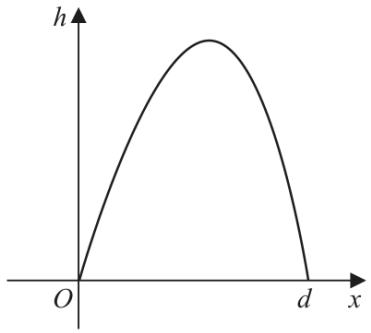


Figure 4

Figure 4 is a graph showing the path of a golf ball after the ball has been hit until it first hits the ground.

The vertical height, h metres, of the ball above the ground has been plotted against the horizontal distance travelled, x metres, measured from where the ball was hit.

The ball travels a horizontal distance of d metres before it first hits the ground.

The ball is modelled as a particle travelling in a vertical plane above horizontal ground.

The path of the ball is modelled by the equation

$$h = 1.5x - 0.5x\,e^{0.02x} \qquad 0 \leqslant x \leqslant d$$

Use the model to answer parts (a), (b) and (c).

(a) Find the value of d, giving your answer to 2 decimal places.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(3)

(b) Show that the maximum value of h occurs when

$$x = 50\ln\left(\frac{150}{x + 50}\right)$$

(4)

Using the iteration formula

$$x_{n+1} = 50\ln\left(\frac{150}{x_n + 50}\right) \qquad \text{with } x_1 = 30$$

(c) (i) find the value of x_2 to 2 decimal places,

(ii) find, by repeated iteration, the horizontal distance travelled by the golf ball before it reaches its maximum height. Give your answer to 2 decimal places.

(3)

1 Part (a)

At the first landing, $h = 0$ and $x = d > 0$:

$$0 = d(1.5 - 0.5e^{0.02d}), \qquad e^{0.02d} = 3.$$

Taking logarithms, $0.02d = \ln 3$, hence $d = 50\ln 3 = 54.930614\ldots$. The root $x = 0$ is the launch point, not the landing point.

$$\boxed{d = 54.93 \text{ m}}$$

2 Part (b)

By the product and chain rules,

$$h'(x) = 1.5 - 0.5e^{0.02x} - 0.01xe^{0.02x}.$$

At a stationary point,

$$1.5 = 0.01(x + 50)e^{0.02x}, \qquad e^{0.02x} = \frac{150}{x + 50}.$$

Since $x + 50 > 0$, logarithms give $0.02x = \ln(150/(x + 50))$. Also $h''(x) = -(0.02 + 0.0002x)e^{0.02x} < 0$ on $[0, d]$. Thus the stationary point is the unique maximum; the endpoints both have zero height.

$$\boxed{x = 50\ln\left(\frac{150}{x + 50}\right)}$$

3 Part (c)

(i) With $x_1 = 30$,

$$x_2 = 50\ln\left(\frac{150}{30 + 50}\right) = 31.43043297\ldots.$$

(ii) Continuing with full intermediate precision gives $x_3 = 30.54431108\ldots$, $x_4 = 31.09139065\ldots$, $x_5 = 30.75292479\ldots$ and $x_6 = 30.96205562\ldots$. Continued iteration converges to $30.88212\ldots$, which lies in $(0, d)$. This is the horizontal distance at maximum height, not the maximum height itself.

$$\boxed{x_2 = 31.43, \quad x_{\text{maximum}} = 30.88 \text{ m}}$$

FINAL ANSWER

(a) $d = 54.93$ m (b) $x = 50\ln\left(\frac{150}{x+50}\right)$ (c) $x_2 = 31.43, \quad x_{\text{maximum}} = 30.88$ m

Question 8

12 marks | Exponentials & Logarithms

Main topic: Log Graphs & Exponential Models

8.

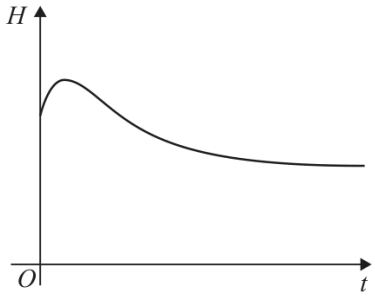


Figure 4

The heart rate of a horse is being monitored.
The heart rate H, measured in beats per minute (bpm), is modelled by the equation

$$H = 32 + 40e^{-0.2t} - 20e^{-0.9t}$$

where t minutes is the time after monitoring began.

Figure 4 is a sketch of H against t.

Use the equation of the model to answer parts (a) to (e).

(a) State the initial heart rate of the horse.

(1)

In the long term, the heart rate of the horse approaches L bpm.

(b) State the value of L.

(1)

The heart rate of the horse reaches its maximum value after T minutes.

(c) Find the value of T, giving your answer to 3 decimal places.

(*Solutions based entirely on calculator technology are not acceptable.*)

(5)

The heart rate of the horse is 37 bpm after M minutes.

(d) Show that M is a solution of the equation

$$t = 5\ln\left(\frac{8}{1 + 4e^{-0.9t}}\right)$$

(2)

Using the iteration formula

$$t_{n+1} = 5\ln\left(\frac{8}{1 + 4e^{-0.9t_n}}\right) \quad \text{with} \quad t_1 = 10$$

(e) (i) find, to 4 decimal places, the value of t_2

(ii) find, to 4 decimal places, the value of M

(3)

1 Part (a)

At $t = 0$, $H = 32 + 40 - 20 = 52$.

$$\boxed{H(0) = 52 \text{ bpm}}$$

2 Part (b)

As $t \to \infty$, both decaying exponential terms tend to zero.

$$\boxed{L = 32}$$

3 Part (c)

$$\frac{dH}{dt} = -8e^{-0.2t} + 18e^{-0.9t}.$$

At the maximum,

$$18e^{-0.9T} = 8e^{-0.2T} \quad \Longrightarrow \quad e^{0.7T} = \frac{9}{4} \quad \Longrightarrow \quad T = \frac{10}{7}\ln\frac{9}{4} = 1.1584717\ldots.$$

Also $H''(t) = 1.6e^{-0.2t} - 16.2e^{-0.9t}$. At the stationary point, $e^{-0.9T} = (4/9)e^{-0.2T}$, so $H''(T) = -5.6e^{-0.2T} < 0$, confirming a maximum.

$$\boxed{T = 1.158 \text{ minutes}}$$

4 Part (d)

For $H = 37$,

$$5 = 40e^{-0.2t} - 20e^{-0.9t} \quad \Longrightarrow \quad 8e^{-0.2t} = 1 + 4e^{-0.9t}.$$

Taking logarithms gives

$$-0.2t = \ln\left(\frac{1 + 4e^{-0.9t}}{8}\right) \quad \Longrightarrow \quad t = -5\ln\left(\frac{1 + 4e^{-0.9t}}{8}\right).$$

Using $-\ln u = \ln(1/u)$ for $u > 0$ then gives

$$t = 5\ln\left(\frac{8}{1 + 4e^{-0.9t}}\right).$$

$$\boxed{t = 5\ln\left(\frac{8}{1 + 4e^{-0.9t}}\right)}$$

5

Part (e)

Starting with $t_1 = 10$,

$$t_2 = 5\ln\left(\frac{8}{1+4e^{-9}}\right) = 10.3947401213\ldots.$$

Continuing without rounding intermediate values gives $t_3 = 10.3954778346\ldots$, $t_4 = 10.3954789826\ldots$ and $t_5 = 10.3954789844\ldots$. Thus (i) $t_2 = 10.3947$, and (ii) $M = 10.3955$ minutes, both to 4 decimal places.

$$\boxed{t_2 = 10.3947, \quad M = 10.3955 \text{ minutes}}$$

✓ **FINAL ANSWER**

(a) $H(0) = 52$ bpm (b) $L = 32$ (c) $T = 1.158$ minutes (d) $t = 5\ln\left(\frac{8}{1+4e^{-0.9t}}\right)$ (e) $t_2 = 10.3947, \quad M = 10.3955$ minutes

Question 9

8 marks | Exponentials & Logarithms

Main topic: Applications of Differentiation

9.

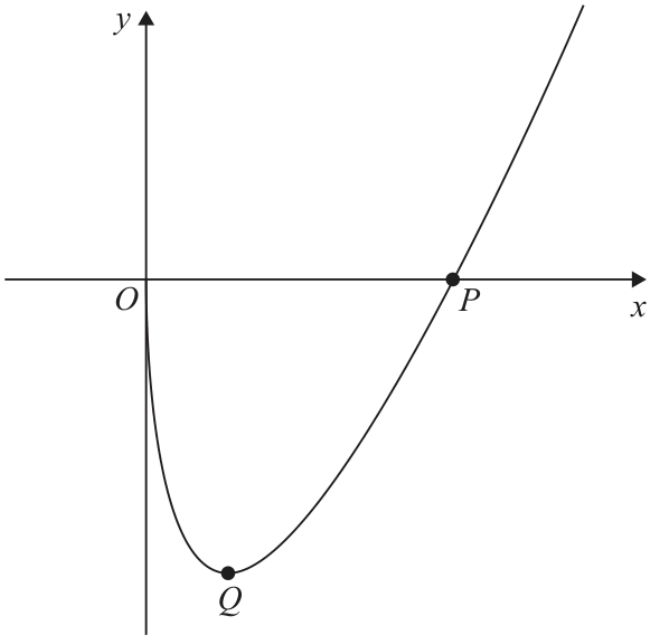


Figure 1

Figure 1 shows a sketch of part of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 6\sqrt{x}\ln(4x) \qquad x > 0$$

The curve cuts the x-axis at point P

(a) State the x coordinate of P

(1)

The point Q, shown in Figure 1, is the stationary point on C

(b) Use calculus to find the exact coordinates of Q

(5)

(c) Hence find the range of the function $\mathrm{g}(x)$ where

$$\mathrm{g}(x) = -2\,\mathrm{f}(x)$$

(2)

1 Part (a)

Since $x > 0$, $6\sqrt{x}$ is positive. Therefore $f(x) = 0$ requires $\ln(4x) = 0$, so $4x = 1$ and $x = 1/4$.

$$\boxed{x_P = \frac{1}{4}}$$

2 Part (b)

Apply the product and chain rules:

$$f'(x) = \frac{3}{\sqrt{x}}\ln(4x) + 6\sqrt{x}\frac{1}{x} = \frac{3}{\sqrt{x}}[\ln(4x) + 2].$$

At Q, set $f'(x) = 0$. Since $3/\sqrt{x} > 0$,

$$\ln(4x) + 2 = 0 \quad \Longrightarrow \quad 4x = e^{-2} \quad \Longrightarrow \quad x = \frac{1}{4e^2}.$$

Then $\sqrt{x} = 1/(2e)$ and $\ln(4x) = -2$, giving $f(x) = 6(1/(2e))(-2) = -6/e$.

$$\boxed{Q = \left(\frac{1}{4e^2}, -\frac{6}{e}\right)}$$

3 Part (c)

The factor $3/\sqrt{x}$ is positive and $\ln(4x) + 2$ increases through zero at Q, so f' changes from negative to positive: Q is the global minimum. As $x \to \infty$, $f(x) \to \infty$, hence the range of f is $[-6/e, \infty)$. Multiplication by -2 reverses the inequality and includes the maximum $12/e$.

$$\boxed{g(x) \leq \frac{12}{e}}$$

FINAL ANSWER

(a) $x_P = \frac{1}{4}$ (b) $Q = \left(\frac{1}{4e^2}, -\frac{6}{e}\right)$ (c) $g(x) \leq \frac{12}{e}$

WMA13/01 MAY/JUNE 2025

Question 6

8 marks | Exponentials & Logarithms | Main topic: Log Graphs & Exponential Models

6: **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The temperature, θ°C, of a computer processor, t minutes after the computer is switched off, is modelled by the equation

$$\theta = 21 + A\mathrm{e}^{-kt}$$

where A and k are positive constants.

Given that the temperature of the processor was 75°C when the computer was switched off,

(a) find the value of A.

(2)

Given also that it takes 5 minutes for the temperature of the processor to decrease from 75°C to 25°C,

(b) find the value of k, giving your answer to 3 significant figures.

(3)

At time T minutes, the temperature of the processor is decreasing at a rate of 9°C per minute.

(c) Find the value of T according to the model, giving your answer to 2 decimal places.

(3)

WORKED SOLUTION

Question 6 · WMA13/01 May/June 2025

Part (a)

At $t = 0$, $75 = 21 + A$, so $\boxed{A = 54}$.

Part (b)

At $t = 5$, $25 = 21 + 54e^{-5k}$, hence

$$e^{-5k} = \frac{2}{27}, \qquad k = \frac{1}{5}\ln\frac{27}{2} = 0.520537937\ldots = \boxed{0.521}.$$

3 Part (c)

Differentiate the temperature model:

$$\frac{d\theta}{dt} = -54ke^{-kt}.$$

A decrease of $9\,^\circ\mathrm{C}$ per minute means $d\theta/dt = -9$, not $+9$. Thus

$$-54ke^{-kT} = -9 \implies e^{-kT} = \frac{1}{6k} \implies T = \frac{\ln(6k)}{k}.$$

Using the unrounded value of k,

$$T = 2.187865\ldots = \boxed{2.19 \text{ minutes}}.$$

FINAL ANSWER

(a) $A = 54$ (b) 0.521 (c) 2.19 minutes

WMA13/01 OCTOBER 2025

Question 6

7 marks | Exponentials & Logarithms | Main topic: Integration

6. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(i) Given $y = \ln(2x^2 + 5)$ find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$

(2)

(ii) A curve C has equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{21x}{3x^2 + k} \qquad x \in \mathbb{R}$$

where k is a positive integer, $k > 1$

Given that

$$\int_1^k \mathrm{f}(x)\,\mathrm{d}x < 7\ln 8$$

find the greatest possible value of k

(5)

WORKED SOLUTION

Question 6 · WMA13/01 October 2025

1 **Part (i)**

Using the chain rule,

$$\frac{dy}{dx} = \frac{1}{2x^2 + 5}(4x) = \frac{4x}{2x^2 + 5}.$$

2 Part (ii)

Treat k as a constant when integrating:

$$\int_1^k \frac{21x}{3x^2+k}\,dx = \left[\frac{7}{2}\ln(3x^2+k)\right]_1^k = \frac{7}{2}\ln\left(\frac{3k^2+k}{k+3}\right).$$

The stated inequality therefore implies

$$\ln\left(\frac{3k^2+k}{k+3}\right) < 2\ln 8 = \ln 64.$$

Since the exponential is increasing and $k+3>0$,

$$3k^2+k < 64(k+3) \Longrightarrow k^2-21k-64<0.$$

The roots are $(21\pm\sqrt{697})/2$, so the positive upper bound is $k<(21+\sqrt{697})/2 = 23.700378782\ldots$. With $k>1$ an integer, the greatest possible value is $k=23$.

FINAL ANSWER

(i) $4*x/(2*x^2+5)$ (ii) 23

WMA13/01 JANUARY 2026

Question 8

4 marks | Exponentials & Logarithms | Main topic: Differentiation Techniques

8. A curve has equation

$$x = \mathrm{e}^{2\tan y} \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a}{x\left(b + (\ln x)^n\right)}$$

where a, b and n are integers to be found.

(4)

WORKED SOLUTION

Question 8 · WMA13/01 January 2026

1 **Solution**

\paragraph{Solution} Differentiate $x = e^{2\tan y}$ with respect to y:

$$\frac{dx}{dy} = 2\sec^2 y\, e^{2\tan y} = 2x\sec^2 y.$$

This is positive on the stated interval, so

$$\frac{dy}{dx} = \frac{1}{2x\sec^2 y}.$$

Taking logarithms of the original relation gives $\ln x = 2\tan y$. Using $\sec^2 y = 1 + \tan^2 y$,

$$\frac{dy}{dx} = \frac{1}{2x\left(1 + \frac{(\ln x)^2}{4}\right)} = \frac{2}{x\left(4 + (\ln x)^2\right)}.$$

Hence $a = 2$, $b = 4$ and $n = 2$.

FINAL ANSWER

(main) $a = 2,\ b = 4,\ n = 2$

WMA13/01 JANUARY 2026

Question 10

9 marks | Exponentials & Logarithms | Main topic: Log Graphs & Exponential Models

10. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

Two populations of insects are being studied over the same period of time.

The number of insects N_A in population A, t years after the start of the study, is modelled by the equation

$$N_A = 8000 + 900e^{0.2t}$$

According to the model,

(a) find the rate of growth in the number of insects in population A exactly 5 years after the start of the study.

(2)

The number of insects N_B in population B, t years after the start of the study, is modelled by the equation

$$N_B = 8000 + Pe^{kt}$$

where P and k are positive constants.

Given that

- there are 10 000 insects in population B at the start of the study
- there are 11 570 insects in population B exactly 4 years after the start of the study
- there are the same number of insects in population A and population B after T years

find, according to the models,

(b) the exact value of P and the value of k to 3 decimal places,

(4)

(c) the value of T, giving your answer to one decimal place.

(3)

WORKED SOLUTION

Question 10 · WMA13/01 January 2026

1 **Part (a)**

$$\frac{dN_A}{dt} = 900(0.2)e^{0.2t} = 180e^{0.2t}.$$

At $t = 5$, the growth rate is $180e = 489.290729\ldots$ insects per year.

2 Part (b)

At $t = 0$, $10000 = 8000 + P$, so $P = 2000$ exactly. Using the population at $t = 4$,

$$11570 = 8000 + 2000e^{4k} \quad \Longrightarrow \quad e^{4k} = \frac{3570}{2000} = 1.785.$$

Taking natural logarithms,

$$k = \frac{\ln 1.785}{4} = 0.144854604\ldots = 0.145 \quad (3\text{ d.p.}).$$

3 Part (c)

At equality, $8000 + 900e^{0.2T} = 8000 + 2000e^{kT}$, hence

$$e^{(0.2-k)T} = \frac{20}{9}, \qquad T = \frac{\ln(20/9)}{0.2 - k}.$$

Use the unrounded value $k = \ln(1.785)/4$:

$$T = 14.480042784\ldots = 14.5 \text{ years (1 decimal place)}.$$

FINAL ANSWER

(a) $180e$ insects per year (b) $P = 2000,\ k = 0.145\ (3\text{ d.p.})$ (c) 14.5 years

Question 7

12 marks | Exponentials & Logarithms | Main topic: Log Graphs & Exponential Models

7. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The selling price of a car is monitored over a 15-year period.

The selling price P_C of the car, in thousands of pounds, t years after monitoring began, is modelled by the equation

$$P_C = 7 + 16e^{-0.1t} \qquad 0 \leqslant t \leqslant 15$$

Given that when $t = T$ the selling price of the car is £15 000

(a) find, according to the model, the value of T. Give the answer to 2 decimal places.

(3)

(b) Find the value of $\frac{dP_C}{dt}$ when $t = 5$. Give the answer to 2 decimal places.

(2)

The selling price of a van is monitored over the same 15-year period.

The selling price P_V of the van, in thousands of pounds, t years after monitoring began, is modelled by the equation

$$P_V = 4 + me^{-0.2t} \qquad 0 \leqslant t \leqslant 15$$

where m is a constant.

Use the equations of these models to answer parts (c) and (d).

Given that $P_V = 36$ when $t = 0$

(c) (i) state the value of m

(ii) find the difference between the initial selling price of the car and the initial selling price of the van.

(3)

When $t = \alpha$, the selling price of the car is equal to the selling price of the van.

(d) Find, using algebra and showing your working, the value of α, giving the answer to 3 significant figures.

(4)

1 Part (a)

Prices in the model are in thousands of pounds, so

$$15 = 7 + 16e^{-0.1T} \quad \Longrightarrow \quad e^{-0.1T} = \frac{1}{2}.$$

Taking logarithms gives $T = 10\ln 2 = 6.9314718\ldots$, hence $\boxed{T = 6.93 \text{ years}}$.

2 Part (b)

$$\frac{dP_C}{dt} = -1.6e^{-0.1t}, \qquad \left.\frac{dP_C}{dt}\right|_{t=5} = -1.6e^{-0.5} = -0.970449\ldots.$$

Thus the answer is $\boxed{-0.97 \text{ thousand pounds per year}}$ to 2 decimal places.

3 Part (c)(i)

At $t = 0$, $36 = 4 + m$, so $\boxed{m = 32}$.

4 Part (c)(ii)

The initial car price is $7 + 16 = 23$ thousand pounds. The difference is $36 - 23 = 13$ thousand pounds, or $\boxed{\pounds 13\,000}$.

5 Part (d)

Equating the models with $m = 32$ gives

$$7 + 16e^{-0.1\alpha} = 4 + 32e^{-0.2\alpha}.$$

Let $u = e^{-0.1\alpha} > 0$. Then $32u^2 - 16u - 3 = 0$, so

$$u = \frac{16 \pm \sqrt{256 + 384}}{64} = \frac{2 \pm \sqrt{10}}{8}.$$

Reject the negative root because an exponential is positive. Hence

$$\alpha = -10\ln\left(\frac{2 + \sqrt{10}}{8}\right) = 4.3806365\ldots, \qquad \boxed{\alpha = 4.38 \text{ years}}.$$

This value is inside the model interval $0 \le t \le 15$.

FINAL ANSWER

(a) $T = 6.93$ years (b) -0.97 thousand pounds per year (c)(i) $m = 32$ (c)(ii) $\pounds 13\,000$ (d) $\alpha = 4.38$ years

TOPIC

Log Graphs & Exponential Models

Every past-paper question on this topic, January 2020 to June 2026.

WMA13/01 JANUARY 2021

Question 8

7 marks | Log Graphs & Exponential Models

8. The percentage, P, of the population of a small country who have access to the internet, is modelled by the equation

$$P = ab^t$$

where a and b are constants and t is the number of years after the start of 2005

Using the data for the years between the start of 2005 and the start of 2010, a graph is plotted of $\log_{10} P$ against t.

The points are found to lie approximately on a straight line with gradient 0.09 and intercept 0.68 on the $\log_{10} P$ axis.

(a) Find, according to the model, the value of a and the value of b, giving your answers to 2 decimal places.

(4)

(b) In the context of the model, give a practical interpretation of the constant a.

(1)

(c) Use the model to estimate the percentage of the population who had access to the internet at the start of 2015

(2)

WORKED SOLUTION

Question 8 · WMA13/01 January 2021

1 Part (a)

Taking logarithms gives $\log_{10} P = \log_{10} a + t \log_{10} b$. Match the intercept and gradient: $\log_{10} a = 0.68$, $\log_{10} b = 0.09$. Thus $a = 10^{0.68} = 4.786300923\ldots$ and $b = 10^{0.09} = 1.230268771\ldots$.

$$\boxed{a = 4.79, \qquad b = 1.23}$$

2 Part (b)

At $t = 0$, $P = a$. Thus a is the percentage of the country's population with internet access at the start of 2005, approximately 4.79%.

$$\boxed{\text{Initial percentage with internet access at the start of 2005}}$$

3 Part (c)

The start of 2015 is $t = 10$ years after the start of 2005. Using the model parameters, $P = 4.79(1.23)^{10} = 37.9652818\ldots\%$, giving 38.0% to three significant figures. Using unrounded parameters gives $10^{0.68+0.09(10)} = 38.0189396\ldots\%$, the same rounded answer.

$$\boxed{38.0\%}$$

FINAL ANSWER

(a) $a = 4.79, \quad b = 1.23$ (b) $\textbf{Initial percentage with internet access at the start of 2005}$ (c) 38.0%

WMA13/01 MAY/JUNE 2021

Question 8

13 marks | Log Graphs & Exponential Models

8. A scientist is studying a population of fish in a lake. The number of fish, N, in the population, t years after the start of the study, is modelled by the equation

$$N = \frac{600e^{0.3t}}{2 + e^{0.3t}} \qquad t \geqslant 0$$

Use the equation of the model to answer parts (a), (b), (c), (d) and (e).

(a) Find the number of fish in the lake at the start of the study.

(1)

(b) Find the upper limit to the number of fish in the lake.

(1)

(c) Find the time, after the start of the study, when there are predicted to be 500 fish in the lake. Give your answer in years and months to the nearest month.

(4)

(d) Show that

$$\frac{dN}{dt} = \frac{Ae^{0.3t}}{(2 + e^{0.3t})^2}$$

where A is a constant to be found.

(3)

Given that when $t = T$, $\frac{dN}{dt} = 8$

(e) find the value of T to one decimal place.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(4)

WORKED SOLUTION

Question 8 · WMA13/01 May/June 2021

1 **Part (a)**

At the start, $t = 0$ and $e^{0.3t} = 1$. Thus $N(0) = 600/(2 + 1) = 200$.

$$\boxed{200 \text{ fish}}$$

2 Part (b)

Divide numerator and denominator by $e^{0.3t}$ to write $N = 600/(1 + 2e^{-0.3t})$. As $t \to \infty$, the denominator tends to 1, so the upper limit is 600 fish.

$$\boxed{600 \text{ fish}}$$

3 Part (c)

Set $N = 500$ and multiply through: $1000 + 500e^{0.3t} = 600e^{0.3t}$, so $e^{0.3t} = 10$. Hence $t = \ln 10/0.3 = 7.675283643\ldots$ years. The fractional year is $12(0.675283643\ldots) = 8.10340372\ldots$ months. Round only the months to the nearest whole month.

$$\boxed{7 \text{ years } 8 \text{ months}}$$

4 Part (d)

The quotient rule gives

$$\frac{dN}{dt} = \frac{180e^{0.3t}(2 + e^{0.3t}) - 600e^{0.3t}(0.3e^{0.3t})}{(2 + e^{0.3t})^2} = \frac{360e^{0.3t}}{(2 + e^{0.3t})^2}.$$

The two $e^{0.6t}$ terms cancel exactly.

$$\boxed{A = 360}$$

5 Part (e)

Let $z = e^{0.3T}$. Since $T \geq 0$, require $z \geq 1$. Then $360z/(2 + z)^2 = 8$ gives $45z = (2 + z)^2$, hence

$$z^2 - 41z + 4 = 0, \qquad z = \frac{41 \pm \sqrt{1665}}{2}.$$

The smaller root lies between 0 and 1 and would give $T < 0$, so reject it. For the larger root,

$$T = \frac{1}{0.3}\ln\left(\frac{41 + \sqrt{1665}}{2}\right) = 12.37061331\ldots.$$

$$\boxed{T = 12.4 \text{ years}}$$

FINAL ANSWER

(a) 200 fish (b) 600 fish (c) 7 years 8 months (d) $A = 360$ (e) $T = 12.4$ years

WMA13/01 OCTOBER 2021

Question 7

6 marks | Log Graphs & Exponential Models

7 The mass, M kg, of a species of tree can be modelled by the equation

$$\log_{10} M = 1.93 \log_{10} r + 0.684$$

where r cm is the base radius of the tree.

The base radius of a particular tree of this species is 45 cm.

According to the model,

(a) find the mass of this tree, giving your answer to 2 significant figures.

(2)

(b) Show that the equation of the model can be written in the form

$$M = pr^q$$

giving the values of the constants p and q to 3 significant figures.

(3)

(c) With reference to the model, interpret the value of the constant p.

(1)

WORKED SOLUTION

Question 7 · WMA13/01 October 2021

1 **Part (a)**

At $r = 45$, $\log_{10} M = 1.93 \log_{10} 45 + 0.684$. Exponentiate to obtain $M = 10^{0.684} 45^{1.93} = 7493.766412\ldots$ kg.

$$\boxed{M = 7.5 \times 10^3 \text{ kg}}$$

2 **Part (b)**

Exponentiate the full equation:

$$M = 10^{1.93 \log_{10} r + 0.684} = 10^{0.684} r^{1.93}.$$

Therefore $p = 10^{0.684} = 4.83058802\ldots$ and $q = 1.93$.

$$\boxed{p = 4.83, \qquad q = 1.93}$$

3

Part (c)

If the base radius is $r = 1$ cm, the model gives $M = p \cdot 1^q = p$ kg. Thus p is the predicted mass in kilograms of a tree of base radius 1 cm.

$$\boxed{\text{Predicted mass (kg) when base radius is 1 cm}}$$

✓ **FINAL ANSWER**

(a) $M = 7.5 \times 10^3$ kg (b) $p = 4.83, \qquad q = 1.93$ (c)

$\text{Predicted mass (kg) when base radius is 1 cm}$

WMA13/01 JANUARY 2022

Question 8

8 marks | Log Graphs & Exponential Models

8. A dose of antibiotics is given to a patient.

The amount of the antibiotic, x milligrams, in the patient's bloodstream t hours after the dose was given, is found to satisfy the equation

$$\log_{10} x = 2.74 - 0.079t$$

(a) Show that this equation can be written in the form

$$x = pq^{-t}$$

where p and q are constants to be found. Give the value of p to the nearest whole number and the value of q to 2 significant figures.

(4)

(b) With reference to the equation in part (a), interpret the value of the constant p.

(1)

When a different dose of the antibiotic is given to another patient, the values of x and t satisfy the equation

$$x = 400 \times 1.4^{-t}$$

(c) Use calculus to find, to 2 significant figures, the value of $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ when $t = 5$

(3)

WORKED SOLUTION

Question 8 · WMA13/01 January 2022

1 **Part (a)**

Exponentiate base 10:

$$x = 10^{2.74-0.079t} = 10^{2.74}(10^{0.079})^{-t}.$$

Thus $p = 10^{2.74} = 549.5408739\ldots$ and $q = 10^{0.079} = 1.199499303\ldots$. Round p to the nearest integer and q to two significant figures.

$$\boxed{p = 550, \qquad q = 1.2}$$

2 Part (b)

At $t = 0$, $x = pq^0 = p$. Thus p is the initial amount of antibiotic in the patient's bloodstream, approximately 550 milligrams immediately after the dose in this model.

$$\boxed{p \text{ represents the initial amount, approximately } 550 \text{ mg}}$$

3 Part (c)

For the second patient, write $x = 400e^{-t\ln 1.4}$. Differentiate to obtain $dx/dt = -400\ln(1.4)\,1.4^{-t}$. At $t = 5$, the derivative is $-25.02470969\ldots \text{ mg h}^{-1}$. The negative sign indicates the amount is decreasing.

$$\boxed{\left.\frac{dx}{dt}\right|_{t=5} = -25 \text{ mg h}^{-1}}$$

FINAL ANSWER

(a) $p = 550, \quad q = 1.2$ (b) p represents the initial amount, approximately 550 mg (c) $\left.\frac{dx}{dt}\right|_{t=5} = -25 \text{ mg h}^{-1}$

WMA13/01 JANUARY 2023

Question 10

9 marks | Log Graphs & Exponential Models

10. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

A population of fruit flies is being studied.

The number of fruit flies, F, in the population, t days after the start of the study, is modelled by the equation

$$F = \frac{350e^{kt}}{9 + e^{kt}}$$

where k is a constant.

Use the equation of the model to answer parts (a), (b) and (c).

(a) Find the number of fruit flies in the population at the start of the study.

(1)

Given that there are 200 fruit flies in the population 15 days after the start of the study,

(b) show that $k = \frac{1}{15}\ln 12$

(3)

Given also that, when $t = T$, the number of fruit flies in the population is increasing at a rate of 10 per day,

(c) find the possible values of T, giving your answers to one decimal place.

(5)

WORKED SOLUTION

Question 10 · WMA13/01 January 2023

1 **Part (a)**

At the start $t = 0$, so $e^{kt} = 1$ and the model gives $F(0) = 350/(9 + 1) = 35$ fruit flies.

$$\boxed{F(0) = 35}$$

2 Part (b)

Using $F(15) = 200$,

$$200 = \frac{350e^{15k}}{9 + e^{15k}} \quad \Longrightarrow \quad 1800 + 200e^{15k} = 350e^{15k} \quad \Longrightarrow \quad e^{15k} = 12.$$

Take natural logs to obtain $15k = \ln 12$, as required.

$$\boxed{k = \frac{\ln 12}{15}}$$

---

3 Part (c)

By the quotient rule,

$$F'(t) = \frac{350ke^{kt}(9 + e^{kt}) - 350ke^{2kt}}{(9 + e^{kt})^2} = \frac{3150ke^{kt}}{(9 + e^{kt})^2}.$$

Set $F'(T) = 10$ and let $u = e^{kT} > 0$. Then

$$315ku = (9 + u)^2 \quad \Longrightarrow \quad u^2 + (18 - 315k)u + 81 = 0.$$

With the exact $k = \ln 12/15$, the quadratic formula gives

$$u = \frac{315k - 18 \pm \sqrt{(315k - 18)^2 - 324}}{2} = 2.5615490777\ldots \text{ or } 31.6214905678\ldots.$$

Both values are positive and at least 1, hence both correspond to $T \geq 0$. Finally $T = \ln u/k$ gives $5.6779528361\ldots$ or $20.8488936858\ldots$ days. To one decimal place the two times are as follows.

$$\boxed{T = 5.7 \text{ days}, \qquad T = 20.8 \text{ days}}$$

FINAL ANSWER

(a) $F(0) = 35$ (b) $k = \frac{\ln 12}{15}$ (c) $T = 5.7$ days, $\qquad T = 20.8$ days

WMA13/01 MAY/JUNE 2023

Question 7

8 marks | Log Graphs & Exponential Models

7. A scientist is studying two different populations of bacteria.

The number of bacteria N in the first population is modelled by the equation

$$N = Ae^{kt} \qquad t \geqslant 0$$

where A and k are positive constants and t is the time in hours from the start of the study.

Given that

- there were 2500 bacteria in this population at the start of the study
- there were 10 000 bacteria 8 hours later

(a) find the exact value of A and the value of k to 4 significant figures.

(3)

The number of bacteria N in the second population is modelled by the equation

$$N = 60\,000e^{-0.6t} \qquad t \geqslant 0$$

where t is the time in hours from the start of the study.

(b) Find the rate of decrease of bacteria in this population exactly 5 hours from the start of the study. Give your answer to 3 significant figures.

(2)

When $t = T$, the number of bacteria in the two different populations was the same.

(c) Find the value of T, giving your answer to 3 significant figures.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(3)

WORKED SOLUTION

Question 7 · WMA13/01 May/June 2023

1 **Part (a)**

At $t = 0$, $2500 = Ae^0$, so $A = 2500$ exactly. At $t = 8$, $10000 = 2500e^{8k}$; hence $e^{8k} = 4$ and $k = \ln 4/8 = 0.1732867951\ldots$. To four significant figures:

$$\boxed{A = 2500, \qquad k = 0.1733}$$

2 Part (b)

For the second population,

$$\frac{dN}{dt} = -36000e^{-0.6t}, \qquad \left.\frac{dN}{dt}\right|_{t=5} = -36000e^{-3} = -1792.334461\ldots.$$

The requested rate of decrease is the positive magnitude, to three significant figures.

$$\boxed{1.79 \times 10^3 \text{ bacteria per hour}}$$

3 Part (c)

Retain the exact value $k = \ln 4/8$. Equating populations gives

$$2500e^{kT} = 60000e^{-0.6T} \quad \Longrightarrow \quad e^{(k+0.6)T} = 24.$$

Take natural logs and solve:

$$T = \frac{\ln 24}{0.6 + \ln 4/8} = 4.109799689\ldots \text{ hours}.$$

$$\boxed{T = 4.11 \text{ hours}}$$

FINAL ANSWER

(a) $A = 2500, \qquad k = 0.1733$ (b) 1.79×10^3 bacteria per hour (c) $T = 4.11$ hours

WMA13/01 OCTOBER 2023

Question 6

6 marks | Log Graphs & Exponential Models

6. An area of sea floor is being monitored.

The area of the sea floor, $S\,\text{km}^2$, covered by coral reefs is modelled by the equation

$$S = pq^t$$

where p and q are constants and t is the number of years after monitoring began.

Given that

$$\log_{10} S = 4.5 - 0.006t$$

(a) find, according to the model, the area of sea floor covered by coral reefs when $t = 2$

(2)

(b) find a complete equation for the model in the form

$$S = pq^t$$

giving the value of p and the value of q each to 3 significant figures.

(3)

(c) With reference to the model, interpret the value of the constant q

(1)

WORKED SOLUTION

Question 6 · WMA13/01 October 2023

1 **Part (a)**

Substitute $t = 2$ into the logarithmic model:

$$\log_{10} S = 4.5 - 0.006(2) = 4.488 \quad \Longrightarrow \quad S = 10^{4.488}.$$

Thus $S = 30760.968\ldots\ \text{km}^2$. To 3 significant figures, the area is $3.08 \times 10^4\ \text{km}^2$.

$$\boxed{S = 3.08 \times 10^4\ \text{km}^2}$$

2 Part (b)

Exponentiating and using index laws,

$$S = 10^{4.5-0.006t} = 10^{4.5}(10^{-0.006})^t.$$

Thus $p = 31622.7766\ldots$ and $q = 0.9862795\ldots$. To 3 significant figures, $p = 3.16 \times 10^4$ and $q = 0.986$. Hence the complete model, with both constants rounded as requested, is $S = 31600(0.986)^t$, where S is in km^2 and t is in years.

$$\boxed{S = 31600(0.986)^t}$$

3 Part (c)

The factor q is the yearly multiplier for the sea-floor area covered by coral reefs. Each year, about 98.6% of the previous year's coral-covered sea-floor area remains. This is a yearly decrease of about 1.4% (or $1.372\ldots\%$ using the unrounded value of q).

$$\boxed{\text{About } 98.6\% \text{ of the coral-covered sea-floor area remains each year.}}$$

FINAL ANSWER

(a) $S = 3.08 \times 10^4\ \text{km}^2$ (b) $S = 31600(0.986)^t$ (c) About 98.6% of the coral-covered sea-floor area remains each year.

Question 8

10 marks | Log Graphs & Exponential Models

8.

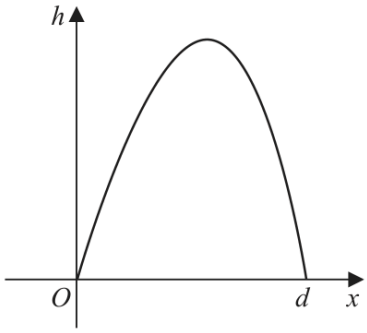


Figure 4

Figure 4 is a graph showing the path of a golf ball after the ball has been hit until it first hits the ground.

The vertical height, h metres, of the ball above the ground has been plotted against the horizontal distance travelled, x metres, measured from where the ball was hit.

The ball travels a horizontal distance of d metres before it first hits the ground.

The ball is modelled as a particle travelling in a vertical plane above horizontal ground.

The path of the ball is modelled by the equation

$$h = 1.5x - 0.5x\,e^{0.02x} \qquad 0 \leqslant x \leqslant d$$

Use the model to answer parts (a), (b) and (c).

(a) Find the value of d, giving your answer to 2 decimal places.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(3)

(b) Show that the maximum value of h occurs when

$$x = 50\ln\left(\frac{150}{x + 50}\right)$$

(4)

Using the iteration formula

$$x_{n+1} = 50\ln\left(\frac{150}{x_n + 50}\right) \qquad \text{with } x_1 = 30$$

(c) (i) find the value of x_2 to 2 decimal places,

(ii) find, by repeated iteration, the horizontal distance travelled by the golf ball before it reaches its maximum height. Give your answer to 2 decimal places.

(3)

1 Part (a)

At the first landing, $h = 0$ and $x = d > 0$:

$$0 = d(1.5 - 0.5e^{0.02d}), \qquad e^{0.02d} = 3.$$

Taking logarithms, $0.02d = \ln 3$, hence $d = 50\ln 3 = 54.930614\ldots$. The root $x = 0$ is the launch point, not the landing point.

$$\boxed{d = 54.93 \text{ m}}$$

2 Part (b)

By the product and chain rules,

$$h'(x) = 1.5 - 0.5e^{0.02x} - 0.01xe^{0.02x}.$$

At a stationary point,

$$1.5 = 0.01(x + 50)e^{0.02x}, \qquad e^{0.02x} = \frac{150}{x + 50}.$$

Since $x + 50 > 0$, logarithms give $0.02x = \ln(150/(x + 50))$. Also $h''(x) = -(0.02 + 0.0002x)e^{0.02x} < 0$ on $[0, d]$. Thus the stationary point is the unique maximum; the endpoints both have zero height.

$$\boxed{x = 50\ln\left(\frac{150}{x + 50}\right)}$$

3 Part (c)

(i) With $x_1 = 30$,

$$x_2 = 50\ln\left(\frac{150}{30 + 50}\right) = 31.43043297\ldots.$$

(ii) Continuing with full intermediate precision gives $x_3 = 30.54431108\ldots$, $x_4 = 31.09139065\ldots$, $x_5 = 30.75292479\ldots$ and $x_6 = 30.96205562\ldots$. Continued iteration converges to $30.88212\ldots$, which lies in $(0, d)$. This is the horizontal distance at maximum height, not the maximum height itself.

$$\boxed{x_2 = 31.43, \quad x_{\text{maximum}} = 30.88 \text{ m}}$$

FINAL ANSWER

(a) $d = 54.93$ m (b) $x = 50\ln\left(\frac{150}{x+50}\right)$ (c) $x_2 = 31.43, \quad x_{\text{maximum}} = 30.88$ m

Question 8

12 marks | Log Graphs & Exponential Models

8.

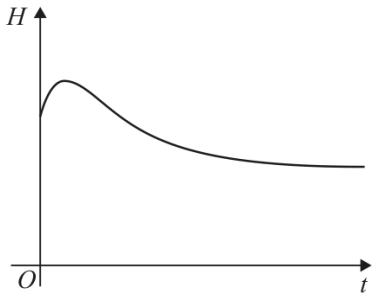


Figure 4

The heart rate of a horse is being monitored.
The heart rate H, measured in beats per minute (bpm), is modelled by the equation

$$H = 32 + 40e^{-0.2t} - 20\,e^{-0.9t}$$

where t minutes is the time after monitoring began.

Figure 4 is a sketch of H against t.

Use the equation of the model to answer parts (a) to (e).

(a) State the initial heart rate of the horse.

(1)

In the long term, the heart rate of the horse approaches L bpm.

(b) State the value of L.

(1)

The heart rate of the horse reaches its maximum value after T minutes.

(c) Find the value of T, giving your answer to 3 decimal places.

(*Solutions based entirely on calculator technology are not acceptable.*)

(5)

The heart rate of the horse is 37 bpm after M minutes.

(d) Show that M is a solution of the equation

$$t = 5\ln\left(\frac{8}{1 + 4e^{-0.9t}}\right)$$

(2)

Using the iteration formula

$$t_{n+1} = 5\ln\left(\frac{8}{1 + 4e^{-0.9t_n}}\right) \quad \text{with} \quad t_1 = 10$$

(e) (i) find, to 4 decimal places, the value of t_2

(ii) find, to 4 decimal places, the value of M

(3)

1 Part (a)

At $t = 0$, $H = 32 + 40 - 20 = 52$.

$$\boxed{H(0) = 52 \text{ bpm}}$$

2 Part (b)

As $t \to \infty$, both decaying exponential terms tend to zero.

$$\boxed{L = 32}$$

3 Part (c)

$$\frac{dH}{dt} = -8e^{-0.2t} + 18e^{-0.9t}.$$

At the maximum,

$$18e^{-0.9T} = 8e^{-0.2T} \quad \Longrightarrow \quad e^{0.7T} = \frac{9}{4} \quad \Longrightarrow \quad T = \frac{10}{7}\ln\frac{9}{4} = 1.1584717\ldots.$$

Also $H''(t) = 1.6e^{-0.2t} - 16.2e^{-0.9t}$. At the stationary point, $e^{-0.9T} = (4/9)e^{-0.2T}$, so $H''(T) = -5.6e^{-0.2T} < 0$, confirming a maximum.

$$\boxed{T = 1.158 \text{ minutes}}$$

4 Part (d)

For $H = 37$,

$$5 = 40e^{-0.2t} - 20e^{-0.9t} \quad \Longrightarrow \quad 8e^{-0.2t} = 1 + 4e^{-0.9t}.$$

Taking logarithms gives

$$-0.2t = \ln\left(\frac{1 + 4e^{-0.9t}}{8}\right) \quad \Longrightarrow \quad t = -5\ln\left(\frac{1 + 4e^{-0.9t}}{8}\right).$$

Using $-\ln u = \ln(1/u)$ for $u > 0$ then gives

$$t = 5\ln\left(\frac{8}{1 + 4e^{-0.9t}}\right).$$

$$\boxed{t = 5\ln\left(\frac{8}{1 + 4e^{-0.9t}}\right)}$$

5 **Part (e)**

Starting with $t_1 = 10$,

$$t_2 = 5\ln\left(\frac{8}{1+4e^{-9}}\right) = 10.3947401213\ldots.$$

Continuing without rounding intermediate values gives $t_3 = 10.3954778346\ldots$, $t_4 = 10.3954789826\ldots$ and $t_5 = 10.3954789844\ldots$. Thus (i) $t_2 = 10.3947$, and (ii) $M = 10.3955$ minutes, both to 4 decimal places.

$$\boxed{t_2 = 10.3947, \quad M = 10.3955 \text{ minutes}}$$

FINAL ANSWER

(a) $H(0) = 52$ bpm (b) $L = 32$ (c) $T = 1.158$ minutes (d) $t = 5\ln\left(\frac{8}{1+4e^{-0.9t}}\right)$ (e) $t_2 = 10.3947, \quad M = 10.3955$ minutes

WMA13/01 MAY/JUNE 2025

Question 6

8 marks | Log Graphs & Exponential Models

6: **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The temperature, θ°C, of a computer processor, t minutes after the computer is switched off, is modelled by the equation

$$\theta = 21 + Ae^{-kt}$$

where A and k are positive constants.

Given that the temperature of the processor was 75°C when the computer was switched off,

(a) find the value of A.

(2)

Given also that it takes 5 minutes for the temperature of the processor to decrease from 75°C to 25°C,

(b) find the value of k, giving your answer to 3 significant figures.

(3)

At time T minutes, the temperature of the processor is decreasing at a rate of 9°C per minute.

(c) Find the value of T according to the model, giving your answer to 2 decimal places.

(3)

WORKED SOLUTION

Question 6 · WMA13/01 May/June 2025

1 **Part (a)**

At $t = 0$, $75 = 21 + A$, so $\boxed{A = 54}$.

2 **Part (b)**

At $t = 5$, $25 = 21 + 54e^{-5k}$, hence

$$e^{-5k} = \frac{2}{27}, \qquad k = \frac{1}{5}\ln\frac{27}{2} = 0.520537937\ldots = \boxed{0.521}.$$

3 Part (c)

Differentiate the temperature model:

$$\frac{d\theta}{dt} = -54ke^{-kt}.$$

A decrease of $9\,^\circ\mathrm{C}$ per minute means $d\theta/dt = -9$, not $+9$. Thus

$$-54ke^{-kT} = -9 \implies e^{-kT} = \frac{1}{6k} \implies T = \frac{\ln(6k)}{k}.$$

Using the unrounded value of k,

$$T = 2.187865\ldots = \boxed{2.19 \text{ minutes}}.$$

FINAL ANSWER

(a) $A = 54$ (b) 0.521 (c) 2.19 minutes

WMA13/01 JANUARY 2026

Question 10

9 marks | Log Graphs & Exponential Models

10. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

Two populations of insects are being studied over the same period of time.

The number of insects N_A in population A, t years after the start of the study, is modelled by the equation

$$N_A = 8000 + 900\mathrm{e}^{0.2t}$$

According to the model,

(a) find the rate of growth in the number of insects in population A exactly 5 years after the start of the study.

(2)

The number of insects N_B in population B, t years after the start of the study, is modelled by the equation

$$N_B = 8000 + P\mathrm{e}^{kt}$$

where P and k are positive constants.

Given that

- there are 10 000 insects in population B at the start of the study
- there are 11 570 insects in population B exactly 4 years after the start of the study
- there are the same number of insects in population A and population B after T years

find, according to the models,

(b) the exact value of P and the value of k to 3 decimal places,

(4)

(c) the value of T, giving your answer to one decimal place.

(3)

WORKED SOLUTION

Question 10 · WMA13/01 January 2026

1 **Part (a)**

$$\frac{dN_A}{dt} = 900(0.2)e^{0.2t} = 180e^{0.2t}.$$

At $t = 5$, the growth rate is $180e = 489.290729\ldots$ insects per year.

2 Part (b)

At $t = 0$, $10000 = 8000 + P$, so $P = 2000$ exactly. Using the population at $t = 4$,

$$11570 = 8000 + 2000e^{4k} \quad \Longrightarrow \quad e^{4k} = \frac{3570}{2000} = 1.785.$$

Taking natural logarithms,

$$k = \frac{\ln 1.785}{4} = 0.144854604\ldots = 0.145 \quad (3\text{ d.p.}).$$

3 Part (c)

At equality, $8000 + 900e^{0.2T} = 8000 + 2000e^{kT}$, hence

$$e^{(0.2-k)T} = \frac{20}{9}, \qquad T = \frac{\ln(20/9)}{0.2 - k}.$$

Use the unrounded value $k = \ln(1.785)/4$:

$$T = 14.480042784\ldots = 14.5 \text{ years (1 decimal place)}.$$

FINAL ANSWER

(a) $180e$ insects per year (b) $P = 2000,\ k = 0.145\ (3\text{ d.p.})$ (c) 14.5 years

Question 7

12 marks | Log Graphs & Exponential Models

7. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The selling price of a car is monitored over a 15-year period.

The selling price P_C of the car, in thousands of pounds, t years after monitoring began, is modelled by the equation

$$P_C = 7 + 16e^{-0.1t} \qquad 0 \leqslant t \leqslant 15$$

Given that when $t = T$ the selling price of the car is £15 000

(a) find, according to the model, the value of T. Give the answer to 2 decimal places.

(3)

(b) Find the value of $\frac{dP_C}{dt}$ when $t = 5$. Give the answer to 2 decimal places.

(2)

The selling price of a van is monitored over the same 15-year period.

The selling price P_V of the van, in thousands of pounds, t years after monitoring began, is modelled by the equation

$$P_V = 4 + me^{-0.2t} \qquad 0 \leqslant t \leqslant 15$$

where m is a constant.

Use the equations of these models to answer parts (c) and (d).

Given that $P_V = 36$ when $t = 0$

(c) (i) state the value of m

(ii) find the difference between the initial selling price of the car and the initial selling price of the van.

(3)

When $t = \alpha$, the selling price of the car is equal to the selling price of the van.

(d) Find, using algebra and showing your working, the value of α, giving the answer to 3 significant figures.

(4)

1 Part (a)

Prices in the model are in thousands of pounds, so

$$15 = 7 + 16e^{-0.1T} \quad \Longrightarrow \quad e^{-0.1T} = \frac{1}{2}.$$

Taking logarithms gives $T = 10\ln 2 = 6.9314718\ldots$, hence $\boxed{T = 6.93 \text{ years}}$.

2 Part (b)

$$\frac{dP_C}{dt} = -1.6e^{-0.1t}, \qquad \left.\frac{dP_C}{dt}\right|_{t=5} = -1.6e^{-0.5} = -0.970449\ldots.$$

Thus the answer is $\boxed{-0.97 \text{ thousand pounds per year}}$ to 2 decimal places.

3 Part (c)(i)

At $t = 0$, $36 = 4 + m$, so $\boxed{m = 32}$.

4 Part (c)(ii)

The initial car price is $7 + 16 = 23$ thousand pounds. The difference is $36 - 23 = 13$ thousand pounds, or $\boxed{\pounds 13\,000}$.

5 Part (d)

Equating the models with $m = 32$ gives

$$7 + 16e^{-0.1\alpha} = 4 + 32e^{-0.2\alpha}.$$

Let $u = e^{-0.1\alpha} > 0$. Then $32u^2 - 16u - 3 = 0$, so

$$u = \frac{16 \pm \sqrt{256 + 384}}{64} = \frac{2 \pm \sqrt{10}}{8}.$$

Reject the negative root because an exponential is positive. Hence

$$\alpha = -10\ln\left(\frac{2 + \sqrt{10}}{8}\right) = 4.3806365\ldots, \qquad \boxed{\alpha = 4.38 \text{ years}}.$$

This value is inside the model interval $0 \le t \le 15$.

FINAL ANSWER

(a) $T = 6.93$ years (b) -0.97 thousand pounds per year (c)(i) $m = 32$ (c)(ii) $\pounds 13\,000$ (d) $\alpha = 4.38$ years

TOPIC

Differentiation Techniques

Every past-paper question on this topic, January 2020 to June 2026.

WMA13/01 OCTOBER 2020

Question 8

9 marks | Differentiation Techniques

8. (i) The curve C has equation $y = \mathrm{g}(x)$ where

$$\mathrm{g}(x) = \mathrm{e}^{3x} \sec 2x \qquad -\frac{\pi}{4} < x < \frac{\pi}{4}$$

(a) Find $\mathrm{g}'(x)$

(2)

(b) Hence find the x coordinate of the stationary point of C.

(3)

(ii) A different curve has equation

$$x = \ln(\sin y) \qquad 0 < y < \frac{\pi}{2}$$

Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^x}{\mathrm{f}(x)}$$

where $\mathrm{f}(x)$ is a function of e^x that should be found.

(4)

WORKED SOLUTION

Question 8 · WMA13/01 October 2020

1 **Part (i)(a)**

By the product and chain rules,

$$g'(x) = 3e^{3x}\sec 2x + 2e^{3x}\sec 2x \tan 2x = e^{3x}\sec 2x(3 + 2\tan 2x).$$

$$\boxed{g'(x) = e^{3x}\sec 2x(3 + 2\tan 2x)}$$

2 Part (i)(b)

For $-\pi/4 < x < \pi/4$, $e^{3x}\sec 2x$ is positive and nonzero. Hence $g'(x) = 0$ precisely when $3 + 2\tan 2x = 0$, or $\tan 2x = -3/2$. Since $-\pi/2 < 2x < \pi/2$, tangent is one-to-one there:

$$x = -\frac{1}{2}\arctan\frac{3}{2} = -0.4913968616\ldots.$$

$$\boxed{x = -\frac{1}{2}\arctan\frac{3}{2} \approx -0.491}$$

---

3 Part (ii)

Differentiate $x = \ln(\sin y)$ with respect to y:

$$\frac{dx}{dy} = \frac{\cos y}{\sin y}, \qquad \frac{dy}{dx} = \frac{\sin y}{\cos y}.$$

The original equation gives $\sin y = e^{x}$. Since $0 < y < \pi/2$, cosine is positive, so $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - e^{2x}}$. Therefore

$$\frac{dy}{dx} = \frac{e^{x}}{\sqrt{1 - e^{2x}}}, \qquad f(x) = \sqrt{1 - e^{2x}}.$$

Here $x < 0$ from the original interval, so the square root is real and nonzero.

$$\boxed{f(x) = \sqrt{1 - e^{2x}}, \qquad x < 0}$$

FINAL ANSWER

(i)(a) $g'(x) = e^{3x}\sec 2x(3 + 2\tan 2x)$ (i)(b) $x = -\frac{1}{2}\arctan\frac{3}{2} \approx -0.491$ (ii) $f(x) = \sqrt{1 - e^{2x}}, \qquad x < 0$

WMA13/01 JANUARY 2021

Question 10

10 marks | Differentiation Techniques

10. The curve C has equation

$$x = 3\sec^2 2y \qquad x > 3 \qquad 0 < y < \frac{\pi}{4}$$

(a) Find $\dfrac{\mathrm{d}x}{\mathrm{d}y}$ in terms of y.

(2)

(b) Hence show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{p}{qx\sqrt{x-3}}$$

where p is irrational and q is an integer, stating the values of p and q.

(3)

(c) Find the equation of the normal to C at the point where $y = \dfrac{\pi}{12}$, giving your answer in the form $y = mx + c$, giving m and c as exact irrational numbers.

(5)

WORKED SOLUTION

Question 10 · WMA13/01 January 2021

1 **Part (a)**

Apply the chain rule twice:

$$\frac{dx}{dy} = 3 \cdot 2\sec 2y\,(2\sec 2y \tan 2y) = 12\sec^2 2y \tan 2y.$$

$$\boxed{\frac{dx}{dy} = 12\sec^2 2y \tan 2y}$$

2 **Part (b)**

From $x = 3\sec^2 2y$, $\sec^2 2y = x/3$ and $\tan^2 2y = x/3 - 1$. Since $0 < 2y < \pi/2$, tangent is positive, so $\tan 2y = \sqrt{(x-3)/3}$. Thus $dx/dy = 4x\sqrt{(x-3)/3} > 0$. Taking the reciprocal gives

$$\frac{dy}{dx} = \frac{\sqrt{3}}{4x\sqrt{x-3}}, \qquad p = \sqrt{3}, \quad q = 4.$$

$$\boxed{p = \sqrt{3}, \quad q = 4}$$

3 **Part (c)**

At $y = \pi/12$, $2y = \pi/6$ and $x = 3(4/3) = 4$. The tangent gradient is $\sqrt{3}/(4 \cdot 4\sqrt{1}) = \sqrt{3}/16$. Therefore the normal gradient is $-16/\sqrt{3} = -16\sqrt{3}/3$. Using the point $(4, \pi/12)$,

$$y - \frac{\pi}{12} = -\frac{16\sqrt{3}}{3}(x - 4) \Longrightarrow y = -\frac{16\sqrt{3}}{3}x + \frac{64\sqrt{3}}{3} + \frac{\pi}{12}.$$

$$\boxed{y = -\frac{16\sqrt{3}}{3}x + \frac{64\sqrt{3}}{3} + \frac{\pi}{12}}$$

✓ **FINAL ANSWER**

(a) $\frac{dx}{dy} = 12\sec^2 2y \tan 2y$ (b) $p = \sqrt{3}, \quad q = 4$ (c) $y = -\frac{16\sqrt{3}}{3}x + \frac{64\sqrt{3}}{3} + \frac{\pi}{12}$

WMA13/01 MAY/JUNE 2021

Question 7

5 marks | Differentiation Techniques

7. Given that

$$x = 6\sin^2 2y \qquad 0 < y < \frac{\pi}{4}$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{A\sqrt{(Bx - x^2)}}$$

where A and B are integers to be found.

(5)

WORKED SOLUTION

Question 7 · WMA13/01 May/June 2021

1 Part (main)

Differentiate x with respect to y:

$$\frac{dx}{dy} = 6 \cdot 2\sin 2y\cos 2y \cdot 2 = 24\sin 2y\cos 2y.$$

Since $0 < 2y < \pi/2$, both sine and cosine are positive. Using $\sin^2 2y = x/6$ and $\cos^2 2y = 1 - x/6$,

$$\frac{dx}{dy} = 24\sqrt{x/6}\sqrt{1 - x/6} = 4\sqrt{6x - x^2}.$$

On $0 < x < 6$ this is nonzero, so take its reciprocal.

$$\boxed{\frac{dy}{dx} = \frac{1}{4\sqrt{6x - x^2}}, \qquad A = 4,\ B = 6}$$

FINAL ANSWER

(main) $\frac{dy}{dx} = \frac{1}{4\sqrt{6x-x^2}}$, $A = 4,\ B = 6$

WMA13/01 OCTOBER 2021

Question 6

8 marks | Differentiation Techniques

6. (i) The curve C_1 has equation

$$y = 3\ln(x^2 - 5) - 4x^2 + 15 \qquad x > \sqrt{5}$$

Show that C_1 has a stationary point at $x = \dfrac{\sqrt{p}}{2}$ where p is a constant to be found.

(4)

(ii) A different curve C_2 has equation

$$y = 4x - 12\sin^2 x$$

(a) Show that, for this curve,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = A + B\sin 2x$$

where A and B are constants to be found.

(b) Hence, state the maximum gradient of this curve.

(4)

WORKED SOLUTION

Question 6 · WMA13/01 October 2021

1 **Part (i)**

Differentiate to obtain $y' = 6x/(x^2 - 5) - 8x$. At a stationary point in $x > \sqrt{5}$, x is nonzero, so $6/(x^2 - 5) = 8$, giving $6 = 8x^2 - 40$ and $x^2 = 23/4$. The original positive domain selects $x = \sqrt{23}/2$, which exceeds $\sqrt{5}$. Hence $p = 23$.

$$\boxed{p = 23, \qquad x = \frac{\sqrt{23}}{2}}$$

2 **Part (ii)**

(a) Differentiate the squared sine with the chain rule: $y' = 4 - 24\sin x\cos x = 4 - 12\sin 2x$. Thus $A = 4$, $B = -12$. (b) Since $-1 \le \sin 2x \le 1$, the largest derivative is attained when $\sin 2x = -1$: $4 - 12(-1) = 16$.

$$\boxed{A = 4, \quad B = -12; \qquad \text{maximum gradient} = 16}$$

FINAL ANSWER

(i) $p = 23, \quad x = \frac{\sqrt{23}}{2}$ (ii) $A = 4, \quad B = -12; \quad$ maximum gradient $= 16$

Question 8

7 marks | Differentiation Techniques

8. A curve C has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \arcsin\left(\frac{1}{2}x\right) \qquad -2 \leqslant x \leqslant 2 \qquad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

(a) Sketch C.

(1)

(b) Given $x = 2\sin y$, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{A - x^2}}$$

where A is a constant to be found.

(3)

The point P lies on C and has y coordinate $\frac{\pi}{4}$

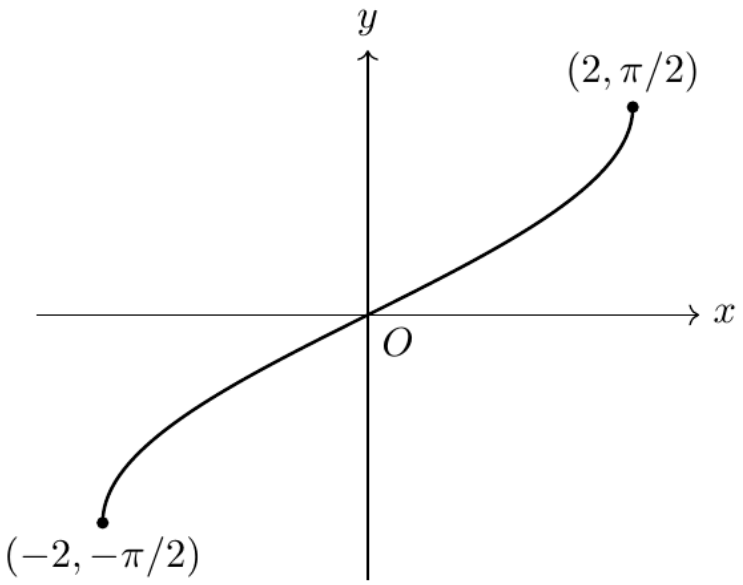
(c) Find the equation of the tangent to C at P. Write your answer in the form $y = mx + c$, where m and c are constants to be found.

(3)

1 Part (a)

The principal arcsine curve increases from $(-2, -\pi/2)$ through $(0, 0)$ to $(2, \pi/2)$. It is concave down on the negative-x half, concave up on the positive-x half, and vertical at both ends.

$$\boxed{y = \arcsin(x/2), \quad -2 \le x \le 2}$$



2 Part (b)

Differentiate the supplied inverse relation: $dx/dy = 2\cos y$. For $-\pi/2 < y < \pi/2$, cosine is positive, so $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2/4}$. Taking the reciprocal gives

$$\frac{dy}{dx} = \frac{1}{2\cos y} = \frac{1}{\sqrt{4 - x^2}}, \qquad -2 < x < 2.$$

Thus $A = 4$; the endpoints have vertical tangents, not finite derivatives.

$$\boxed{A = 4, \qquad \frac{dy}{dx} = \frac{1}{\sqrt{4 - x^2}}}$$

3 Part (c)

At $y = \pi/4$, $x = 2\sin(\pi/4) = \sqrt{2}$. The gradient is $1/\sqrt{4 - 2} = 1/\sqrt{2}$. Hence $y - \pi/4 = (x - \sqrt{2})/\sqrt{2}$, so $y = x/\sqrt{2} + \pi/4 - 1$.

$$\boxed{y = \frac{x}{\sqrt{2}} + \frac{\pi}{4} - 1}$$

FINAL ANSWER

(a) $y = \arcsin(x/2), \quad -2 \le x \le 2$ (b) $A = 4, \quad \frac{dy}{dx} = \frac{1}{\sqrt{4-x^2}}$ (c) $y = \frac{x}{\sqrt{2}} + \frac{\pi}{4} - 1$

Question 10

7 marks | Differentiation Techniques

10.

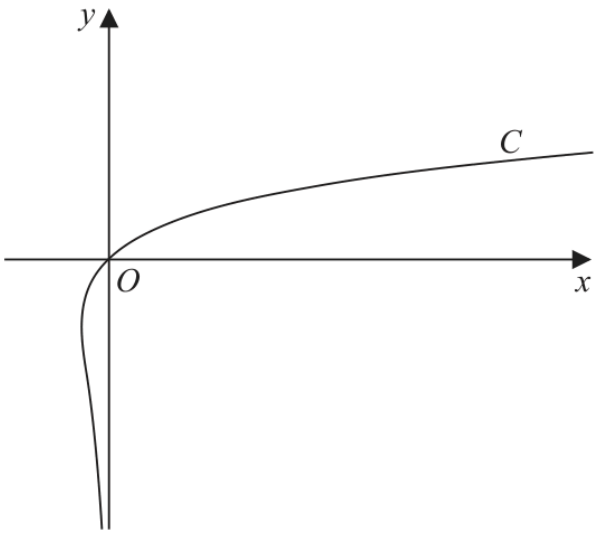


Figure 3

Figure 3 shows a sketch of the curve C with equation

$$x = ye^{2y} \qquad y \in \mathbb{R}$$

(a) Show that

$$\frac{dy}{dx} = \frac{y}{x(1+2y)}$$

(4)

Given that the straight line with equation $x = k$, where k is a constant, cuts C at exactly two points,

(b) find the range of possible values for k.

(3)

1 Part (a)

Differentiate x with respect to y by the product rule:

$$\frac{dx}{dy} = e^{2y} + 2ye^{2y} = e^{2y}(1+2y).$$

Where $dx/dy \neq 0$, take the reciprocal:

$$\frac{dy}{dx} = \frac{e^{-2y}}{1+2y}.$$

For $y \neq 0$, the original relation gives $e^{2y} = x/y$, hence the required form $dy/dx = y/[x(1+2y)]$. At the origin, that last quotient is $0/0$; the preceding uncancelled derivative gives $dy/dx = 1$. At $y = -1/2$ there is no finite dy/dx.

$$\boxed{\frac{dy}{dx} = \frac{y}{x(1+2y)} \quad \text{where defined}}$$

2 Part (b)

Consider $x = h(y) = ye^{2y}$. Its derivative $h'(y) = e^{2y}(1+2y)$ vanishes at $y = -1/2$. It is negative before this y value and positive after it, so the minimum x value is $h(-1/2) = -1/(2e)$. As $y \to -\infty$, $h(y) \to 0$ from below; also $h(0) = 0$ and h then increases without bound. Thus a level $x = k$ meets both negative-y branches precisely between the minimum and zero. At the minimum the two intersections coincide; at zero there is just the finite point $y = 0$.

$$\boxed{-\frac{1}{2e} < k < 0}$$

FINAL ANSWER

(a) $\frac{dy}{dx} = \frac{y}{x(1+2y)}$ where defined (b) $-\frac{1}{2e} < k < 0$

WMA13/01 OCTOBER 2022

Question 6

6 marks | Differentiation Techniques

6.

$$y = \frac{2 + 3\sin x}{\cos x + \sin x}$$

Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a\tan x + b\sec x + c}{\sec x + 2\sin x}$$

where a, b and c are integers to be found.

(6)

WORKED SOLUTION

Question 6 · WMA13/01 October 2022

1 Part (main)

With $y = (2 + 3\sin x)/(\cos x + \sin x)$, the quotient rule gives

$$\frac{dy}{dx} = \frac{3\cos x(\cos x + \sin x) - (2 + 3\sin x)(\cos x - \sin x)}{(\cos x + \sin x)^2}.$$

Expand the numerator and cancel the opposite mixed terms:

$$3\cos^2 x + 3\sin^2 x - 2\cos x + 2\sin x = 3 - 2\cos x + 2\sin x.$$

The denominator is $1 + 2\sin x\cos x$. Divide numerator and denominator by $\cos x$ to obtain

$$\frac{dy}{dx} = \frac{2\tan x + 3\sec x - 2}{\sec x + 2\sin x}.$$

This displayed form is used where its trigonometric terms are defined; the preceding quotient form also covers points with $\cos x = 0$ when the original denominator is nonzero.

$$\boxed{a = 2, \quad b = 3, \quad c = -2}$$

FINAL ANSWER

(main) $a = 2, \quad b = 3, \quad c = -2$

Question 10

8 marks | Differentiation Techniques

10.

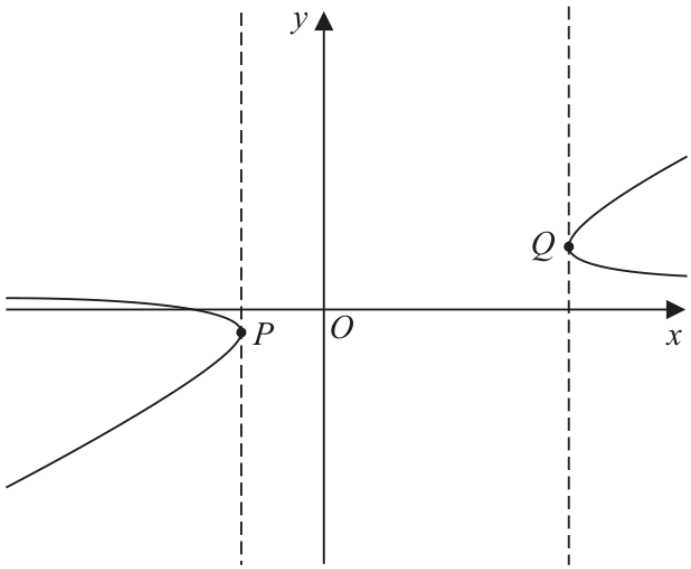


Figure 4

Figure 4 shows a sketch of the curve with equation

$$x = \frac{2y^2 + 6}{3y - 3}$$

(a) Find $\frac{dx}{dy}$ giving your answer as a fully simplified fraction.

(4)

The tangents at points P and Q on the curve are parallel to the y-axis, as shown in Figure 4.

(b) Use the answer to part (a) to find the equations of these two tangents.

(4)

1 Part (a)

The quotient rule gives, for $y \neq 1$,

$$\frac{dx}{dy} = \frac{4y(3y-3) - 3(2y^2+6)}{(3y-3)^2} = \frac{6y^2 - 12y - 18}{9(y-1)^2} = \frac{2(y-3)(y+1)}{3(y-1)^2}.$$

$$\boxed{\frac{dx}{dy} = \frac{2(y-3)(y+1)}{3(y-1)^2}, \quad y \neq 1}$$

2 Part (b)

A tangent parallel to the y-axis is vertical, so $dx/dy = 0$. The numerator gives $(y-3)(y+1) = 0$, hence $y = 3$ or $y = -1$; neither makes the denominator zero. Then

$$x(3) = \frac{18+6}{9-3} = 4, \qquad x(-1) = \frac{2+6}{-3-3} = -\frac{4}{3}.$$

Thus $Q = (4, 3)$ and $P = (-4/3, -1)$, with the vertical tangent equations below.

$$\boxed{x = -\frac{4}{3}, \qquad x = 4}$$

FINAL ANSWER

(a) $\frac{dx}{dy} = \frac{2(y-3)(y+1)}{3(y-1)^2}, \quad y \neq 1$ (b) $x = -\frac{4}{3}, \qquad x = 4$

Question 10

9 marks | Differentiation Techniques

10. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

A curve C has equation

$$x = \sin^2 4y \qquad 0 \leqslant y \leqslant \frac{\pi}{8} \qquad 0 \leqslant x \leqslant 1$$

The point P with x coordinate $\frac{1}{4}$ lies on C

(a) Find the exact y coordinate of P

(2)

(b) Find $\frac{dx}{dy}$

(2)

(c) Hence show that $\frac{dy}{dx}$ can be written in the form

$$\frac{dy}{dx} = \frac{1}{\sqrt{q + r(x + s)^2}}$$

where q, r and s are constants to be found.

(3)

Using the answer to part (c),

(d) (i) state the x coordinate of the point where the value of $\frac{dy}{dx}$ is a minimum,

(ii) state the value of $\frac{dy}{dx}$ at this point.

(2)

1 Part (a)

At $x = 1/4$, $\sin^2 4y = 1/4$. Since $0 \le 4y \le \pi/2$,

$$\sin 4y = \frac{1}{2} \quad \Longrightarrow \quad 4y = \frac{\pi}{6}.$$

Only this branch lies in the given range of y.

$$\boxed{y = \frac{\pi}{24}}$$

2 Part (b)

Differentiate $x = (\sin 4y)^2$ with respect to y:

$$\frac{dx}{dy} = 2\sin 4y\,(4\cos 4y).$$

$$\boxed{\frac{dx}{dy} = 8\sin 4y\cos 4y}$$

3 Part (c)

On the interior of the given interval, $\sin 4y = \sqrt{x}$ and $\cos 4y = \sqrt{1-x}$ are positive. Taking the reciprocal,

$$\frac{dy}{dx} = \frac{1}{8\sqrt{x}\sqrt{1-x}} = \frac{1}{\sqrt{64x - 64x^2}}.$$

Complete the square:

$$64x - 64x^2 = 16 - 64\left(x - \frac{1}{2}\right)^2.$$

Thus $dy/dx = (q + r(x+s)^2)^{-1/2}$ has the constants below, with $0 < x < 1$ for a finite derivative; the endpoints have vertical tangents.

$$\boxed{q = 16, \qquad r = -64, \qquad s = -\frac{1}{2}}$$

4 Part (d)

The derivative is positive. Its minimum occurs when the square root in the denominator is largest. Since $(x - 1/2)^2 \ge 0$, the largest radicand is 16, attained at $x = 1/2$. Hence the smallest gradient is $1/\sqrt{16}$.

$$\boxed{\text{(i) } x = \frac{1}{2}, \qquad \text{(ii) } \min \frac{dy}{dx} = \frac{1}{4}}$$

FINAL ANSWER

(a) $y = \frac{\pi}{24}$ (b) $\frac{dx}{dy} = 8\sin 4y \cos 4y$ (c) $q = 16, \qquad r = -64, \qquad s = -\frac{1}{2}$ (d)
(i) $x = \frac{1}{2}, \qquad$ (ii) $\min \frac{dy}{dx} = \frac{1}{4}$

Question 10

9 marks | Differentiation Techniques

10.

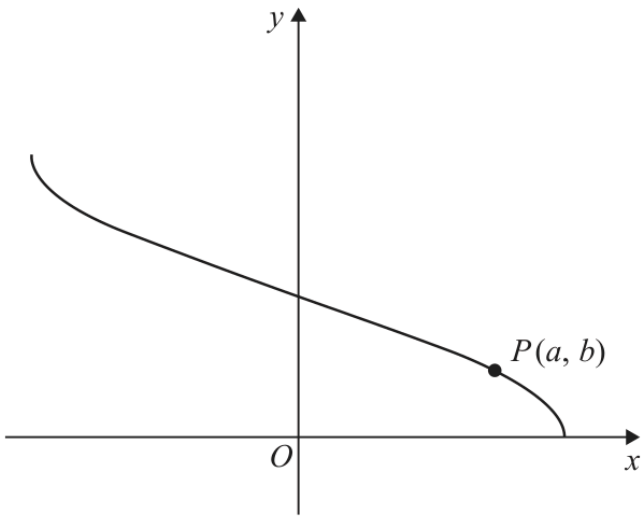


Figure 2

Figure 2 shows a sketch of the curve with equation

$$x = 3\cos 2y \qquad -3 \leqslant x \leqslant 3 \qquad 0 \leqslant y \leqslant \frac{\pi}{2}$$

(a) Find $\dfrac{\mathrm{d}x}{\mathrm{d}y}$ in terms of y.

(2)

(b) Hence show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{k}{\sqrt{9 - x^2}}$$

where k is a constant to be found.

(3)

The point $P(a, b)$ lies on the curve and is shown in Figure 2.

Given that

- the gradient of the curve at P is $-\dfrac{1}{4}$
- both a and b are positive

(c) find the exact values of a and b.

(4)

1

Part (a)

Differentiate $x = 3\cos 2y$ with respect to y using the chain rule:

$$\frac{dx}{dy} = 3(-\sin 2y)\cdot 2 = -6\sin 2y.$$

$$\boxed{\frac{dx}{dy} = -6\sin 2y}$$

2

Part (b)

For $0 < y < \pi/2$, $\sin 2y > 0$ and

$$\frac{dy}{dx} = \frac{1}{dx/dy} = -\frac{1}{6\sin 2y}.$$

Since $\cos 2y = x/3$ and the sine is positive on this interval,

$$\sin 2y = \sqrt{1-\cos^2 2y} = \sqrt{1 - x^2/9} = \frac{\sqrt{9-x^2}}{3}.$$

Consequently $dy/dx = -1/(2\sqrt{9-x^2})$, valid for $-3 < x < 3$. The endpoint tangents are vertical; there is no finite derivative there.

$$\boxed{\frac{dy}{dx} = -\frac{1}{2\sqrt{9-x^2}}, \qquad k = -\frac{1}{2}}$$

3

Part (c)

Set the gradient equal to $-1/4$:

$$-\frac{1}{2\sqrt{9-a^2}} = -\frac{1}{4} \quad\Longrightarrow\quad \sqrt{9-a^2} = 2 \quad\Longrightarrow\quad a^2 = 5.$$

Because $a > 0$, take $a = \sqrt{5}$. Substitute into $a = 3\cos 2b$:

$$\cos 2b = \frac{\sqrt{5}}{3} \quad\Longrightarrow\quad b = \frac{1}{2}\arccos\left(\frac{\sqrt{5}}{3}\right).$$

Here $0 < 2b < \pi/2$ since both coordinates are positive, so the principal arccos value is the required one.

$$\boxed{a = \sqrt{5}, \qquad b = \frac{1}{2}\arccos\left(\frac{\sqrt{5}}{3}\right)}$$

FINAL ANSWER

(a) $\frac{dx}{dy} = -6\sin 2y$ (b) $\frac{dy}{dx} = -\frac{1}{2\sqrt{9-x^2}}, \qquad k = -\frac{1}{2}$ (c) $a = \sqrt{5}, \qquad b = \frac{1}{2}\arccos\left(\frac{\sqrt{5}}{3}\right)$

Question 9

11 marks | Differentiation Techniques

9:

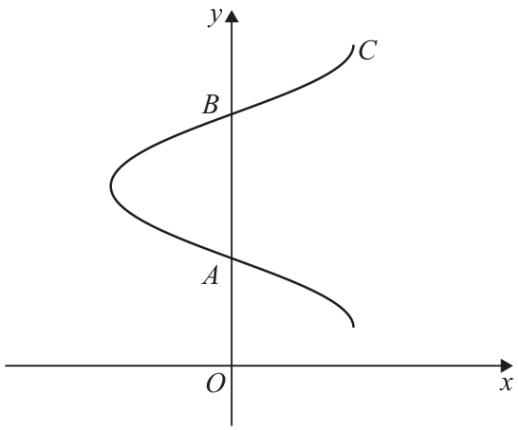


Figure 2

Figure 2 shows a sketch of the curve C with equation

$$x = \frac{2}{3}\sin\left(3y + \frac{\pi}{4}\right) \qquad \frac{\pi}{12} < y < \frac{3\pi}{4}$$

The curve intersects the y-axis at the points A and B as shown.

(a) Find the exact value of the y coordinate of

- point A
- point B

(3)

(b) Show that

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{p - qx^2}$$

where p and q are integers to be found.

(4)

The **normal** to C at A and the **tangent** to C at B intersect at the point D.

Using

- the answer to part (b)
- the sketch of curve C in Figure 2

(c) find, in simplest form, the exact x coordinate of D.

(4)

1 Part (a)

On the y axis, $x = 0$, so $\sin(3y + \pi/4) = 0$. Since $\pi/12 < y < 3\pi/4$, we have $\pi/2 < 3y + \pi/4 < 5\pi/2$. Therefore the possible angles are π and 2π, giving

$$\boxed{y_A = \frac{\pi}{4}, \qquad y_B = \frac{7\pi}{12}}.$$

2 Part (b)

Differentiate x with respect to y, then take the reciprocal:

$$\frac{dx}{dy} = 2\cos(3y + \pi/4), \qquad \frac{dy}{dx} = \frac{1}{2\cos(3y + \pi/4)}.$$

Using $\sin(3y + \pi/4) = 3x/2$,

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4(1 - \sin^2(3y + \pi/4))} = \boxed{\frac{1}{4 - 9x^2}}.$$

Thus $\boxed{p = 4,\ q = 9}$. The derivative formula applies where the tangent is not vertical.

3 Part (c)

At $x = 0$, part (b) gives $|dy/dx| = 1/2$. The sketch shows a negative tangent gradient at A and a positive one at B. The normal at A therefore has gradient 2, while the tangent at B has gradient $1/2$:

$$y = 2x + \frac{\pi}{4}, \qquad y = \frac{x}{2} + \frac{7\pi}{12}.$$

At their intersection,

$$\frac{3}{2}x = \frac{7\pi}{12} - \frac{\pi}{4} = \frac{\pi}{3} \quad \Longrightarrow \quad \boxed{x_D = \frac{2\pi}{9}}.$$

FINAL ANSWER

(a) $y_A = \frac{\pi}{4}, \quad y_B = \frac{7\pi}{12}$ (b) $\frac{1}{4-9x^2}$, $p = 4,\ q = 9$ (c) $x_D = \frac{2\pi}{9}$

WMA13/01 JANUARY 2026

Question 8

4 marks | Differentiation Techniques

8. A curve has equation

$$x = e^{2\tan y} \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Show that

$$\frac{dy}{dx} = \frac{a}{x\left(b + (\ln x)^n\right)}$$

where a, b and n are integers to be found.

(4)

WORKED SOLUTION

Question 8 · WMA13/01 January 2026

1 **Solution**

\paragraph{Solution} Differentiate $x = e^{2\tan y}$ with respect to y:

$$\frac{dx}{dy} = 2\sec^2 y\, e^{2\tan y} = 2x\sec^2 y.$$

This is positive on the stated interval, so

$$\frac{dy}{dx} = \frac{1}{2x\sec^2 y}.$$

Taking logarithms of the original relation gives $\ln x = 2\tan y$. Using $\sec^2 y = 1 + \tan^2 y$,

$$\frac{dy}{dx} = \frac{1}{2x\left(1 + \frac{(\ln x)^2}{4}\right)} = \frac{2}{x\left(4 + (\ln x)^2\right)}.$$

Hence $a = 2$, $b = 4$ and $n = 2$.

FINAL ANSWER

(main) $a = 2,\ b = 4,\ n = 2$

WMA13/01 MAY/JUNE 2026

Question 6

6 marks | Differentiation Techniques

6. The curve C has equation

$$x = \left(3\sin 2y - \frac{6y}{\pi}\right)^2$$

Given that

- the point P with y coordinate $\frac{\pi}{4}$ lies on C
- the tangent to C at P cuts the x-axis at the point Q

use calculus to find the x coordinate of Q

(6)

WORKED SOLUTION

Question 6 · WMA13/01 May/June 2026

1 **Solution**

At $y = \pi/4$,

$$x = \left(3\sin\frac{\pi}{2} - \frac{6(\pi/4)}{\pi}\right)^2 = \left(3 - \frac{3}{2}\right)^2 = \frac{9}{4}.$$

Differentiate x with respect to y using the chain rule:

$$\frac{dx}{dy} = 2\left(3\sin 2y - \frac{6y}{\pi}\right)\left(6\cos 2y - \frac{6}{\pi}\right).$$

At P, $dx/dy = 2(3/2)(-6/\pi) = -18/\pi$. Since this is nonzero, the tangent gradient is $\frac{dy}{dx} = -\frac{\pi}{18}$, and its equation is

$$y - \frac{\pi}{4} = -\frac{\pi}{18}\left(x - \frac{9}{4}\right).$$

Put $y = 0$ for its intersection with the x-axis:

$$x - \frac{9}{4} = \frac{18}{4}, \qquad \boxed{x_Q = \frac{27}{4}}.$$

✓ **FINAL ANSWER**

$x_Q = \frac{27}{4}$

Question 7

11 marks | Differentiation Techniques

Main topic: Applications of Differentiation

7.

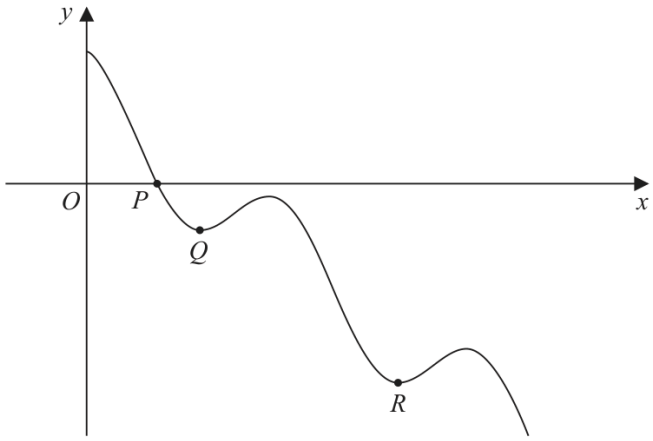


Figure 3

Figure 3 shows a sketch of part of the curve with equation

$$y = 2\cos 3x - 3x + 4 \qquad x > 0$$

where x is measured in radians.

The curve crosses the x-axis at the point P, as shown in Figure 3.

Given that the x coordinate of P is α,

(a) show that α lies between 0.8 and 0.9

(2)

The iteration formula

$$x_{n+1} = \frac{1}{3}\arccos(1.5x_n - 2)$$

can be used to find an approximate value for α.

(b) Using this iteration formula with $x_1 = 0.8$ find, to 4 decimal places, the value of

(i) x_2

(ii) x_5

(3)

The point Q and the point R are local minimum points on the curve, as shown in Figure 3.

Given that the x coordinates of Q and R are β and λ respectively, and that they are the two smallest values of x at which local minima occur,

(c) find, using calculus, the exact value of β and the exact value of λ.

(6)

1 Part (a)

Let $f(x) = 2\cos 3x - 3x + 4$, using radians. Then

$$f(0.8) = 2\cos 2.4 - 2.4 + 4 = 0.1252125\ldots > 0,$$
$$f(0.9) = 2\cos 2.7 - 2.7 + 4 = -0.5081443\ldots < 0.$$

The function is continuous on $[0.8, 0.9]$, so this change of sign shows that there is a root between the endpoints. Moreover, $f'(x) = -6\sin 3x - 3 < 0$ on this interval, so this root is unique there. Thus the crossing shown at P has $\boxed{0.8 < \alpha < 0.9}$.

2 Part (b)(i)

Using the inverse cosine in radians,

$$x_2 = \frac{1}{3}\arccos(1.5(0.8) - 2) = \frac{1}{3}\arccos(-0.8) = 0.8326971\ldots.$$

Hence $\boxed{x_2 = 0.8327}$ to 4 decimal places.

3 Part (b)(ii)

Continue the iteration, retaining the full calculator values at each step:

$$x_3 = \frac{1}{3}\arccos(1.5x_2 - 2) = 0.8067674\ldots,$$
$$x_4 = \frac{1}{3}\arccos(1.5x_3 - 2) = 0.8271196\ldots,$$
$$x_5 = \frac{1}{3}\arccos(1.5x_4 - 2) = 0.8110216\ldots.$$

Therefore $\boxed{x_5 = 0.8110}$ to 4 decimal places.

4 Part (c)

Differentiate the curve equation:

$$\frac{dy}{dx} = -6\sin 3x - 3, \qquad \frac{d^2y}{dx^2} = -18\cos 3x.$$

At a stationary point,

$$-6\sin 3x - 3 = 0 \quad \Longrightarrow \quad \sin 3x = -\frac{1}{2}.$$

For $x > 0$, the stationary points therefore belong to the two families

$$3x = \frac{7\pi}{6} + 2n\pi \quad \text{or} \quad 3x = \frac{11\pi}{6} + 2n\pi, \qquad n = 0, 1, 2, \ldots.$$

In the first family, $\cos 3x = -\frac{\sqrt{3}}{2}$, giving $y'' = 9\sqrt{3} > 0$, so these are local minima. In the second family, $\cos 3x = \frac{\sqrt{3}}{2}$ and $y'' = -9\sqrt{3} < 0$, so these are local maxima and are excluded. Thus all positive minimum coordinates are

$$x = \frac{7\pi}{18} + \frac{2n\pi}{3}, \qquad n = 0, 1, 2, \ldots.$$

The two smallest are obtained with $n = 0$ and $n = 1$:

$$\boxed{\beta = \frac{7\pi}{18}}, \qquad \boxed{\lambda = \frac{19\pi}{18}}.$$

FINAL ANSWER

(a) $0.8 < \alpha < 0.9$ (b)(i) $x_2 = 0.8327$ (b)(ii) $x_5 = 0.8110$ (c) $\beta = \frac{7\pi}{18}$, $\lambda = \frac{19\pi}{18}$

Question 9

14 marks | Differentiation Techniques | Main topic: Applications of Differentiation

9. (a) Given that

$$\frac{x^4 - x^3 - 10x^2 + 3x - 9}{x^2 - x - 12} \equiv x^2 + P + \frac{Q}{x - 4} \qquad x > -3$$

find the value of the constant P and show that $Q = 5$

(4)

The curve C has equation $y = \mathrm{g}(x)$, where

$$\mathrm{g}(x) = \frac{x^4 - x^3 - 10x^2 + 3x - 9}{x^2 - x - 12} \qquad -3 < x < 3.5 \qquad x \in \mathbb{R}$$

(b) Find the equation of the tangent to C at the point where $x = 2$
Give your answer in the form $y = mx + c$, where m and c are constants to be found.

(5)

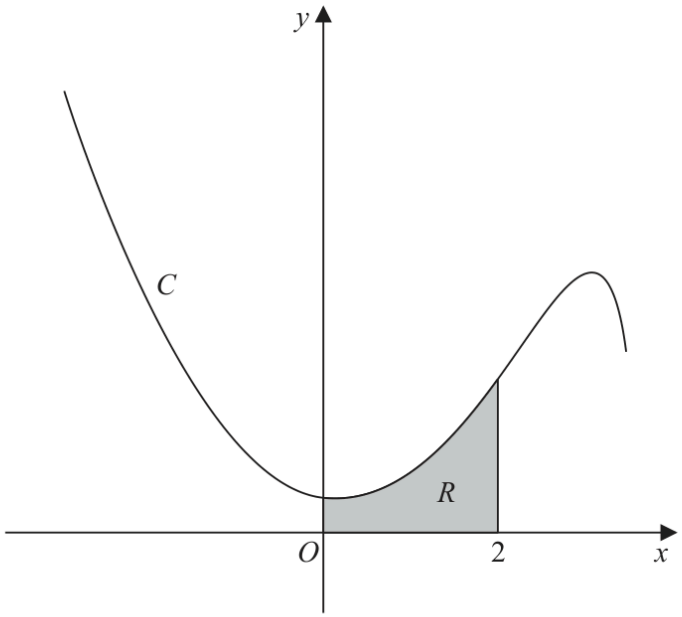


Figure 4

Figure 4 shows a sketch of the curve C.
The region R, shown shaded in Figure 4, is bounded by C, the y-axis, the x-axis and the line with equation $x = 2$

(c) Find the exact area of R, writing your answer in the form $a + b\ln 2$, where a and b are constants to be found.

(5)

1 Part (a)

Factor the denominator as $(x-4)(x+3)$ and multiply the proposed identity through by it:

$$x^4 - x^3 - 10x^2 + 3x - 9 = (x^2 + P)(x^2 - x - 12) + Q(x+3).$$

The right side expands to $x^4 - x^3 + (P-12)x^2 + (Q-P)x + 3Q - 12P$. Comparing coefficients gives $P - 12 = -10$, so $P = 2$, and $Q - P = 3$, so $Q = 5$. The constant check is $3(5) - 12(2) = -9$, as required. The identity holds where the original fraction is defined: the stated $x > -3$ still excludes $x = 4$.

$$\boxed{P = 2, \qquad Q = 5}$$

2 Part (b)

From (a), $g(x) = x^2 + 2 + 5/(x-4)$ on the curve domain. Thus $g'(x) = 2x - 5/(x-4)^2$. At $x = 2$, $g(2) = 4 + 2 - 5/2 = 7/2$ and $g'(2) = 4 - 5/4 = 11/4$. The tangent through $(2, 7/2)$ is

$$y - \frac{7}{2} = \frac{11}{4}(x-2) \quad \Longrightarrow \quad y = \frac{11}{4}x - 2.$$

$$\boxed{y = \frac{11}{4}x - 2}$$

3 Part (c)

The source shading is above the x-axis from $x = 0$ to $x = 2$, so integrate g over those bounds.

$$\text{Area}(R) = \int_0^2 \left(x^2 + 2 + \frac{5}{x-4}\right) dx = \left[\frac{x^3}{3} + 2x + 5\ln|x-4|\right]_0^2.$$

The absolute value is essential because $x - 4 < 0$ throughout this interval. Evaluating gives

$$\frac{8}{3} + 4 + 5\ln 2 - 5\ln 4 = \frac{20}{3} - 5\ln 2.$$

Thus $a = 20/3$ and $b = -5$ in the requested form.

$$\boxed{\text{Area}(R) = \frac{20}{3} - 5\ln 2}$$

FINAL ANSWER

(a) $P = 2, \quad Q = 5$ (b) $y = \frac{11}{4}x - 2$ (c) $\text{Area}(R) = \frac{20}{3} - 5\ln 2$

WMA13/01 JANUARY 2021

Question 6

8 marks | Differentiation Techniques | Main topic: Numerical Methods

6.

$$f(x) = x\cos\left(\frac{x}{3}\right) \qquad x > 0$$

(a) Find $f'(x)$

(2)

(b) Show that the equation $f'(x) = 0$ can be written as

$$x = k\arctan\left(\frac{k}{x}\right)$$

where k is an integer to be found.

(2)

(c) Starting with $x_1 = 2.5$ use the iteration formula

$$x_{n+1} = k\arctan\left(\frac{k}{x_n}\right)$$

with the value of k found in part (b), to calculate the values of x_2 and x_6 giving your answers to 3 decimal places.

(2)

(d) Using a suitable interval and a suitable function that should be stated, show that a root of $f'(x) = 0$ is 2.581 correct to 3 decimal places.

(2)

WORKED SOLUTION

Question 6 · WMA13/01 January 2021

1 **Part (a)**

By the product and chain rules, $f'(x) = \cos(x/3) - (x/3)\sin(x/3)$.

$$\boxed{f'(x) = \cos(x/3) - \frac{x}{3}\sin(x/3)}$$

2 Part (b)

Set $f'(x) = 0$: $\cos(x/3) = (x/3)\sin(x/3)$. At a solution, cosine cannot vanish (that would leave a nonzero right side), so $\tan(x/3) = 3/x$. On the principal positive branch $0 < x/3 < \pi/2$, this gives

$$x = 3\arctan(3/x), \qquad k = 3.$$

This is the branch used by the supplied iteration for the root near 2.581. The original domain $x > 0$ also admits later tangent branches: in general $x = 3\arctan(3/x) + 3n\pi$. The principal-branch rearrangement must not be interpreted as listing every positive stationary point.

$$\boxed{k = 3; \quad x = 3\arctan(3/x) \text{ on the selected branch}}$$

---

3 Part (c)

Using radians and $x_1 = 2.5$, the successive values are

n	1	2	3	4	5	6
x_n	2.5	2.628174152	2.554102174	2.596526871	2.572101407	2.586122337

Keep full precision between steps and round the requested terms only.

$$\boxed{x_2 = 2.628, \qquad x_6 = 2.586}$$

---

4 Part (d)

Define $F(x) = \cos(x/3) - (x/3)\sin(x/3)$, which is continuous near 2.581. The rounding interval endpoints give

$$F(2.5805) = 0.0003467157\ldots > 0, \qquad F(2.5815) = -0.0003456897\ldots < 0.$$

By continuity a root lies strictly between these endpoints; every number in this open interval rounds to 2.581 to three decimal places.

$$\boxed{x = 2.581 \text{ correct to 3 d.p.}}$$

✓ FINAL ANSWER

(a) $f'(x) = \cos(x/3) - \frac{x}{3}\sin(x/3)$ (b) $k = 3; \quad x = 3\arctan(3/x)$ on the selected branch (c) $x_2 = 2.628, \qquad x_6 = 2.586$ (d) $x = 2.581$ correct to 3 d.p.

WMA13/01 JANUARY 2021

Question 9

4 marks | Differentiation Techniques | Main topic: Integration

9. Find

(i) $\displaystyle\int \frac{3x-2}{3x^2-4x+5}\,\mathrm{d}x$ **(2)**

(ii) $\displaystyle\int \frac{\mathrm{e}^{2x}}{(\mathrm{e}^{2x}-1)^3}\,\mathrm{d}x \qquad x \neq 0$ **(2)**

WORKED SOLUTION

Question 9 · WMA13/01 January 2021

1 Part (i)

The denominator derivative is $6x-4=2(3x-2)$. Hence reverse the logarithmic chain rule:

$$\int \frac{3x-2}{3x^2-4x+5}\,dx=\frac{1}{2}\ln(3x^2-4x+5)+C.$$

The log argument is always positive because $3x^2-4x+5=3(x-2/3)^2+11/3$.

$$\boxed{\frac{1}{2}\ln(3x^2-4x+5)+C}$$

2 Part (ii)

Put $u=e^{2x}-1$, so $du=2e^{2x}\,dx$. Then $\int e^{2x}(e^{2x}-1)^{-3}\,dx=(1/2)\int u^{-3}\,du=-1/(4u^2)+C$. The source excludes $x=0$, where the integrand is undefined.

$$\boxed{-\frac{1}{4(e^{2x}-1)^2}+C}$$

FINAL ANSWER

(i) $\frac{1}{2}\ln(3x^2-4x+5)+C$ (ii) $-\frac{1}{4(e^{2x}-1)^2}+C$

WMA13/01 MAY/JUNE 2021

Question 8

13 marks | Differentiation Techniques | Main topic: Log Graphs & Exponential Models

8. A scientist is studying a population of fish in a lake. The number of fish, N, in the population, t years after the start of the study, is modelled by the equation

$$N = \frac{600e^{0.3t}}{2 + e^{0.3t}} \qquad t \geqslant 0$$

Use the equation of the model to answer parts (a), (b), (c), (d) and (e).

(a) Find the number of fish in the lake at the start of the study.

(1)

(b) Find the upper limit to the number of fish in the lake.

(1)

(c) Find the time, after the start of the study, when there are predicted to be 500 fish in the lake. Give your answer in years and months to the nearest month.

(4)

(d) Show that

$$\frac{dN}{dt} = \frac{Ae^{0.3t}}{(2 + e^{0.3t})^2}$$

where A is a constant to be found.

(3)

Given that when $t = T$, $\frac{dN}{dt} = 8$

(e) find the value of T to one decimal place.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(4)

WORKED SOLUTION

Question 8 · WMA13/01 May/June 2021

Part (a)

At the start, $t = 0$ and $e^{0.3t} = 1$. Thus $N(0) = 600/(2+1) = 200$.

$$\boxed{200 \text{ fish}}$$

2 Part (b)

Divide numerator and denominator by $e^{0.3t}$ to write $N = 600/(1 + 2e^{-0.3t})$. As $t \to \infty$, the denominator tends to 1, so the upper limit is 600 fish.

$$\boxed{600 \text{ fish}}$$

3 Part (c)

Set $N = 500$ and multiply through: $1000 + 500e^{0.3t} = 600e^{0.3t}$, so $e^{0.3t} = 10$. Hence $t = \ln 10/0.3 = 7.675283643\ldots$ years. The fractional year is $12(0.675283643\ldots) = 8.10340372\ldots$ months. Round only the months to the nearest whole month.

$$\boxed{7 \text{ years } 8 \text{ months}}$$

4 Part (d)

The quotient rule gives

$$\frac{dN}{dt} = \frac{180e^{0.3t}(2 + e^{0.3t}) - 600e^{0.3t}(0.3e^{0.3t})}{(2 + e^{0.3t})^2} = \frac{360e^{0.3t}}{(2 + e^{0.3t})^2}.$$

The two $e^{0.6t}$ terms cancel exactly.

$$\boxed{A = 360}$$

5 Part (e)

Let $z = e^{0.3T}$. Since $T \geq 0$, require $z \geq 1$. Then $360z/(2 + z)^2 = 8$ gives $45z = (2 + z)^2$, hence

$$z^2 - 41z + 4 = 0, \qquad z = \frac{41 \pm \sqrt{1665}}{2}.$$

The smaller root lies between 0 and 1 and would give $T < 0$, so reject it. For the larger root,

$$T = \frac{1}{0.3}\ln\left(\frac{41 + \sqrt{1665}}{2}\right) = 12.37061331\ldots.$$

$$\boxed{T = 12.4 \text{ years}}$$

FINAL ANSWER

(a) 200 fish (b) 600 fish (c) 7 years 8 months (d) $A = 360$ (e) $T = 12.4$ years

Question 9

9 marks | Differentiation Techniques | Main topic: Numerical Methods

9.

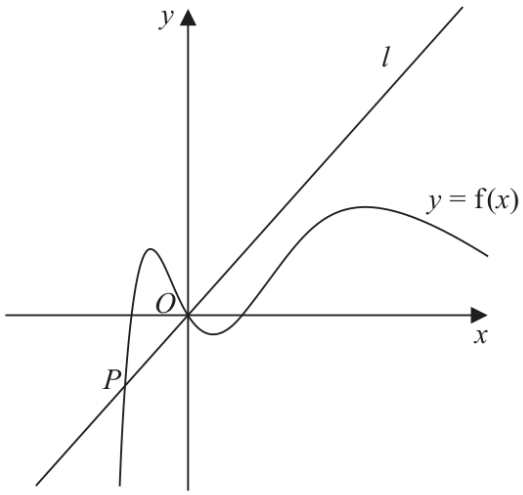


Figure 3

Figure 3 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x(x^2 - 4)\mathrm{e}^{-\frac{1}{2}x}$$

(a) Find $\mathrm{f}'(x)$.

(2)

The line l is the normal to the curve at O and meets the curve again at the point P.

The point P lies in the 3rd quadrant, as shown in Figure 3.

(b) Show that the x coordinate of P is a solution of the equation

$$x = -\frac{1}{2}\sqrt{16 + \mathrm{e}^{\frac{1}{2}x}}$$

(4)

(c) Using the iterative formula

$$x_{n+1} = -\frac{1}{2}\sqrt{16 + \mathrm{e}^{\frac{1}{2}x_n}} \quad \text{with } x_1 = -2$$

find, to 4 decimal places,

(i) the value of x_2

(ii) the x coordinate of P.

(3)

1 Part (a)

Write $f(x) = (x^3 - 4x)e^{-x/2}$. By the product rule,

$$f'(x) = (3x^2 - 4)e^{-x/2} - \frac{1}{2}(x^3 - 4x)e^{-x/2} = e^{-x/2}\left(-\frac{x^3}{2} + 3x^2 + 2x - 4\right).$$

$$\boxed{f'(x) = e^{-x/2}(-x^3/2 + 3x^2 + 2x - 4)}$$

2 Part (b)

At O, $f'(0) = -4$, so the normal gradient is $1/4$ and its equation is $y = x/4$. At the other intersection P, x is nonzero; equating curve and normal and dividing by x gives $(x^2 - 4)e^{-x/2} = 1/4$. Thus $x^2 = 4 + e^{x/2}/4 = (16 + e^{x/2})/4$. Since P lies in the third quadrant, select the negative square root.

$$\boxed{x = -\frac{1}{2}\sqrt{16 + e^{x/2}}}$$

3 Part (c)

Starting from $x_1 = -2$, $x_2 = -\frac{1}{2}\sqrt{16 + e^{-1}} = -2.0228617996\ldots$. Further unrounded iterates are $x_3 = -2.0226034084\ldots$, $x_4 = -2.0226063126\ldots$ and $x_5 = -2.0226062799\ldots$. The continuing sequence settles at $-2.0226062803\ldots$.

$$\boxed{\text{(i) } x_2 = -2.0229; \qquad \text{(ii) } x_P = -2.0226}$$

FINAL ANSWER

(a) $f'(x) = e^{-x/2}(-x^3/2 + 3x^2 + 2x - 4)$ (b) $x = -\frac{1}{2}\sqrt{16 + e^{x/2}}$ (c) $\text{(i) } x_2 = -2.0229; \qquad \text{(ii) } x_P = -2.0226$

WMA13/01 JANUARY 2022

Question 6

11 marks | Differentiation Techniques | Main topic: Functions, Composition & Inverses

6. The function f is defined by

$$\mathrm{f}(x) = \frac{5x - 3}{x - 4} \qquad x > 4$$

(a) Show, by using calculus, that f is a decreasing function.

(3)

(b) Find f^{-1}

(3)

(c) (i) Show that $\mathrm{ff}(x) = \dfrac{ax + b}{x + c}$ where a, b and c are constants to be found.

(ii) Deduce the range of ff.

(5)

WORKED SOLUTION

Question 6 · WMA13/01 January 2022

1 Part (a)

By the quotient rule,

$$f'(x) = \frac{5(x-4) - (5x-3)}{(x-4)^2} = \frac{-17}{(x-4)^2}.$$

For every $x > 4$, the denominator is positive and the numerator is negative. Therefore $f'(x) < 0$ throughout its domain, so f is decreasing.

$$\boxed{f \text{ is decreasing for } x > 4}$$

2 Part (b)

Set $y = (5x - 3)/(x - 4)$. Then $yx - 4y = 5x - 3$, giving $x(y - 5) = 4y - 3$ and $x = (4y - 3)/(y - 5)$. Exchange the variable names. Since $f(x) = 5 + 17/(x - 4)$ on $x > 4$, its range is $(5, \infty)$, which becomes the inverse's domain.

$$\boxed{f^{-1}(x) = \frac{4x - 3}{x - 5}, \qquad x > 5}$$

3 **Part (c)**

(i) For $x > 4$, $f(x) > 5 > 4$, so the second application is defined.

$$ff(x) = \frac{5\left(\frac{5x-3}{x-4}\right) - 3}{\left(\frac{5x-3}{x-4}\right) - 4} = \frac{25x - 15 - 3x + 12}{5x - 3 - 4x + 16} = \frac{22x - 3}{x + 13}.$$

Thus $a = 22$, $b = -3$, $c = 13$. (ii) Rewrite $ff(x) = 22 - 289/(x + 13)$ with the original domain $x > 4$. It is strictly increasing, tends to 5 as $x \to 4^+$ and to 22 as $x \to \infty$. Neither limiting value is attained.

$$\boxed{ff(x) = \frac{22x - 3}{x + 13}, \qquad 5 < ff(x) < 22}$$

FINAL ANSWER

(a) f **is decreasing for** $x > 4$ (b) $f^{-1}(x) = \frac{4x-3}{x-5}, \qquad x > 5$ (c) $ff(x) = \frac{22x-3}{x+13}, \qquad 5 < ff(x) < 22$

WMA13/01 JANUARY 2022

Question 8

8 marks | Differentiation Techniques | Main topic: Log Graphs & Exponential Models

8. A dose of antibiotics is given to a patient.

The amount of the antibiotic, x milligrams, in the patient's bloodstream t hours after the dose was given, is found to satisfy the equation

$$\log_{10} x = 2.74 - 0.079t$$

(a) Show that this equation can be written in the form

$$x = pq^{-t}$$

where p and q are constants to be found. Give the value of p to the nearest whole number and the value of q to 2 significant figures.

(4)

(b) With reference to the equation in part (a), interpret the value of the constant p.

(1)

When a different dose of the antibiotic is given to another patient, the values of x and t satisfy the equation

$$x = 400 \times 1.4^{-t}$$

(c) Use calculus to find, to 2 significant figures, the value of $\dfrac{dx}{dt}$ when $t = 5$

(3)

WORKED SOLUTION

Question 8 · WMA13/01 January 2022

1 **Part (a)**

Exponentiate base 10:

$$x = 10^{2.74-0.079t} = 10^{2.74}(10^{0.079})^{-t}.$$

Thus $p = 10^{2.74} = 549.5408739\ldots$ and $q = 10^{0.079} = 1.199499303\ldots$. Round p to the nearest integer and q to two significant figures.

$$\boxed{p = 550, \qquad q = 1.2}$$

2 Part (b)

At $t = 0$, $x = pq^0 = p$. Thus p is the initial amount of antibiotic in the patient's bloodstream, approximately 550 milligrams immediately after the dose in this model.

$$\boxed{p \text{ represents the initial amount, approximately } 550 \text{ mg}}$$

3 Part (c)

For the second patient, write $x = 400e^{-t\ln 1.4}$. Differentiate to obtain $dx/dt = -400\ln(1.4)\,1.4^{-t}$. At $t = 5$, the derivative is $-25.02470969\ldots \text{ mg}\,\text{h}^{-1}$. The negative sign indicates the amount is decreasing.

$$\boxed{\left.\frac{dx}{dt}\right|_{t=5} = -25 \text{ mg}\,\text{h}^{-1}}$$

FINAL ANSWER

(a) $p = 550, \quad q = 1.2$ (b) p represents the initial amount, approximately 550 mg (c) $\left.\frac{dx}{dt}\right|_{t=5} = -25 \text{ mg}\,\text{h}^{-1}$

Question 6

10 marks | Differentiation Techniques | Main topic: Applications of Differentiation

6.

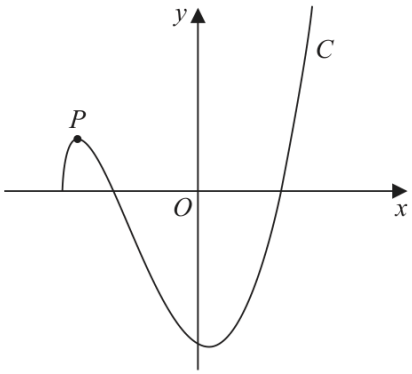


Figure 3

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

The function f is defined by

$$f(x) = 5(x^2 - 2)(4x + 9)^{\frac{1}{2}} \qquad x \geqslant -\frac{9}{4}$$

(a) Show that

$$f'(x) = \frac{k(5x^2 + 9x - 2)}{(4x + 9)^{\frac{1}{2}}}$$

where k is an integer to be found.

(4)

(b) Hence, find the values of x for which $f'(x) = 0$

(1)

Figure 3 shows a sketch of the curve C with equation $y = f(x)$.

The curve has a local maximum at the point P

(c) Find the exact coordinates of P

(2)

The function g is defined by

$$g(x) = 2f(x) + 4 \qquad -\frac{9}{4} \leqslant x \leqslant 0$$

(d) Find the range of g

(3)

1 Part (a)

Use product and chain rules:

$$f'(x) = 10x(4x+9)^{1/2} + 10(x^2-2)(4x+9)^{-1/2}.$$

With a common denominator,

$$f'(x) = \frac{10x(4x+9) + 10(x^2-2)}{\sqrt{4x+9}} = \frac{10(5x^2+9x-2)}{\sqrt{4x+9}}.$$

This derivative formula applies for $x > -9/4$; the endpoint derivative is not finite.

$$\boxed{k = 10}$$

2 Part (b)

The denominator is positive in the derivative domain. Thus $5x^2+9x-2 = (5x-1)(x+2) = 0$.

$$\boxed{x = -2, \quad \frac{1}{5}}$$

3 Part (c)

The derivative changes from positive to negative at $x = -2$, so this is the local maximum P. Substitute into f: $f(-2) = 5(4-2)\sqrt{-8+9} = 10$.

$$\boxed{P = (-2, 10)}$$

4 Part (d)

On $[-9/4, 0]$, f rises from $f(-9/4) = 0$ to $f(-2) = 10$, then decreases to $f(0) = -30$. The other stationary point $1/5$ is outside this restricted interval. Thus the minimum and maximum of f on it are -30 and 10. Applying the positive vertical transformation $g = 2f + 4$ gives endpoints $2(-30)+4 = -56$ and $2(10)+4 = 24$, both attained.

$$\boxed{-56 \le g(x) \le 24}$$

FINAL ANSWER

(a) $k = 10$ (b) $x = -2, \quad \frac{1}{5}$ (c) $P = (-2, 10)$ (d) $-56 \le g(x) \le 24$

Question 8

12 marks | Differentiation Techniques | Main topic: Numerical Methods

8.

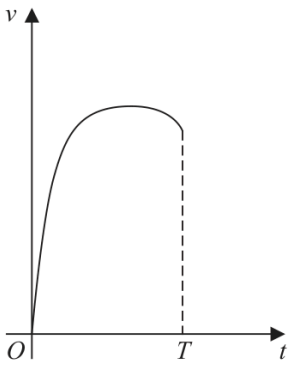


Figure 4

Figure 4 is a graph showing the velocity of a sprinter during a 100 m race.

The sprinter's velocity during the race, $v\,\text{m s}^{-1}$, is modelled by the equation

$$v = 12 - \text{e}^{t-10} - 12\text{e}^{-0.75t} \qquad t \geqslant 0$$

where t seconds is the time after the sprinter begins to run.

According to the model,

(a) find, using calculus, the sprinter's maximum velocity during the race.

(5)

Given that the sprinter runs 100 m in T seconds, such that

$$\int_0^T v\,\text{d}t = 100$$

(b) show that T is a solution of the equation

$$T = \frac{1}{12}\left(116 - 16\text{e}^{-0.75T} + \text{e}^{T-10} - \text{e}^{-10}\right)$$

(4)

The iteration formula

$$T_{n+1} = \frac{1}{12}\left(116 - 16\text{e}^{-0.75T_n} + \text{e}^{T_n-10} - \text{e}^{-10}\right)$$

is used to find an approximate value for T

Using this iteration formula with $T_1 = 10$

(c) find, to 4 decimal places,

(i) the value of T_2

(ii) the time taken by the sprinter to run the race, according to the model.

(3)

1 Part (a)

Differentiate the velocity:

$$v'(t) = -e^{t-10} + 9e^{-0.75t}.$$

At a stationary velocity, $e^{t-10} = 9e^{-0.75t}$, so $e^{1.75t} = 9e^{10}$ and $t = (10 + \ln 9)/1.75 = 6.9698426\ldots$ seconds. The second derivative $-e^{t-10} - 6.75e^{-0.75t}$ is negative, so this is a maximum. Substitute this unrounded time into v to get $11.88728125\ldots$ metres per second.

$$\boxed{v_{\max} = 11.9\ \mathrm{m\,s^{-1}}}$$

2 Part (b)

Integrating the velocity gives $12t - e^{t-10} + 16e^{-0.75t}$. Evaluate between 0 and T:

$$100 = 12T - e^{T-10} + 16e^{-0.75T} - (-e^{-10} + 16).$$

Rearrange to $12T = 116 - 16e^{-0.75T} + e^{T-10} - e^{-10}$ and divide by 12.

$$\boxed{T = \frac{1}{12}(116 - 16e^{-0.75T} + e^{T-10} - e^{-10})}$$

3 Part (c)

With $T_1 = 10$, substitution gives $T_2 = 9.7492587708\ldots$. Continuing with unrounded values gives $T_3 = 9.7306248365\ldots$, $T_4 = 9.7294150526\ldots$, $T_5 = 9.7293372715\ldots$, $T_6 = 9.7293322738\ldots$.

Further iteration settles at $9.7293319307\ldots$ seconds. Thus (i) $T_2 = 9.7493$ and (ii) the race time is 9.7293 seconds, to four decimal places.

$$\boxed{T_2 = 9.7493, \qquad T = 9.7293\ \mathrm{s}}$$

FINAL ANSWER

(a) $v_{\max} = 11.9\ \mathrm{m\,s^{-1}}$ (b) $T = \frac{1}{12}(116 - 16e^{-0.75T} + e^{T-10} - e^{-10})$ (c) $T_2 = 9.7493, \qquad T = 9.7293\ \mathrm{s}$

Question 9

9 marks | Differentiation Techniques | Main topic: Trig Equations & R-Form

9.

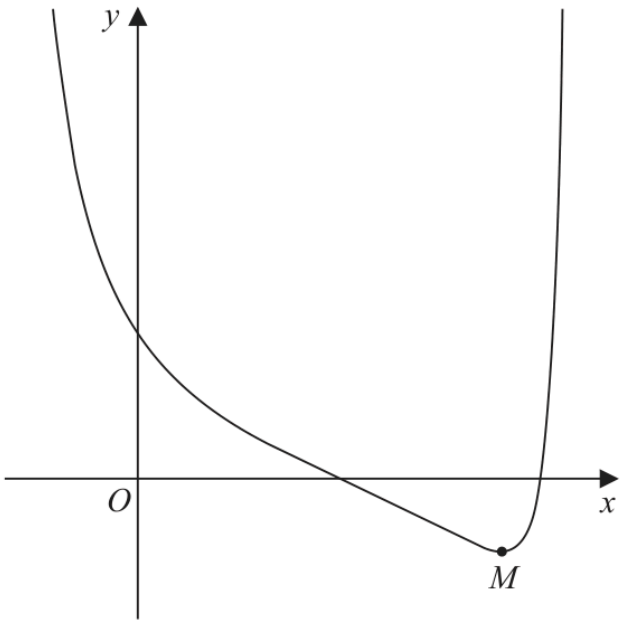


Figure 5

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 5 shows the curve with equation

$$y = \frac{1 + 2\cos x}{1 + \sin x} \qquad -\frac{\pi}{2} < x < \frac{3\pi}{2}$$

The point M, shown in Figure 5, is the minimum point on the curve.

(a) Show that the x coordinate of M is a solution of the equation

$$2\sin x + \cos x = -2$$

(4)

(b) Hence find, to 3 significant figures, the x coordinate of M.

(5)

1 Part (a)

Differentiate by the quotient rule:

$$y' = \frac{-2\sin x(1+\sin x) - (1+2\cos x)\cos x}{(1+\sin x)^2} = \frac{-2\sin x - \cos x - 2(\sin^2 x + \cos^2 x)}{(1+\sin x)^2} = \frac{-2\sin x - \cos x - 2}{(1+\sin x)^2}.$$

Inside the stated interval, $1+\sin x > 0$. At the interior minimum M, $y' = 0$, so its numerator is zero, giving the required equation.

$$\boxed{2\sin x + \cos x = -2}$$

2 Part (b)

Write $2\sin x + \cos x = \sqrt{5}\sin(x+\beta)$, where $\cos\beta = 2/\sqrt{5}$, $\sin\beta = 1/\sqrt{5}$ and $\beta = \arctan(1/2)$. The equation becomes $\sin(x+\beta) = -2/\sqrt{5}$. Let $\gamma = \arcsin(2/\sqrt{5})$, so $\gamma + \beta = \pi/2$. The solutions in the closed boundary range include $x = -\pi/2$, $3\pi/2$, and

$$x = \pi + \gamma - \beta = 3.7850937624\ldots.$$

Both boundary values are outside the original open domain (and make its denominator zero). The sole interior stationary point is M; the derivative changes from negative to positive there.

$$\boxed{x_M = 3.79}$$

FINAL ANSWER

(a) $2\sin x + \cos x = -2$ (b) $x_M = 3.79$

WMA13/01 JANUARY 2023

Question 7

9 marks | Differentiation Techniques | Main topic: Applications of Differentiation

7. The curve C has equation

$$x = 3\tan\left(y - \frac{\pi}{6}\right) \qquad x \in \mathbb{R} \quad -\frac{\pi}{3} < y < \frac{2\pi}{3}$$

(a) Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a}{x^2 + b}$$

where a and b are integers to be found.

(4)

The point P with y coordinate $\frac{\pi}{3}$ lies on C.

Given that the tangent to C at P crosses the x-axis at the point Q.

(b) find, in simplest form, the exact x coordinate of Q.

(5)

WORKED SOLUTION

Question 7 · WMA13/01 January 2023

1 **Part (a)**

Put $u = y - \pi/6$. Since $x = 3\tan u$,

$$\frac{dx}{dy} = 3\sec^2 u = 3(1 + \tan^2 u) = 3\left(1 + \frac{x^2}{9}\right) = \frac{x^2 + 9}{3}.$$

This is positive for all real x, so we may take the reciprocal: $dy/dx = 3/(x^2 + 9)$. Thus the required integers are $a = 3$, $b = 9$.

$$\boxed{\frac{dy}{dx} = \frac{3}{x^2 + 9}, \qquad a = 3, \quad b = 9}$$

2 **Part (b)**

At $y = \pi/3$, $x = 3\tan(\pi/3 - \pi/6) = 3\tan(\pi/6) = \sqrt{3}$. The tangent gradient is $3/((\sqrt{3})^2 + 9) = 1/4$. Its point-gradient equation is

$$y - \frac{\pi}{3} = \frac{1}{4}(x - \sqrt{3}).$$

At its x-axis crossing $y = 0$, $-4\pi/3 = x - \sqrt{3}$.

$$\boxed{x_Q = \sqrt{3} - \frac{4\pi}{3}}$$

✓ **FINAL ANSWER**

(a) $\frac{dy}{dx} = \frac{3}{x^2+9}$, $a = 3$, $b = 9$ (b) $x_Q = \sqrt{3} - \frac{4\pi}{3}$

Question 9

11 marks | Differentiation Techniques | Main topic: Numerical Methods

9.

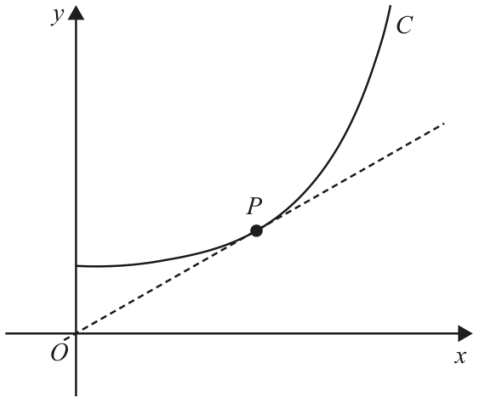


Figure 3

Figure 3 shows a sketch of part of the curve C with equation

$$y = \sqrt{3 + 4e^{x^2}} \qquad x \geqslant 0$$

(a) Find $\frac{dy}{dx}$, giving your answer in simplest form.

(2)

The point P with x coordinate α lies on C.

Given that the tangent to C at P passes through the origin, as shown in Figure 3,

(b) show that $x = \alpha$ is a solution of the equation

$$4x^2e^{x^2} - 4e^{x^2} - 3 = 0$$

(3)

(c) Hence show that α lies between 1 and 2

(2)

(d) Show that the equation in part (b) can be written in the form

$$x = \frac{1}{2}\sqrt{4 + 3e^{-x^2}}$$

(1)

The iteration formula

$$x_{n+1} = \frac{1}{2}\sqrt{4 + 3e^{-x_n^2}}$$

with $x_1 = 1$ is used to find an approximation for α.

(e) Use the iteration formula to find, to 4 decimal places, the value of

(i) x_3

(ii) α

(3)

1 Part (a)

Differentiate $y = (3 + 4e^{x^2})^{1/2}$ using the chain rule:

$$\frac{dy}{dx} = \frac{1}{2}(3 + 4e^{x^2})^{-1/2} \cdot 4e^{x^2} \cdot 2x = \frac{4xe^{x^2}}{\sqrt{3 + 4e^{x^2}}}.$$

$$\boxed{\frac{dy}{dx} = \frac{4xe^{x^2}}{\sqrt{3 + 4e^{x^2}}}}$$

2 Part (b)

Since the tangent at $P = (\alpha, \sqrt{3 + 4e^{\alpha^2}})$ passes through the origin, its gradient is y_P/α, with $\alpha > 0$. Equate this to the derivative:

$$\frac{4\alpha e^{\alpha^2}}{\sqrt{3 + 4e^{\alpha^2}}} = \frac{\sqrt{3 + 4e^{\alpha^2}}}{\alpha} \quad \Longrightarrow \quad 4\alpha^2 e^{\alpha^2} = 3 + 4e^{\alpha^2}.$$

Consequently $x = \alpha$ solves the displayed equation.

$$\boxed{4x^2 e^{x^2} - 4e^{x^2} - 3 = 0}$$

3 Part (c)

Let $h(x) = 4x^2 e^{x^2} - 4e^{x^2} - 3$, a continuous function. Then $h(1) = -3 < 0$ and $h(2) = 12e^4 - 3 = 652.1778\ldots > 0$, so a root lies between 1 and 2. Moreover $h'(x) = 8x^3 e^{x^2} > 0$ for $x > 0$, so this is the unique positive root corresponding to the tangent abscissa α.

$$\boxed{1 < \alpha < 2}$$

4 Part (d)

From $4x^2 e^{x^2} = 4e^{x^2} + 3$, divide by the positive e^{x^2} to get $4x^2 = 4 + 3e^{-x^2}$. The source domain has $x \geq 0$, so take the positive square root.

$$\boxed{x = \frac{1}{2}\sqrt{4 + 3e^{-x^2}}}$$

5 **Part (e)**

With $x_1 = 1$, the unrounded iterates start

$$x_2 = 1.1295616764\ldots, \quad x_3 = 1.0997193683\ldots, \quad x_4 = 1.1062486441\ldots, \quad x_5 = 1.1048012899\ldots.$$

Continue: $x_6 = 1.1051212304\ldots$, $x_7 = 1.1050504628\ldots$, $x_8 = 1.1050661137\ldots$, $x_9 = 1.1050626523\ldots$; further iteration settles at $1.1050632792\ldots$. Therefore (i) $x_3 = 1.0997$ and (ii) $\alpha = 1.1051$, both to four decimal places.

$$\boxed{x_3 = 1.0997, \qquad \alpha = 1.1051}$$

✓ **FINAL ANSWER**

(a) $\frac{dy}{dx} = \frac{4xe^{x^2}}{\sqrt{3+4e^{x^2}}}$ (b) $4x^2e^{x^2} - 4e^{x^2} - 3 = 0$ (c) $1 < \alpha < 2$ (d) $x = \frac{1}{2}\sqrt{4 + 3e^{-x^2}}$ (e) $x_3 = 1.0997, \qquad \alpha = 1.1051$

WMA13/01 JANUARY 2023

Question 10

9 marks | Differentiation Techniques | Main topic: Log Graphs & Exponential Models

10. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

A population of fruit flies is being studied.

The number of fruit flies, F, in the population, t days after the start of the study, is modelled by the equation

$$F = \frac{350e^{kt}}{9 + e^{kt}}$$

where k is a constant.

Use the equation of the model to answer parts (a), (b) and (c).

(a) Find the number of fruit flies in the population at the start of the study.

(1)

Given that there are 200 fruit flies in the population 15 days after the start of the study,

(b) show that $k = \frac{1}{15}\ln 12$

(3)

Given also that, when $t = T$, the number of fruit flies in the population is increasing at a rate of 10 per day,

(c) find the possible values of T, giving your answers to one decimal place.

(5)

WORKED SOLUTION

Question 10 · WMA13/01 January 2023

1 **Part (a)**

At the start $t = 0$, so $e^{kt} = 1$ and the model gives $F(0) = 350/(9 + 1) = 35$ fruit flies.

$$\boxed{F(0) = 35}$$

2 Part (b)

Using $F(15) = 200$,

$$200 = \frac{350e^{15k}}{9 + e^{15k}} \quad \Longrightarrow \quad 1800 + 200e^{15k} = 350e^{15k} \quad \Longrightarrow \quad e^{15k} = 12.$$

Take natural logs to obtain $15k = \ln 12$, as required.

$$\boxed{k = \frac{\ln 12}{15}}$$

3 Part (c)

By the quotient rule,

$$F'(t) = \frac{350ke^{kt}(9 + e^{kt}) - 350ke^{2kt}}{(9 + e^{kt})^2} = \frac{3150ke^{kt}}{(9 + e^{kt})^2}.$$

Set $F'(T) = 10$ and let $u = e^{kT} > 0$. Then

$$315ku = (9 + u)^2 \quad \Longrightarrow \quad u^2 + (18 - 315k)u + 81 = 0.$$

With the exact $k = \ln 12/15$, the quadratic formula gives

$$u = \frac{315k - 18 \pm \sqrt{(315k - 18)^2 - 324}}{2} = 2.5615490777\ldots \text{ or } 31.6214905678\ldots.$$

Both values are positive and at least 1, hence both correspond to $T \geq 0$. Finally $T = \ln u/k$ gives $5.6779528361\ldots$ or $20.8488936858\ldots$ days. To one decimal place the two times are as follows.

$$\boxed{T = 5.7 \text{ days}, \qquad T = 20.8 \text{ days}}$$

FINAL ANSWER

(a) $F(0) = 35$ (b) $k = \frac{\ln 12}{15}$ (c) $T = 5.7$ days, $\qquad T = 20.8$ days

WMA13/01 MAY/JUNE 2023

Question 7

8 marks | Differentiation Techniques | Main topic: Log Graphs & Exponential Models

7. A scientist is studying two different populations of bacteria.

The number of bacteria N in the first population is modelled by the equation

$$N = Ae^{kt} \qquad t \geqslant 0$$

where A and k are positive constants and t is the time in hours from the start of the study.

Given that
- there were 2500 bacteria in this population at the start of the study
- there were 10 000 bacteria 8 hours later

(a) find the exact value of A and the value of k to 4 significant figures.

(3)

The number of bacteria N in the second population is modelled by the equation

$$N = 60\,000e^{-0.6t} \qquad t \geqslant 0$$

where t is the time in hours from the start of the study.

(b) Find the rate of decrease of bacteria in this population exactly 5 hours from the start of the study. Give your answer to 3 significant figures.

(2)

When $t = T$, the number of bacteria in the two different populations was the same.

(c) Find the value of T, giving your answer to 3 significant figures.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(3)

WORKED SOLUTION

Question 7 · WMA13/01 May/June 2023

1 **Part (a)**

At $t = 0$, $2500 = Ae^0$, so $A = 2500$ exactly. At $t = 8$, $10000 = 2500e^{8k}$; hence $e^{8k} = 4$ and $k = \ln 4/8 = 0.1732867951\ldots$. To four significant figures:

$$\boxed{A = 2500, \qquad k = 0.1733}$$

2 Part (b)

For the second population,

$$\frac{dN}{dt} = -36000e^{-0.6t}, \qquad \left.\frac{dN}{dt}\right|_{t=5} = -36000e^{-3} = -1792.334461\ldots.$$

The requested rate of decrease is the positive magnitude, to three significant figures.

$$\boxed{1.79 \times 10^3 \text{ bacteria per hour}}$$

3 Part (c)

Retain the exact value $k = \ln 4/8$. Equating populations gives

$$2500e^{kT} = 60000e^{-0.6T} \quad \Longrightarrow \quad e^{(k+0.6)T} = 24.$$

Take natural logs and solve:

$$T = \frac{\ln 24}{0.6 + \ln 4/8} = 4.109799689\ldots \text{ hours}.$$

$$\boxed{T = 4.11 \text{ hours}}$$

FINAL ANSWER

(a) $A = 2500, \qquad k = 0.1733$ (b) 1.79×10^3 bacteria per hour (c) $T = 4.11$ hours

Question 8

9 marks | Differentiation Techniques | Main topic: Applications of Differentiation

8.

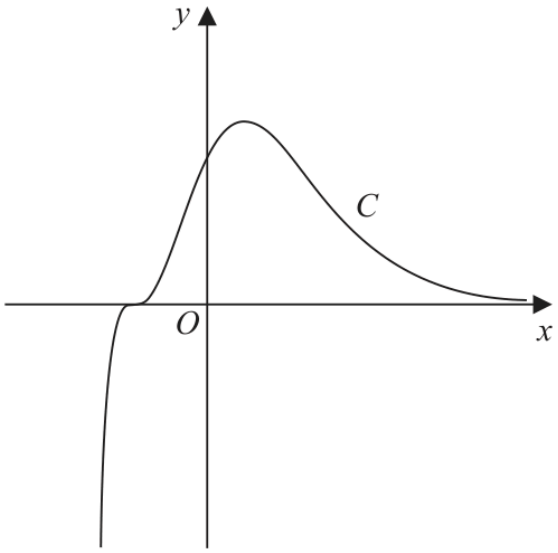


Figure 3

Figure 3 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = (2x + 1)^3 \mathrm{e}^{-4x}$$

(a) Show that

$$\mathrm{f}'(x) = A(2x + 1)^2 (1 - 4x)\, \mathrm{e}^{-4x}$$

where A is a constant to be found.

(4)

(b) Hence find the exact coordinates of the two stationary points on C.

(3)

The function g is defined by

$$\mathrm{g}(x) = 8\,\mathrm{f}(x - 2)$$

(c) Find the coordinates of the maximum stationary point on the curve with equation $y = \mathrm{g}(x)$.

(2)

1 Part (a)

The product rule gives

$$f'(x) = 6(2x+1)^2e^{-4x} - 4(2x+1)^3e^{-4x} = (2x+1)^2e^{-4x}[6 - 4(2x+1)] = 2(2x+1)^2(1-4x)e^{-4x}.$$

Comparison with the stated form gives $A = 2$.

$$\boxed{A = 2}$$

2 Part (b)

Since $e^{-4x} > 0$, $f'(x) = 0$ means $2x + 1 = 0$ or $1 - 4x = 0$. Thus $x = -1/2$ or $x = 1/4$.
Substitution gives $f(-1/2) = 0$ and $f(1/4) = (3/2)^3e^{-1} = 27/(8e)$.

$$\boxed{\left(-\frac{1}{2}, 0\right), \quad \left(\frac{1}{4}, \frac{27}{8e}\right)}$$

3 Part (c)

The derivative changes from positive to negative at $x = 1/4$, so this is the maximum; at $x = -1/2$ it does not change sign. The transformation $g(x) = 8f(x-2)$ sends (x, y) to $(x+2, 8y)$. Hence the maximum stationary point is $(1/4 + 2, 8 \cdot 27/(8e)) = (9/4, 27/e)$.

$$\boxed{\left(\frac{9}{4}, \frac{27}{e}\right)}$$

FINAL ANSWER

(a) $A = 2$ (b) $\left(-\frac{1}{2}, 0\right), \quad \left(\frac{1}{4}, \frac{27}{8e}\right)$ (c) $\left(\frac{9}{4}, \frac{27}{e}\right)$

Question 7

10 marks | Differentiation Techniques

Main topic: Functions, Composition & Inverses

7.

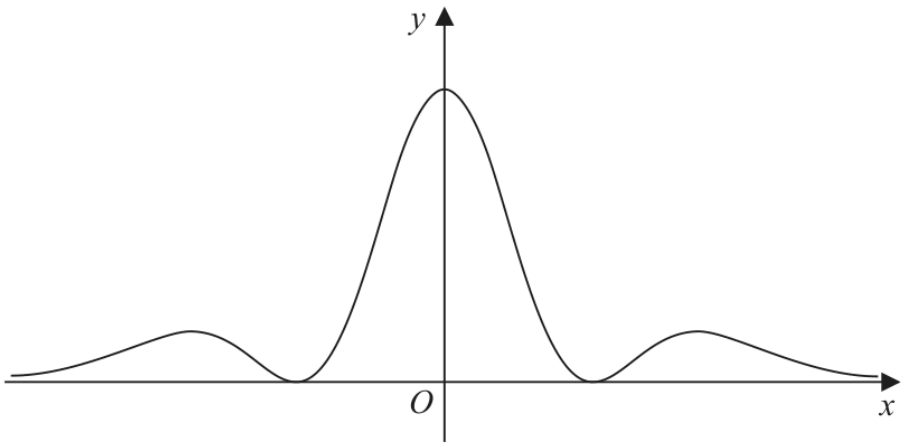


Figure 1

Figure 1 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \mathrm{e}^{-x^2}\left(2x^2 - 3\right)^2$$

(a) Find the range of f

(2)

(b) Show that

$$\mathrm{f}'(x) = 2x\left(2x^2 - 3\right)\mathrm{e}^{-x^2}\left(A - Bx^2\right)$$

where A and B are constants to be found.

(4)

Given that the line $y = k$, where k is a constant, $k > 0$, intersects the curve at exactly two distinct points,

(c) find the exact range of values of k

(4)

1 Part (a)

The exponential is positive and the squared factor is nonnegative, so $f(x) \geq 0$. The value zero is attained at $x = \pm\sqrt{3/2}$. The graph has its highest point at $x = 0$, where $f(0) = 9$; the outer maxima are lower. Thus both endpoints of the range are included.

$$\boxed{0 \leq f(x) \leq 9}$$

2 Part (b)

For $f(x) = e^{-x^2}(2x^2 - 3)^2$, product and chain rules give

$$f'(x) = -2xe^{-x^2}(2x^2 - 3)^2 + e^{-x^2}\,2(2x^2 - 3)\,4x.$$

Factoring yields

$$f'(x) = 2x(2x^2 - 3)e^{-x^2}\big(-(2x^2 - 3) + 4\big) = 2x(2x^2 - 3)e^{-x^2}(7 - 2x^2).$$

$$\boxed{A = 7, \qquad B = 2}$$

3 Part (c)

The stationary abscissae are 0, $\pm\sqrt{3/2}$ and $\pm\sqrt{7/2}$. At the outer local maxima,

$$f\left(\pm\sqrt{7/2}\right) = e^{-7/2}(7 - 3)^2 = 16e^{-7/2}.$$

The central maximum is 9 and the two minima are zero. Also $f(x) \to 0$ as $x \to \pm\infty$. A positive horizontal line above both outer maxima but below 9 meets only the two sides of the central peak. At $k = 16e^{-7/2}$ it also touches both outer peaks, giving four distinct intersections; at $k = 9$ there is only one. Both bounds must be excluded.

$$\boxed{16e^{-7/2} < k < 9}$$

FINAL ANSWER

(a) $0 \leq f(x) \leq 9$ (b) $A = 7, \qquad B = 2$ (c) $16e^{-7/2} < k < 9$

Question 6

7 marks | Differentiation Techniques

Main topic: Applications of Differentiation

6.

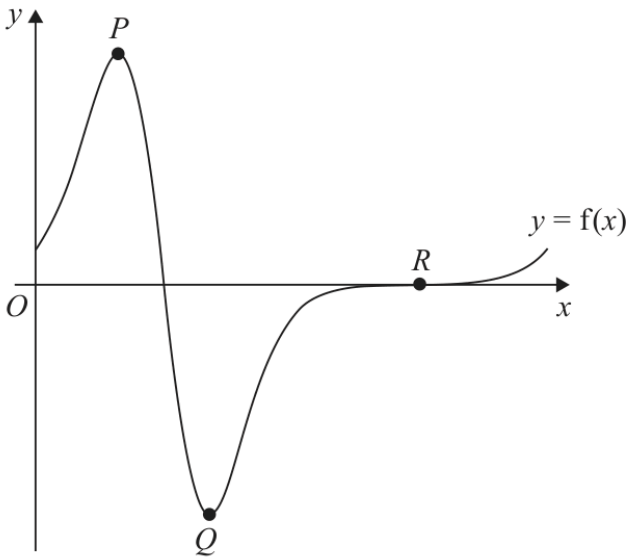


Figure 1

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 1 shows a sketch of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 2\mathrm{e}^{3\sin x}\cos x \qquad 0 \leqslant x \leqslant 2\pi$$

The curve intersects the x-axis at point R, as shown in Figure 1.

(a) State the coordinates of R

(1)

The curve has two turning points, at point P and point Q, also shown in Figure 1.

(b) Show that, at points P and Q,

$$a\sin^2 x + b\sin x + c = 0$$

where a, b and c are integers to be found.

(4)

(c) Hence find the x coordinate of point Q, giving your answer to 3 decimal places.

(2)

1 Part (a)

Since $2e^{3\sin x} > 0$, zeros occur when $\cos x = 0$. The labelled point R is the right-hand crossing, at $x = 3\pi/2$.

$$\boxed{R = \left(\frac{3\pi}{2}, 0\right)}$$

2 Part (b)

For $f(x) = 2e^{3\sin x}\cos x$, the product and chain rules give

$$f'(x) = 2\left(3e^{3\sin x}\cos^2 x - e^{3\sin x}\sin x\right) = 2e^{3\sin x}(3\cos^2 x - \sin x).$$

At a turning point the derivative is zero. The exponential factor is nonzero, so $3\cos^2 x - \sin x = 0$. Substitute $\cos^2 x = 1 - \sin^2 x$ and rearrange: $3 - 3\sin^2 x - \sin x = 0$.

$$\boxed{3\sin^2 x + \sin x - 3 = 0, \quad a = 3,\ b = 1,\ c = -3}$$

3 Part (c)

Solving $3s^2 + s - 3 = 0$ for $s = \sin x$ gives

$$s = \frac{-1 \pm \sqrt{37}}{6}.$$

The negative root is less than -1, so it is impossible for a real sine. Let $\beta = \arcsin\left((\sqrt{37} - 1)/6\right)$ in the first quadrant. The labelled minimum Q lies in the second quadrant, hence $x_Q = \pi - \beta = 2.131037293\ldots$.

$$\boxed{x_Q = 2.131}$$

FINAL ANSWER

(a) $R = \left(\frac{3\pi}{2}, 0\right)$ (b) $3\sin^2 x + \sin x - 3 = 0, \quad a = 3,\ b = 1,\ c = -3$ (c) $x_Q = 2.131$

Question 7

12 marks | Differentiation Techniques | Main topic: Applications of Differentiation

7. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

The curve C has equation

$$y = \frac{16}{9(3x - k)} \qquad x \neq \frac{k}{3}$$

where k is a positive constant not equal to 3

(a) Find $\frac{dy}{dx}$ giving your answer in simplest form in terms of k.

(2)

The point P with x coordinate 1 lies on C.

Given that the gradient of the curve at P is -12

(b) find the two possible values of k

(3)

Given also that $k < 3$

(c) find the equation of the normal to C at P, writing your answer in the form $ax + by + c = 0$, where a, b and c are integers to be found.

(3)

(d) show, using algebraic integration that,

$$\int_1^3 \frac{16}{9(3x - k)} dx = \lambda \ln 10$$

where λ is a constant to be found.

(4)

1 Part (a)

Write $y = (16/9)(3x-k)^{-1}$. The chain rule gives

$$\frac{dy}{dx} = \frac{16}{9}(-1)(3x-k)^{-2}\cdot 3.$$

$$\boxed{\frac{dy}{dx} = -\frac{16}{3(3x-k)^2}}$$

2 Part (b)

At $x = 1$, the given gradient yields

$$-\frac{16}{3(3-k)^2} = -12 \quad\Longrightarrow\quad (3-k)^2 = \frac{4}{9}.$$

Hence $3-k = \pm 2/3$. Both resulting k-values are positive and different from 3.

$$\boxed{k = \frac{7}{3} \text{ or } \frac{11}{3}}$$

3 Part (c)

Since $k < 3$, use $k = 7/3$. At $x = 1$, $y = 16/[9(3-7/3)] = 8/3$. The tangent gradient is -12, so the normal gradient is $1/12$:

$$y - \frac{8}{3} = \frac{1}{12}(x-1), \qquad 12y - 32 = x - 1.$$

$$\boxed{x - 12y + 31 = 0}$$

4 Part (d)

With $k = 7/3$, the denominator does not vanish on $[1,3]$. Using the reciprocal-linear antiderivative,

$$\int_1^3 \frac{16}{9(3x-7/3)}\,dx = \left[\frac{16}{27}\ln|3x-7/3|\right]_1^3 = \frac{16}{27}\left(\ln\frac{20}{3} - \ln\frac{2}{3}\right) = \frac{16}{27}\ln\left(\frac{20/3}{2/3}\right).$$

$$\boxed{\frac{16}{27}\ln 10, \quad \lambda = \frac{16}{27}}$$

FINAL ANSWER

(a) $\frac{dy}{dx} = -\frac{16}{3(3x-k)^2}$ (b) $k = \frac{7}{3}$ or $\frac{11}{3}$ (c) $x - 12y + 31 = 0$ (d) $\frac{16}{27}\ln 10, \quad \lambda = \frac{16}{27}$

Question 6

9 marks | Differentiation Techniques | Main topic: Applications of Differentiation

6.

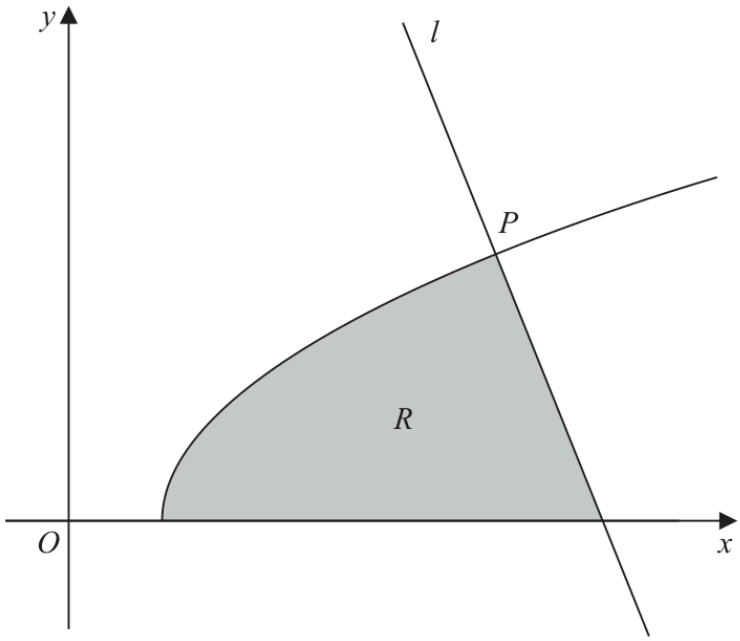


Figure 3

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 3 shows a sketch of part of the curve with equation

$$y = \sqrt{4x - 7}$$

The line l, shown in Figure 3, is the normal to the curve at the point $P(8, 5)$

(a) Use calculus to show that an equation of l is

$$5x + 2y - 50 = 0$$

(5)

The region R, shown shaded in Figure 3, is bounded by the curve, the x-axis and l.

(b) Use algebraic integration to find the exact area of R.

(4)

1 Part (a)

For $y = (4x-7)^{1/2}$,

$$\frac{dy}{dx} = \frac{1}{2}(4x-7)^{-1/2}\cdot 4 = \frac{2}{\sqrt{4x-7}}.$$

At $P=(8,5)$, the tangent gradient is $2/5$, so the normal gradient is $-5/2$. The point-gradient equation is

$$y-5 = -\frac{5}{2}(x-8), \qquad 2y-10 = -5x+40.$$

$$\boxed{5x+2y-50=0}$$

2 Part (b)

The curve meets the x-axis at $x=7/4$. The normal meets it at $x=10$. Split the shaded region at P, into a curve area from $7/4$ to 8 and a triangle from 8 to 10:

$$\int_{7/4}^{8}\sqrt{4x-7}\,dx = \left[\frac{(4x-7)^{3/2}}{6}\right]_{7/4}^{8} = \frac{125}{6}.$$

The triangle has area $\frac{1}{2}(10-8)\cdot 5 = 5$. Hence the total area is $125/6+5$.

$$\boxed{\text{Area}(R) = \frac{155}{6}}$$

FINAL ANSWER

(a) $5x+2y-50=0$ (b) $\text{Area}(R)=\frac{155}{6}$

Question 8

10 marks | Differentiation Techniques

Main topic: Log Graphs & Exponential Models

8.

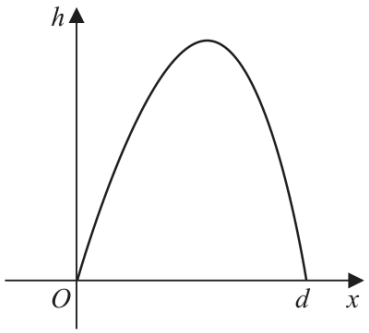


Figure 4

Figure 4 is a graph showing the path of a golf ball after the ball has been hit until it first hits the ground.

The vertical height, h metres, of the ball above the ground has been plotted against the horizontal distance travelled, x metres, measured from where the ball was hit.

The ball travels a horizontal distance of d metres before it first hits the ground.

The ball is modelled as a particle travelling in a vertical plane above horizontal ground.

The path of the ball is modelled by the equation

$$h = 1.5x - 0.5x\,e^{0.02x} \qquad 0 \leqslant x \leqslant d$$

Use the model to answer parts (a), (b) and (c).

(a) Find the value of d, giving your answer to 2 decimal places.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(3)

(b) Show that the maximum value of h occurs when

$$x = 50\ln\left(\frac{150}{x + 50}\right)$$

(4)

Using the iteration formula

$$x_{n+1} = 50\ln\left(\frac{150}{x_n + 50}\right) \qquad \text{with } x_1 = 30$$

(c) (i) find the value of x_2 to 2 decimal places,

(ii) find, by repeated iteration, the horizontal distance travelled by the golf ball before it reaches its maximum height. Give your answer to 2 decimal places.

(3)

1 Part (a)

At the first landing, $h = 0$ and $x = d > 0$:

$$0 = d(1.5 - 0.5e^{0.02d}), \qquad e^{0.02d} = 3.$$

Taking logarithms, $0.02d = \ln 3$, hence $d = 50\ln 3 = 54.930614\ldots$. The root $x = 0$ is the launch point, not the landing point.

$$\boxed{d = 54.93 \text{ m}}$$

2 Part (b)

By the product and chain rules,

$$h'(x) = 1.5 - 0.5e^{0.02x} - 0.01xe^{0.02x}.$$

At a stationary point,

$$1.5 = 0.01(x + 50)e^{0.02x}, \qquad e^{0.02x} = \frac{150}{x + 50}.$$

Since $x + 50 > 0$, logarithms give $0.02x = \ln(150/(x + 50))$. Also $h''(x) = -(0.02 + 0.0002x)e^{0.02x} < 0$ on $[0, d]$. Thus the stationary point is the unique maximum; the endpoints both have zero height.

$$\boxed{x = 50\ln\left(\frac{150}{x + 50}\right)}$$

3 Part (c)

(i) With $x_1 = 30$,

$$x_2 = 50\ln\left(\frac{150}{30 + 50}\right) = 31.43043297\ldots.$$

(ii) Continuing with full intermediate precision gives $x_3 = 30.54431108\ldots$, $x_4 = 31.09139065\ldots$, $x_5 = 30.75292479\ldots$ and $x_6 = 30.96205562\ldots$. Continued iteration converges to $30.88212\ldots$, which lies in $(0, d)$. This is the horizontal distance at maximum height, not the maximum height itself.

$$\boxed{x_2 = 31.43, \quad x_{\text{maximum}} = 30.88 \text{ m}}$$

FINAL ANSWER

(a) $d = 54.93$ m (b) $x = 50\ln\left(\frac{150}{x+50}\right)$ (c) $x_2 = 31.43, \quad x_{\text{maximum}} = 30.88$ m

Question 9

10 marks | Differentiation Techniques | Main topic: Applications of Differentiation

9.

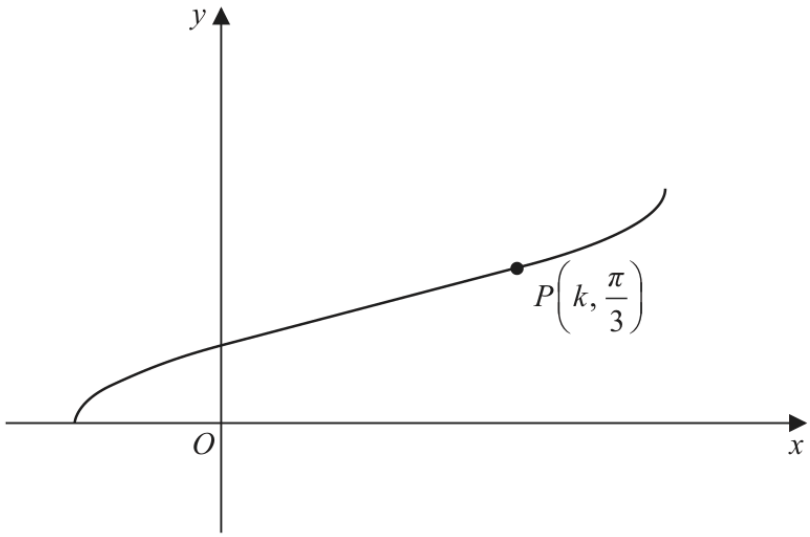


Figure 5

The curve shown in Figure 5 has equation

$$x = 4\sin^2 y - 1 \qquad 0 \leqslant y \leqslant \frac{\pi}{2}$$

The point $P\left(k, \frac{\pi}{3}\right)$ lies on the curve.

(a) Verify that $k = 2$

(1)

(b) (i) Find $\frac{\mathrm{d}x}{\mathrm{d}y}$ in terms of y

(ii) Hence show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\sqrt{x+1}\sqrt{3-x}}$

(6)

The normal to the curve at P cuts the x-axis at the point N.

(c) Find the exact area of triangle OPN, where O is the origin.

Give your answer in the form $a\pi + b\pi^2$ where a and b are constants.

(3)

1 Part (a)

At $y = \pi/3$, $x = 4(\sqrt{3}/2)^2 - 1 = 3 - 1 = 2$.

$$\boxed{k = 2}$$

2 Part (b)

(i) Differentiating with respect to y gives

$$\frac{dx}{dy} = 8\sin y \cos y.$$

(ii) For $0 < y < \pi/2$, both sine and cosine are positive, and $-1 < x < 3$. From $x + 1 = 4\sin^2 y$,

$$\sin y = \frac{\sqrt{x+1}}{2}, \qquad \cos y = \sqrt{1 - \frac{x+1}{4}} = \frac{\sqrt{3-x}}{2}.$$

Consequently $dx/dy = 2\sqrt{x+1}\sqrt{3-x}$. Taking the reciprocal proves the required formula on $-1 < x < 3$. At $x = -1$ and $x = 3$ the denominator is zero: these are endpoint vertical tangents, not finite derivatives.

$$\boxed{\frac{dx}{dy} = 8\sin y\cos y, \qquad \frac{dy}{dx} = \frac{1}{2\sqrt{x+1}\sqrt{3-x}}}$$

3 Part (c)

At $P = (2, \pi/3)$ the tangent gradient is $1/(2\sqrt{3})$, so the normal gradient is $-2\sqrt{3}$. Thus the normal is $y - \pi/3 = -2\sqrt{3}(x-2)$. Setting $y = 0$ gives $N = (2 + \pi/(6\sqrt{3}), 0)$. Triangle OPN has horizontal base $ON = 2 + \pi/(6\sqrt{3})$ and perpendicular height $\pi/3$:

$$\text{Area}(OPN) = \frac{1}{2}\left(2 + \frac{\pi}{6\sqrt{3}}\right)\frac{\pi}{3} = \frac{\pi}{3} + \frac{\pi^2}{36\sqrt{3}}.$$

$$\boxed{\text{Area}(OPN) = \frac{1}{3}\pi + \frac{\sqrt{3}}{108}\pi^2}$$

FINAL ANSWER

(a) $k = 2$ (b) $\frac{dx}{dy} = 8\sin y\cos y$, $\frac{dy}{dx} = \frac{1}{2\sqrt{x+1}\sqrt{3-x}}$ (c) $\text{Area}(OPN) = \frac{1}{3}\pi + \frac{\sqrt{3}}{108}\pi^2$

Question 7

10 marks | Differentiation Techniques | Main topic: Graph Transformations

7.

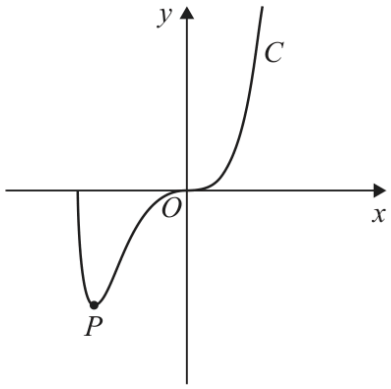


Figure 3

The curve C has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x^3\sqrt{4x+7} \qquad x \geqslant -\frac{7}{4}$$

(a) Show that

$$\mathrm{f}'(x) = \frac{kx^2(2x+3)}{\sqrt{4x+7}}$$

where k is a constant to be found.

(4)

The point P, shown in Figure 3, is the minimum turning point on C.

(b) Find the coordinates of P.

(2)

(c) Hence find the range of the function g defined by

$$\mathrm{g}(x) = -4\,\mathrm{f}(x) \qquad x \geqslant -\frac{7}{4}$$

(2)

The point Q with coordinates $\left(\frac{1}{2}, \frac{3}{8}\right)$ lies on C.

(d) Find the coordinates of the point to which Q is mapped when C is transformed to the curve with equation

$$y = 40\,\mathrm{f}\left(x - \frac{3}{2}\right) - 8$$

(2)

1 Part (a)

For $x > -7/4$, apply the product and chain rules:

$$f'(x) = 3x^2\sqrt{4x+7} + x^3\frac{2}{\sqrt{4x+7}} = \frac{3x^2(4x+7)+2x^3}{\sqrt{4x+7}} = \frac{7x^2(2x+3)}{\sqrt{4x+7}}.$$

$$\boxed{k = 7}$$

2 Part (b)

The interior stationary values satisfy $x^2(2x+3)=0$, so $x=0$ or $x=-3/2$. Near $x=-3/2$, the derivative changes from negative to positive; near $x=0$ it stays positive on both sides. Thus the minimum turning point is at $x=-3/2$.

$$f(-3/2) = (-3/2)^3\sqrt{-6+7} = -27/8.$$

$$\boxed{P = \left(-\frac{3}{2}, -\frac{27}{8}\right)}$$

3 Part (c)

The endpoint has $f(-7/4)=0$, the global minimum is $-27/8$ and $f(x)$ tends to infinity as x increases. Hence the range of f is $[-27/8, \infty)$. Multiplication by -4 reverses the inequality and makes the upper endpoint $27/2$.

$$\boxed{g(x) \le \frac{27}{2}}$$

4 Part (d)

The transformation is a shift right by $3/2$, vertical stretch by 40, then shift down by 8:

$$(x, y) \mapsto (x + 3/2,\, 40y - 8).$$

Therefore $Q = (1/2, 3/8)$ maps to $(1/2 + 3/2,\ 40\cdot 3/8 - 8)$.

$$\boxed{Q \mapsto (2, 7)}$$

FINAL ANSWER

(a) $k = 7$ (b) $P = \left(-\frac{3}{2}, -\frac{27}{8}\right)$ (c) $g(x) \le \frac{27}{2}$ (d) $Q \mapsto (2, 7)$

Question 8

12 marks | Differentiation Techniques | Main topic: Log Graphs & Exponential Models

8.

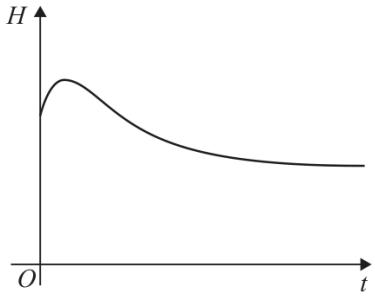


Figure 4

The heart rate of a horse is being monitored.
The heart rate H, measured in beats per minute (bpm), is modelled by the equation

$$H = 32 + 40e^{-0.2t} - 20e^{-0.9t}$$

where t minutes is the time after monitoring began.

Figure 4 is a sketch of H against t.

Use the equation of the model to answer parts (a) to (e).

(a) State the initial heart rate of the horse.

(1)

In the long term, the heart rate of the horse approaches L bpm.

(b) State the value of L.

(1)

The heart rate of the horse reaches its maximum value after T minutes.

(c) Find the value of T, giving your answer to 3 decimal places.

(*Solutions based entirely on calculator technology are not acceptable.*)

(5)

The heart rate of the horse is 37 bpm after M minutes.

(d) Show that M is a solution of the equation

$$t = 5\ln\left(\frac{8}{1 + 4e^{-0.9t}}\right)$$

(2)

Using the iteration formula

$$t_{n+1} = 5\ln\left(\frac{8}{1 + 4e^{-0.9t_n}}\right) \quad \text{with} \quad t_1 = 10$$

(e) (i) find, to 4 decimal places, the value of t_2

(ii) find, to 4 decimal places, the value of M

(3)

1 Part (a)

At $t = 0$, $H = 32 + 40 - 20 = 52$.

$$\boxed{H(0) = 52 \text{ bpm}}$$

2 Part (b)

As $t \to \infty$, both decaying exponential terms tend to zero.

$$\boxed{L = 32}$$

3 Part (c)

$$\frac{dH}{dt} = -8e^{-0.2t} + 18e^{-0.9t}.$$

At the maximum,

$$18e^{-0.9T} = 8e^{-0.2T} \quad \Longrightarrow \quad e^{0.7T} = \frac{9}{4} \quad \Longrightarrow \quad T = \frac{10}{7}\ln\frac{9}{4} = 1.1584717\ldots.$$

Also $H''(t) = 1.6e^{-0.2t} - 16.2e^{-0.9t}$. At the stationary point, $e^{-0.9T} = (4/9)e^{-0.2T}$, so $H''(T) = -5.6e^{-0.2T} < 0$, confirming a maximum.

$$\boxed{T = 1.158 \text{ minutes}}$$

4 Part (d)

For $H = 37$,

$$5 = 40e^{-0.2t} - 20e^{-0.9t} \quad \Longrightarrow \quad 8e^{-0.2t} = 1 + 4e^{-0.9t}.$$

Taking logarithms gives

$$-0.2t = \ln\left(\frac{1 + 4e^{-0.9t}}{8}\right) \quad \Longrightarrow \quad t = -5\ln\left(\frac{1 + 4e^{-0.9t}}{8}\right).$$

Using $-\ln u = \ln(1/u)$ for $u > 0$ then gives

$$t = 5\ln\left(\frac{8}{1 + 4e^{-0.9t}}\right).$$

$$\boxed{t = 5\ln\left(\frac{8}{1 + 4e^{-0.9t}}\right)}$$

5 **Part (e)**

Starting with $t_1 = 10$,

$$t_2 = 5\ln\left(\frac{8}{1+4e^{-9}}\right) = 10.3947401213\ldots.$$

Continuing without rounding intermediate values gives $t_3 = 10.3954778346\ldots$, $t_4 = 10.3954789826\ldots$ and $t_5 = 10.3954789844\ldots$. Thus (i) $t_2 = 10.3947$, and (ii) $M = 10.3955$ minutes, both to 4 decimal places.

$$\boxed{t_2 = 10.3947, \quad M = 10.3955 \text{ minutes}}$$

✓ **FINAL ANSWER**

(a) $H(0) = 52$ bpm (b) $L = 32$ (c) $T = 1.158$ minutes (d) $t = 5\ln\left(\frac{8}{1+4e^{-0.9t}}\right)$ (e) $t_2 = 10.3947, \quad M = 10.3955$ minutes

Question 9

14 marks | Differentiation Techniques | Main topic: Integration

9.

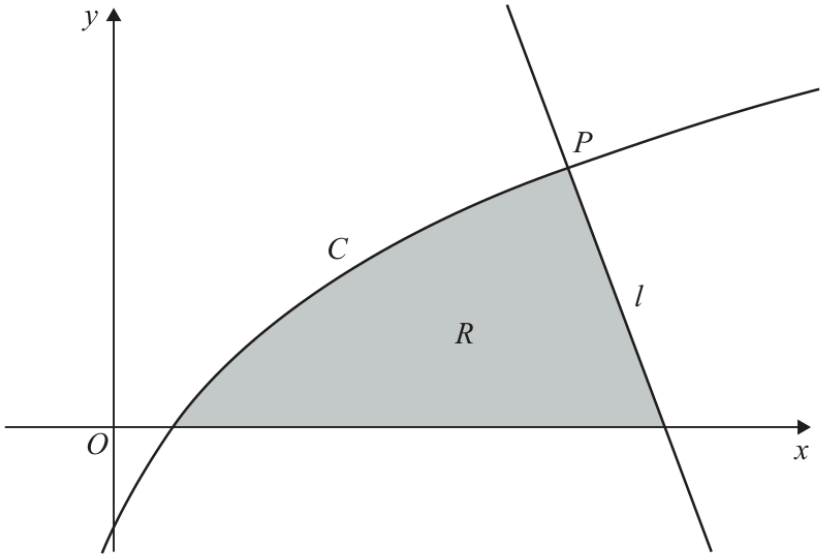


Figure 5

Figure 5 shows a sketch of part of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{6x^2 + 4x - 2}{2x + 1} \qquad x > -\frac{1}{2}$$

(a) Find $\mathrm{f}'(x)$, giving the answer in simplest form.

(3)

The line l is the normal to C at the point $P(2, 6)$

(b) Show that an equation for l is

$$16y + 5x = 106$$

(3)

(c) Write $\mathrm{f}(x)$ in the form $Ax + B + \dfrac{D}{2x + 1}$ where A, B and D are constants.

(3)

The region R, shown shaded in Figure 5, is bounded by C, l and the x-axis.

(d) Use algebraic integration to find the exact area of R, giving your answer in the form $P + Q\ln 3$, where P and Q are rational constants.

(*Solutions based entirely on calculator technology are not acceptable.*)

(5)

1 Part (a)

With $u = 6x^2 + 4x - 2$ and $v = 2x + 1$, the quotient rule gives

$$f'(x) = \frac{(12x + 4)(2x + 1) - 2(6x^2 + 4x - 2)}{(2x + 1)^2} = \frac{12x^2 + 12x + 8}{(2x + 1)^2}.$$

$$\boxed{f'(x) = \frac{4(3x^2 + 3x + 2)}{(2x + 1)^2}}$$

2 Part (b)

At $x = 2$, $f'(2) = 80/25 = 16/5$, so the normal gradient is $-5/16$. Through $P = (2, 6)$,

$$y - 6 = -\frac{5}{16}(x - 2) \quad \Longrightarrow \quad 16y - 96 = -5x + 10.$$

$$\boxed{16y + 5x = 106}$$

3 Part (c)

Multiplying the proposed identity by $2x + 1$ gives

$$6x^2 + 4x - 2 = (Ax + B)(2x + 1) + D = 2Ax^2 + (A + 2B)x + B + D.$$

Compare coefficients: $2A = 6$, $A + 2B = 4$, $B + D = -2$. Thus $f(x) = 3x + \frac{1}{2} - \frac{5}{2(2x+1)}$.

$$\boxed{A = 3, \quad B = \frac{1}{2}, \quad D = -\frac{5}{2}}$$

4 Part (d)

The curve meets the x-axis when $6x^2 + 4x - 2 = 2(3x - 1)(x + 1) = 0$. The root $x = -1$ is outside $x > -1/2$, leaving $x = 1/3$. The normal meets the axis at $x = 106/5$. Split the shaded region at P: a curve area from $1/3$ to 2, then a triangle.

$$\int_{1/3}^{2} f(x)\,dx = \left[\frac{3}{2}x^2 + \frac{1}{2}x - \frac{5}{4}\ln(2x + 1)\right]_{1/3}^{2} = 7 - \frac{1}{3} - \frac{5}{4}\ln\left(\frac{5}{5/3}\right) = \frac{20}{3} - \frac{5}{4}\ln 3.$$

The triangle has area $\frac{1}{2}(106/5 - 2) \cdot 6 = 288/5$. Adding gives the exact area of R.

$$\boxed{\text{Area}(R) = \frac{964}{15} - \frac{5}{4}\ln 3}$$

FINAL ANSWER

(a) $f'(x) = \frac{4(3x^2+3x+2)}{(2x+1)^2}$ (b) $16y + 5x = 106$ (c) $A = 3, \quad B = \frac{1}{2}, \quad D = -\frac{5}{2}$ (d) $\text{Area}(R) = \frac{964}{15} - \frac{5}{4}\ln 3$

Question 9

8 marks | Differentiation Techniques | Main topic: Applications of Differentiation

9.

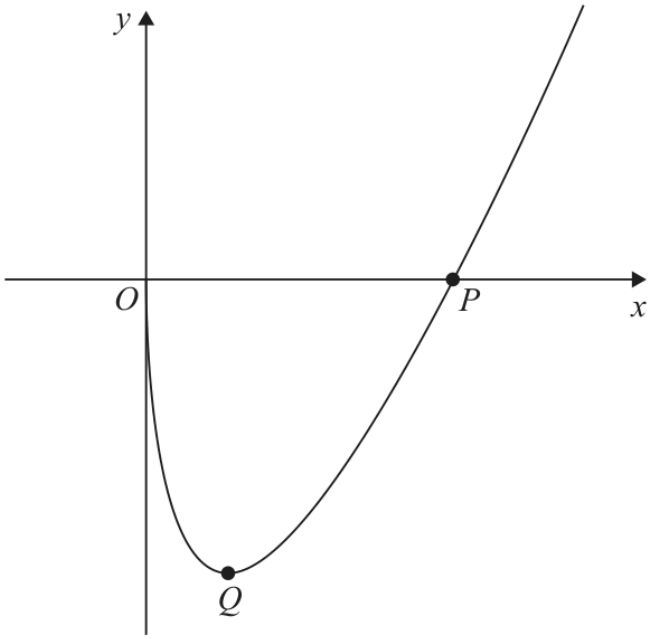


Figure 1

Figure 1 shows a sketch of part of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 6\sqrt{x}\ln(4x) \qquad x > 0$$

The curve cuts the x-axis at point P

(a) State the x coordinate of P

(1)

The point Q, shown in Figure 1, is the stationary point on C

(b) Use calculus to find the exact coordinates of Q

(5)

(c) Hence find the range of the function $\mathrm{g}(x)$ where

$$\mathrm{g}(x) = -2\,\mathrm{f}(x)$$

(2)

1 Part (a)

Since $x > 0$, $6\sqrt{x}$ is positive. Therefore $f(x) = 0$ requires $\ln(4x) = 0$, so $4x = 1$ and $x = 1/4$.

$$\boxed{x_P = \frac{1}{4}}$$

2 Part (b)

Apply the product and chain rules:

$$f'(x) = \frac{3}{\sqrt{x}}\ln(4x) + 6\sqrt{x}\frac{1}{x} = \frac{3}{\sqrt{x}}[\ln(4x) + 2].$$

At Q, set $f'(x) = 0$. Since $3/\sqrt{x} > 0$,

$$\ln(4x) + 2 = 0 \quad \Longrightarrow \quad 4x = e^{-2} \quad \Longrightarrow \quad x = \frac{1}{4e^2}.$$

Then $\sqrt{x} = 1/(2e)$ and $\ln(4x) = -2$, giving $f(x) = 6(1/(2e))(-2) = -6/e$.

$$\boxed{Q = \left(\frac{1}{4e^2}, -\frac{6}{e}\right)}$$

3 Part (c)

The factor $3/\sqrt{x}$ is positive and $\ln(4x) + 2$ increases through zero at Q, so f' changes from negative to positive: Q is the global minimum. As $x \to \infty$, $f(x) \to \infty$, hence the range of f is $[-6/e, \infty)$. Multiplication by -2 reverses the inequality and includes the maximum $12/e$.

$$\boxed{g(x) \leq \frac{12}{e}}$$

FINAL ANSWER

(a) $x_P = \frac{1}{4}$ (b) $Q = \left(\frac{1}{4e^2}, -\frac{6}{e}\right)$ (c) $g(x) \leq \frac{12}{e}$

WMA13/01 MAY/JUNE 2025

Question 6

8 marks | Differentiation Techniques | Main topic: Log Graphs & Exponential Models

6: **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The temperature, θ°C, of a computer processor, t minutes after the computer is switched off, is modelled by the equation

$$\theta = 21 + A\mathrm{e}^{-kt}$$

where A and k are positive constants.

Given that the temperature of the processor was 75°C when the computer was switched off,

(a) find the value of A.

(2)

Given also that it takes 5 minutes for the temperature of the processor to decrease from 75°C to 25°C,

(b) find the value of k, giving your answer to 3 significant figures.

(3)

At time T minutes, the temperature of the processor is decreasing at a rate of 9°C per minute.

(c) Find the value of T according to the model, giving your answer to 2 decimal places.

(3)

WORKED SOLUTION

Question 6 · WMA13/01 May/June 2025

Part (a)

At $t = 0$, $75 = 21 + A$, so $\boxed{A = 54}$.

Part (b)

At $t = 5$, $25 = 21 + 54e^{-5k}$, hence

$$e^{-5k} = \frac{2}{27}, \qquad k = \frac{1}{5}\ln\frac{27}{2} = 0.520537937\ldots = \boxed{0.521}.$$

3 Part (c)

Differentiate the temperature model:

$$\frac{d\theta}{dt} = -54ke^{-kt}.$$

A decrease of $9\,^\circ\mathrm{C}$ per minute means $d\theta/dt = -9$, not $+9$. Thus

$$-54ke^{-kT} = -9 \implies e^{-kT} = \frac{1}{6k} \implies T = \frac{\ln(6k)}{k}.$$

Using the unrounded value of k,

$$T = 2.187865\ldots = \boxed{2.19 \text{ minutes}}.$$

FINAL ANSWER

(a) $A = 54$ (b) 0.521 (c) 2.19 minutes

Question 7

9 marks | Differentiation Techniques | Main topic: Numerical Methods

7: A continuous curve has equation

$$y = e^{-x^2}\sin 3x \qquad 0 \leqslant x \leqslant \frac{\pi}{3}$$

The curve has a stationary point at the point P.

(a) Show, using calculus, that the x coordinate of P is a solution of the equation

$$x = \frac{1}{3}\arctan\left(\frac{3}{2x}\right)$$

(4)

Using the iteration formula

$$x_{n+1} = \frac{1}{3}\arctan\left(\frac{3}{2x_n}\right) \qquad x_1 = 0.4$$

(b) find the value of

(i) x_2

(ii) x_4

giving your answers to 4 decimal places.

(3)

(c) Using a suitable interval and a suitable function which should be stated, show that the x coordinate of P is 0.430 correct to 3 decimal places.

(2)

1 Part (a)

Apply the product and chain rules:

$$\frac{dy}{dx} = e^{-x^2}\left(3\cos 3x - 2x\sin 3x\right).$$

Since $e^{-x^2} > 0$, a stationary point satisfies

$$3\cos 3x = 2x\sin 3x.$$

Neither endpoint $x = 0, \pi/3$ is stationary. In the interior $\sin 3x > 0$ and $x > 0$, so $\cos 3x > 0$ at a stationary point and $0 < 3x < \pi/2$. Therefore division and the principal inverse tangent are valid:

$$\tan 3x = \frac{3}{2x} \quad \Longrightarrow \quad \boxed{x = \frac{1}{3}\arctan\left(\frac{3}{2x}\right)}.$$

2 Part (b)

Work in radians and retain calculator precision between iterations:

$$x_2 = \frac{1}{3}\arctan\frac{3}{0.8} = 0.4367313117\ldots,$$

$$x_3 = 0.4291581882\ldots, \qquad x_4 = 0.4307116849\ldots.$$

Thus (i) $\boxed{x_2 = 0.4367}$ and (ii) $\boxed{x_4 = 0.4307}$.

3 Part (c)

Let $h(x) = 3\cos 3x - 2x\sin 3x$, which is continuous. At the rounding boundaries,

$$h(0.4295) = 0.0106861\ldots > 0, \qquad h(0.4305) = -0.000598027\ldots < 0.$$

The sign change places a root strictly between 0.4295 and 0.4305. On $0 < x < \pi/6$, $h'(x) = -11\sin 3x - 6x\cos 3x < 0$, so the stationary root is unique; for $\pi/6 \le x \le \pi/3$, $h(x) < 0$. Hence the x coordinate of P is $\boxed{0.430 \text{ to } 3 \text{ decimal places}}$.

FINAL ANSWER

(a) $x = \frac{1}{3}\arctan\left(\frac{3}{2x}\right)$ (b) $x_2 = 0.4367$, $x_4 = 0.4307$ (c) 0.430 to 3 decimal places

WMA13/01 OCTOBER 2025

Question 6

7 marks | Differentiation Techniques | Main topic: Integration

6. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(i) Given $y = \ln(2x^2 + 5)$ find $\dfrac{dy}{dx}$

(2)

(ii) A curve C has equation $y = f(x)$ where

$$f(x) = \frac{21x}{3x^2 + k} \qquad x \in \mathbb{R}$$

where k is a positive integer, $k > 1$

Given that

$$\int_1^k f(x)\,dx < 7\ln 8$$

find the greatest possible value of k

(5)

WORKED SOLUTION

Question 6 · WMA13/01 October 2025

1 **Part (i)**

Using the chain rule,

$$\frac{dy}{dx} = \frac{1}{2x^2 + 5}(4x) = \frac{4x}{2x^2 + 5}.$$

2 Part (ii)

Treat k as a constant when integrating:

$$\int_1^k \frac{21x}{3x^2+k}\,dx = \left[\frac{7}{2}\ln(3x^2+k)\right]_1^k = \frac{7}{2}\ln\left(\frac{3k^2+k}{k+3}\right).$$

The stated inequality therefore implies

$$\ln\left(\frac{3k^2+k}{k+3}\right) < 2\ln 8 = \ln 64.$$

Since the exponential is increasing and $k+3>0$,

$$3k^2+k < 64(k+3) \Longrightarrow k^2-21k-64<0.$$

The roots are $(21\pm\sqrt{697})/2$, so the positive upper bound is $k<(21+\sqrt{697})/2 = 23.700378782\ldots$. With $k>1$ an integer, the greatest possible value is $k=23$.

FINAL ANSWER

(i) $4*x/(2*x^2+5)$ (ii) 23

WMA13/01 OCTOBER 2025

Question 7

7 marks | Differentiation Techniques | Main topic: Applications of Differentiation

7. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

A curve C has equation

$$y = 2x\mathrm{e}^{x^2+(3k-2)x}$$

where k is a constant.

Given that C has two distinct turning points, find the range of possible values of k.

(7)

WORKED SOLUTION

Question 7 · WMA13/01 October 2025

1 Solution

\paragraph{Solution} Put $u = x^2 + (3k-2)x$, so $u' = 2x + 3k - 2$. By the product and chain rules,

$$\frac{dy}{dx} = 2e^u + 2xe^u(2x + 3k - 2) = 2e^u\big(2x^2 + (3k-2)x + 1\big).$$

As $2e^u > 0$, stationary points satisfy

$$2x^2 + (3k-2)x + 1 = 0.$$

For two distinct real roots, the discriminant must be positive:

$$(3k-2)^2 - 8 > 0 \iff |3k-2| > 2\sqrt{2}.$$

Consequently

$$k < \frac{2-2\sqrt{2}}{3} \quad \text{or} \quad k > \frac{2+2\sqrt{2}}{3}.$$

For these values the quadratic changes sign at both simple roots, so both stationary points are genuine turning points. Equality gives a repeated root with no derivative sign change and is excluded.

FINAL ANSWER

(main) $(-infinity, (2 - 2 * sqrt(2))/3) union ((2 + 2 * sqrt(2))/3, infinity)$

WMA13/01 JANUARY 2026

Question 10

9 marks | Differentiation Techniques | Main topic: Log Graphs & Exponential Models

10. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

Two populations of insects are being studied over the same period of time.

The number of insects N_A in population A, t years after the start of the study, is modelled by the equation

$$N_A = 8000 + 900e^{0.2t}$$

According to the model,

(a) find the rate of growth in the number of insects in population A exactly 5 years after the start of the study.

(2)

The number of insects N_B in population B, t years after the start of the study, is modelled by the equation

$$N_B = 8000 + Pe^{kt}$$

where P and k are positive constants.

Given that

- there are 10 000 insects in population B at the start of the study
- there are 11 570 insects in population B exactly 4 years after the start of the study
- there are the same number of insects in population A and population B after T years

find, according to the models,

(b) the exact value of P and the value of k to 3 decimal places,

(4)

(c) the value of T, giving your answer to one decimal place.

(3)

WORKED SOLUTION

Question 10 · WMA13/01 January 2026

1 **Part (a)**

$$\frac{dN_A}{dt} = 900(0.2)e^{0.2t} = 180e^{0.2t}.$$

At $t = 5$, the growth rate is $180e = 489.290729\ldots$ insects per year.

2 Part (b)

At $t = 0$, $10000 = 8000 + P$, so $P = 2000$ exactly. Using the population at $t = 4$,

$$11570 = 8000 + 2000e^{4k} \quad \Longrightarrow \quad e^{4k} = \frac{3570}{2000} = 1.785.$$

Taking natural logarithms,

$$k = \frac{\ln 1.785}{4} = 0.144854604\ldots = 0.145 \quad (3\text{ d.p.}).$$

---

3 Part (c)

At equality, $8000 + 900e^{0.2T} = 8000 + 2000e^{kT}$, hence

$$e^{(0.2-k)T} = \frac{20}{9}, \qquad T = \frac{\ln(20/9)}{0.2 - k}.$$

Use the unrounded value $k = \ln(1.785)/4$:

$$T = 14.480042784\ldots = 14.5 \text{ years (1 decimal place)}.$$

FINAL ANSWER

(a) $180e$ insects per year (b) $P = 2000,\ k = 0.145\ (3\text{ d.p.})$ (c) 14.5 years

Question 7

12 marks | Differentiation Techniques | Main topic: Log Graphs & Exponential Models

7. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The selling price of a car is monitored over a 15-year period.

The selling price P_C of the car, in thousands of pounds, t years after monitoring began, is modelled by the equation

$$P_C = 7 + 16e^{-0.1t} \qquad 0 \leqslant t \leqslant 15$$

Given that when $t = T$ the selling price of the car is £15 000

(a) find, according to the model, the value of T. Give the answer to 2 decimal places.

(3)

(b) Find the value of $\frac{dP_C}{dt}$ when $t = 5$. Give the answer to 2 decimal places.

(2)

The selling price of a van is monitored over the same 15-year period.

The selling price P_V of the van, in thousands of pounds, t years after monitoring began, is modelled by the equation

$$P_V = 4 + me^{-0.2t} \qquad 0 \leqslant t \leqslant 15$$

where m is a constant.

Use the equations of these models to answer parts (c) and (d).

Given that $P_V = 36$ when $t = 0$

(c) (i) state the value of m

(ii) find the difference between the initial selling price of the car and the initial selling price of the van.

(3)

When $t = \alpha$, the selling price of the car is equal to the selling price of the van.

(d) Find, using algebra and showing your working, the value of α, giving the answer to 3 significant figures.

(4)

1 **Part (a)**

Prices in the model are in thousands of pounds, so

$$15 = 7 + 16e^{-0.1T} \quad \Longrightarrow \quad e^{-0.1T} = \frac{1}{2}.$$

Taking logarithms gives $T = 10\ln 2 = 6.9314718\ldots$, hence $\boxed{T = 6.93 \text{ years}}$.

2 **Part (b)**

$$\frac{dP_C}{dt} = -1.6e^{-0.1t}, \qquad \left.\frac{dP_C}{dt}\right|_{t=5} = -1.6e^{-0.5} = -0.970449\ldots.$$

Thus the answer is $\boxed{-0.97 \text{ thousand pounds per year}}$ to 2 decimal places.

3 **Part (c)(i)**

At $t = 0$, $36 = 4 + m$, so $\boxed{m = 32}$.

4 **Part (c)(ii)**

The initial car price is $7 + 16 = 23$ thousand pounds. The difference is $36 - 23 = 13$ thousand pounds, or $\boxed{\pounds 13\,000}$.

5 **Part (d)**

Equating the models with $m = 32$ gives

$$7 + 16e^{-0.1\alpha} = 4 + 32e^{-0.2\alpha}.$$

Let $u = e^{-0.1\alpha} > 0$. Then $32u^2 - 16u - 3 = 0$, so

$$u = \frac{16 \pm \sqrt{256 + 384}}{64} = \frac{2 \pm \sqrt{10}}{8}.$$

Reject the negative root because an exponential is positive. Hence

$$\alpha = -10\ln\left(\frac{2 + \sqrt{10}}{8}\right) = 4.3806365\ldots, \qquad \boxed{\alpha = 4.38 \text{ years}}.$$

This value is inside the model interval $0 \le t \le 15$.

✓ **FINAL ANSWER**

(a) $T = 6.93$ years (b) -0.97 thousand pounds per year (c)(i) $m = 32$ (c)(ii) $\pounds 13\,000$ (d) $\alpha = 4.38$ years

TOPIC

Applications of Differentiation

Every past-paper question on this topic, January 2020 to June 2026.

Question 7

11 marks | Applications of Differentiation

7.

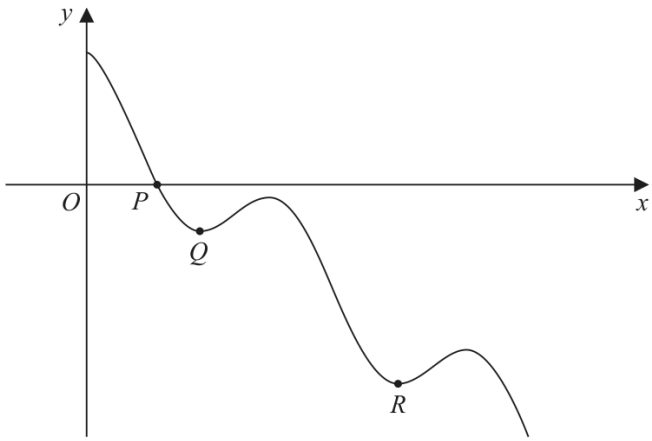


Figure 3

Figure 3 shows a sketch of part of the curve with equation

$$y = 2\cos 3x - 3x + 4 \qquad x > 0$$

where x is measured in radians.

The curve crosses the x-axis at the point P, as shown in Figure 3.

Given that the x coordinate of P is α,

(a) show that α lies between 0.8 and 0.9

(2)

The iteration formula

$$x_{n+1} = \frac{1}{3}\arccos(1.5x_n - 2)$$

can be used to find an approximate value for α.

(b) Using this iteration formula with $x_1 = 0.8$ find, to 4 decimal places, the value of

(i) x_2

(ii) x_5

(3)

The point Q and the point R are local minimum points on the curve, as shown in Figure 3.

Given that the x coordinates of Q and R are β and λ respectively, and that they are the two smallest values of x at which local minima occur,

(c) find, using calculus, the exact value of β and the exact value of λ.

(6)

1 Part (a)

Let $f(x) = 2\cos 3x - 3x + 4$, using radians. Then

$$f(0.8) = 2\cos 2.4 - 2.4 + 4 = 0.1252125\ldots > 0,$$
$$f(0.9) = 2\cos 2.7 - 2.7 + 4 = -0.5081443\ldots < 0.$$

The function is continuous on $[0.8, 0.9]$, so this change of sign shows that there is a root between the endpoints. Moreover, $f'(x) = -6\sin 3x - 3 < 0$ on this interval, so this root is unique there. Thus the crossing shown at P has $\boxed{0.8 < \alpha < 0.9}$.

2 Part (b)(i)

Using the inverse cosine in radians,

$$x_2 = \frac{1}{3}\arccos(1.5(0.8) - 2) = \frac{1}{3}\arccos(-0.8) = 0.8326971\ldots.$$

Hence $\boxed{x_2 = 0.8327}$ to 4 decimal places.

3 Part (b)(ii)

Continue the iteration, retaining the full calculator values at each step:

$$x_3 = \frac{1}{3}\arccos(1.5x_2 - 2) = 0.8067674\ldots,$$
$$x_4 = \frac{1}{3}\arccos(1.5x_3 - 2) = 0.8271196\ldots,$$
$$x_5 = \frac{1}{3}\arccos(1.5x_4 - 2) = 0.8110216\ldots.$$

Therefore $\boxed{x_5 = 0.8110}$ to 4 decimal places.

4 Part (c)

Differentiate the curve equation:

$$\frac{dy}{dx} = -6\sin 3x - 3, \qquad \frac{d^2y}{dx^2} = -18\cos 3x.$$

At a stationary point,

$$-6\sin 3x - 3 = 0 \quad \Longrightarrow \quad \sin 3x = -\frac{1}{2}.$$

For $x > 0$, the stationary points therefore belong to the two families

$$3x = \frac{7\pi}{6} + 2n\pi \quad \text{or} \quad 3x = \frac{11\pi}{6} + 2n\pi, \qquad n = 0, 1, 2, \ldots.$$

In the first family, $\cos 3x = -\frac{\sqrt{3}}{2}$, giving $y'' = 9\sqrt{3} > 0$, so these are local minima. In the second family, $\cos 3x = \frac{\sqrt{3}}{2}$ and $y'' = -9\sqrt{3} < 0$, so these are local maxima and are excluded. Thus all positive minimum coordinates are

$$x = \frac{7\pi}{18} + \frac{2n\pi}{3}, \qquad n = 0, 1, 2, \ldots.$$

The two smallest are obtained with $n = 0$ and $n = 1$:

$$\boxed{\beta = \frac{7\pi}{18}}, \qquad \boxed{\lambda = \frac{19\pi}{18}}.$$

FINAL ANSWER

(a) $0.8 < \alpha < 0.9$ (b)(i) $x_2 = 0.8327$ (b)(ii) $x_5 = 0.8110$ (c) $\beta = \frac{7\pi}{18}$, $\lambda = \frac{19\pi}{18}$

Question 9

14 marks | Applications of Differentiation

9. (a) Given that

$$\frac{x^4 - x^3 - 10x^2 + 3x - 9}{x^2 - x - 12} \equiv x^2 + P + \frac{Q}{x - 4} \qquad x > -3$$

find the value of the constant P and show that $Q = 5$

(4)

The curve C has equation $y = \mathrm{g}(x)$, where

$$\mathrm{g}(x) = \frac{x^4 - x^3 - 10x^2 + 3x - 9}{x^2 - x - 12} \qquad -3 < x < 3.5 \qquad x \in \mathbb{R}$$

(b) Find the equation of the tangent to C at the point where $x = 2$
Give your answer in the form $y = mx + c$, where m and c are constants to be found.

(5)

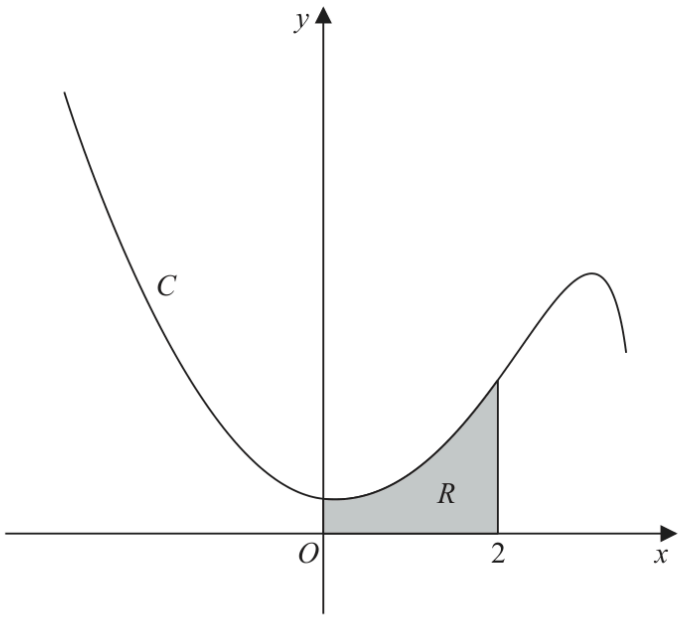


Figure 4

Figure 4 shows a sketch of the curve C.
The region R, shown shaded in Figure 4, is bounded by C, the y-axis, the x-axis and the line with equation $x = 2$

(c) Find the exact area of R, writing your answer in the form $a + b\ln 2$, where a and b are constants to be found.

(5)

1 Part (a)

Factor the denominator as $(x-4)(x+3)$ and multiply the proposed identity through by it:

$$x^4 - x^3 - 10x^2 + 3x - 9 = (x^2 + P)(x^2 - x - 12) + Q(x+3).$$

The right side expands to $x^4 - x^3 + (P-12)x^2 + (Q-P)x + 3Q - 12P$. Comparing coefficients gives $P - 12 = -10$, so $P = 2$, and $Q - P = 3$, so $Q = 5$. The constant check is $3(5) - 12(2) = -9$, as required. The identity holds where the original fraction is defined: the stated $x > -3$ still excludes $x = 4$.

$$\boxed{P = 2, \qquad Q = 5}$$

2 Part (b)

From (a), $g(x) = x^2 + 2 + 5/(x-4)$ on the curve domain. Thus $g'(x) = 2x - 5/(x-4)^2$. At $x = 2$, $g(2) = 4 + 2 - 5/2 = 7/2$ and $g'(2) = 4 - 5/4 = 11/4$. The tangent through $(2, 7/2)$ is

$$y - \frac{7}{2} = \frac{11}{4}(x-2) \quad \Longrightarrow \quad y = \frac{11}{4}x - 2.$$

$$\boxed{y = \frac{11}{4}x - 2}$$

3 Part (c)

The source shading is above the x-axis from $x = 0$ to $x = 2$, so integrate g over those bounds.

$$\text{Area}(R) = \int_0^2 \left(x^2 + 2 + \frac{5}{x-4}\right) dx = \left[\frac{x^3}{3} + 2x + 5\ln|x-4|\right]_0^2.$$

The absolute value is essential because $x - 4 < 0$ throughout this interval. Evaluating gives

$$\frac{8}{3} + 4 + 5\ln 2 - 5\ln 4 = \frac{20}{3} - 5\ln 2.$$

Thus $a = 20/3$ and $b = -5$ in the requested form.

$$\boxed{\text{Area}(R) = \frac{20}{3} - 5\ln 2}$$

FINAL ANSWER

(a) $P = 2, \quad Q = 5$ (b) $y = \frac{11}{4}x - 2$ (c) $\text{Area}(R) = \frac{20}{3} - 5\ln 2$

Question 6

10 marks | Applications of Differentiation

6.

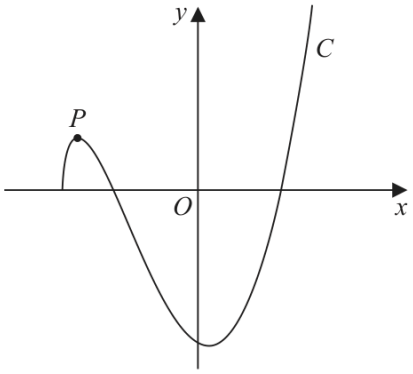


Figure 3

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

The function f is defined by

$$f(x) = 5(x^2 - 2)(4x + 9)^{\frac{1}{2}} \qquad x \geqslant -\frac{9}{4}$$

(a) Show that

$$f'(x) = \frac{k(5x^2 + 9x - 2)}{(4x + 9)^{\frac{1}{2}}}$$

where k is an integer to be found.

(4)

(b) Hence, find the values of x for which $f'(x) = 0$

(1)

Figure 3 shows a sketch of the curve C with equation $y = f(x)$.

The curve has a local maximum at the point P

(c) Find the exact coordinates of P

(2)

The function g is defined by

$$g(x) = 2f(x) + 4 \qquad -\frac{9}{4} \leqslant x \leqslant 0$$

(d) Find the range of g

(3)

1 Part (a)

Use product and chain rules:

$$f'(x) = 10x(4x+9)^{1/2} + 10(x^2-2)(4x+9)^{-1/2}.$$

With a common denominator,

$$f'(x) = \frac{10x(4x+9) + 10(x^2-2)}{\sqrt{4x+9}} = \frac{10(5x^2+9x-2)}{\sqrt{4x+9}}.$$

This derivative formula applies for $x > -9/4$; the endpoint derivative is not finite.

$$\boxed{k = 10}$$

2 Part (b)

The denominator is positive in the derivative domain. Thus $5x^2+9x-2 = (5x-1)(x+2) = 0$.

$$\boxed{x = -2, \quad \frac{1}{5}}$$

3 Part (c)

The derivative changes from positive to negative at $x = -2$, so this is the local maximum P. Substitute into f: $f(-2) = 5(4-2)\sqrt{-8+9} = 10$.

$$\boxed{P = (-2, 10)}$$

4 Part (d)

On $[-9/4, 0]$, f rises from $f(-9/4) = 0$ to $f(-2) = 10$, then decreases to $f(0) = -30$. The other stationary point $1/5$ is outside this restricted interval. Thus the minimum and maximum of f on it are -30 and 10. Applying the positive vertical transformation $g = 2f + 4$ gives endpoints $2(-30)+4 = -56$ and $2(10)+4 = 24$, both attained.

$$\boxed{-56 \le g(x) \le 24}$$

FINAL ANSWER

(a) $k = 10$ (b) $x = -2, \quad \frac{1}{5}$ (c) $P = (-2, 10)$ (d) $-56 \le g(x) \le 24$

WMA13/01 JANUARY 2023

Question 7

9 marks | Applications of Differentiation

7. The curve C has equation

$$x = 3\tan\left(y - \frac{\pi}{6}\right) \qquad x \in \mathbb{R} \quad -\frac{\pi}{3} < y < \frac{2\pi}{3}$$

(a) Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a}{x^2 + b}$$

where a and b are integers to be found.

(4)

The point P with y coordinate $\frac{\pi}{3}$ lies on C.

Given that the tangent to C at P crosses the x-axis at the point Q.

(b) find, in simplest form, the exact x coordinate of Q.

(5)

WORKED SOLUTION

Question 7 · WMA13/01 January 2023

1 **Part (a)**

Put $u = y - \pi/6$. Since $x = 3\tan u$,

$$\frac{dx}{dy} = 3\sec^2 u = 3(1 + \tan^2 u) = 3\left(1 + \frac{x^2}{9}\right) = \frac{x^2 + 9}{3}.$$

This is positive for all real x, so we may take the reciprocal: $dy/dx = 3/(x^2 + 9)$. Thus the required integers are $a = 3$, $b = 9$.

$$\boxed{\frac{dy}{dx} = \frac{3}{x^2 + 9}, \qquad a = 3, \quad b = 9}$$

2 **Part (b)**

At $y = \pi/3$, $x = 3\tan(\pi/3 - \pi/6) = 3\tan(\pi/6) = \sqrt{3}$. The tangent gradient is $3/((\sqrt{3})^2 + 9) = 1/4$. Its point-gradient equation is

$$y - \frac{\pi}{3} = \frac{1}{4}(x - \sqrt{3}).$$

At its x-axis crossing $y = 0$, $-4\pi/3 = x - \sqrt{3}$.

$$\boxed{x_Q = \sqrt{3} - \frac{4\pi}{3}}$$

✓ **FINAL ANSWER**

(a) $\frac{dy}{dx} = \frac{3}{x^2+9}$, $a = 3$, $b = 9$ (b) $x_Q = \sqrt{3} - \frac{4\pi}{3}$

Question 8

9 marks | Applications of Differentiation

8.

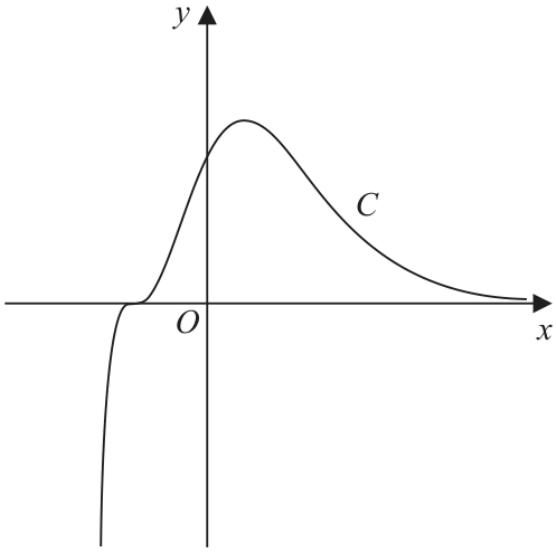


Figure 3

Figure 3 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = (2x + 1)^3 \mathrm{e}^{-4x}$$

(a) Show that

$$\mathrm{f}'(x) = A(2x + 1)^2 (1 - 4x)\, \mathrm{e}^{-4x}$$

where A is a constant to be found.

(4)

(b) Hence find the exact coordinates of the two stationary points on C.

(3)

The function g is defined by

$$\mathrm{g}(x) = 8\,\mathrm{f}(x - 2)$$

(c) Find the coordinates of the maximum stationary point on the curve with equation $y = \mathrm{g}(x)$.

(2)

1 Part (a)

The product rule gives

$$f'(x) = 6(2x+1)^2e^{-4x} - 4(2x+1)^3e^{-4x} = (2x+1)^2e^{-4x}[6-4(2x+1)] = 2(2x+1)^2(1-4x)e^{-4x}.$$

Comparison with the stated form gives $A = 2$.

$$\boxed{A = 2}$$

2 Part (b)

Since $e^{-4x} > 0$, $f'(x) = 0$ means $2x+1=0$ or $1-4x=0$. Thus $x = -1/2$ or $x = 1/4$.
Substitution gives $f(-1/2) = 0$ and $f(1/4) = (3/2)^3e^{-1} = 27/(8e)$.

$$\boxed{\left(-\frac{1}{2}, 0\right), \quad \left(\frac{1}{4}, \frac{27}{8e}\right)}$$

3 Part (c)

The derivative changes from positive to negative at $x = 1/4$, so this is the maximum; at $x = -1/2$ it does not change sign. The transformation $g(x) = 8f(x-2)$ sends (x, y) to $(x+2, 8y)$. Hence the maximum stationary point is $(1/4 + 2, 8 \cdot 27/(8e)) = (9/4, 27/e)$.

$$\boxed{\left(\frac{9}{4}, \frac{27}{e}\right)}$$

FINAL ANSWER

(a) $A = 2$ (b) $\left(-\frac{1}{2}, 0\right), \quad \left(\frac{1}{4}, \frac{27}{8e}\right)$ (c) $\left(\frac{9}{4}, \frac{27}{e}\right)$

Question 6

7 marks | Applications of Differentiation

6.

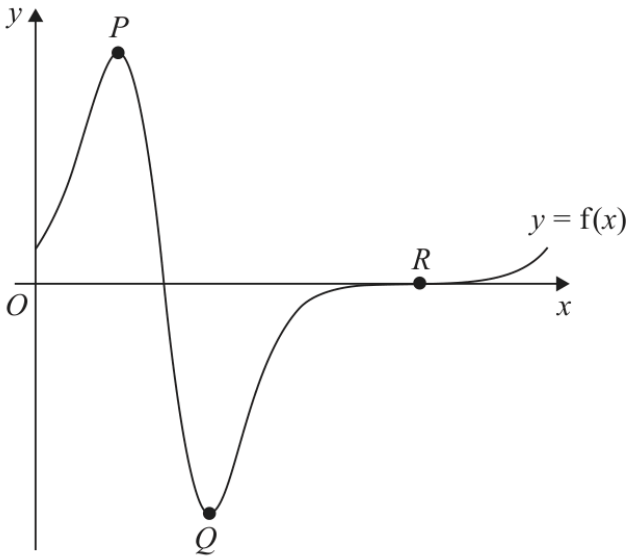


Figure 1

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 1 shows a sketch of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 2\mathrm{e}^{3\sin x}\cos x \qquad 0 \leqslant x \leqslant 2\pi$$

The curve intersects the x-axis at point R, as shown in Figure 1.

(a) State the coordinates of R

(1)

The curve has two turning points, at point P and point Q, also shown in Figure 1.

(b) Show that, at points P and Q,

$$a\sin^2 x + b\sin x + c = 0$$

where a, b and c are integers to be found.

(4)

(c) Hence find the x coordinate of point Q, giving your answer to 3 decimal places.

(2)

1 Part (a)

Since $2e^{3\sin x} > 0$, zeros occur when $\cos x = 0$. The labelled point R is the right-hand crossing, at $x = 3\pi/2$.

$$\boxed{R = \left(\frac{3\pi}{2}, 0\right)}$$

2 Part (b)

For $f(x) = 2e^{3\sin x}\cos x$, the product and chain rules give

$$f'(x) = 2\left(3e^{3\sin x}\cos^2 x - e^{3\sin x}\sin x\right) = 2e^{3\sin x}(3\cos^2 x - \sin x).$$

At a turning point the derivative is zero. The exponential factor is nonzero, so $3\cos^2 x - \sin x = 0$. Substitute $\cos^2 x = 1 - \sin^2 x$ and rearrange: $3 - 3\sin^2 x - \sin x = 0$.

$$\boxed{3\sin^2 x + \sin x - 3 = 0, \quad a = 3,\ b = 1,\ c = -3}$$

3 Part (c)

Solving $3s^2 + s - 3 = 0$ for $s = \sin x$ gives

$$s = \frac{-1 \pm \sqrt{37}}{6}.$$

The negative root is less than -1, so it is impossible for a real sine. Let $\beta = \arcsin\left((\sqrt{37} - 1)/6\right)$ in the first quadrant. The labelled minimum Q lies in the second quadrant, hence $x_Q = \pi - \beta = 2.131037293\ldots$.

$$\boxed{x_Q = 2.131}$$

FINAL ANSWER

(a) $R = \left(\frac{3\pi}{2}, 0\right)$ (b) $3\sin^2 x + \sin x - 3 = 0, \quad a = 3,\ b = 1,\ c = -3$ (c) $x_Q = 2.131$

Question 7

12 marks | Applications of Differentiation

7. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

The curve C has equation

$$y = \frac{16}{9(3x - k)} \qquad x \neq \frac{k}{3}$$

where k is a positive constant not equal to 3

(a) Find $\frac{dy}{dx}$ giving your answer in simplest form in terms of k.

(2)

The point P with x coordinate 1 lies on C.

Given that the gradient of the curve at P is -12

(b) find the two possible values of k

(3)

Given also that $k < 3$

(c) find the equation of the normal to C at P, writing your answer in the form $ax + by + c = 0$, where a, b and c are integers to be found.

(3)

(d) show, using algebraic integration that,

$$\int_1^3 \frac{16}{9(3x - k)} dx = \lambda \ln 10$$

where λ is a constant to be found.

(4)

1 Part (a)

Write $y = (16/9)(3x-k)^{-1}$. The chain rule gives

$$\frac{dy}{dx} = \frac{16}{9}(-1)(3x-k)^{-2}\cdot 3.$$

$$\boxed{\frac{dy}{dx} = -\frac{16}{3(3x-k)^2}}$$

2 Part (b)

At $x = 1$, the given gradient yields

$$-\frac{16}{3(3-k)^2} = -12 \quad\Longrightarrow\quad (3-k)^2 = \frac{4}{9}.$$

Hence $3-k = \pm 2/3$. Both resulting k-values are positive and different from 3.

$$\boxed{k = \frac{7}{3} \text{ or } \frac{11}{3}}$$

3 Part (c)

Since $k < 3$, use $k = 7/3$. At $x = 1$, $y = 16/[9(3-7/3)] = 8/3$. The tangent gradient is -12, so the normal gradient is $1/12$:

$$y - \frac{8}{3} = \frac{1}{12}(x-1), \qquad 12y - 32 = x - 1.$$

$$\boxed{x - 12y + 31 = 0}$$

4 Part (d)

With $k = 7/3$, the denominator does not vanish on $[1,3]$. Using the reciprocal-linear antiderivative,

$$\int_1^3 \frac{16}{9(3x-7/3)}\,dx = \left[\frac{16}{27}\ln|3x-7/3|\right]_1^3 = \frac{16}{27}\left(\ln\frac{20}{3} - \ln\frac{2}{3}\right) = \frac{16}{27}\ln\left(\frac{20/3}{2/3}\right).$$

$$\boxed{\frac{16}{27}\ln 10, \quad \lambda = \frac{16}{27}}$$

FINAL ANSWER

(a) $\frac{dy}{dx} = -\frac{16}{3(3x-k)^2}$ (b) $k = \frac{7}{3}$ or $\frac{11}{3}$ (c) $x - 12y + 31 = 0$ (d) $\frac{16}{27}\ln 10, \quad \lambda = \frac{16}{27}$

Question 6

9 marks | Applications of Differentiation

6.

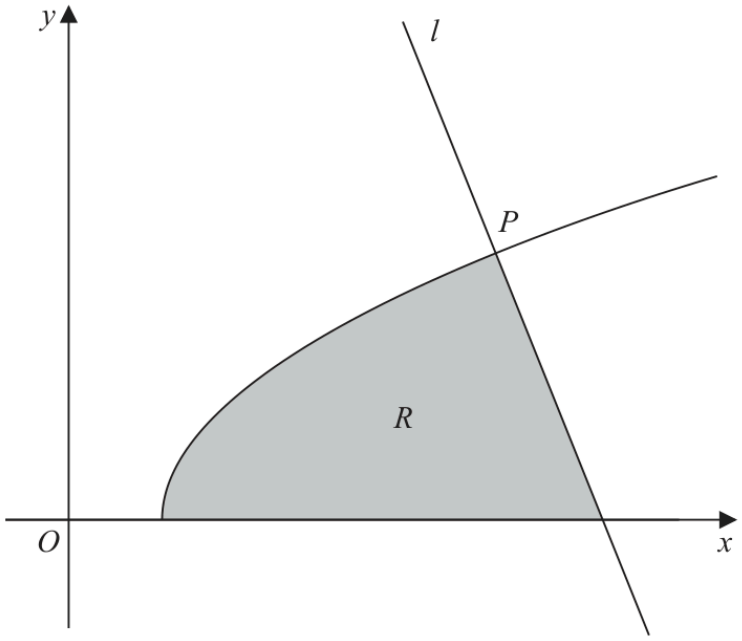


Figure 3

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 3 shows a sketch of part of the curve with equation

$$y = \sqrt{4x - 7}$$

The line l, shown in Figure 3, is the normal to the curve at the point $P(8, 5)$

(a) Use calculus to show that an equation of l is

$$5x + 2y - 50 = 0$$

(5)

The region R, shown shaded in Figure 3, is bounded by the curve, the x-axis and l.

(b) Use algebraic integration to find the exact area of R.

(4)

1 Part (a)

For $y = (4x-7)^{1/2}$,

$$\frac{dy}{dx} = \frac{1}{2}(4x-7)^{-1/2}\cdot 4 = \frac{2}{\sqrt{4x-7}}.$$

At $P=(8,5)$, the tangent gradient is $2/5$, so the normal gradient is $-5/2$. The point-gradient equation is

$$y-5 = -\frac{5}{2}(x-8), \qquad 2y-10 = -5x+40.$$

$$\boxed{5x+2y-50=0}$$

2 Part (b)

The curve meets the x-axis at $x=7/4$. The normal meets it at $x=10$. Split the shaded region at P, into a curve area from $7/4$ to 8 and a triangle from 8 to 10:

$$\int_{7/4}^{8}\sqrt{4x-7}\,dx = \left[\frac{(4x-7)^{3/2}}{6}\right]_{7/4}^{8} = \frac{125}{6}.$$

The triangle has area $\frac{1}{2}(10-8)\cdot 5 = 5$. Hence the total area is $125/6+5$.

$$\boxed{\text{Area}(R) = \frac{155}{6}}$$

FINAL ANSWER

(a) $5x+2y-50=0$ (b) $\text{Area}(R)=\frac{155}{6}$

Question 9

10 marks | Applications of Differentiation

9.

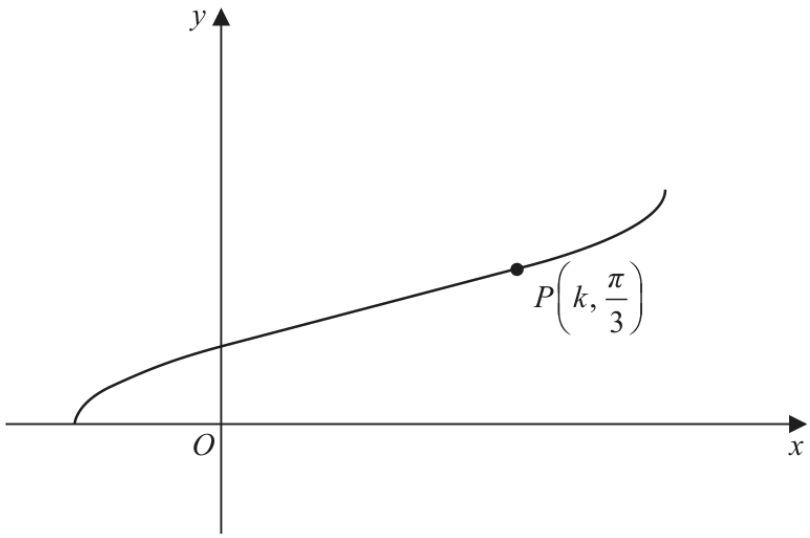


Figure 5

The curve shown in Figure 5 has equation

$$x = 4\sin^2 y - 1 \qquad 0 \leqslant y \leqslant \frac{\pi}{2}$$

The point $P\left(k, \frac{\pi}{3}\right)$ lies on the curve.

(a) Verify that $k = 2$

(1)

(b) (i) Find $\frac{dx}{dy}$ in terms of y

(ii) Hence show that $\frac{dy}{dx} = \frac{1}{2\sqrt{x+1}\sqrt{3-x}}$

(6)

The normal to the curve at P cuts the x-axis at the point N.

(c) Find the exact area of triangle OPN, where O is the origin.

Give your answer in the form $a\pi + b\pi^2$ where a and b are constants.

(3)

1 Part (a)

At $y = \pi/3$, $x = 4(\sqrt{3}/2)^2 - 1 = 3 - 1 = 2$.

$$\boxed{k = 2}$$

2 Part (b)

(i) Differentiating with respect to y gives

$$\frac{dx}{dy} = 8\sin y \cos y.$$

(ii) For $0 < y < \pi/2$, both sine and cosine are positive, and $-1 < x < 3$. From $x + 1 = 4\sin^2 y$,

$$\sin y = \frac{\sqrt{x+1}}{2}, \qquad \cos y = \sqrt{1 - \frac{x+1}{4}} = \frac{\sqrt{3-x}}{2}.$$

Consequently $dx/dy = 2\sqrt{x+1}\sqrt{3-x}$. Taking the reciprocal proves the required formula on $-1 < x < 3$. At $x = -1$ and $x = 3$ the denominator is zero: these are endpoint vertical tangents, not finite derivatives.

$$\boxed{\frac{dx}{dy} = 8\sin y\cos y, \qquad \frac{dy}{dx} = \frac{1}{2\sqrt{x+1}\sqrt{3-x}}}$$

3 Part (c)

At $P = (2, \pi/3)$ the tangent gradient is $1/(2\sqrt{3})$, so the normal gradient is $-2\sqrt{3}$. Thus the normal is $y - \pi/3 = -2\sqrt{3}(x - 2)$. Setting $y = 0$ gives $N = (2 + \pi/(6\sqrt{3}), 0)$. Triangle OPN has horizontal base $ON = 2 + \pi/(6\sqrt{3})$ and perpendicular height $\pi/3$:

$$\text{Area}(OPN) = \frac{1}{2}\left(2 + \frac{\pi}{6\sqrt{3}}\right)\frac{\pi}{3} = \frac{\pi}{3} + \frac{\pi^2}{36\sqrt{3}}.$$

$$\boxed{\text{Area}(OPN) = \frac{1}{3}\pi + \frac{\sqrt{3}}{108}\pi^2}$$

FINAL ANSWER

(a) $k = 2$ (b) $\frac{dx}{dy} = 8\sin y\cos y$, $\frac{dy}{dx} = \frac{1}{2\sqrt{x+1}\sqrt{3-x}}$ (c) $\text{Area}(OPN) = \frac{1}{3}\pi + \frac{\sqrt{3}}{108}\pi^2$

Question 9

8 marks | Applications of Differentiation

9.

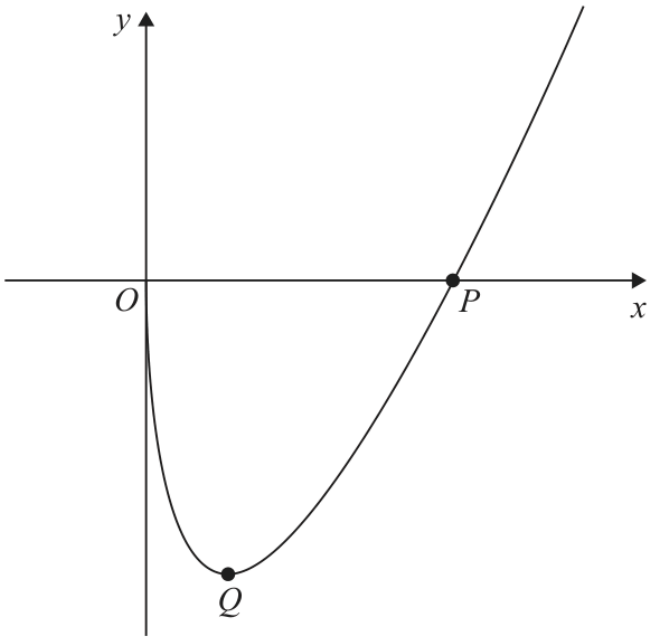


Figure 1

Figure 1 shows a sketch of part of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 6\sqrt{x}\ln(4x) \qquad x > 0$$

The curve cuts the x-axis at point P

(a) State the x coordinate of P

(1)

The point Q, shown in Figure 1, is the stationary point on C

(b) Use calculus to find the exact coordinates of Q

(5)

(c) Hence find the range of the function $\mathrm{g}(x)$ where

$$\mathrm{g}(x) = -2\,\mathrm{f}(x)$$

(2)

1 Part (a)

Since $x > 0$, $6\sqrt{x}$ is positive. Therefore $f(x) = 0$ requires $\ln(4x) = 0$, so $4x = 1$ and $x = 1/4$.

$$\boxed{x_P = \frac{1}{4}}$$

2 Part (b)

Apply the product and chain rules:

$$f'(x) = \frac{3}{\sqrt{x}}\ln(4x) + 6\sqrt{x}\frac{1}{x} = \frac{3}{\sqrt{x}}[\ln(4x) + 2].$$

At Q, set $f'(x) = 0$. Since $3/\sqrt{x} > 0$,

$$\ln(4x) + 2 = 0 \quad \Longrightarrow \quad 4x = e^{-2} \quad \Longrightarrow \quad x = \frac{1}{4e^2}.$$

Then $\sqrt{x} = 1/(2e)$ and $\ln(4x) = -2$, giving $f(x) = 6(1/(2e))(-2) = -6/e$.

$$\boxed{Q = \left(\frac{1}{4e^2}, -\frac{6}{e}\right)}$$

3 Part (c)

The factor $3/\sqrt{x}$ is positive and $\ln(4x) + 2$ increases through zero at Q, so f' changes from negative to positive: Q is the global minimum. As $x \to \infty$, $f(x) \to \infty$, hence the range of f is $[-6/e, \infty)$. Multiplication by -2 reverses the inequality and includes the maximum $12/e$.

$$\boxed{g(x) \leq \frac{12}{e}}$$

FINAL ANSWER

(a) $x_P = \frac{1}{4}$ (b) $Q = \left(\frac{1}{4e^2}, -\frac{6}{e}\right)$ (c) $g(x) \leq \frac{12}{e}$

WMA13/01 OCTOBER 2025

Question 7

7 marks | Applications of Differentiation

7. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

A curve C has equation

$$y = 2x\mathrm{e}^{x^2+(3k-2)x}$$

where k is a constant.

Given that C has two distinct turning points, find the range of possible values of k.

(7)

WORKED SOLUTION

Question 7 · WMA13/01 October 2025

1 Solution

\paragraph{Solution} Put $u = x^2 + (3k-2)x$, so $u' = 2x + 3k - 2$. By the product and chain rules,

$$\frac{dy}{dx} = 2e^u + 2xe^u(2x + 3k - 2) = 2e^u\big(2x^2 + (3k-2)x + 1\big).$$

As $2e^u > 0$, stationary points satisfy

$$2x^2 + (3k-2)x + 1 = 0.$$

For two distinct real roots, the discriminant must be positive:

$$(3k-2)^2 - 8 > 0 \iff |3k-2| > 2\sqrt{2}.$$

Consequently

$$k < \frac{2-2\sqrt{2}}{3} \quad \text{or} \quad k > \frac{2+2\sqrt{2}}{3}.$$

For these values the quadratic changes sign at both simple roots, so both stationary points are genuine turning points. Equality gives a repeated root with no derivative sign change and is excluded.

FINAL ANSWER

(main) $(-infinity, (2 - 2 * sqrt(2))/3) union ((2 + 2 * sqrt(2))/3, infinity)$

WMA13/01 OCTOBER 2020

Question 8

9 marks | Applications of Differentiation | Main topic: Differentiation Techniques

8. (i) The curve C has equation $y = \mathrm{g}(x)$ where

$$\mathrm{g}(x) = \mathrm{e}^{3x} \sec 2x \qquad -\frac{\pi}{4} < x < \frac{\pi}{4}$$

(a) Find $\mathrm{g}'(x)$

(2)

(b) Hence find the x coordinate of the stationary point of C.

(3)

(ii) A different curve has equation

$$x = \ln(\sin y) \qquad 0 < y < \frac{\pi}{2}$$

Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^x}{\mathrm{f}(x)}$$

where $\mathrm{f}(x)$ is a function of e^x that should be found.

(4)

WORKED SOLUTION

Question 8 · WMA13/01 October 2020

1 **Part (i)(a)**

By the product and chain rules,

$$g'(x) = 3e^{3x}\sec 2x + 2e^{3x}\sec 2x \tan 2x = e^{3x}\sec 2x(3 + 2\tan 2x).$$

$$\boxed{g'(x) = e^{3x}\sec 2x(3 + 2\tan 2x)}$$

2 Part (i)(b)

For $-\pi/4 < x < \pi/4$, $e^{3x}\sec 2x$ is positive and nonzero. Hence $g'(x) = 0$ precisely when $3 + 2\tan 2x = 0$, or $\tan 2x = -3/2$. Since $-\pi/2 < 2x < \pi/2$, tangent is one-to-one there:

$$x = -\frac{1}{2}\arctan\frac{3}{2} = -0.4913968616\ldots.$$

$$\boxed{x = -\frac{1}{2}\arctan\frac{3}{2} \approx -0.491}$$

---

3 Part (ii)

Differentiate $x = \ln(\sin y)$ with respect to y:

$$\frac{dx}{dy} = \frac{\cos y}{\sin y}, \qquad \frac{dy}{dx} = \frac{\sin y}{\cos y}.$$

The original equation gives $\sin y = e^{x}$. Since $0 < y < \pi/2$, cosine is positive, so $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - e^{2x}}$. Therefore

$$\frac{dy}{dx} = \frac{e^{x}}{\sqrt{1 - e^{2x}}}, \qquad f(x) = \sqrt{1 - e^{2x}}.$$

Here $x < 0$ from the original interval, so the square root is real and nonzero.

$$\boxed{f(x) = \sqrt{1 - e^{2x}}, \qquad x < 0}$$

FINAL ANSWER

(i)(a) $g'(x) = e^{3x}\sec 2x(3 + 2\tan 2x)$ (i)(b) $x = -\frac{1}{2}\arctan\frac{3}{2} \approx -0.491$ (ii) $f(x) = \sqrt{1 - e^{2x}}, \qquad x < 0$

WMA13/01 JANUARY 2021

Question 6

8 marks | Applications of Differentiation | Main topic: Numerical Methods

6.

$$f(x) = x\cos\left(\frac{x}{3}\right) \qquad x > 0$$

(a) Find $f'(x)$

(2)

(b) Show that the equation $f'(x) = 0$ can be written as

$$x = k\arctan\left(\frac{k}{x}\right)$$

where k is an integer to be found.

(2)

(c) Starting with $x_1 = 2.5$ use the iteration formula

$$x_{n+1} = k\arctan\left(\frac{k}{x_n}\right)$$

with the value of k found in part (b), to calculate the values of x_2 and x_6 giving your answers to 3 decimal places.

(2)

(d) Using a suitable interval and a suitable function that should be stated, show that a root of $f'(x) = 0$ is 2.581 correct to 3 decimal places.

(2)

WORKED SOLUTION

Question 6 · WMA13/01 January 2021

1 **Part (a)**

By the product and chain rules, $f'(x) = \cos(x/3) - (x/3)\sin(x/3)$.

$$\boxed{f'(x) = \cos(x/3) - \frac{x}{3}\sin(x/3)}$$

2 Part (b)

Set $f'(x) = 0$: $\cos(x/3) = (x/3)\sin(x/3)$. At a solution, cosine cannot vanish (that would leave a nonzero right side), so $\tan(x/3) = 3/x$. On the principal positive branch $0 < x/3 < \pi/2$, this gives

$$x = 3\arctan(3/x), \qquad k = 3.$$

This is the branch used by the supplied iteration for the root near 2.581. The original domain $x > 0$ also admits later tangent branches: in general $x = 3\arctan(3/x) + 3n\pi$. The principal-branch rearrangement must not be interpreted as listing every positive stationary point.

$$\boxed{k = 3; \quad x = 3\arctan(3/x) \text{ on the selected branch}}$$

3 Part (c)

Using radians and $x_1 = 2.5$, the successive values are

n	1	2	3	4	5	6
x_n	2.5	2.628174152	2.554102174	2.596526871	2.572101407	2.586122337

Keep full precision between steps and round the requested terms only.

$$\boxed{x_2 = 2.628, \qquad x_6 = 2.586}$$

4 Part (d)

Define $F(x) = \cos(x/3) - (x/3)\sin(x/3)$, which is continuous near 2.581. The rounding interval endpoints give

$$F(2.5805) = 0.0003467157\ldots > 0, \qquad F(2.5815) = -0.0003456897\ldots < 0.$$

By continuity a root lies strictly between these endpoints; every number in this open interval rounds to 2.581 to three decimal places.

$$\boxed{x = 2.581 \text{ correct to 3 d.p.}}$$

✓ FINAL ANSWER

(a) $f'(x) = \cos(x/3) - \frac{x}{3}\sin(x/3)$ (b) $k = 3; \quad x = 3\arctan(3/x)$ on the selected branch (c) $x_2 = 2.628, \qquad x_6 = 2.586$ (d) $x = 2.581$ correct to 3 d.p.

WMA13/01 JANUARY 2021

Question 10

10 marks | Applications of Differentiation | Main topic: Differentiation Techniques

10. The curve C has equation

$$x = 3\sec^2 2y \qquad x > 3 \qquad 0 < y < \frac{\pi}{4}$$

(a) Find $\dfrac{\mathrm{d}x}{\mathrm{d}y}$ in terms of y.

(2)

(b) Hence show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{p}{qx\sqrt{x-3}}$$

where p is irrational and q is an integer, stating the values of p and q.

(3)

(c) Find the equation of the normal to C at the point where $y = \dfrac{\pi}{12}$, giving your answer in the form $y = mx + c$, giving m and c as exact irrational numbers.

(5)

WORKED SOLUTION

Question 10 · WMA13/01 January 2021

1 **Part (a)**

Apply the chain rule twice:

$$\frac{dx}{dy} = 3 \cdot 2\sec 2y\,(2\sec 2y\tan 2y) = 12\sec^2 2y\tan 2y.$$

$$\boxed{\frac{dx}{dy} = 12\sec^2 2y\tan 2y}$$

2 **Part (b)**

From $x = 3\sec^2 2y$, $\sec^2 2y = x/3$ and $\tan^2 2y = x/3 - 1$. Since $0 < 2y < \pi/2$, tangent is positive, so $\tan 2y = \sqrt{(x-3)/3}$. Thus $dx/dy = 4x\sqrt{(x-3)/3} > 0$. Taking the reciprocal gives

$$\frac{dy}{dx} = \frac{\sqrt{3}}{4x\sqrt{x-3}}, \qquad p = \sqrt{3}, \quad q = 4.$$

$$\boxed{p = \sqrt{3}, \quad q = 4}$$

3 **Part (c)**

At $y = \pi/12$, $2y = \pi/6$ and $x = 3(4/3) = 4$. The tangent gradient is $\sqrt{3}/(4 \cdot 4\sqrt{1}) = \sqrt{3}/16$. Therefore the normal gradient is $-16/\sqrt{3} = -16\sqrt{3}/3$. Using the point $(4, \pi/12)$,

$$y - \frac{\pi}{12} = -\frac{16\sqrt{3}}{3}(x - 4) \Longrightarrow y = -\frac{16\sqrt{3}}{3}x + \frac{64\sqrt{3}}{3} + \frac{\pi}{12}.$$

$$\boxed{y = -\frac{16\sqrt{3}}{3}x + \frac{64\sqrt{3}}{3} + \frac{\pi}{12}}$$

✓ **FINAL ANSWER**

(a) $\frac{dx}{dy} = 12\sec^2 2y \tan 2y$ (b) $p = \sqrt{3}, \quad q = 4$ (c) $y = -\frac{16\sqrt{3}}{3}x + \frac{64\sqrt{3}}{3} + \frac{\pi}{12}$

WMA13/01 MAY/JUNE 2021

Question 8

13 marks | Applications of Differentiation | Main topic: Log Graphs & Exponential Models

8. A scientist is studying a population of fish in a lake. The number of fish, N, in the population, t years after the start of the study, is modelled by the equation

$$N = \frac{600e^{0.3t}}{2 + e^{0.3t}} \qquad t \geqslant 0$$

Use the equation of the model to answer parts (a), (b), (c), (d) and (e).

(a) Find the number of fish in the lake at the start of the study.

(1)

(b) Find the upper limit to the number of fish in the lake.

(1)

(c) Find the time, after the start of the study, when there are predicted to be 500 fish in the lake. Give your answer in years and months to the nearest month.

(4)

(d) Show that

$$\frac{dN}{dt} = \frac{Ae^{0.3t}}{(2 + e^{0.3t})^2}$$

where A is a constant to be found.

(3)

Given that when $t = T$, $\frac{dN}{dt} = 8$

(e) find the value of T to one decimal place.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(4)

WORKED SOLUTION

Question 8 · WMA13/01 May/June 2021

Part (a)

At the start, $t = 0$ and $e^{0.3t} = 1$. Thus $N(0) = 600/(2+1) = 200$.

$$\boxed{200 \text{ fish}}$$

2 Part (b)

Divide numerator and denominator by $e^{0.3t}$ to write $N = 600/(1 + 2e^{-0.3t})$. As $t \to \infty$, the denominator tends to 1, so the upper limit is 600 fish.

$$\boxed{600 \text{ fish}}$$

3 Part (c)

Set $N = 500$ and multiply through: $1000 + 500e^{0.3t} = 600e^{0.3t}$, so $e^{0.3t} = 10$. Hence $t = \ln 10/0.3 = 7.675283643\ldots$ years. The fractional year is $12(0.675283643\ldots) = 8.10340372\ldots$ months. Round only the months to the nearest whole month.

$$\boxed{7 \text{ years } 8 \text{ months}}$$

4 Part (d)

The quotient rule gives

$$\frac{dN}{dt} = \frac{180e^{0.3t}(2 + e^{0.3t}) - 600e^{0.3t}(0.3e^{0.3t})}{(2 + e^{0.3t})^2} = \frac{360e^{0.3t}}{(2 + e^{0.3t})^2}.$$

The two $e^{0.6t}$ terms cancel exactly.

$$\boxed{A = 360}$$

5 Part (e)

Let $z = e^{0.3T}$. Since $T \geq 0$, require $z \geq 1$. Then $360z/(2 + z)^2 = 8$ gives $45z = (2 + z)^2$, hence

$$z^2 - 41z + 4 = 0, \qquad z = \frac{41 \pm \sqrt{1665}}{2}.$$

The smaller root lies between 0 and 1 and would give $T < 0$, so reject it. For the larger root,

$$T = \frac{1}{0.3}\ln\left(\frac{41 + \sqrt{1665}}{2}\right) = 12.37061331\ldots.$$

$$\boxed{T = 12.4 \text{ years}}$$

FINAL ANSWER

(a) 200 fish (b) 600 fish (c) 7 years 8 months (d) $A = 360$ (e) $T = 12.4$ years

WMA13/01 OCTOBER 2021

Question 6

8 marks | Applications of Differentiation | Main topic: Differentiation Techniques

6. (i) The curve C_1 has equation

$$y = 3\ln(x^2 - 5) - 4x^2 + 15 \qquad x > \sqrt{5}$$

Show that C_1 has a stationary point at $x = \dfrac{\sqrt{p}}{2}$ where p is a constant to be found.

(4)

(ii) A different curve C_2 has equation

$$y = 4x - 12\sin^2 x$$

(a) Show that, for this curve,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = A + B\sin 2x$$

where A and B are constants to be found.

(b) Hence, state the maximum gradient of this curve.

(4)

WORKED SOLUTION

Question 6 · WMA13/01 October 2021

1 **Part (i)**

Differentiate to obtain $y' = 6x/(x^2 - 5) - 8x$. At a stationary point in $x > \sqrt{5}$, x is nonzero, so $6/(x^2 - 5) = 8$, giving $6 = 8x^2 - 40$ and $x^2 = 23/4$. The original positive domain selects $x = \sqrt{23}/2$, which exceeds $\sqrt{5}$. Hence $p = 23$.

$$\boxed{p = 23, \qquad x = \frac{\sqrt{23}}{2}}$$

2 **Part (ii)**

(a) Differentiate the squared sine with the chain rule: $y' = 4 - 24\sin x\cos x = 4 - 12\sin 2x$. Thus $A = 4$, $B = -12$. (b) Since $-1 \le \sin 2x \le 1$, the largest derivative is attained when $\sin 2x = -1$: $4 - 12(-1) = 16$.

$$\boxed{A = 4, \quad B = -12; \qquad \text{maximum gradient} = 16}$$

FINAL ANSWER

(i) $p = 23, \quad x = \frac{\sqrt{23}}{2}$ (ii) $A = 4, \quad B = -12; \quad$ maximum gradient $= 16$

WMA13/01 OCTOBER 2021

Question 8

7 marks | Applications of Differentiation | Main topic: Differentiation Techniques

8. A curve C has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \arcsin\left(\frac{1}{2}x\right) \qquad -2 \leqslant x \leqslant 2 \qquad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

(a) Sketch C.

(1)

(b) Given $x = 2\sin y$, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{A - x^2}}$$

where A is a constant to be found.

(3)

The point P lies on C and has y coordinate $\frac{\pi}{4}$

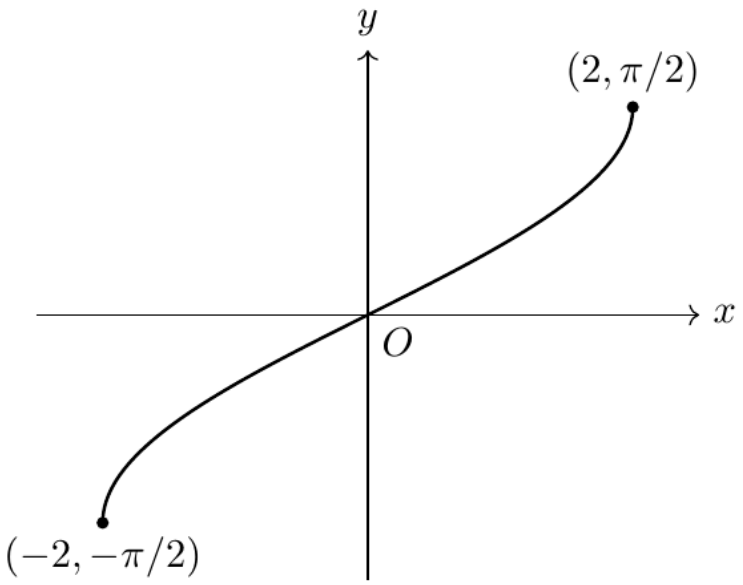
(c) Find the equation of the tangent to C at P. Write your answer in the form $y = mx + c$, where m and c are constants to be found.

(3)

1 Part (a)

The principal arcsine curve increases from $(-2, -\pi/2)$ through $(0, 0)$ to $(2, \pi/2)$. It is concave down on the negative-x half, concave up on the positive-x half, and vertical at both ends.

$$\boxed{y = \arcsin(x/2), \quad -2 \le x \le 2}$$



2 Part (b)

Differentiate the supplied inverse relation: $dx/dy = 2\cos y$. For $-\pi/2 < y < \pi/2$, cosine is positive, so $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2/4}$. Taking the reciprocal gives

$$\frac{dy}{dx} = \frac{1}{2\cos y} = \frac{1}{\sqrt{4 - x^2}}, \qquad -2 < x < 2.$$

Thus $A = 4$; the endpoints have vertical tangents, not finite derivatives.

$$\boxed{A = 4, \qquad \frac{dy}{dx} = \frac{1}{\sqrt{4 - x^2}}}$$

3 Part (c)

At $y = \pi/4$, $x = 2\sin(\pi/4) = \sqrt{2}$. The gradient is $1/\sqrt{4 - 2} = 1/\sqrt{2}$. Hence $y - \pi/4 = (x - \sqrt{2})/\sqrt{2}$, so $y = x/\sqrt{2} + \pi/4 - 1$.

$$\boxed{y = \frac{x}{\sqrt{2}} + \frac{\pi}{4} - 1}$$

FINAL ANSWER

(a) $y = \arcsin(x/2), \quad -2 \le x \le 2$ (b) $A = 4, \quad \frac{dy}{dx} = \frac{1}{\sqrt{4-x^2}}$ (c) $y = \frac{x}{\sqrt{2}} + \frac{\pi}{4} - 1$

Question 9

9 marks | Applications of Differentiation | Main topic: Numerical Methods

9.

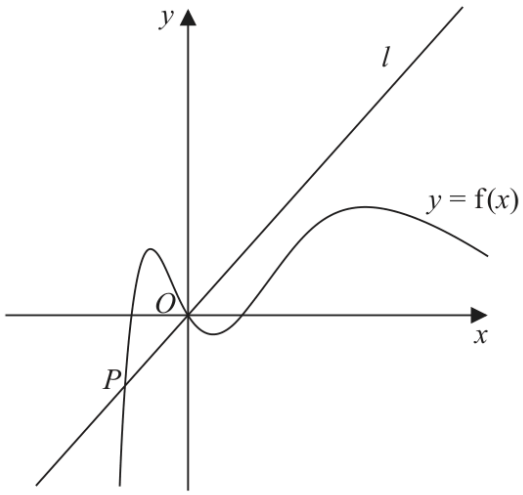


Figure 3

Figure 3 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x(x^2 - 4)\mathrm{e}^{-\frac{1}{2}x}$$

(a) Find $\mathrm{f}'(x)$.

(2)

The line l is the normal to the curve at O and meets the curve again at the point P.

The point P lies in the 3rd quadrant, as shown in Figure 3.

(b) Show that the x coordinate of P is a solution of the equation

$$x = -\frac{1}{2}\sqrt{16 + \mathrm{e}^{\frac{1}{2}x}}$$

(4)

(c) Using the iterative formula

$$x_{n+1} = -\frac{1}{2}\sqrt{16 + \mathrm{e}^{\frac{1}{2}x_n}} \quad \text{with } x_1 = -2$$

find, to 4 decimal places,

(i) the value of x_2

(ii) the x coordinate of P.

(3)

1 Part (a)

Write $f(x) = (x^3 - 4x)e^{-x/2}$. By the product rule,

$$f'(x) = (3x^2 - 4)e^{-x/2} - \frac{1}{2}(x^3 - 4x)e^{-x/2} = e^{-x/2}\left(-\frac{x^3}{2} + 3x^2 + 2x - 4\right).$$

$$\boxed{f'(x) = e^{-x/2}(-x^3/2 + 3x^2 + 2x - 4)}$$

2 Part (b)

At O, $f'(0) = -4$, so the normal gradient is $1/4$ and its equation is $y = x/4$. At the other intersection P, x is nonzero; equating curve and normal and dividing by x gives $(x^2 - 4)e^{-x/2} = 1/4$. Thus $x^2 = 4 + e^{x/2}/4 = (16 + e^{x/2})/4$. Since P lies in the third quadrant, select the negative square root.

$$\boxed{x = -\frac{1}{2}\sqrt{16 + e^{x/2}}}$$

3 Part (c)

Starting from $x_1 = -2$, $x_2 = -\frac{1}{2}\sqrt{16 + e^{-1}} = -2.0228617996\ldots$. Further unrounded iterates are $x_3 = -2.0226034084\ldots$, $x_4 = -2.0226063126\ldots$ and $x_5 = -2.0226062799\ldots$. The continuing sequence settles at $-2.0226062803\ldots$.

$$\boxed{\text{(i) } x_2 = -2.0229; \qquad \text{(ii) } x_P = -2.0226}$$

✓ FINAL ANSWER

(a) $f'(x) = e^{-x/2}(-x^3/2 + 3x^2 + 2x - 4)$ (b) $x = -\frac{1}{2}\sqrt{16 + e^{x/2}}$ (c) $\text{(i) } x_2 = -2.0229; \qquad \text{(ii) } x_P = -2.0226$

WMA13/01 JANUARY 2022

Question 6

11 marks | Applications of Differentiation | Main topic: Functions, Composition & Inverses

6. The function f is defined by

$$\mathrm{f}(x) = \frac{5x - 3}{x - 4} \qquad x > 4$$

(a) Show, by using calculus, that f is a decreasing function.

(3)

(b) Find f^{-1}

(3)

(c) (i) Show that $\mathrm{ff}(x) = \dfrac{ax + b}{x + c}$ where a, b and c are constants to be found.

(ii) Deduce the range of ff.

(5)

WORKED SOLUTION

Question 6 · WMA13/01 January 2022

1 Part (a)

By the quotient rule,

$$f'(x) = \frac{5(x-4) - (5x-3)}{(x-4)^2} = \frac{-17}{(x-4)^2}.$$

For every $x > 4$, the denominator is positive and the numerator is negative. Therefore $f'(x) < 0$ throughout its domain, so f is decreasing.

$$\boxed{f \text{ is decreasing for } x > 4}$$

2 Part (b)

Set $y = (5x - 3)/(x - 4)$. Then $yx - 4y = 5x - 3$, giving $x(y - 5) = 4y - 3$ and $x = (4y - 3)/(y - 5)$. Exchange the variable names. Since $f(x) = 5 + 17/(x - 4)$ on $x > 4$, its range is $(5, \infty)$, which becomes the inverse's domain.

$$\boxed{f^{-1}(x) = \frac{4x - 3}{x - 5}, \qquad x > 5}$$

3 **Part (c)**

(i) For $x > 4$, $f(x) > 5 > 4$, so the second application is defined.

$$ff(x) = \frac{5\left(\frac{5x-3}{x-4}\right) - 3}{\left(\frac{5x-3}{x-4}\right) - 4} = \frac{25x - 15 - 3x + 12}{5x - 3 - 4x + 16} = \frac{22x - 3}{x + 13}.$$

Thus $a = 22$, $b = -3$, $c = 13$. (ii) Rewrite $ff(x) = 22 - 289/(x + 13)$ with the original domain $x > 4$. It is strictly increasing, tends to 5 as $x \to 4^+$ and to 22 as $x \to \infty$. Neither limiting value is attained.

$$\boxed{ff(x) = \frac{22x - 3}{x + 13}, \qquad 5 < ff(x) < 22}$$

FINAL ANSWER

(a) f **is decreasing for** $x > 4$ (b) $f^{-1}(x) = \frac{4x-3}{x-5}, \qquad x > 5$ (c) $ff(x) = \frac{22x-3}{x+13}, \qquad 5 < ff(x) < 22$

WMA13/01 JANUARY 2022

Question 8

8 marks | Applications of Differentiation | Main topic: Log Graphs & Exponential Models

8. A dose of antibiotics is given to a patient.

The amount of the antibiotic, x milligrams, in the patient's bloodstream t hours after the dose was given, is found to satisfy the equation

$$\log_{10} x = 2.74 - 0.079t$$

(a) Show that this equation can be written in the form

$$x = pq^{-t}$$

where p and q are constants to be found. Give the value of p to the nearest whole number and the value of q to 2 significant figures.

(4)

(b) With reference to the equation in part (a), interpret the value of the constant p.

(1)

When a different dose of the antibiotic is given to another patient, the values of x and t satisfy the equation

$$x = 400 \times 1.4^{-t}$$

(c) Use calculus to find, to 2 significant figures, the value of $\dfrac{dx}{dt}$ when $t = 5$

(3)

WORKED SOLUTION

Question 8 · WMA13/01 January 2022

1 **Part (a)**

Exponentiate base 10:

$$x = 10^{2.74-0.079t} = 10^{2.74}(10^{0.079})^{-t}.$$

Thus $p = 10^{2.74} = 549.5408739\ldots$ and $q = 10^{0.079} = 1.199499303\ldots$. Round p to the nearest integer and q to two significant figures.

$$\boxed{p = 550, \qquad q = 1.2}$$

2 Part (b)

At $t = 0$, $x = pq^0 = p$. Thus p is the initial amount of antibiotic in the patient's bloodstream, approximately 550 milligrams immediately after the dose in this model.

$$\boxed{p \text{ represents the initial amount, approximately } 550 \text{ mg}}$$

3 Part (c)

For the second patient, write $x = 400e^{-t\ln 1.4}$. Differentiate to obtain $dx/dt = -400\ln(1.4)\,1.4^{-t}$. At $t = 5$, the derivative is $-25.02470969\ldots \text{ mg}\,\text{h}^{-1}$. The negative sign indicates the amount is decreasing.

$$\boxed{\left.\frac{dx}{dt}\right|_{t=5} = -25 \text{ mg}\,\text{h}^{-1}}$$

FINAL ANSWER

(a) $p = 550, \quad q = 1.2$ (b) p represents the initial amount, approximately 550 mg (c) $\left.\frac{dx}{dt}\right|_{t=5} = -25 \text{ mg}\,\text{h}^{-1}$

Question 10

7 marks | Applications of Differentiation | Main topic: Differentiation Techniques

10.

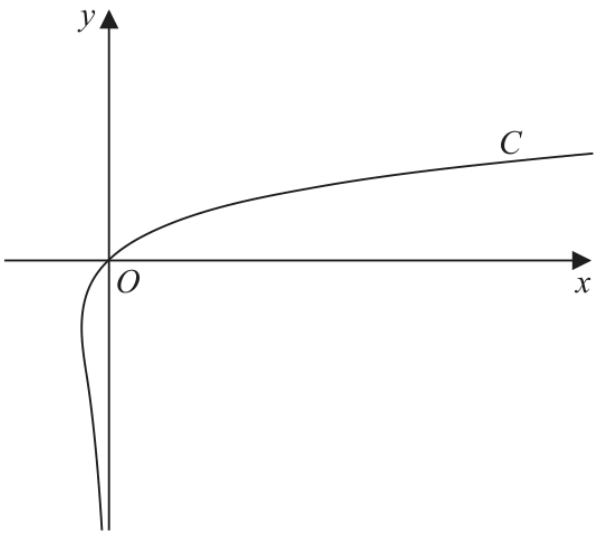


Figure 3

Figure 3 shows a sketch of the curve C with equation

$$x = ye^{2y} \qquad y \in \mathbb{R}$$

(a) Show that

$$\frac{dy}{dx} = \frac{y}{x(1+2y)}$$

(4)

Given that the straight line with equation $x = k$, where k is a constant, cuts C at exactly two points,

(b) find the range of possible values for k.

(3)

1 Part (a)

Differentiate x with respect to y by the product rule:

$$\frac{dx}{dy} = e^{2y} + 2ye^{2y} = e^{2y}(1+2y).$$

Where $dx/dy \neq 0$, take the reciprocal:

$$\frac{dy}{dx} = \frac{e^{-2y}}{1+2y}.$$

For $y \neq 0$, the original relation gives $e^{2y} = x/y$, hence the required form $dy/dx = y/[x(1+2y)]$. At the origin, that last quotient is $0/0$; the preceding uncancelled derivative gives $dy/dx = 1$. At $y = -1/2$ there is no finite dy/dx.

$$\boxed{\frac{dy}{dx} = \frac{y}{x(1+2y)} \quad \text{where defined}}$$

2 Part (b)

Consider $x = h(y) = ye^{2y}$. Its derivative $h'(y) = e^{2y}(1+2y)$ vanishes at $y = -1/2$. It is negative before this y value and positive after it, so the minimum x value is $h(-1/2) = -1/(2e)$. As $y \to -\infty$, $h(y) \to 0$ from below; also $h(0) = 0$ and h then increases without bound. Thus a level $x = k$ meets both negative-y branches precisely between the minimum and zero. At the minimum the two intersections coincide; at zero there is just the finite point $y = 0$.

$$\boxed{-\frac{1}{2e} < k < 0}$$

FINAL ANSWER

(a) $\frac{dy}{dx} = \frac{y}{x(1+2y)}$ where defined (b) $-\frac{1}{2e} < k < 0$

Question 8

12 marks | Applications of Differentiation | Main topic: Numerical Methods

8.

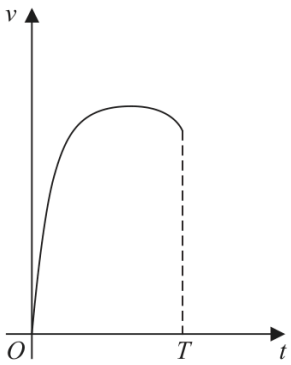


Figure 4

Figure 4 is a graph showing the velocity of a sprinter during a 100 m race.

The sprinter's velocity during the race, $v\,\text{m s}^{-1}$, is modelled by the equation

$$v = 12 - \text{e}^{t-10} - 12\text{e}^{-0.75t} \qquad t \geqslant 0$$

where t seconds is the time after the sprinter begins to run.

According to the model,

(a) find, using calculus, the sprinter's maximum velocity during the race.

(5)

Given that the sprinter runs 100 m in T seconds, such that

$$\int_0^T v\,\text{d}t = 100$$

(b) show that T is a solution of the equation

$$T = \frac{1}{12}\left(116 - 16\text{e}^{-0.75T} + \text{e}^{T-10} - \text{e}^{-10}\right)$$

(4)

The iteration formula

$$T_{n+1} = \frac{1}{12}\left(116 - 16\text{e}^{-0.75T_n} + \text{e}^{T_n-10} - \text{e}^{-10}\right)$$

is used to find an approximate value for T

Using this iteration formula with $T_1 = 10$

(c) find, to 4 decimal places,

(i) the value of T_2

(ii) the time taken by the sprinter to run the race, according to the model.

(3)

1 Part (a)

Differentiate the velocity:

$$v'(t) = -e^{t-10} + 9e^{-0.75t}.$$

At a stationary velocity, $e^{t-10} = 9e^{-0.75t}$, so $e^{1.75t} = 9e^{10}$ and $t = (10 + \ln 9)/1.75 = 6.9698426\ldots$ seconds. The second derivative $-e^{t-10} - 6.75e^{-0.75t}$ is negative, so this is a maximum. Substitute this unrounded time into v to get $11.88728125\ldots$ metres per second.

$$\boxed{v_{\max} = 11.9\ \mathrm{m\,s^{-1}}}$$

2 Part (b)

Integrating the velocity gives $12t - e^{t-10} + 16e^{-0.75t}$. Evaluate between 0 and T:

$$100 = 12T - e^{T-10} + 16e^{-0.75T} - (-e^{-10} + 16).$$

Rearrange to $12T = 116 - 16e^{-0.75T} + e^{T-10} - e^{-10}$ and divide by 12.

$$\boxed{T = \frac{1}{12}(116 - 16e^{-0.75T} + e^{T-10} - e^{-10})}$$

3 Part (c)

With $T_1 = 10$, substitution gives $T_2 = 9.7492587708\ldots$. Continuing with unrounded values gives $T_3 = 9.7306248365\ldots$, $T_4 = 9.7294150526\ldots$, $T_5 = 9.7293372715\ldots$, $T_6 = 9.7293322738\ldots$.

Further iteration settles at $9.7293319307\ldots$ seconds. Thus (i) $T_2 = 9.7493$ and (ii) the race time is 9.7293 seconds, to four decimal places.

$$\boxed{T_2 = 9.7493, \qquad T = 9.7293\ \mathrm{s}}$$

FINAL ANSWER

(a) $v_{\max} = 11.9\ \mathrm{m\,s^{-1}}$ (b) $T = \frac{1}{12}(116 - 16e^{-0.75T} + e^{T-10} - e^{-10})$ (c) $T_2 = 9.7493, \qquad T = 9.7293\ \mathrm{s}$

Question 9

9 marks | Applications of Differentiation | Main topic: Trig Equations & R-Form

9.

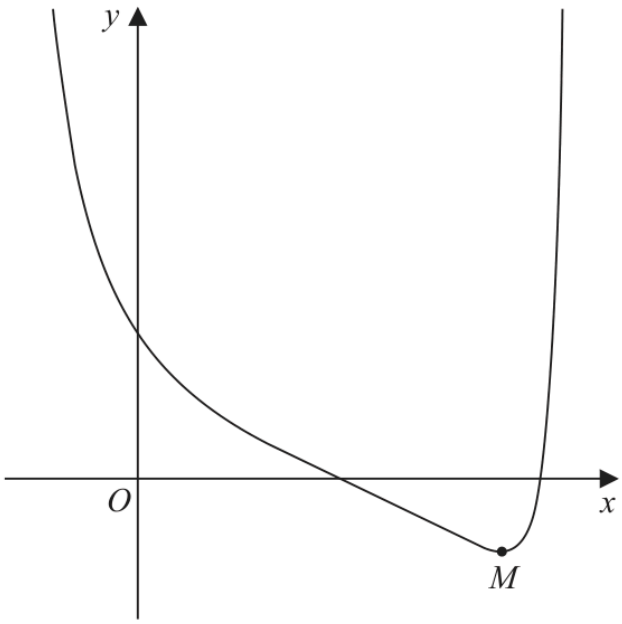


Figure 5

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 5 shows the curve with equation

$$y = \frac{1 + 2\cos x}{1 + \sin x} \qquad -\frac{\pi}{2} < x < \frac{3\pi}{2}$$

The point M, shown in Figure 5, is the minimum point on the curve.

(a) Show that the x coordinate of M is a solution of the equation

$$2\sin x + \cos x = -2$$

(4)

(b) Hence find, to 3 significant figures, the x coordinate of M.

(5)

1 Part (a)

Differentiate by the quotient rule:

$$y' = \frac{-2\sin x(1+\sin x) - (1+2\cos x)\cos x}{(1+\sin x)^2} = \frac{-2\sin x - \cos x - 2(\sin^2 x + \cos^2 x)}{(1+\sin x)^2} = \frac{-2\sin x - \cos x - 2}{(1+\sin x)^2}.$$

Inside the stated interval, $1+\sin x > 0$. At the interior minimum M, $y' = 0$, so its numerator is zero, giving the required equation.

$$\boxed{2\sin x + \cos x = -2}$$

2 Part (b)

Write $2\sin x + \cos x = \sqrt{5}\sin(x+\beta)$, where $\cos\beta = 2/\sqrt{5}$, $\sin\beta = 1/\sqrt{5}$ and $\beta = \arctan(1/2)$. The equation becomes $\sin(x+\beta) = -2/\sqrt{5}$. Let $\gamma = \arcsin(2/\sqrt{5})$, so $\gamma + \beta = \pi/2$. The solutions in the closed boundary range include $x = -\pi/2$, $3\pi/2$, and

$$x = \pi + \gamma - \beta = 3.7850937624\ldots.$$

Both boundary values are outside the original open domain (and make its denominator zero). The sole interior stationary point is M; the derivative changes from negative to positive there.

$$\boxed{x_M = 3.79}$$

FINAL ANSWER

(a) $2\sin x + \cos x = -2$ (b) $x_M = 3.79$

Question 9

11 marks | Applications of Differentiation | Main topic: Numerical Methods

9.

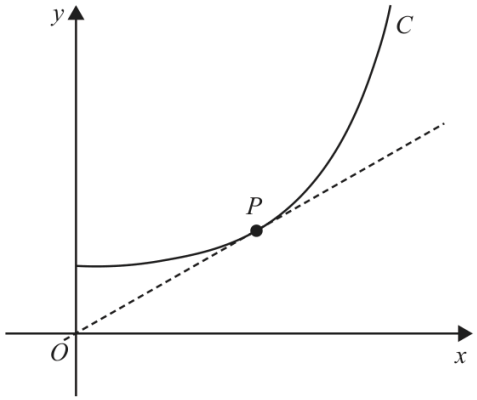


Figure 3

Figure 3 shows a sketch of part of the curve C with equation

$$y = \sqrt{3 + 4e^{x^2}} \qquad x \geqslant 0$$

(a) Find $\frac{dy}{dx}$, giving your answer in simplest form.

(2)

The point P with x coordinate α lies on C.

Given that the tangent to C at P passes through the origin, as shown in Figure 3,

(b) show that $x = \alpha$ is a solution of the equation

$$4x^2e^{x^2} - 4e^{x^2} - 3 = 0$$

(3)

(c) Hence show that α lies between 1 and 2

(2)

(d) Show that the equation in part (b) can be written in the form

$$x = \frac{1}{2}\sqrt{4 + 3e^{-x^2}}$$

(1)

The iteration formula

$$x_{n+1} = \frac{1}{2}\sqrt{4 + 3e^{-x_n^2}}$$

with $x_1 = 1$ is used to find an approximation for α.

(e) Use the iteration formula to find, to 4 decimal places, the value of

(i) x_3

(ii) α

(3)

1 Part (a)

Differentiate $y = (3 + 4e^{x^2})^{1/2}$ using the chain rule:

$$\frac{dy}{dx} = \frac{1}{2}(3 + 4e^{x^2})^{-1/2} \cdot 4e^{x^2} \cdot 2x = \frac{4xe^{x^2}}{\sqrt{3 + 4e^{x^2}}}.$$

$$\boxed{\frac{dy}{dx} = \frac{4xe^{x^2}}{\sqrt{3 + 4e^{x^2}}}}$$

2 Part (b)

Since the tangent at $P = (\alpha, \sqrt{3 + 4e^{\alpha^2}})$ passes through the origin, its gradient is y_P/α, with $\alpha > 0$. Equate this to the derivative:

$$\frac{4\alpha e^{\alpha^2}}{\sqrt{3 + 4e^{\alpha^2}}} = \frac{\sqrt{3 + 4e^{\alpha^2}}}{\alpha} \quad \Longrightarrow \quad 4\alpha^2 e^{\alpha^2} = 3 + 4e^{\alpha^2}.$$

Consequently $x = \alpha$ solves the displayed equation.

$$\boxed{4x^2 e^{x^2} - 4e^{x^2} - 3 = 0}$$

3 Part (c)

Let $h(x) = 4x^2 e^{x^2} - 4e^{x^2} - 3$, a continuous function. Then $h(1) = -3 < 0$ and $h(2) = 12e^4 - 3 = 652.1778\ldots > 0$, so a root lies between 1 and 2. Moreover $h'(x) = 8x^3 e^{x^2} > 0$ for $x > 0$, so this is the unique positive root corresponding to the tangent abscissa α.

$$\boxed{1 < \alpha < 2}$$

4 Part (d)

From $4x^2 e^{x^2} = 4e^{x^2} + 3$, divide by the positive e^{x^2} to get $4x^2 = 4 + 3e^{-x^2}$. The source domain has $x \geq 0$, so take the positive square root.

$$\boxed{x = \frac{1}{2}\sqrt{4 + 3e^{-x^2}}}$$

5

Part (e)

With $x_1 = 1$, the unrounded iterates start

$$x_2 = 1.1295616764\ldots, \quad x_3 = 1.0997193683\ldots, \quad x_4 = 1.1062486441\ldots, \quad x_5 = 1.1048012899\ldots.$$

Continue: $x_6 = 1.1051212304\ldots$, $x_7 = 1.1050504628\ldots$, $x_8 = 1.1050661137\ldots$, $x_9 = 1.1050626523\ldots$; further iteration settles at $1.1050632792\ldots$. Therefore (i) $x_3 = 1.0997$ and (ii) $\alpha = 1.1051$, both to four decimal places.

$$\boxed{x_3 = 1.0997, \qquad \alpha = 1.1051}$$

✓ **FINAL ANSWER**

(a) $\frac{dy}{dx} = \frac{4xe^{x^2}}{\sqrt{3+4e^{x^2}}}$ (b) $4x^2e^{x^2} - 4e^{x^2} - 3 = 0$ (c) $1 < \alpha < 2$ (d) $x = \frac{1}{2}\sqrt{4 + 3e^{-x^2}}$ (e) $x_3 = 1.0997, \qquad \alpha = 1.1051$

WMA13/01 JANUARY 2023

Question 10

9 marks | Applications of Differentiation | Main topic: Log Graphs & Exponential Models

10. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

A population of fruit flies is being studied.

The number of fruit flies, F, in the population, t days after the start of the study, is modelled by the equation

$$F = \frac{350e^{kt}}{9 + e^{kt}}$$

where k is a constant.

Use the equation of the model to answer parts (a), (b) and (c).

(a) Find the number of fruit flies in the population at the start of the study.

(1)

Given that there are 200 fruit flies in the population 15 days after the start of the study,

(b) show that $k = \frac{1}{15}\ln 12$

(3)

Given also that, when $t = T$, the number of fruit flies in the population is increasing at a rate of 10 per day,

(c) find the possible values of T, giving your answers to one decimal place.

(5)

WORKED SOLUTION

Question 10 · WMA13/01 January 2023

1 **Part (a)**

At the start $t = 0$, so $e^{kt} = 1$ and the model gives $F(0) = 350/(9 + 1) = 35$ fruit flies.

$$\boxed{F(0) = 35}$$

2 Part (b)

Using $F(15) = 200$,

$$200 = \frac{350e^{15k}}{9 + e^{15k}} \quad \Longrightarrow \quad 1800 + 200e^{15k} = 350e^{15k} \quad \Longrightarrow \quad e^{15k} = 12.$$

Take natural logs to obtain $15k = \ln 12$, as required.

$$\boxed{k = \frac{\ln 12}{15}}$$

3 Part (c)

By the quotient rule,

$$F'(t) = \frac{350ke^{kt}(9 + e^{kt}) - 350ke^{2kt}}{(9 + e^{kt})^2} = \frac{3150ke^{kt}}{(9 + e^{kt})^2}.$$

Set $F'(T) = 10$ and let $u = e^{kT} > 0$. Then

$$315ku = (9 + u)^2 \quad \Longrightarrow \quad u^2 + (18 - 315k)u + 81 = 0.$$

With the exact $k = \ln 12/15$, the quadratic formula gives

$$u = \frac{315k - 18 \pm \sqrt{(315k - 18)^2 - 324}}{2} = 2.5615490777\ldots \text{ or } 31.6214905678\ldots.$$

Both values are positive and at least 1, hence both correspond to $T \geq 0$. Finally $T = \ln u/k$ gives $5.6779528361\ldots$ or $20.8488936858\ldots$ days. To one decimal place the two times are as follows.

$$\boxed{T = 5.7 \text{ days}, \qquad T = 20.8 \text{ days}}$$

FINAL ANSWER

(a) $F(0) = 35$ (b) $k = \frac{\ln 12}{15}$ (c) $T = 5.7$ days, $\qquad T = 20.8$ days

WMA13/01 MAY/JUNE 2023

Question 7

8 marks | Applications of Differentiation | Main topic: Log Graphs & Exponential Models

7. A scientist is studying two different populations of bacteria.

The number of bacteria N in the first population is modelled by the equation

$$N = Ae^{kt} \qquad t \geqslant 0$$

where A and k are positive constants and t is the time in hours from the start of the study.

Given that
- there were 2500 bacteria in this population at the start of the study
- there were 10 000 bacteria 8 hours later

(a) find the exact value of A and the value of k to 4 significant figures.

(3)

The number of bacteria N in the second population is modelled by the equation

$$N = 60\,000e^{-0.6t} \qquad t \geqslant 0$$

where t is the time in hours from the start of the study.

(b) Find the rate of decrease of bacteria in this population exactly 5 hours from the start of the study. Give your answer to 3 significant figures.

(2)

When $t = T$, the number of bacteria in the two different populations was the same.

(c) Find the value of T, giving your answer to 3 significant figures.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(3)

WORKED SOLUTION

Question 7 · WMA13/01 May/June 2023

1 **Part (a)**

At $t = 0$, $2500 = Ae^0$, so $A = 2500$ exactly. At $t = 8$, $10000 = 2500e^{8k}$; hence $e^{8k} = 4$ and $k = \ln 4/8 = 0.1732867951\ldots$. To four significant figures:

$$\boxed{A = 2500, \qquad k = 0.1733}$$

2 Part (b)

For the second population,

$$\frac{dN}{dt} = -36000e^{-0.6t}, \qquad \left.\frac{dN}{dt}\right|_{t=5} = -36000e^{-3} = -1792.334461\ldots.$$

The requested rate of decrease is the positive magnitude, to three significant figures.

$$\boxed{1.79 \times 10^3 \text{ bacteria per hour}}$$

3 Part (c)

Retain the exact value $k = \ln 4/8$. Equating populations gives

$$2500e^{kT} = 60000e^{-0.6T} \quad \Longrightarrow \quad e^{(k+0.6)T} = 24.$$

Take natural logs and solve:

$$T = \frac{\ln 24}{0.6 + \ln 4/8} = 4.109799689\ldots \text{ hours}.$$

$$\boxed{T = 4.11 \text{ hours}}$$

FINAL ANSWER

(a) $A = 2500, \qquad k = 0.1733$ (b) 1.79×10^3 bacteria per hour (c) $T = 4.11$ hours

Question 10

8 marks | Applications of Differentiation | Main topic: Differentiation Techniques

10.

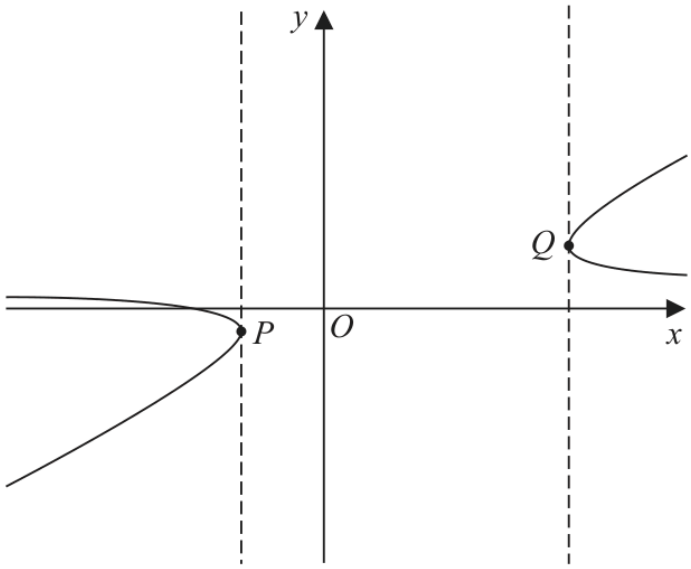


Figure 4

Figure 4 shows a sketch of the curve with equation

$$x = \frac{2y^2 + 6}{3y - 3}$$

(a) Find $\frac{dx}{dy}$ giving your answer as a fully simplified fraction.

(4)

The tangents at points P and Q on the curve are parallel to the y-axis, as shown in Figure 4.

(b) Use the answer to part (a) to find the equations of these two tangents.

(4)

1 Part (a)

The quotient rule gives, for $y \neq 1$,

$$\frac{dx}{dy} = \frac{4y(3y-3) - 3(2y^2+6)}{(3y-3)^2} = \frac{6y^2 - 12y - 18}{9(y-1)^2} = \frac{2(y-3)(y+1)}{3(y-1)^2}.$$

$$\boxed{\frac{dx}{dy} = \frac{2(y-3)(y+1)}{3(y-1)^2}, \quad y \neq 1}$$

2 Part (b)

A tangent parallel to the y-axis is vertical, so $dx/dy = 0$. The numerator gives $(y-3)(y+1) = 0$, hence $y = 3$ or $y = -1$; neither makes the denominator zero. Then

$$x(3) = \frac{18+6}{9-3} = 4, \qquad x(-1) = \frac{2+6}{-3-3} = -\frac{4}{3}.$$

Thus $Q = (4, 3)$ and $P = (-4/3, -1)$, with the vertical tangent equations below.

$$\boxed{x = -\frac{4}{3}, \qquad x = 4}$$

FINAL ANSWER

(a) $\frac{dx}{dy} = \frac{2(y-3)(y+1)}{3(y-1)^2}, \quad y \neq 1$ (b) $x = -\frac{4}{3}, \qquad x = 4$

Question 7

10 marks | Applications of Differentiation | Main topic: Functions, Composition & Inverses

7.

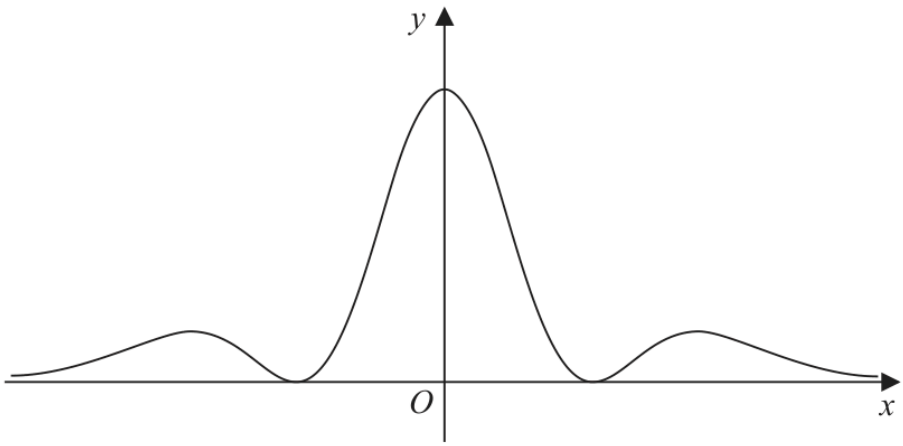


Figure 1

Figure 1 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \mathrm{e}^{-x^2}(2x^2 - 3)^2$$

(a) Find the range of f

(2)

(b) Show that

$$\mathrm{f}'(x) = 2x(2x^2 - 3)\mathrm{e}^{-x^2}(A - Bx^2)$$

where A and B are constants to be found.

(4)

Given that the line $y = k$, where k is a constant, $k > 0$, intersects the curve at exactly two distinct points,

(c) find the exact range of values of k

(4)

1 Part (a)

The exponential is positive and the squared factor is nonnegative, so $f(x) \geq 0$. The value zero is attained at $x = \pm\sqrt{3/2}$. The graph has its highest point at $x = 0$, where $f(0) = 9$; the outer maxima are lower. Thus both endpoints of the range are included.

$$\boxed{0 \leq f(x) \leq 9}$$

2 Part (b)

For $f(x) = e^{-x^2}(2x^2 - 3)^2$, product and chain rules give

$$f'(x) = -2xe^{-x^2}(2x^2 - 3)^2 + e^{-x^2}\,2(2x^2 - 3)\,4x.$$

Factoring yields

$$f'(x) = 2x(2x^2 - 3)e^{-x^2}\big(-(2x^2 - 3) + 4\big) = 2x(2x^2 - 3)e^{-x^2}(7 - 2x^2).$$

$$\boxed{A = 7, \qquad B = 2}$$

3 Part (c)

The stationary abscissae are 0, $\pm\sqrt{3/2}$ and $\pm\sqrt{7/2}$. At the outer local maxima,

$$f\left(\pm\sqrt{7/2}\right) = e^{-7/2}(7 - 3)^2 = 16e^{-7/2}.$$

The central maximum is 9 and the two minima are zero. Also $f(x) \to 0$ as $x \to \pm\infty$. A positive horizontal line above both outer maxima but below 9 meets only the two sides of the central peak. At $k = 16e^{-7/2}$ it also touches both outer peaks, giving four distinct intersections; at $k = 9$ there is only one. Both bounds must be excluded.

$$\boxed{16e^{-7/2} < k < 9}$$

FINAL ANSWER

(a) $0 \leq f(x) \leq 9$ (b) $A = 7, \qquad B = 2$ (c) $16e^{-7/2} < k < 9$

Question 10

9 marks | Applications of Differentiation | Main topic: Differentiation Techniques

10. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

A curve C has equation

$$x = \sin^2 4y \qquad 0 \leqslant y \leqslant \frac{\pi}{8} \qquad 0 \leqslant x \leqslant 1$$

The point P with x coordinate $\frac{1}{4}$ lies on C

(a) Find the exact y coordinate of P

(2)

(b) Find $\frac{\mathrm{d}x}{\mathrm{d}y}$

(2)

(c) Hence show that $\frac{\mathrm{d}y}{\mathrm{d}x}$ can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{q + r(x + s)^2}}$$

where q, r and s are constants to be found.

(3)

Using the answer to part (c),

(d) (i) state the x coordinate of the point where the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ is a minimum,

(ii) state the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at this point.

(2)

1 Part (a)

At $x = 1/4$, $\sin^2 4y = 1/4$. Since $0 \le 4y \le \pi/2$,

$$\sin 4y = \frac{1}{2} \quad \Longrightarrow \quad 4y = \frac{\pi}{6}.$$

Only this branch lies in the given range of y.

$$\boxed{y = \frac{\pi}{24}}$$

2 Part (b)

Differentiate $x = (\sin 4y)^2$ with respect to y:

$$\frac{dx}{dy} = 2\sin 4y\,(4\cos 4y).$$

$$\boxed{\frac{dx}{dy} = 8\sin 4y\cos 4y}$$

3 Part (c)

On the interior of the given interval, $\sin 4y = \sqrt{x}$ and $\cos 4y = \sqrt{1-x}$ are positive. Taking the reciprocal,

$$\frac{dy}{dx} = \frac{1}{8\sqrt{x}\sqrt{1-x}} = \frac{1}{\sqrt{64x - 64x^2}}.$$

Complete the square:

$$64x - 64x^2 = 16 - 64\left(x - \frac{1}{2}\right)^2.$$

Thus $dy/dx = (q + r(x+s)^2)^{-1/2}$ has the constants below, with $0 < x < 1$ for a finite derivative; the endpoints have vertical tangents.

$$\boxed{q = 16, \qquad r = -64, \qquad s = -\frac{1}{2}}$$

4 Part (d)

The derivative is positive. Its minimum occurs when the square root in the denominator is largest. Since $(x - 1/2)^2 \ge 0$, the largest radicand is 16, attained at $x = 1/2$. Hence the smallest gradient is $1/\sqrt{16}$.

$$\boxed{\text{(i) } x = \frac{1}{2}, \qquad \text{(ii) } \min \frac{dy}{dx} = \frac{1}{4}}$$

FINAL ANSWER

(a) $y = \frac{\pi}{24}$ (b) $\frac{dx}{dy} = 8\sin 4y \cos 4y$ (c) $q = 16, \qquad r = -64, \qquad s = -\frac{1}{2}$ (d)
(i) $x = \frac{1}{2}$, (ii) $\min \frac{dy}{dx} = \frac{1}{4}$

Question 8

10 marks | Applications of Differentiation | Main topic: Log Graphs & Exponential Models

8.

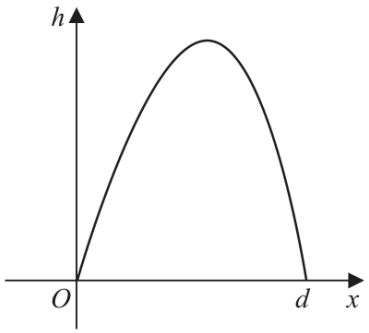


Figure 4

Figure 4 is a graph showing the path of a golf ball after the ball has been hit until it first hits the ground.

The vertical height, h metres, of the ball above the ground has been plotted against the horizontal distance travelled, x metres, measured from where the ball was hit.

The ball travels a horizontal distance of d metres before it first hits the ground.

The ball is modelled as a particle travelling in a vertical plane above horizontal ground.

The path of the ball is modelled by the equation

$$h = 1.5x - 0.5x\,e^{0.02x} \qquad 0 \leqslant x \leqslant d$$

Use the model to answer parts (a), (b) and (c).

(a) Find the value of d, giving your answer to 2 decimal places.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(3)

(b) Show that the maximum value of h occurs when

$$x = 50\ln\left(\frac{150}{x + 50}\right)$$

(4)

Using the iteration formula

$$x_{n+1} = 50\ln\left(\frac{150}{x_n + 50}\right) \qquad \text{with } x_1 = 30$$

(c) (i) find the value of x_2 to 2 decimal places,

(ii) find, by repeated iteration, the horizontal distance travelled by the golf ball before it reaches its maximum height. Give your answer to 2 decimal places.

(3)

1 Part (a)

At the first landing, $h = 0$ and $x = d > 0$:

$$0 = d(1.5 - 0.5e^{0.02d}), \qquad e^{0.02d} = 3.$$

Taking logarithms, $0.02d = \ln 3$, hence $d = 50\ln 3 = 54.930614\ldots$. The root $x = 0$ is the launch point, not the landing point.

$$\boxed{d = 54.93 \text{ m}}$$

2 Part (b)

By the product and chain rules,

$$h'(x) = 1.5 - 0.5e^{0.02x} - 0.01xe^{0.02x}.$$

At a stationary point,

$$1.5 = 0.01(x + 50)e^{0.02x}, \qquad e^{0.02x} = \frac{150}{x + 50}.$$

Since $x + 50 > 0$, logarithms give $0.02x = \ln(150/(x + 50))$. Also $h''(x) = -(0.02 + 0.0002x)e^{0.02x} < 0$ on $[0, d]$. Thus the stationary point is the unique maximum; the endpoints both have zero height.

$$\boxed{x = 50\ln\left(\frac{150}{x + 50}\right)}$$

3 Part (c)

(i) With $x_1 = 30$,

$$x_2 = 50\ln\left(\frac{150}{30 + 50}\right) = 31.43043297\ldots.$$

(ii) Continuing with full intermediate precision gives $x_3 = 30.54431108\ldots$, $x_4 = 31.09139065\ldots$, $x_5 = 30.75292479\ldots$ and $x_6 = 30.96205562\ldots$. Continued iteration converges to $30.88212\ldots$, which lies in $(0, d)$. This is the horizontal distance at maximum height, not the maximum height itself.

$$\boxed{x_2 = 31.43, \quad x_{\text{maximum}} = 30.88 \text{ m}}$$

FINAL ANSWER

(a) $d = 54.93$ m (b) $x = 50\ln\left(\frac{150}{x+50}\right)$ (c) $x_2 = 31.43, \quad x_{\text{maximum}} = 30.88$ m

Question 7

10 marks | Applications of Differentiation | Main topic: Graph Transformations

7.

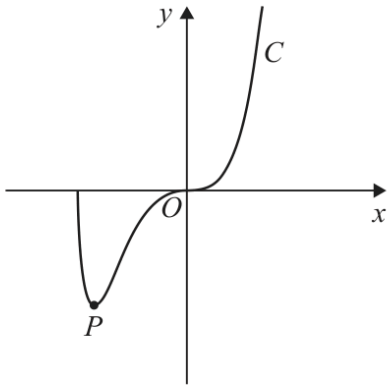


Figure 3

The curve C has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x^3\sqrt{4x+7} \qquad x \geqslant -\frac{7}{4}$$

(a) Show that

$$\mathrm{f}'(x) = \frac{kx^2(2x+3)}{\sqrt{4x+7}}$$

where k is a constant to be found.

(4)

The point P, shown in Figure 3, is the minimum turning point on C.

(b) Find the coordinates of P.

(2)

(c) Hence find the range of the function g defined by

$$\mathrm{g}(x) = -4\,\mathrm{f}(x) \qquad x \geqslant -\frac{7}{4}$$

(2)

The point Q with coordinates $\left(\frac{1}{2}, \frac{3}{8}\right)$ lies on C.

(d) Find the coordinates of the point to which Q is mapped when C is transformed to the curve with equation

$$y = 40\,\mathrm{f}\left(x - \frac{3}{2}\right) - 8$$

(2)

1 Part (a)

For $x > -7/4$, apply the product and chain rules:

$$f'(x) = 3x^2\sqrt{4x+7} + x^3\frac{2}{\sqrt{4x+7}} = \frac{3x^2(4x+7)+2x^3}{\sqrt{4x+7}} = \frac{7x^2(2x+3)}{\sqrt{4x+7}}.$$

$$\boxed{k = 7}$$

2 Part (b)

The interior stationary values satisfy $x^2(2x+3)=0$, so $x=0$ or $x=-3/2$. Near $x=-3/2$, the derivative changes from negative to positive; near $x=0$ it stays positive on both sides. Thus the minimum turning point is at $x=-3/2$.

$$f(-3/2) = (-3/2)^3\sqrt{-6+7} = -27/8.$$

$$\boxed{P = \left(-\frac{3}{2}, -\frac{27}{8}\right)}$$

3 Part (c)

The endpoint has $f(-7/4)=0$, the global minimum is $-27/8$ and $f(x)$ tends to infinity as x increases. Hence the range of f is $[-27/8, \infty)$. Multiplication by -4 reverses the inequality and makes the upper endpoint $27/2$.

$$\boxed{g(x) \le \frac{27}{2}}$$

4 Part (d)

The transformation is a shift right by $3/2$, vertical stretch by 40, then shift down by 8:

$$(x, y) \mapsto (x + 3/2,\, 40y - 8).$$

Therefore $Q = (1/2, 3/8)$ maps to $(1/2 + 3/2,\ 40\cdot 3/8 - 8)$.

$$\boxed{Q \mapsto (2, 7)}$$

FINAL ANSWER

(a) $k = 7$ (b) $P = \left(-\frac{3}{2}, -\frac{27}{8}\right)$ (c) $g(x) \le \frac{27}{2}$ (d) $Q \mapsto (2, 7)$

Question 8

12 marks | Applications of Differentiation | Main topic: Log Graphs & Exponential Models

8.

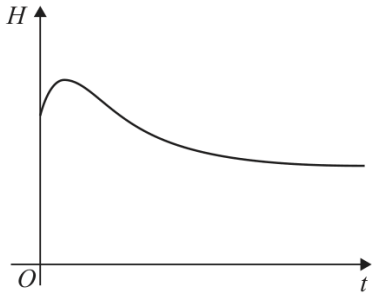


Figure 4

The heart rate of a horse is being monitored.
The heart rate H, measured in beats per minute (bpm), is modelled by the equation

$$H = 32 + 40\mathrm{e}^{-0.2t} - 20\,\mathrm{e}^{-0.9t}$$

where t minutes is the time after monitoring began.

Figure 4 is a sketch of H against t.

Use the equation of the model to answer parts (a) to (e).

(a) State the initial heart rate of the horse.

(1)

In the long term, the heart rate of the horse approaches L bpm.

(b) State the value of L.

(1)

The heart rate of the horse reaches its maximum value after T minutes.

(c) Find the value of T, giving your answer to 3 decimal places.

(*Solutions based entirely on calculator technology are not acceptable.*)

(5)

The heart rate of the horse is 37 bpm after M minutes.

(d) Show that M is a solution of the equation

$$t = 5\ln\left(\frac{8}{1 + 4\mathrm{e}^{-0.9t}}\right)$$

(2)

Using the iteration formula

$$t_{n+1} = 5\ln\left(\frac{8}{1 + 4\mathrm{e}^{-0.9t_n}}\right) \quad \text{with} \quad t_1 = 10$$

(e) (i) find, to 4 decimal places, the value of t_2

(ii) find, to 4 decimal places, the value of M

(3)

1 Part (a)

At $t = 0$, $H = 32 + 40 - 20 = 52$.

$$\boxed{H(0) = 52 \text{ bpm}}$$

2 Part (b)

As $t \to \infty$, both decaying exponential terms tend to zero.

$$\boxed{L = 32}$$

3 Part (c)

$$\frac{dH}{dt} = -8e^{-0.2t} + 18e^{-0.9t}.$$

At the maximum,

$$18e^{-0.9T} = 8e^{-0.2T} \quad \Longrightarrow \quad e^{0.7T} = \frac{9}{4} \quad \Longrightarrow \quad T = \frac{10}{7}\ln\frac{9}{4} = 1.1584717\ldots.$$

Also $H''(t) = 1.6e^{-0.2t} - 16.2e^{-0.9t}$. At the stationary point, $e^{-0.9T} = (4/9)e^{-0.2T}$, so $H''(T) = -5.6e^{-0.2T} < 0$, confirming a maximum.

$$\boxed{T = 1.158 \text{ minutes}}$$

4 Part (d)

For $H = 37$,

$$5 = 40e^{-0.2t} - 20e^{-0.9t} \quad \Longrightarrow \quad 8e^{-0.2t} = 1 + 4e^{-0.9t}.$$

Taking logarithms gives

$$-0.2t = \ln\left(\frac{1 + 4e^{-0.9t}}{8}\right) \quad \Longrightarrow \quad t = -5\ln\left(\frac{1 + 4e^{-0.9t}}{8}\right).$$

Using $-\ln u = \ln(1/u)$ for $u > 0$ then gives

$$t = 5\ln\left(\frac{8}{1 + 4e^{-0.9t}}\right).$$

$$\boxed{t = 5\ln\left(\frac{8}{1 + 4e^{-0.9t}}\right)}$$

5

Part (e)

Starting with $t_1 = 10$,

$$t_2 = 5\ln\left(\frac{8}{1+4e^{-9}}\right) = 10.3947401213\ldots.$$

Continuing without rounding intermediate values gives $t_3 = 10.3954778346\ldots$, $t_4 = 10.3954789826\ldots$ and $t_5 = 10.3954789844\ldots$. Thus (i) $t_2 = 10.3947$, and (ii) $M = 10.3955$ minutes, both to 4 decimal places.

$$\boxed{t_2 = 10.3947, \quad M = 10.3955 \text{ minutes}}$$

✓ **FINAL ANSWER**

(a) $H(0) = 52$ bpm (b) $L = 32$ (c) $T = 1.158$ minutes (d) $t = 5\ln\left(\frac{8}{1+4e^{-0.9t}}\right)$ (e) $t_2 = 10.3947, \quad M = 10.3955$ minutes

Question 9

14 marks | Applications of Differentiation | Main topic: Integration

9.

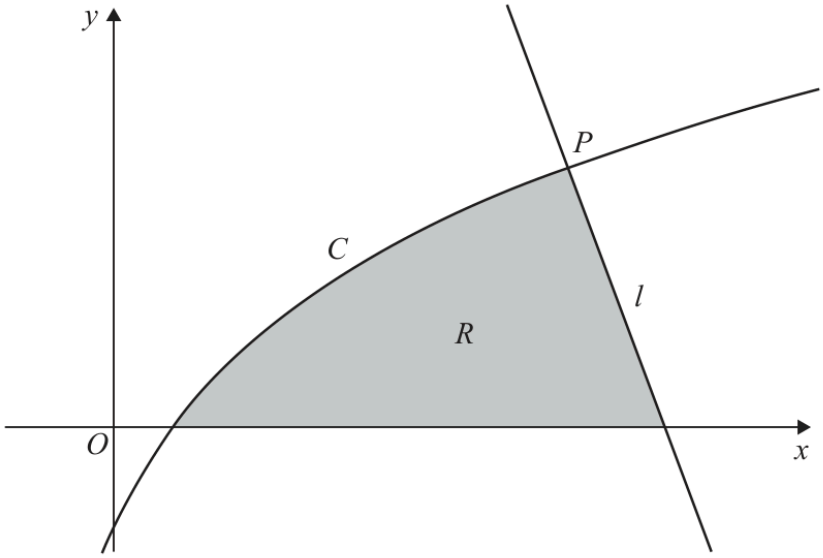


Figure 5

Figure 5 shows a sketch of part of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{6x^2 + 4x - 2}{2x + 1} \qquad x > -\frac{1}{2}$$

(a) Find $\mathrm{f}'(x)$, giving the answer in simplest form.

(3)

The line l is the normal to C at the point $P(2, 6)$

(b) Show that an equation for l is

$$16y + 5x = 106$$

(3)

(c) Write $\mathrm{f}(x)$ in the form $Ax + B + \dfrac{D}{2x + 1}$ where A, B and D are constants.

(3)

The region R, shown shaded in Figure 5, is bounded by C, l and the x-axis.

(d) Use algebraic integration to find the exact area of R, giving your answer in the form $P + Q\ln 3$, where P and Q are rational constants.

(*Solutions based entirely on calculator technology are not acceptable.*)

(5)

1 Part (a)

With $u = 6x^2 + 4x - 2$ and $v = 2x + 1$, the quotient rule gives

$$f'(x) = \frac{(12x + 4)(2x + 1) - 2(6x^2 + 4x - 2)}{(2x + 1)^2} = \frac{12x^2 + 12x + 8}{(2x + 1)^2}.$$

$$\boxed{f'(x) = \frac{4(3x^2 + 3x + 2)}{(2x + 1)^2}}$$

2 Part (b)

At $x = 2$, $f'(2) = 80/25 = 16/5$, so the normal gradient is $-5/16$. Through $P = (2, 6)$,

$$y - 6 = -\frac{5}{16}(x - 2) \quad \Longrightarrow \quad 16y - 96 = -5x + 10.$$

$$\boxed{16y + 5x = 106}$$

3 Part (c)

Multiplying the proposed identity by $2x + 1$ gives

$$6x^2 + 4x - 2 = (Ax + B)(2x + 1) + D = 2Ax^2 + (A + 2B)x + B + D.$$

Compare coefficients: $2A = 6$, $A + 2B = 4$, $B + D = -2$. Thus $f(x) = 3x + \frac{1}{2} - \frac{5}{2(2x+1)}$.

$$\boxed{A = 3, \quad B = \frac{1}{2}, \quad D = -\frac{5}{2}}$$

4 Part (d)

The curve meets the x-axis when $6x^2 + 4x - 2 = 2(3x - 1)(x + 1) = 0$. The root $x = -1$ is outside $x > -1/2$, leaving $x = 1/3$. The normal meets the axis at $x = 106/5$. Split the shaded region at P: a curve area from $1/3$ to 2, then a triangle.

$$\int_{1/3}^{2} f(x)\,dx = \left[\frac{3}{2}x^2 + \frac{1}{2}x - \frac{5}{4}\ln(2x + 1)\right]_{1/3}^{2} = 7 - \frac{1}{3} - \frac{5}{4}\ln\left(\frac{5}{5/3}\right) = \frac{20}{3} - \frac{5}{4}\ln 3.$$

The triangle has area $\frac{1}{2}(106/5 - 2) \cdot 6 = 288/5$. Adding gives the exact area of R.

$$\boxed{\text{Area}(R) = \frac{964}{15} - \frac{5}{4}\ln 3}$$

FINAL ANSWER

(a) $f'(x) = \frac{4(3x^2+3x+2)}{(2x+1)^2}$ (b) $16y + 5x = 106$ (c) $A = 3, \quad B = \frac{1}{2}, \quad D = -\frac{5}{2}$ (d) $\text{Area}(R) = \frac{964}{15} - \frac{5}{4}\ln 3$

Question 10

9 marks | Applications of Differentiation | Main topic: Differentiation Techniques

10.

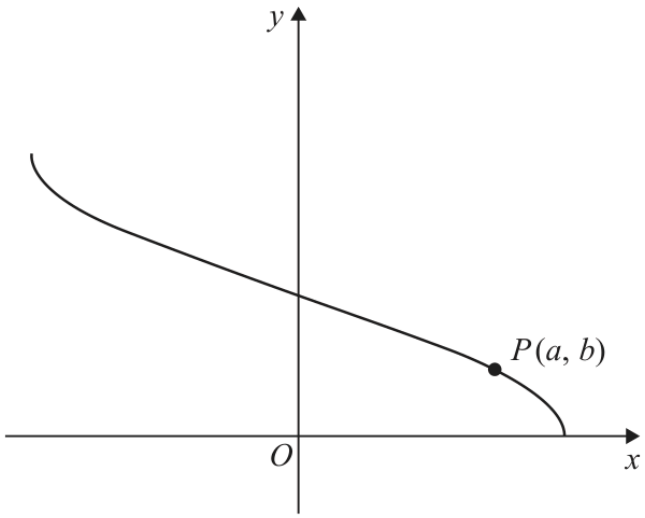


Figure 2

Figure 2 shows a sketch of the curve with equation

$$x = 3\cos 2y \qquad -3 \leqslant x \leqslant 3 \qquad 0 \leqslant y \leqslant \frac{\pi}{2}$$

(a) Find $\frac{\mathrm{d}x}{\mathrm{d}y}$ in terms of y.

(2)

(b) Hence show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{k}{\sqrt{9 - x^2}}$$

where k is a constant to be found.

(3)

The point $P(a, b)$ lies on the curve and is shown in Figure 2.

Given that

- the gradient of the curve at P is $-\frac{1}{4}$
- both a and b are positive

(c) find the exact values of a and b.

(4)

1

Part (a)

Differentiate $x = 3\cos 2y$ with respect to y using the chain rule:

$$\frac{dx}{dy} = 3(-\sin 2y)\cdot 2 = -6\sin 2y.$$

$$\boxed{\frac{dx}{dy} = -6\sin 2y}$$

2

Part (b)

For $0 < y < \pi/2$, $\sin 2y > 0$ and

$$\frac{dy}{dx} = \frac{1}{dx/dy} = -\frac{1}{6\sin 2y}.$$

Since $\cos 2y = x/3$ and the sine is positive on this interval,

$$\sin 2y = \sqrt{1-\cos^2 2y} = \sqrt{1 - x^2/9} = \frac{\sqrt{9-x^2}}{3}.$$

Consequently $dy/dx = -1/(2\sqrt{9-x^2})$, valid for $-3 < x < 3$. The endpoint tangents are vertical; there is no finite derivative there.

$$\boxed{\frac{dy}{dx} = -\frac{1}{2\sqrt{9-x^2}}, \qquad k = -\frac{1}{2}}$$

3

Part (c)

Set the gradient equal to $-1/4$:

$$-\frac{1}{2\sqrt{9-a^2}} = -\frac{1}{4} \quad\Longrightarrow\quad \sqrt{9-a^2} = 2 \quad\Longrightarrow\quad a^2 = 5.$$

Because $a > 0$, take $a = \sqrt{5}$. Substitute into $a = 3\cos 2b$:

$$\cos 2b = \frac{\sqrt{5}}{3} \quad\Longrightarrow\quad b = \frac{1}{2}\arccos\left(\frac{\sqrt{5}}{3}\right).$$

Here $0 < 2b < \pi/2$ since both coordinates are positive, so the principal arccos value is the required one.

$$\boxed{a = \sqrt{5}, \qquad b = \frac{1}{2}\arccos\left(\frac{\sqrt{5}}{3}\right)}$$

FINAL ANSWER

(a) $\frac{dx}{dy} = -6\sin 2y$ (b) $\frac{dy}{dx} = -\frac{1}{2\sqrt{9-x^2}}$, $\quad k = -\frac{1}{2}$ (c) $a = \sqrt{5}$, $\quad b = \frac{1}{2}\arccos\left(\frac{\sqrt{5}}{3}\right)$

WMA13/01 MAY/JUNE 2025

Question 6

8 marks | Applications of Differentiation | Main topic: Log Graphs & Exponential Models

6: **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The temperature, θ°C, of a computer processor, t minutes after the computer is switched off, is modelled by the equation

$$\theta = 21 + A\mathrm{e}^{-kt}$$

where A and k are positive constants.

Given that the temperature of the processor was 75°C when the computer was switched off,

(a) find the value of A.

(2)

Given also that it takes 5 minutes for the temperature of the processor to decrease from 75°C to 25°C,

(b) find the value of k, giving your answer to 3 significant figures.

(3)

At time T minutes, the temperature of the processor is decreasing at a rate of 9°C per minute.

(c) Find the value of T according to the model, giving your answer to 2 decimal places.

(3)

WORKED SOLUTION

Question 6 · WMA13/01 May/June 2025

Part (a)

At $t = 0$, $75 = 21 + A$, so $\boxed{A = 54}$.

Part (b)

At $t = 5$, $25 = 21 + 54e^{-5k}$, hence

$$e^{-5k} = \frac{2}{27}, \qquad k = \frac{1}{5}\ln\frac{27}{2} = 0.520537937\ldots = \boxed{0.521}.$$

3 Part (c)

Differentiate the temperature model:

$$\frac{d\theta}{dt} = -54ke^{-kt}.$$

A decrease of $9\,^\circ\text{C}$ per minute means $d\theta/dt = -9$, not $+9$. Thus

$$-54ke^{-kT} = -9 \implies e^{-kT} = \frac{1}{6k} \implies T = \frac{\ln(6k)}{k}.$$

Using the unrounded value of k,

$$T = 2.187865\ldots = \boxed{2.19 \text{ minutes}}.$$

FINAL ANSWER

(a) $A = 54$ (b) 0.521 (c) 2.19 minutes

Question 7

9 marks | Applications of Differentiation | Main topic: Numerical Methods

7: A continuous curve has equation

$$y = e^{-x^2} \sin 3x \qquad 0 \leqslant x \leqslant \frac{\pi}{3}$$

The curve has a stationary point at the point P.

(a) Show, using calculus, that the x coordinate of P is a solution of the equation

$$x = \frac{1}{3} \arctan\left(\frac{3}{2x}\right)$$

(4)

Using the iteration formula

$$x_{n+1} = \frac{1}{3} \arctan\left(\frac{3}{2x_n}\right) \qquad x_1 = 0.4$$

(b) find the value of

(i) x_2

(ii) x_4

giving your answers to 4 decimal places.

(3)

(c) Using a suitable interval and a suitable function which should be stated, show that the x coordinate of P is 0.430 correct to 3 decimal places.

(2)

1 Part (a)

Apply the product and chain rules:

$$\frac{dy}{dx} = e^{-x^2}\left(3\cos 3x - 2x\sin 3x\right).$$

Since $e^{-x^2} > 0$, a stationary point satisfies

$$3\cos 3x = 2x\sin 3x.$$

Neither endpoint $x = 0, \pi/3$ is stationary. In the interior $\sin 3x > 0$ and $x > 0$, so $\cos 3x > 0$ at a stationary point and $0 < 3x < \pi/2$. Therefore division and the principal inverse tangent are valid:

$$\tan 3x = \frac{3}{2x} \quad \Longrightarrow \quad \boxed{x = \frac{1}{3}\arctan\left(\frac{3}{2x}\right)}.$$

2 Part (b)

Work in radians and retain calculator precision between iterations:

$$x_2 = \frac{1}{3}\arctan\frac{3}{0.8} = 0.4367313117\ldots,$$

$$x_3 = 0.4291581882\ldots, \qquad x_4 = 0.4307116849\ldots.$$

Thus (i) $\boxed{x_2 = 0.4367}$ and (ii) $\boxed{x_4 = 0.4307}$.

3 Part (c)

Let $h(x) = 3\cos 3x - 2x\sin 3x$, which is continuous. At the rounding boundaries,

$$h(0.4295) = 0.0106861\ldots > 0, \qquad h(0.4305) = -0.000598027\ldots < 0.$$

The sign change places a root strictly between 0.4295 and 0.4305. On $0 < x < \pi/6$, $h'(x) = -11\sin 3x - 6x\cos 3x < 0$, so the stationary root is unique; for $\pi/6 \le x \le \pi/3$, $h(x) < 0$. Hence the x coordinate of P is $\boxed{0.430 \text{ to } 3 \text{ decimal places}}$.

FINAL ANSWER

(a) $x = \frac{1}{3}\arctan\left(\frac{3}{2x}\right)$ (b) $x_2 = 0.4367$, $x_4 = 0.4307$ (c) 0.430 to 3 decimal places

Question 9

11 marks | Applications of Differentiation | Main topic: Differentiation Techniques

9:

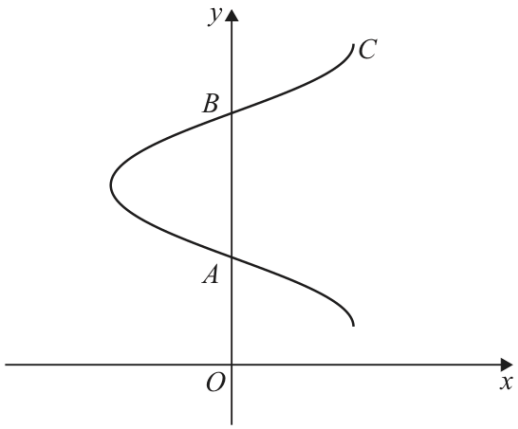


Figure 2

Figure 2 shows a sketch of the curve C with equation

$$x = \frac{2}{3}\sin\left(3y + \frac{\pi}{4}\right) \qquad \frac{\pi}{12} < y < \frac{3\pi}{4}$$

The curve intersects the y-axis at the points A and B as shown.

(a) Find the exact value of the y coordinate of

- point A
- point B

(3)

(b) Show that

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{p - qx^2}$$

where p and q are integers to be found.

(4)

The **normal** to C at A and the **tangent** to C at B intersect at the point D.

Using

- the answer to part (b)
- the sketch of curve C in Figure 2

(c) find, in simplest form, the exact x coordinate of D.

(4)

1 Part (a)

On the y axis, $x = 0$, so $\sin(3y + \pi/4) = 0$. Since $\pi/12 < y < 3\pi/4$, we have $\pi/2 < 3y + \pi/4 < 5\pi/2$. Therefore the possible angles are π and 2π, giving

$$\boxed{y_A = \frac{\pi}{4}, \qquad y_B = \frac{7\pi}{12}}.$$

2 Part (b)

Differentiate x with respect to y, then take the reciprocal:

$$\frac{dx}{dy} = 2\cos(3y + \pi/4), \qquad \frac{dy}{dx} = \frac{1}{2\cos(3y + \pi/4)}.$$

Using $\sin(3y + \pi/4) = 3x/2$,

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4(1 - \sin^2(3y + \pi/4))} = \boxed{\frac{1}{4 - 9x^2}}.$$

Thus $\boxed{p = 4,\ q = 9}$. The derivative formula applies where the tangent is not vertical.

3 Part (c)

At $x = 0$, part (b) gives $|dy/dx| = 1/2$. The sketch shows a negative tangent gradient at A and a positive one at B. The normal at A therefore has gradient 2, while the tangent at B has gradient $1/2$:

$$y = 2x + \frac{\pi}{4}, \qquad y = \frac{x}{2} + \frac{7\pi}{12}.$$

At their intersection,

$$\frac{3}{2}x = \frac{7\pi}{12} - \frac{\pi}{4} = \frac{\pi}{3} \quad \Longrightarrow \quad \boxed{x_D = \frac{2\pi}{9}}.$$

FINAL ANSWER

(a) $y_A = \frac{\pi}{4}, \quad y_B = \frac{7\pi}{12}$ (b) $\frac{1}{4-9x^2}$, $p = 4,\ q = 9$ (c) $x_D = \frac{2\pi}{9}$

WMA13/01 JANUARY 2026

Question 10

9 marks | Applications of Differentiation | Main topic: Log Graphs & Exponential Models

10. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

Two populations of insects are being studied over the same period of time.

The number of insects N_A in population A, t years after the start of the study, is modelled by the equation

$$N_A = 8000 + 900e^{0.2t}$$

According to the model,

(a) find the rate of growth in the number of insects in population A exactly 5 years after the start of the study.

(2)

The number of insects N_B in population B, t years after the start of the study, is modelled by the equation

$$N_B = 8000 + Pe^{kt}$$

where P and k are positive constants.

Given that

- there are 10 000 insects in population B at the start of the study
- there are 11 570 insects in population B exactly 4 years after the start of the study
- there are the same number of insects in population A and population B after T years

find, according to the models,

(b) the exact value of P and the value of k to 3 decimal places,

(4)

(c) the value of T, giving your answer to one decimal place.

(3)

WORKED SOLUTION

Question 10 · WMA13/01 January 2026

1 **Part (a)**

$$\frac{dN_A}{dt} = 900(0.2)e^{0.2t} = 180e^{0.2t}.$$

At $t = 5$, the growth rate is $180e = 489.290729\ldots$ insects per year.

2 Part (b)

At $t = 0$, $10000 = 8000 + P$, so $P = 2000$ exactly. Using the population at $t = 4$,

$$11570 = 8000 + 2000e^{4k} \quad \Longrightarrow \quad e^{4k} = \frac{3570}{2000} = 1.785.$$

Taking natural logarithms,

$$k = \frac{\ln 1.785}{4} = 0.144854604\ldots = 0.145 \quad (3\text{ d.p.}).$$

3 Part (c)

At equality, $8000 + 900e^{0.2T} = 8000 + 2000e^{kT}$, hence

$$e^{(0.2-k)T} = \frac{20}{9}, \qquad T = \frac{\ln(20/9)}{0.2 - k}.$$

Use the unrounded value $k = \ln(1.785)/4$:

$$T = 14.480042784\ldots = 14.5 \text{ years (1 decimal place)}.$$

FINAL ANSWER

(a) $180e$ insects per year (b) $P = 2000,\ k = 0.145\ (3\text{ d.p.})$ (c) 14.5 years

WMA13/01 MAY/JUNE 2026

Question 6

6 marks | Applications of Differentiation | Main topic: Differentiation Techniques

6. The curve C has equation

$$x = \left(3\sin 2y - \frac{6y}{\pi}\right)^2$$

Given that

- the point P with y coordinate $\frac{\pi}{4}$ lies on C
- the tangent to C at P cuts the x-axis at the point Q

use calculus to find the x coordinate of Q

(6)

WORKED SOLUTION

Question 6 · WMA13/01 May/June 2026

1 **Solution**

At $y = \pi/4$,

$$x = \left(3\sin\frac{\pi}{2} - \frac{6(\pi/4)}{\pi}\right)^2 = \left(3 - \frac{3}{2}\right)^2 = \frac{9}{4}.$$

Differentiate x with respect to y using the chain rule:

$$\frac{dx}{dy} = 2\left(3\sin 2y - \frac{6y}{\pi}\right)\left(6\cos 2y - \frac{6}{\pi}\right).$$

At P, $dx/dy = 2(3/2)(-6/\pi) = -18/\pi$. Since this is nonzero, the tangent gradient is $\frac{dy}{dx} = -\frac{\pi}{18}$, and its equation is

$$y - \frac{\pi}{4} = -\frac{\pi}{18}\left(x - \frac{9}{4}\right).$$

Put $y = 0$ for its intersection with the x-axis:

$$x - \frac{9}{4} = \frac{18}{4}, \qquad \boxed{x_Q = \frac{27}{4}}.$$

FINAL ANSWER

$x_Q = \frac{27}{4}$

Question 7

12 marks | Applications of Differentiation | Main topic: Log Graphs & Exponential Models

7. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The selling price of a car is monitored over a 15-year period.

The selling price P_C of the car, in thousands of pounds, t years after monitoring began, is modelled by the equation

$$P_C = 7 + 16e^{-0.1t} \qquad 0 \leqslant t \leqslant 15$$

Given that when $t = T$ the selling price of the car is £15 000

(a) find, according to the model, the value of T. Give the answer to 2 decimal places.

(3)

(b) Find the value of $\frac{dP_C}{dt}$ when $t = 5$. Give the answer to 2 decimal places.

(2)

The selling price of a van is monitored over the same 15-year period.

The selling price P_V of the van, in thousands of pounds, t years after monitoring began, is modelled by the equation

$$P_V = 4 + me^{-0.2t} \qquad 0 \leqslant t \leqslant 15$$

where m is a constant.

Use the equations of these models to answer parts (c) and (d).

Given that $P_V = 36$ when $t = 0$

(c) (i) state the value of m

(ii) find the difference between the initial selling price of the car and the initial selling price of the van.

(3)

When $t = \alpha$, the selling price of the car is equal to the selling price of the van.

(d) Find, using algebra and showing your working, the value of α, giving the answer to 3 significant figures.

(4)

1 Part (a)

Prices in the model are in thousands of pounds, so

$$15 = 7 + 16e^{-0.1T} \quad \Longrightarrow \quad e^{-0.1T} = \frac{1}{2}.$$

Taking logarithms gives $T = 10\ln 2 = 6.9314718\ldots$, hence $\boxed{T = 6.93 \text{ years}}$.

2 Part (b)

$$\frac{dP_C}{dt} = -1.6e^{-0.1t}, \qquad \left.\frac{dP_C}{dt}\right|_{t=5} = -1.6e^{-0.5} = -0.970449\ldots.$$

Thus the answer is $\boxed{-0.97 \text{ thousand pounds per year}}$ to 2 decimal places.

3 Part (c)(i)

At $t = 0$, $36 = 4 + m$, so $\boxed{m = 32}$.

4 Part (c)(ii)

The initial car price is $7 + 16 = 23$ thousand pounds. The difference is $36 - 23 = 13$ thousand pounds, or $\boxed{\pounds 13\,000}$.

5 Part (d)

Equating the models with $m = 32$ gives

$$7 + 16e^{-0.1\alpha} = 4 + 32e^{-0.2\alpha}.$$

Let $u = e^{-0.1\alpha} > 0$. Then $32u^2 - 16u - 3 = 0$, so

$$u = \frac{16 \pm \sqrt{256 + 384}}{64} = \frac{2 \pm \sqrt{10}}{8}.$$

Reject the negative root because an exponential is positive. Hence

$$\alpha = -10\ln\left(\frac{2 + \sqrt{10}}{8}\right) = 4.3806365\ldots, \qquad \boxed{\alpha = 4.38 \text{ years}}.$$

This value is inside the model interval $0 \le t \le 15$.

FINAL ANSWER

(a) $T = 6.93$ years (b) -0.97 thousand pounds per year (c)(i) $m = 32$ (c)(ii) $\pounds 13\,000$ (d) $\alpha = 4.38$ years

TOPIC

Integration

Every past-paper question on this topic, January 2020 to June 2026.

WMA13/01 JANUARY 2020

Question 8

10 marks | Integration

8. (i) Find, using algebraic integration, the exact value of

$$\int_3^{42} \frac{2}{3x-1}\,\mathrm{d}x$$

giving your answer in simplest form.

(4)

(ii) $$\mathrm{h}(x) = \frac{2x^3 - 7x^2 + 8x + 1}{(x-1)^2} \qquad x > 1$$

Given $\mathrm{h}(x) = Ax + B + \frac{C}{(x-1)^2}$ where A, B and C are constants to be found, find

$$\int \mathrm{h}(x)\,\mathrm{d}x$$

(6)

WORKED SOLUTION

Question 8 · WMA13/01 January 2020

1 **Part (i)**

Since $3x - 1 > 0$ throughout $3 \le x \le 42$, an antiderivative is

$$\int \frac{2}{3x-1}\,dx = \frac{2}{3}\ln(3x-1) + K.$$

The factor $1/3$ compensates for the derivative of $3x - 1$. Thus

$$\begin{aligned}\int_3^{42} \frac{2}{3x-1}\,dx &= \left[\frac{2}{3}\ln(3x-1)\right]_3^{42} \\ &= \frac{2}{3}\left(\ln 125 - \ln 8\right) \\ &= \frac{2}{3}\ln\left(\frac{125}{8}\right) = \frac{2}{3}\ln\left(\left(\frac{5}{2}\right)^3\right).\end{aligned}$$

Using $\ln(a^3) = 3\ln a$ for $a > 0$, the exact value in simplest form is

$$\boxed{2\ln\left(\frac{5}{2}\right).}$$

2 **Part (ii)**

Multiply the given decomposition by $(x-1)^2$:

$$\begin{aligned} 2x^3 - 7x^2 + 8x + 1 &= (Ax + B)(x-1)^2 + C \\ &= Ax^3 + (B - 2A)x^2 + (A - 2B)x + (B + C). \end{aligned}$$

Comparing coefficients gives

$$A = 2, \qquad B - 2A = -7 \implies B = -3, \qquad B + C = 1 \implies C = 4.$$

The coefficient of x also agrees, since $A - 2B = 2 - 2(-3) = 8$. Hence

$$\boxed{A = 2, \qquad B = -3, \qquad C = 4,} \qquad h(x) = 2x - 3 + 4(x-1)^{-2}.$$

Integrating each term,

$$\begin{aligned} \int h(x)\,dx &= \int \left(2x - 3 + 4(x-1)^{-2}\right) dx \\ &= x^2 - 3x - 4(x-1)^{-1} + K. \end{aligned}$$

Therefore, for $x > 1$,

$$\boxed{\int h(x)\,dx = x^2 - 3x - \frac{4}{x-1} + K,}$$

where K is an arbitrary constant of integration, distinct from the coefficient $C = 4$.

FINAL ANSWER

(i) $2\ln\left(\frac{5}{2}\right)$ (ii) $A = 2, \qquad B = -3, \qquad C = 4$, $\int h(x)\,dx = x^2 - 3x - \frac{4}{x-1} + K$

WMA13/01 JANUARY 2021

Question 9

4 marks | Integration

9. Find

(i) $\displaystyle\int \frac{3x-2}{3x^2-4x+5}\,\mathrm{d}x$ **(2)**

(ii) $\displaystyle\int \frac{\mathrm{e}^{2x}}{(\mathrm{e}^{2x}-1)^3}\,\mathrm{d}x \qquad x \neq 0$ **(2)**

WORKED SOLUTION

Question 9 · WMA13/01 January 2021

1 Part (i)

The denominator derivative is $6x-4=2(3x-2)$. Hence reverse the logarithmic chain rule:

$$\int \frac{3x-2}{3x^2-4x+5}\,dx=\frac{1}{2}\ln(3x^2-4x+5)+C.$$

The log argument is always positive because $3x^2-4x+5=3(x-2/3)^2+11/3$.

$$\boxed{\frac{1}{2}\ln(3x^2-4x+5)+C}$$

2 Part (ii)

Put $u=e^{2x}-1$, so $du=2e^{2x}\,dx$. Then $\int e^{2x}(e^{2x}-1)^{-3}\,dx=(1/2)\int u^{-3}\,du=-1/(4u^2)+C$. The source excludes $x=0$, where the integrand is undefined.

$$\boxed{-\frac{1}{4(e^{2x}-1)^2}+C}$$

FINAL ANSWER

(i) $\frac{1}{2}\ln(3x^2-4x+5)+C$ (ii) $-\frac{1}{4(e^{2x}-1)^2}+C$

WMA13/01 OCTOBER 2021

Question 10

7 marks | Integration

10.

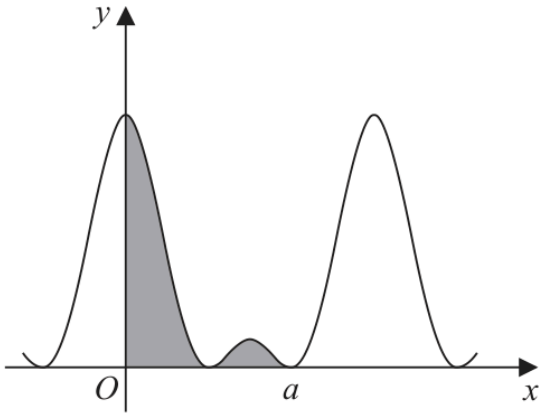


Figure 4

Figure 4 shows a sketch of part of the curve with equation

$$y = (1 + 2\cos 2x)^2$$

(a) Show that

$$(1 + 2\cos 2x)^2 \equiv p + q\cos 2x + r\cos 4x$$

where p, q and r are constants to be found.

(2)

The curve touches the positive x-axis for the second time when $x = a$, as shown in Figure 4.

The regions bounded by the curve, the y-axis and the x-axis up to $x = a$ are shown shaded in Figure 4.

(b) Find, using algebraic integration and making your method clear, the exact total area of the shaded regions. Write your answer in simplest form.

(5)

WORKED SOLUTION

Question 10 · WMA13/01 October 2021

1 **Part (a)**

Expand and use $\cos^2 2x = (1 + \cos 4x)/2$:

$$(1 + 2\cos 2x)^2 = 1 + 4\cos 2x + 4\cos^2 2x = 3 + 4\cos 2x + 2\cos 4x.$$

$$\boxed{p = 3, \quad q = 4, \quad r = 2}$$

2 **Part (b)**

The zeros satisfy $\cos 2x = -1/2$. The first two positive values are $x = \pi/3$ and $x = 2\pi/3$, so $a = 2\pi/3$. The squared curve is nonnegative, so integrate directly for the total shaded area:

$$\mathcal{A} = \int_0^{2\pi/3} (3 + 4\cos 2x + 2\cos 4x)\,dx = \left[3x + 2\sin 2x + \tfrac{1}{2}\sin 4x\right]_0^{2\pi/3}.$$

Use $\sin(4\pi/3) = -\sqrt{3}/2$ and $\sin(8\pi/3) = \sqrt{3}/2$ to obtain $\mathcal{A} = 2\pi - \sqrt{3} + \sqrt{3}/4 = 2\pi - 3\sqrt{3}/4$.

$$\boxed{2\pi - \frac{3\sqrt{3}}{4}}$$

FINAL ANSWER

(a) $p = 3, \quad q = 4, \quad r = 2$ (b) $2\pi - \frac{3\sqrt{3}}{4}$

WMA13/01 JANUARY 2023

Question 8

5 marks | Integration

8. Find, in simplest form,

$$\int (2\cos x - \sin x)^2 \, \mathrm{d}x$$

(5)

WORKED SOLUTION

Question 8 · WMA13/01 January 2023

1 Part (main)

Expand the square first:

$$(2\cos x - \sin x)^2 = 4\cos^2 x - 4\sin x \cos x + \sin^2 x.$$

Use $\cos^2 x = (1 + \cos 2x)/2$, $\sin^2 x = (1 - \cos 2x)/2$, and $2\sin x \cos x = \sin 2x$. The integrand becomes

$$\frac{5}{2} + \frac{3}{2}\cos 2x - 2\sin 2x.$$

Integrating with the angle-2 chain factors gives $5x/2 + 3\sin 2x/4 + \cos 2x + K$.

$$\boxed{\int (2\cos x - \sin x)^2 \, dx = \frac{5}{2}x + \frac{3}{4}\sin 2x + \cos 2x + K}$$

FINAL ANSWER

(main) $\int (2\cos x - \sin x)^2 \, dx = \frac{5}{2}x + \frac{3}{4}\sin 2x + \cos 2x + K$

Question 9

14 marks | Integration

9.

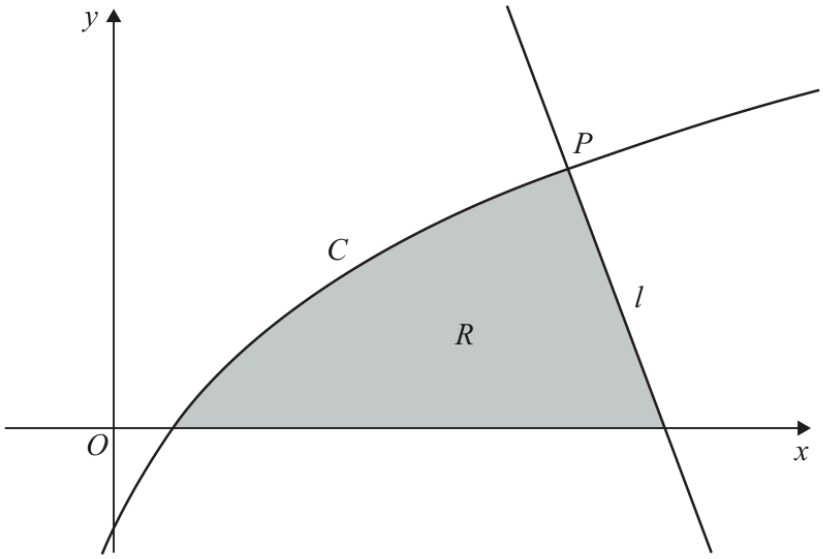


Figure 5

Figure 5 shows a sketch of part of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{6x^2 + 4x - 2}{2x + 1} \qquad x > -\frac{1}{2}$$

(a) Find $\mathrm{f}'(x)$, giving the answer in simplest form.

(3)

The line l is the normal to C at the point $P(2, 6)$

(b) Show that an equation for l is

$$16y + 5x = 106$$

(3)

(c) Write $\mathrm{f}(x)$ in the form $Ax + B + \dfrac{D}{2x + 1}$ where A, B and D are constants.

(3)

The region R, shown shaded in Figure 5, is bounded by C, l and the x-axis.

(d) Use algebraic integration to find the exact area of R, giving your answer in the form $P + Q\ln 3$, where P and Q are rational constants.

(*Solutions based entirely on calculator technology are not acceptable.*)

(5)

1 Part (a)

With $u = 6x^2 + 4x - 2$ and $v = 2x + 1$, the quotient rule gives

$$f'(x) = \frac{(12x + 4)(2x + 1) - 2(6x^2 + 4x - 2)}{(2x + 1)^2} = \frac{12x^2 + 12x + 8}{(2x + 1)^2}.$$

$$\boxed{f'(x) = \frac{4(3x^2 + 3x + 2)}{(2x + 1)^2}}$$

2 Part (b)

At $x = 2$, $f'(2) = 80/25 = 16/5$, so the normal gradient is $-5/16$. Through $P = (2, 6)$,

$$y - 6 = -\frac{5}{16}(x - 2) \quad \Longrightarrow \quad 16y - 96 = -5x + 10.$$

$$\boxed{16y + 5x = 106}$$

3 Part (c)

Multiplying the proposed identity by $2x + 1$ gives

$$6x^2 + 4x - 2 = (Ax + B)(2x + 1) + D = 2Ax^2 + (A + 2B)x + B + D.$$

Compare coefficients: $2A = 6$, $A + 2B = 4$, $B + D = -2$. Thus $f(x) = 3x + \frac{1}{2} - \frac{5}{2(2x+1)}$.

$$\boxed{A = 3, \quad B = \frac{1}{2}, \quad D = -\frac{5}{2}}$$

4 Part (d)

The curve meets the x-axis when $6x^2 + 4x - 2 = 2(3x - 1)(x + 1) = 0$. The root $x = -1$ is outside $x > -1/2$, leaving $x = 1/3$. The normal meets the axis at $x = 106/5$. Split the shaded region at P: a curve area from $1/3$ to 2, then a triangle.

$$\int_{1/3}^{2} f(x)\,dx = \left[\frac{3}{2}x^2 + \frac{1}{2}x - \frac{5}{4}\ln(2x + 1)\right]_{1/3}^{2} = 7 - \frac{1}{3} - \frac{5}{4}\ln\left(\frac{5}{5/3}\right) = \frac{20}{3} - \frac{5}{4}\ln 3.$$

The triangle has area $\frac{1}{2}(106/5 - 2) \cdot 6 = 288/5$. Adding gives the exact area of R.

$$\boxed{\text{Area}(R) = \frac{964}{15} - \frac{5}{4}\ln 3}$$

FINAL ANSWER

(a) $f'(x) = \frac{4(3x^2+3x+2)}{(2x+1)^2}$ (b) $16y + 5x = 106$ (c) $A = 3, \quad B = \frac{1}{2}, \quad D = -\frac{5}{2}$ (d) $\text{Area}(R) = \frac{964}{15} - \frac{5}{4}\ln 3$

WMA13/01 OCTOBER 2025

Question 6

7 marks | Integration

6. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(i) Given $y = \ln(2x^2 + 5)$ find $\dfrac{dy}{dx}$

(2)

(ii) A curve C has equation $y = f(x)$ where

$$f(x) = \frac{21x}{3x^2 + k} \qquad x \in \mathbb{R}$$

where k is a positive integer, $k > 1$

Given that

$$\int_1^k f(x)\,dx < 7\ln 8$$

find the greatest possible value of k

(5)

WORKED SOLUTION

Question 6 · WMA13/01 October 2025

1 **Part (i)**

Using the chain rule,

$$\frac{dy}{dx} = \frac{1}{2x^2 + 5}(4x) = \frac{4x}{2x^2 + 5}.$$

2 Part (ii)

Treat k as a constant when integrating:

$$\int_1^k \frac{21x}{3x^2+k}\,dx = \left[\frac{7}{2}\ln(3x^2+k)\right]_1^k = \frac{7}{2}\ln\left(\frac{3k^2+k}{k+3}\right).$$

The stated inequality therefore implies

$$\ln\left(\frac{3k^2+k}{k+3}\right) < 2\ln 8 = \ln 64.$$

Since the exponential is increasing and $k+3>0$,

$$3k^2+k < 64(k+3) \Longrightarrow k^2-21k-64<0.$$

The roots are $(21\pm\sqrt{697})/2$, so the positive upper bound is $k<(21+\sqrt{697})/2 = 23.700378782\ldots$. With $k>1$ an integer, the greatest possible value is $k=23$.

FINAL ANSWER

(i) $4*x/(2*x^2+5)$ (ii) 23

Question 9

14 marks | Integration

Main topic: Applications of Differentiation

9. (a) Given that

$$\frac{x^4 - x^3 - 10x^2 + 3x - 9}{x^2 - x - 12} \equiv x^2 + P + \frac{Q}{x - 4} \qquad x > -3$$

find the value of the constant P and show that $Q = 5$

(4)

The curve C has equation $y = \mathrm{g}(x)$, where

$$\mathrm{g}(x) = \frac{x^4 - x^3 - 10x^2 + 3x - 9}{x^2 - x - 12} \qquad -3 < x < 3.5 \qquad x \in \mathbb{R}$$

(b) Find the equation of the tangent to C at the point where $x = 2$
Give your answer in the form $y = mx + c$, where m and c are constants to be found.

(5)

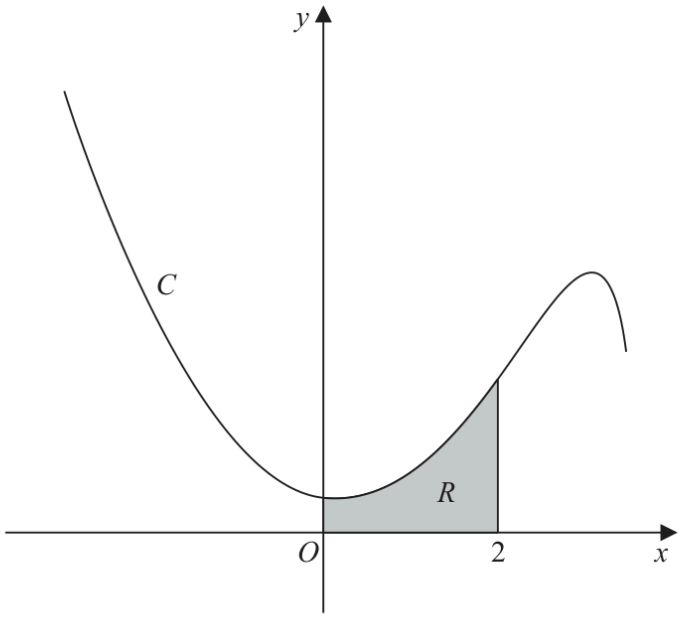


Figure 4

Figure 4 shows a sketch of the curve C.
The region R, shown shaded in Figure 4, is bounded by C, the y-axis, the x-axis and the line with equation $x = 2$

(c) Find the exact area of R, writing your answer in the form $a + b\ln 2$, where a and b are constants to be found.

(5)

1 Part (a)

Factor the denominator as $(x-4)(x+3)$ and multiply the proposed identity through by it:

$$x^4 - x^3 - 10x^2 + 3x - 9 = (x^2 + P)(x^2 - x - 12) + Q(x+3).$$

The right side expands to $x^4 - x^3 + (P-12)x^2 + (Q-P)x + 3Q - 12P$. Comparing coefficients gives $P - 12 = -10$, so $P = 2$, and $Q - P = 3$, so $Q = 5$. The constant check is $3(5) - 12(2) = -9$, as required. The identity holds where the original fraction is defined: the stated $x > -3$ still excludes $x = 4$.

$$\boxed{P = 2, \qquad Q = 5}$$

2 Part (b)

From (a), $g(x) = x^2 + 2 + 5/(x-4)$ on the curve domain. Thus $g'(x) = 2x - 5/(x-4)^2$. At $x = 2$, $g(2) = 4 + 2 - 5/2 = 7/2$ and $g'(2) = 4 - 5/4 = 11/4$. The tangent through $(2, 7/2)$ is

$$y - \frac{7}{2} = \frac{11}{4}(x-2) \quad \Longrightarrow \quad y = \frac{11}{4}x - 2.$$

$$\boxed{y = \frac{11}{4}x - 2}$$

3 Part (c)

The source shading is above the x-axis from $x = 0$ to $x = 2$, so integrate g over those bounds.

$$\text{Area}(R) = \int_0^2 \left(x^2 + 2 + \frac{5}{x-4}\right) dx = \left[\frac{x^3}{3} + 2x + 5\ln|x-4|\right]_0^2.$$

The absolute value is essential because $x - 4 < 0$ throughout this interval. Evaluating gives

$$\frac{8}{3} + 4 + 5\ln 2 - 5\ln 4 = \frac{20}{3} - 5\ln 2.$$

Thus $a = 20/3$ and $b = -5$ in the requested form.

$$\boxed{\text{Area}(R) = \frac{20}{3} - 5\ln 2}$$

FINAL ANSWER

(a) $P = 2, \quad Q = 5$ (b) $y = \frac{11}{4}x - 2$ (c) $\text{Area}(R) = \frac{20}{3} - 5\ln 2$

Question 8

12 marks | Integration | Main topic: Numerical Methods

8.

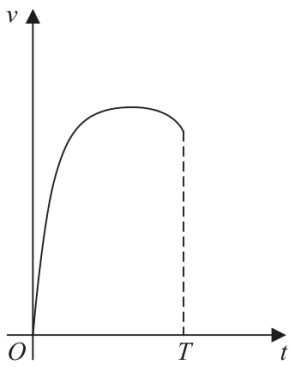


Figure 4

Figure 4 is a graph showing the velocity of a sprinter during a 100 m race.

The sprinter's velocity during the race, $v\,\text{m s}^{-1}$, is modelled by the equation

$$v = 12 - \text{e}^{t-10} - 12\text{e}^{-0.75t} \qquad t \geqslant 0$$

where t seconds is the time after the sprinter begins to run.

According to the model,

(a) find, using calculus, the sprinter's maximum velocity during the race.

(5)

Given that the sprinter runs 100 m in T seconds, such that

$$\int_0^T v\,\text{d}t = 100$$

(b) show that T is a solution of the equation

$$T = \frac{1}{12}\left(116 - 16\text{e}^{-0.75T} + \text{e}^{T-10} - \text{e}^{-10}\right)$$

(4)

The iteration formula

$$T_{n+1} = \frac{1}{12}\left(116 - 16\text{e}^{-0.75T_n} + \text{e}^{T_n-10} - \text{e}^{-10}\right)$$

is used to find an approximate value for T

Using this iteration formula with $T_1 = 10$

(c) find, to 4 decimal places,

(i) the value of T_2

(ii) the time taken by the sprinter to run the race, according to the model.

(3)

1 Part (a)

Differentiate the velocity:

$$v'(t) = -e^{t-10} + 9e^{-0.75t}.$$

At a stationary velocity, $e^{t-10} = 9e^{-0.75t}$, so $e^{1.75t} = 9e^{10}$ and $t = (10 + \ln 9)/1.75 = 6.9698426\ldots$ seconds. The second derivative $-e^{t-10} - 6.75e^{-0.75t}$ is negative, so this is a maximum. Substitute this unrounded time into v to get $11.88728125\ldots$ metres per second.

$$\boxed{v_{\max} = 11.9\ \mathrm{m\,s^{-1}}}$$

2 Part (b)

Integrating the velocity gives $12t - e^{t-10} + 16e^{-0.75t}$. Evaluate between 0 and T:

$$100 = 12T - e^{T-10} + 16e^{-0.75T} - (-e^{-10} + 16).$$

Rearrange to $12T = 116 - 16e^{-0.75T} + e^{T-10} - e^{-10}$ and divide by 12.

$$\boxed{T = \frac{1}{12}(116 - 16e^{-0.75T} + e^{T-10} - e^{-10})}$$

3 Part (c)

With $T_1 = 10$, substitution gives $T_2 = 9.7492587708\ldots$. Continuing with unrounded values gives $T_3 = 9.7306248365\ldots$, $T_4 = 9.7294150526\ldots$, $T_5 = 9.7293372715\ldots$, $T_6 = 9.7293322738\ldots$.

Further iteration settles at $9.7293319307\ldots$ seconds. Thus (i) $T_2 = 9.7493$ and (ii) the race time is 9.7293 seconds, to four decimal places.

$$\boxed{T_2 = 9.7493, \qquad T = 9.7293\ \mathrm{s}}$$

FINAL ANSWER

(a) $v_{\max} = 11.9\ \mathrm{m\,s^{-1}}$ (b) $T = \frac{1}{12}(116 - 16e^{-0.75T} + e^{T-10} - e^{-10})$ (c) $T_2 = 9.7493, \qquad T = 9.7293\ \mathrm{s}$

WMA13/01 MAY/JUNE 2023

Question 9

10 marks | Integration

Main topic: Trig Identities & Exact Values

9. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that

$$\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} \equiv \operatorname{cosec} x \qquad x \neq \frac{n\pi}{2} \qquad n \in \mathbb{Z}$$

(3)

(b) Hence solve, for $0 < \theta < \frac{\pi}{2}$

$$\left(\frac{\cos 2\theta}{\sin \theta} + \frac{\sin 2\theta}{\cos \theta}\right)^2 = 6\cot\theta - 4$$

giving your answers to 3 significant figures as appropriate.

(5)

(c) Using the result from part (a), or otherwise, find the exact value of

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x}\right) \cot x \, dx$$

(2)

WORKED SOLUTION

Question 9 · WMA13/01 May/June 2023

1 Part (a)

Using a common denominator,

$$\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} = \frac{\cos 2x \cos x + \sin 2x \sin x}{\sin x \cos x} = \frac{\cos(2x - x)}{\sin x \cos x} = \frac{1}{\sin x} = \csc x.$$

The stated exclusion $x \neq n\pi/2$ ensures both denominators are nonzero.

$$\boxed{\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} = \csc x}$$

2 Part (b)

By part (a), the squared bracket is $\csc^2\theta$. Using $\csc^2\theta = 1 + \cot^2\theta$ and setting $u = \cot\theta$ gives

$$1 + u^2 = 6u - 4 \quad \Longrightarrow \quad u^2 - 6u + 5 = (u-1)(u-5) = 0.$$

On $0 < \theta < \pi/2$, both positive cotangents are permitted: $u = 1$ gives $\theta = \pi/4$, while $u = 5$ gives $\tan\theta = 1/5$, $\theta = 0.1973955598\ldots$ radians.

$$\boxed{\theta = \frac{\pi}{4}, \quad 0.197 \text{ radians}}$$

---

3 Part (c)

The bracket in part (a) equals $\csc x$. Therefore

$$\int_{\pi/6}^{\pi/4} \csc x \cot x \, dx = \left[-\csc x\right]_{\pi/6}^{\pi/4} = -\sqrt{2} - (-2) = 2 - \sqrt{2}.$$

$$\boxed{2 - \sqrt{2}}$$

FINAL ANSWER

(a) $\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} = \csc x$ (b) $\theta = \frac{\pi}{4}, \quad 0.197$ radians (c) $2 - \sqrt{2}$

Question 7

12 marks | Integration

Main topic: Applications of Differentiation

7. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

The curve C has equation

$$y = \frac{16}{9(3x - k)} \qquad x \neq \frac{k}{3}$$

where k is a positive constant not equal to 3

(a) Find $\frac{dy}{dx}$ giving your answer in simplest form in terms of k.

(2)

The point P with x coordinate 1 lies on C.

Given that the gradient of the curve at P is -12

(b) find the two possible values of k

(3)

Given also that $k < 3$

(c) find the equation of the normal to C at P, writing your answer in the form $ax + by + c = 0$, where a, b and c are integers to be found.

(3)

(d) show, using algebraic integration that,

$$\int_1^3 \frac{16}{9(3x - k)} dx = \lambda \ln 10$$

where λ is a constant to be found.

(4)

1 Part (a)

Write $y = (16/9)(3x-k)^{-1}$. The chain rule gives

$$\frac{dy}{dx} = \frac{16}{9}(-1)(3x-k)^{-2}\cdot 3.$$

$$\boxed{\frac{dy}{dx} = -\frac{16}{3(3x-k)^2}}$$

2 Part (b)

At $x = 1$, the given gradient yields

$$-\frac{16}{3(3-k)^2} = -12 \quad\Longrightarrow\quad (3-k)^2 = \frac{4}{9}.$$

Hence $3-k = \pm 2/3$. Both resulting k-values are positive and different from 3.

$$\boxed{k = \frac{7}{3} \text{ or } \frac{11}{3}}$$

3 Part (c)

Since $k < 3$, use $k = 7/3$. At $x = 1$, $y = 16/[9(3-7/3)] = 8/3$. The tangent gradient is -12, so the normal gradient is $1/12$:

$$y - \frac{8}{3} = \frac{1}{12}(x-1), \qquad 12y - 32 = x - 1.$$

$$\boxed{x - 12y + 31 = 0}$$

4 Part (d)

With $k = 7/3$, the denominator does not vanish on $[1,3]$. Using the reciprocal-linear antiderivative,

$$\int_1^3 \frac{16}{9(3x-7/3)}\,dx = \left[\frac{16}{27}\ln|3x-7/3|\right]_1^3 = \frac{16}{27}\left(\ln\frac{20}{3} - \ln\frac{2}{3}\right) = \frac{16}{27}\ln\left(\frac{20/3}{2/3}\right).$$

$$\boxed{\frac{16}{27}\ln 10, \quad \lambda = \frac{16}{27}}$$

FINAL ANSWER

(a) $\frac{dy}{dx} = -\frac{16}{3(3x-k)^2}$ (b) $k = \frac{7}{3}$ or $\frac{11}{3}$ (c) $x - 12y + 31 = 0$ (d) $\frac{16}{27}\ln 10, \quad \lambda = \frac{16}{27}$

Question 6

9 marks | Integration

Main topic: Applications of Differentiation

6.

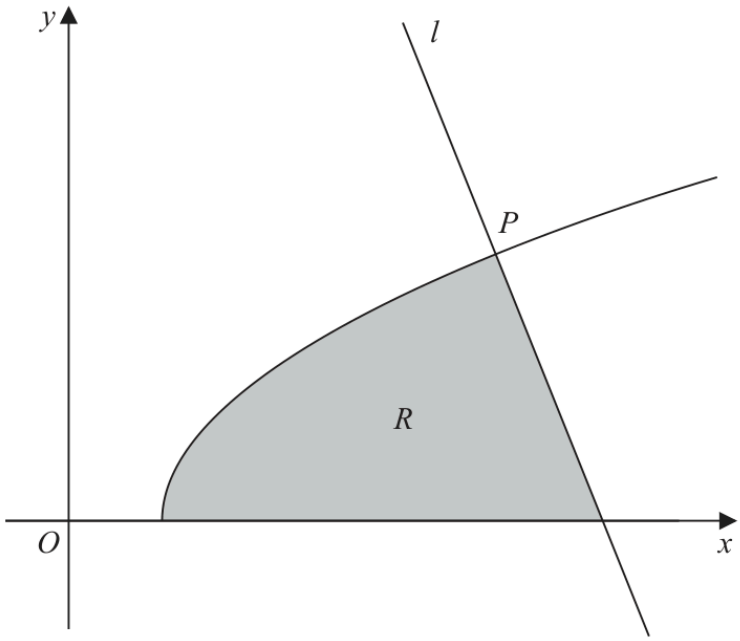


Figure 3

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 3 shows a sketch of part of the curve with equation

$$y = \sqrt{4x - 7}$$

The line l, shown in Figure 3, is the normal to the curve at the point $P(8, 5)$

(a) Use calculus to show that an equation of l is

$$5x + 2y - 50 = 0$$

(5)

The region R, shown shaded in Figure 3, is bounded by the curve, the x-axis and l.

(b) Use algebraic integration to find the exact area of R.

(4)

1 Part (a)

For $y = (4x-7)^{1/2}$,

$$\frac{dy}{dx} = \frac{1}{2}(4x-7)^{-1/2}\cdot 4 = \frac{2}{\sqrt{4x-7}}.$$

At $P = (8,5)$, the tangent gradient is $2/5$, so the normal gradient is $-5/2$. The point-gradient equation is

$$y - 5 = -\frac{5}{2}(x-8), \qquad 2y - 10 = -5x + 40.$$

$$\boxed{5x + 2y - 50 = 0}$$

2 Part (b)

The curve meets the x-axis at $x = 7/4$. The normal meets it at $x = 10$. Split the shaded region at P, into a curve area from $7/4$ to 8 and a triangle from 8 to 10:

$$\int_{7/4}^{8} \sqrt{4x-7}\,dx = \left[\frac{(4x-7)^{3/2}}{6}\right]_{7/4}^{8} = \frac{125}{6}.$$

The triangle has area $\frac{1}{2}(10-8)\cdot 5 = 5$. Hence the total area is $125/6 + 5$.

$$\boxed{\text{Area}(R) = \frac{155}{6}}$$

FINAL ANSWER

(a) $5x + 2y - 50 = 0$ (b) $\text{Area}(R) = \frac{155}{6}$

Question 9

11 marks | Integration | Main topic: Modulus

9.

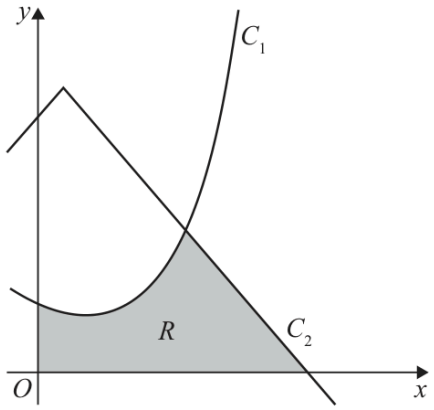


Figure 2

In this question you must show all stages of your working.
Solutions relying entirely on calculator technology are not acceptable.

(a) Find $\int \left(\frac{11}{2x+5} + 4e^{x-3} \right) dx$

(2)

Figure 2 shows a sketch of part of the curve C_1 and part of the graph C_2
The curve C_1 has equation

$$y = \frac{11}{2x+5} + 4e^{x-3}$$

The graph C_2 has equation

$$y = a - |bx - 1|$$

where a and b are positive integers.

(b) State, in terms of a and/or b, the coordinates of the maximum point of C_2

(2)

Given that

- C_1 intersects C_2 at $x = 3$
- C_2 crosses the y-axis at $(0, 9)$

(c) find the value of a and the value of b

(3)

The region R, shown shaded in Figure 2, is bounded by C_1, C_2, the x-axis and the y-axis.

(d) Find the area of R, giving the answer to 3 significant figures.

(4)

1 Part (a)

$$\int \left(\frac{11}{2x+5} + 4e^{x-3} \right) dx = \boxed{\frac{11}{2}\ln|2x+5| + 4e^{x-3} + C}.$$

The absolute value allows both intervals on which the integrand is defined.

2 Part (b)

The largest value of $a - |bx - 1|$ occurs when $bx - 1 = 0$. The maximum point is therefore $\boxed{(1/b, a)}$.

3 Part (c)

At $x = 0$, $9 = a - 1$, so $\boxed{a = 10}$. At the intersection $x = 3$, the first curve has height $11/11 + 4 = 5$. Thus $10 - |3b - 1| = 5$, giving $3b - 1 = 5$ or -5. These give $b = 2$ or $-4/3$; only the positive integer is allowed, so $\boxed{b = 2}$.

4 Part (d)

For $x \geq 3$, the required branch of C_2 is $y = 10 - (2x - 1) = 11 - 2x$. Its x-intercept is $x = 11/2$. The shaded area is the area under C_1 from 0 to 3 plus the triangle from 3 to $11/2$:

$$\begin{aligned} A &= \int_0^3 \left(\frac{11}{2x+5} + 4e^{x-3} \right) dx + \frac{1}{2}\left(\frac{11}{2} - 3\right)5 \\ &= \left[\frac{11}{2}\ln(2x+5) + 4e^{x-3} \right]_0^3 + \frac{25}{4} \\ &= \frac{11}{2}\ln\frac{11}{5} + 4(1 - e^{-3}) + \frac{25}{4} = 14.3873672\ldots. \end{aligned}$$

Hence $\boxed{A = 14.4 \text{ square units}}$ to 3 significant figures.

FINAL ANSWER

(a) $\frac{11}{2}\ln|2x+5| + 4e^{x-3} + C$ (b) $(1/b, a)$ (c) $a = 10$, $b = 2$ (d) $A = 14.4$ square units

TOPIC

Numerical Methods

Every past-paper question on this topic, January 2020 to June 2026.

Question 6

9 marks | Numerical Methods

6.

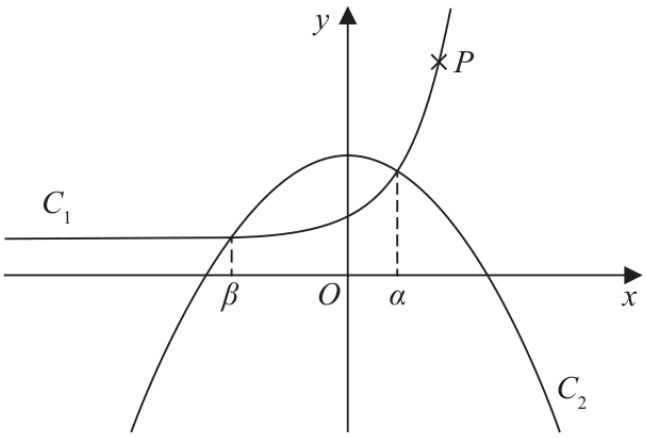


Figure 3

Figure 3 shows a sketch of curve C_1 with equation $y = 5e^{x-1} + 3$

and curve C_2 with equation $y = 10 - x^2$

The point P lies on C_1 and has y coordinate 18

(a) Find the x coordinate of P, writing your answer in the form $\ln k$, where k is a constant to be found.

(3)

The curve C_1 meets the curve C_2 at $x = \alpha$ and at $x = \beta$, as shown in Figure 3.

(b) Using a suitable interval and a suitable function that should be stated, show that to 3 decimal places $\alpha = 1.134$

(3)

The iterative equation

$$x_{n+1} = -\sqrt{7 - 5e^{x_n - 1}}$$

is used to find an approximation to β.

Using this iterative formula with $x_1 = -3$

(c) find the value of x_2 and the value of β, giving each answer to 6 decimal places.

(3)

1 Part (a)

At P, $18 = 5e^{x-1} + 3$, so $e^{x-1} = 3$. Taking natural logarithms gives $x - 1 = \ln 3$, hence $x = 1 + \ln 3 = \ln e + \ln 3 = \ln(3e)$.

$$\boxed{x = \ln(3e), \qquad k = 3e}$$

2 Part (b)

Define $F(x) = 5e^{x-1} + x^2 - 7$; its zeros are the intersection abscissae, and F is continuous. At the endpoints of the three-decimal rounding interval,

$$F(1.1335) = -0.0010714193\ldots < 0, \qquad F(1.1345) = 0.0069135450\ldots > 0.$$

Continuity guarantees a root between them. Moreover $F'(x) = 5e^{x-1} + 2x > 0$ on this positive interval, so that root is unique there. It is the positive intersection α, and rounds to 1.134 to three decimal places.

$$\boxed{\alpha = 1.134 \text{ to 3 d.p.}}$$

3 Part (c)

Start at $x_1 = -3$ and keep full precision in $x_{n+1} = -\sqrt{7 - 5e^{x_n - 1}}$:

n	x_n
1	-3
2	-2.628387681746
3	-2.620535277934
4	-2.620335527988
5	-2.620330426051
6	-2.620330295726
7	-2.620330292397

The later iterates agree to six decimal places. The negative root corresponds to β in the figure.

$$\boxed{x_2 = -2.628388, \qquad \beta = -2.620330}$$

FINAL ANSWER

(a) $x = \ln(3e), \qquad k = 3e$ (b) $\alpha = 1.134$ to 3 d.p. (c) $x_2 = -2.628388, \qquad \beta = -2.620330$

WMA13/01 JANUARY 2021

Question 6

8 marks | Numerical Methods

6.

$$f(x) = x\cos\left(\frac{x}{3}\right) \qquad x > 0$$

(a) Find $f'(x)$

(2)

(b) Show that the equation $f'(x) = 0$ can be written as

$$x = k\arctan\left(\frac{k}{x}\right)$$

where k is an integer to be found.

(2)

(c) Starting with $x_1 = 2.5$ use the iteration formula

$$x_{n+1} = k\arctan\left(\frac{k}{x_n}\right)$$

with the value of k found in part (b), to calculate the values of x_2 and x_6 giving your answers to 3 decimal places.

(2)

(d) Using a suitable interval and a suitable function that should be stated, show that a root of $f'(x) = 0$ is 2.581 correct to 3 decimal places.

(2)

WORKED SOLUTION

Question 6 · WMA13/01 January 2021

1 **Part (a)**

By the product and chain rules, $f'(x) = \cos(x/3) - (x/3)\sin(x/3)$.

$$\boxed{f'(x) = \cos(x/3) - \frac{x}{3}\sin(x/3)}$$

2 Part (b)

Set $f'(x) = 0$: $\cos(x/3) = (x/3)\sin(x/3)$. At a solution, cosine cannot vanish (that would leave a nonzero right side), so $\tan(x/3) = 3/x$. On the principal positive branch $0 < x/3 < \pi/2$, this gives

$$x = 3\arctan(3/x), \qquad k = 3.$$

This is the branch used by the supplied iteration for the root near 2.581. The original domain $x > 0$ also admits later tangent branches: in general $x = 3\arctan(3/x) + 3n\pi$. The principal-branch rearrangement must not be interpreted as listing every positive stationary point.

$$\boxed{k = 3; \quad x = 3\arctan(3/x) \text{ on the selected branch}}$$

---

3 Part (c)

Using radians and $x_1 = 2.5$, the successive values are

n	1	2	3	4	5	6
x_n	2.5	2.628174152	2.554102174	2.596526871	2.572101407	2.586122337

Keep full precision between steps and round the requested terms only.

$$\boxed{x_2 = 2.628, \qquad x_6 = 2.586}$$

---

4 Part (d)

Define $F(x) = \cos(x/3) - (x/3)\sin(x/3)$, which is continuous near 2.581. The rounding interval endpoints give

$$F(2.5805) = 0.0003467157\ldots > 0, \qquad F(2.5815) = -0.0003456897\ldots < 0.$$

By continuity a root lies strictly between these endpoints; every number in this open interval rounds to 2.581 to three decimal places.

$$\boxed{x = 2.581 \text{ correct to 3 d.p.}}$$

✓ FINAL ANSWER

(a) $f'(x) = \cos(x/3) - \frac{x}{3}\sin(x/3)$ (b) $k = 3; \quad x = 3\arctan(3/x)$ on the selected branch (c) $x_2 = 2.628, \qquad x_6 = 2.586$ (d) $x = 2.581$ correct to 3 d.p.

Question 9

9 marks | Numerical Methods

9.

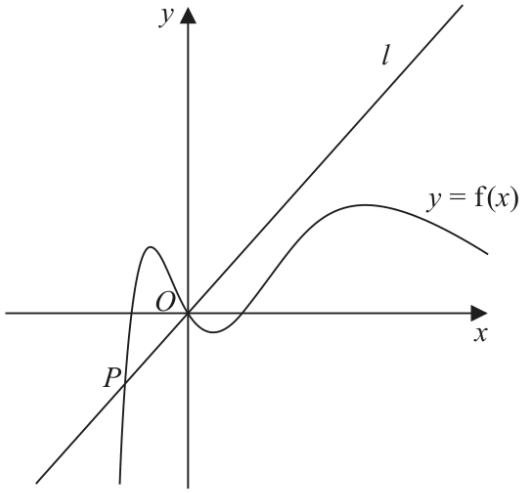


Figure 3

Figure 3 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x(x^2 - 4)\mathrm{e}^{-\frac{1}{2}x}$$

(a) Find $\mathrm{f}'(x)$.

(2)

The line l is the normal to the curve at O and meets the curve again at the point P.

The point P lies in the 3rd quadrant, as shown in Figure 3.

(b) Show that the x coordinate of P is a solution of the equation

$$x = -\frac{1}{2}\sqrt{16 + \mathrm{e}^{\frac{1}{2}x}}$$

(4)

(c) Using the iterative formula

$$x_{n+1} = -\frac{1}{2}\sqrt{16 + \mathrm{e}^{\frac{1}{2}x_n}} \quad \text{with } x_1 = -2$$

find, to 4 decimal places,

(i) the value of x_2

(ii) the x coordinate of P.

(3)

1 Part (a)

Write $f(x) = (x^3 - 4x)e^{-x/2}$. By the product rule,

$$f'(x) = (3x^2 - 4)e^{-x/2} - \frac{1}{2}(x^3 - 4x)e^{-x/2} = e^{-x/2}\left(-\frac{x^3}{2} + 3x^2 + 2x - 4\right).$$

$$\boxed{f'(x) = e^{-x/2}(-x^3/2 + 3x^2 + 2x - 4)}$$

2 Part (b)

At O, $f'(0) = -4$, so the normal gradient is $1/4$ and its equation is $y = x/4$. At the other intersection P, x is nonzero; equating curve and normal and dividing by x gives $(x^2 - 4)e^{-x/2} = 1/4$. Thus $x^2 = 4 + e^{x/2}/4 = (16 + e^{x/2})/4$. Since P lies in the third quadrant, select the negative square root.

$$\boxed{x = -\frac{1}{2}\sqrt{16 + e^{x/2}}}$$

3 Part (c)

Starting from $x_1 = -2$, $x_2 = -\frac{1}{2}\sqrt{16 + e^{-1}} = -2.0228617996\ldots$. Further unrounded iterates are $x_3 = -2.0226034084\ldots$, $x_4 = -2.0226063126\ldots$ and $x_5 = -2.0226062799\ldots$. The continuing sequence settles at $-2.0226062803\ldots$.

$$\boxed{\text{(i) } x_2 = -2.0229; \qquad \text{(ii) } x_P = -2.0226}$$

✓ FINAL ANSWER

(a) $f'(x) = e^{-x/2}(-x^3/2 + 3x^2 + 2x - 4)$ (b) $x = -\frac{1}{2}\sqrt{16 + e^{x/2}}$ (c) $\text{(i) } x_2 = -2.0229; \qquad \text{(ii) } x_P = -2.0226$

Question 8

12 marks | Numerical Methods

8.

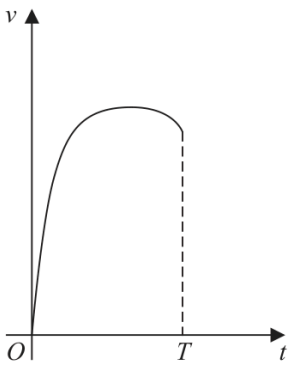


Figure 4

Figure 4 is a graph showing the velocity of a sprinter during a 100 m race.

The sprinter's velocity during the race, $v\,\text{m s}^{-1}$, is modelled by the equation

$$v = 12 - \text{e}^{t-10} - 12\text{e}^{-0.75t} \qquad t \geqslant 0$$

where t seconds is the time after the sprinter begins to run.

According to the model,

(a) find, using calculus, the sprinter's maximum velocity during the race.

(5)

Given that the sprinter runs 100 m in T seconds, such that

$$\int_0^T v\,\text{d}t = 100$$

(b) show that T is a solution of the equation

$$T = \frac{1}{12}\left(116 - 16\text{e}^{-0.75T} + \text{e}^{T-10} - \text{e}^{-10}\right)$$

(4)

The iteration formula

$$T_{n+1} = \frac{1}{12}\left(116 - 16\text{e}^{-0.75T_n} + \text{e}^{T_n-10} - \text{e}^{-10}\right)$$

is used to find an approximate value for T

Using this iteration formula with $T_1 = 10$

(c) find, to 4 decimal places,

(i) the value of T_2

(ii) the time taken by the sprinter to run the race, according to the model.

(3)

1 Part (a)

Differentiate the velocity:

$$v'(t) = -e^{t-10} + 9e^{-0.75t}.$$

At a stationary velocity, $e^{t-10} = 9e^{-0.75t}$, so $e^{1.75t} = 9e^{10}$ and $t = (10 + \ln 9)/1.75 = 6.9698426\ldots$ seconds. The second derivative $-e^{t-10} - 6.75e^{-0.75t}$ is negative, so this is a maximum. Substitute this unrounded time into v to get $11.88728125\ldots$ metres per second.

$$\boxed{v_{\max} = 11.9\ \mathrm{m\,s^{-1}}}$$

2 Part (b)

Integrating the velocity gives $12t - e^{t-10} + 16e^{-0.75t}$. Evaluate between 0 and T:

$$100 = 12T - e^{T-10} + 16e^{-0.75T} - (-e^{-10} + 16).$$

Rearrange to $12T = 116 - 16e^{-0.75T} + e^{T-10} - e^{-10}$ and divide by 12.

$$\boxed{T = \frac{1}{12}(116 - 16e^{-0.75T} + e^{T-10} - e^{-10})}$$

3 Part (c)

With $T_1 = 10$, substitution gives $T_2 = 9.7492587708\ldots$. Continuing with unrounded values gives $T_3 = 9.7306248365\ldots$, $T_4 = 9.7294150526\ldots$, $T_5 = 9.7293372715\ldots$, $T_6 = 9.7293322738\ldots$.

Further iteration settles at $9.7293319307\ldots$ seconds. Thus (i) $T_2 = 9.7493$ and (ii) the race time is 9.7293 seconds, to four decimal places.

$$\boxed{T_2 = 9.7493, \qquad T = 9.7293\ \mathrm{s}}$$

FINAL ANSWER

(a) $v_{\max} = 11.9\ \mathrm{m\,s^{-1}}$ (b) $T = \frac{1}{12}(116 - 16e^{-0.75T} + e^{T-10} - e^{-10})$ (c) $T_2 = 9.7493, \qquad T = 9.7293\ \mathrm{s}$

Question 9

11 marks | Numerical Methods

9.

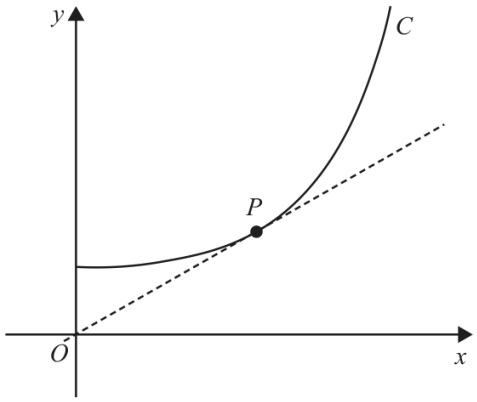


Figure 3

Figure 3 shows a sketch of part of the curve C with equation

$$y = \sqrt{3 + 4e^{x^2}} \qquad x \geqslant 0$$

(a) Find $\dfrac{dy}{dx}$, giving your answer in simplest form.

(2)

The point P with x coordinate α lies on C.

Given that the tangent to C at P passes through the origin, as shown in Figure 3,

(b) show that $x = \alpha$ is a solution of the equation

$$4x^2e^{x^2} - 4e^{x^2} - 3 = 0$$

(3)

(c) Hence show that α lies between 1 and 2

(2)

(d) Show that the equation in part (b) can be written in the form

$$x = \frac{1}{2}\sqrt{4 + 3e^{-x^2}}$$

(1)

The iteration formula

$$x_{n+1} = \frac{1}{2}\sqrt{4 + 3e^{-x_n^2}}$$

with $x_1 = 1$ is used to find an approximation for α.

(e) Use the iteration formula to find, to 4 decimal places, the value of

(i) x_3

(ii) α

(3)

1 Part (a)

Differentiate $y = (3 + 4e^{x^2})^{1/2}$ using the chain rule:

$$\frac{dy}{dx} = \frac{1}{2}(3 + 4e^{x^2})^{-1/2} \cdot 4e^{x^2} \cdot 2x = \frac{4xe^{x^2}}{\sqrt{3 + 4e^{x^2}}}.$$

$$\boxed{\frac{dy}{dx} = \frac{4xe^{x^2}}{\sqrt{3 + 4e^{x^2}}}}$$

2 Part (b)

Since the tangent at $P = (\alpha, \sqrt{3 + 4e^{\alpha^2}})$ passes through the origin, its gradient is y_P/α, with $\alpha > 0$. Equate this to the derivative:

$$\frac{4\alpha e^{\alpha^2}}{\sqrt{3 + 4e^{\alpha^2}}} = \frac{\sqrt{3 + 4e^{\alpha^2}}}{\alpha} \quad \Longrightarrow \quad 4\alpha^2 e^{\alpha^2} = 3 + 4e^{\alpha^2}.$$

Consequently $x = \alpha$ solves the displayed equation.

$$\boxed{4x^2 e^{x^2} - 4e^{x^2} - 3 = 0}$$

3 Part (c)

Let $h(x) = 4x^2 e^{x^2} - 4e^{x^2} - 3$, a continuous function. Then $h(1) = -3 < 0$ and $h(2) = 12e^4 - 3 = 652.1778\ldots > 0$, so a root lies between 1 and 2. Moreover $h'(x) = 8x^3 e^{x^2} > 0$ for $x > 0$, so this is the unique positive root corresponding to the tangent abscissa α.

$$\boxed{1 < \alpha < 2}$$

4 Part (d)

From $4x^2 e^{x^2} = 4e^{x^2} + 3$, divide by the positive e^{x^2} to get $4x^2 = 4 + 3e^{-x^2}$. The source domain has $x \geq 0$, so take the positive square root.

$$\boxed{x = \frac{1}{2}\sqrt{4 + 3e^{-x^2}}}$$

5 **Part (e)**

With $x_1 = 1$, the unrounded iterates start

$$x_2 = 1.1295616764\ldots, \quad x_3 = 1.0997193683\ldots, \quad x_4 = 1.1062486441\ldots, \quad x_5 = 1.1048012899\ldots.$$

Continue: $x_6 = 1.1051212304\ldots$, $x_7 = 1.1050504628\ldots$, $x_8 = 1.1050661137\ldots$, $x_9 = 1.1050626523\ldots$; further iteration settles at $1.1050632792\ldots$. Therefore (i) $x_3 = 1.0997$ and (ii) $\alpha = 1.1051$, both to four decimal places.

$$\boxed{x_3 = 1.0997, \qquad \alpha = 1.1051}$$

✓ **FINAL ANSWER**

(a) $\frac{dy}{dx} = \frac{4xe^{x^2}}{\sqrt{3+4e^{x^2}}}$ (b) $4x^2e^{x^2} - 4e^{x^2} - 3 = 0$ (c) $1 < \alpha < 2$ (d) $x = \frac{1}{2}\sqrt{4 + 3e^{-x^2}}$ (e) $x_3 = 1.0997, \qquad \alpha = 1.1051$

Question 7

9 marks | Numerical Methods

7: A continuous curve has equation

$$y = e^{-x^2} \sin 3x \qquad 0 \leqslant x \leqslant \frac{\pi}{3}$$

The curve has a stationary point at the point P.

(a) Show, using calculus, that the x coordinate of P is a solution of the equation

$$x = \frac{1}{3} \arctan\left(\frac{3}{2x}\right)$$

(4)

Using the iteration formula

$$x_{n+1} = \frac{1}{3} \arctan\left(\frac{3}{2x_n}\right) \qquad x_1 = 0.4$$

(b) find the value of

(i) x_2

(ii) x_4

giving your answers to 4 decimal places.

(3)

(c) Using a suitable interval and a suitable function which should be stated, show that the x coordinate of P is 0.430 correct to 3 decimal places.

(2)

1 Part (a)

Apply the product and chain rules:

$$\frac{dy}{dx} = e^{-x^2}\left(3\cos 3x - 2x\sin 3x\right).$$

Since $e^{-x^2} > 0$, a stationary point satisfies

$$3\cos 3x = 2x\sin 3x.$$

Neither endpoint $x = 0, \pi/3$ is stationary. In the interior $\sin 3x > 0$ and $x > 0$, so $\cos 3x > 0$ at a stationary point and $0 < 3x < \pi/2$. Therefore division and the principal inverse tangent are valid:

$$\tan 3x = \frac{3}{2x} \quad \Longrightarrow \quad \boxed{x = \frac{1}{3}\arctan\left(\frac{3}{2x}\right)}.$$

2 Part (b)

Work in radians and retain calculator precision between iterations:

$$x_2 = \frac{1}{3}\arctan\frac{3}{0.8} = 0.4367313117\ldots,$$

$$x_3 = 0.4291581882\ldots, \qquad x_4 = 0.4307116849\ldots.$$

Thus (i) $\boxed{x_2 = 0.4367}$ and (ii) $\boxed{x_4 = 0.4307}$.

3 Part (c)

Let $h(x) = 3\cos 3x - 2x\sin 3x$, which is continuous. At the rounding boundaries,

$$h(0.4295) = 0.0106861\ldots > 0, \qquad h(0.4305) = -0.000598027\ldots < 0.$$

The sign change places a root strictly between 0.4295 and 0.4305. On $0 < x < \pi/6$, $h'(x) = -11\sin 3x - 6x\cos 3x < 0$, so the stationary root is unique; for $\pi/6 \le x \le \pi/3$, $h(x) < 0$. Hence the x coordinate of P is $\boxed{0.430 \text{ to } 3 \text{ decimal places}}$.

FINAL ANSWER

(a) $x = \frac{1}{3}\arctan\left(\frac{3}{2x}\right)$ (b) $x_2 = 0.4367$, $x_4 = 0.4307$ (c) 0.430 to 3 decimal places

Question 7

11 marks | Numerical Methods

Main topic: Applications of Differentiation

7.

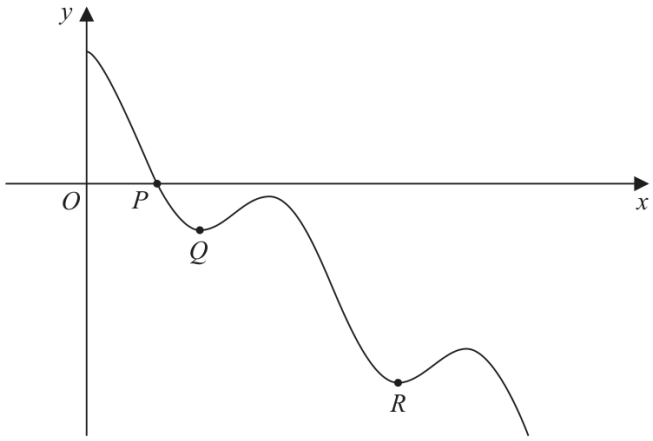


Figure 3

Figure 3 shows a sketch of part of the curve with equation

$$y = 2\cos 3x - 3x + 4 \qquad x > 0$$

where x is measured in radians.

The curve crosses the x-axis at the point P, as shown in Figure 3.

Given that the x coordinate of P is α,

(a) show that α lies between 0.8 and 0.9

(2)

The iteration formula

$$x_{n+1} = \frac{1}{3}\arccos(1.5x_n - 2)$$

can be used to find an approximate value for α.

(b) Using this iteration formula with $x_1 = 0.8$ find, to 4 decimal places, the value of

(i) x_2

(ii) x_5

(3)

The point Q and the point R are local minimum points on the curve, as shown in Figure 3.

Given that the x coordinates of Q and R are β and λ respectively, and that they are the two smallest values of x at which local minima occur,

(c) find, using calculus, the exact value of β and the exact value of λ.

(6)

1 Part (a)

Let $f(x) = 2\cos 3x - 3x + 4$, using radians. Then

$$f(0.8) = 2\cos 2.4 - 2.4 + 4 = 0.1252125\ldots > 0,$$
$$f(0.9) = 2\cos 2.7 - 2.7 + 4 = -0.5081443\ldots < 0.$$

The function is continuous on $[0.8, 0.9]$, so this change of sign shows that there is a root between the endpoints. Moreover, $f'(x) = -6\sin 3x - 3 < 0$ on this interval, so this root is unique there. Thus the crossing shown at P has $\boxed{0.8 < \alpha < 0.9}$.

2 Part (b)(i)

Using the inverse cosine in radians,

$$x_2 = \frac{1}{3}\arccos(1.5(0.8) - 2) = \frac{1}{3}\arccos(-0.8) = 0.8326971\ldots.$$

Hence $\boxed{x_2 = 0.8327}$ to 4 decimal places.

3 Part (b)(ii)

Continue the iteration, retaining the full calculator values at each step:

$$x_3 = \frac{1}{3}\arccos(1.5x_2 - 2) = 0.8067674\ldots,$$
$$x_4 = \frac{1}{3}\arccos(1.5x_3 - 2) = 0.8271196\ldots,$$
$$x_5 = \frac{1}{3}\arccos(1.5x_4 - 2) = 0.8110216\ldots.$$

Therefore $\boxed{x_5 = 0.8110}$ to 4 decimal places.

4 Part (c)

Differentiate the curve equation:

$$\frac{dy}{dx} = -6\sin 3x - 3, \qquad \frac{d^2y}{dx^2} = -18\cos 3x.$$

At a stationary point,

$$-6\sin 3x - 3 = 0 \quad \Longrightarrow \quad \sin 3x = -\frac{1}{2}.$$

For $x > 0$, the stationary points therefore belong to the two families

$$3x = \frac{7\pi}{6} + 2n\pi \quad \text{or} \quad 3x = \frac{11\pi}{6} + 2n\pi, \qquad n = 0, 1, 2, \ldots.$$

In the first family, $\cos 3x = -\frac{\sqrt{3}}{2}$, giving $y'' = 9\sqrt{3} > 0$, so these are local minima. In the second family, $\cos 3x = \frac{\sqrt{3}}{2}$ and $y'' = -9\sqrt{3} < 0$, so these are local maxima and are excluded. Thus all positive minimum coordinates are

$$x = \frac{7\pi}{18} + \frac{2n\pi}{3}, \qquad n = 0, 1, 2, \ldots.$$

The two smallest are obtained with $n = 0$ and $n = 1$:

$$\boxed{\beta = \frac{7\pi}{18}}, \qquad \boxed{\lambda = \frac{19\pi}{18}}.$$

FINAL ANSWER

(a) $0.8 < \alpha < 0.9$ (b)(i) $x_2 = 0.8327$ (b)(ii) $x_5 = 0.8110$ (c) $\beta = \frac{7\pi}{18}$, $\lambda = \frac{19\pi}{18}$

Question 8

10 marks | Numerical Methods

Main topic: Log Graphs & Exponential Models

8.

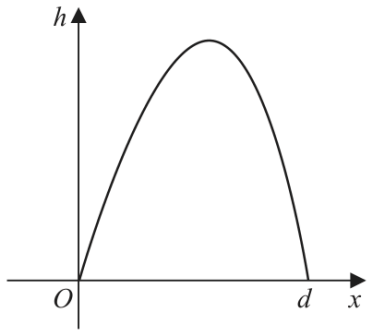


Figure 4

Figure 4 is a graph showing the path of a golf ball after the ball has been hit until it first hits the ground.

The vertical height, h metres, of the ball above the ground has been plotted against the horizontal distance travelled, x metres, measured from where the ball was hit.

The ball travels a horizontal distance of d metres before it first hits the ground.

The ball is modelled as a particle travelling in a vertical plane above horizontal ground.

The path of the ball is modelled by the equation

$$h = 1.5x - 0.5x\,e^{0.02x} \qquad 0 \leqslant x \leqslant d$$

Use the model to answer parts (a), (b) and (c).

(a) Find the value of d, giving your answer to 2 decimal places.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(3)

(b) Show that the maximum value of h occurs when

$$x = 50\ln\left(\frac{150}{x + 50}\right)$$

(4)

Using the iteration formula

$$x_{n+1} = 50\ln\left(\frac{150}{x_n + 50}\right) \qquad \text{with } x_1 = 30$$

(c) (i) find the value of x_2 to 2 decimal places,

(ii) find, by repeated iteration, the horizontal distance travelled by the golf ball before it reaches its maximum height. Give your answer to 2 decimal places.

(3)

1 Part (a)

At the first landing, $h = 0$ and $x = d > 0$:

$$0 = d(1.5 - 0.5e^{0.02d}), \qquad e^{0.02d} = 3.$$

Taking logarithms, $0.02d = \ln 3$, hence $d = 50\ln 3 = 54.930614\ldots$. The root $x = 0$ is the launch point, not the landing point.

$$\boxed{d = 54.93 \text{ m}}$$

2 Part (b)

By the product and chain rules,

$$h'(x) = 1.5 - 0.5e^{0.02x} - 0.01xe^{0.02x}.$$

At a stationary point,

$$1.5 = 0.01(x + 50)e^{0.02x}, \qquad e^{0.02x} = \frac{150}{x + 50}.$$

Since $x + 50 > 0$, logarithms give $0.02x = \ln(150/(x + 50))$. Also $h''(x) = -(0.02 + 0.0002x)e^{0.02x} < 0$ on $[0, d]$. Thus the stationary point is the unique maximum; the endpoints both have zero height.

$$\boxed{x = 50\ln\left(\frac{150}{x + 50}\right)}$$

3 Part (c)

(i) With $x_1 = 30$,

$$x_2 = 50\ln\left(\frac{150}{30 + 50}\right) = 31.43043297\ldots.$$

(ii) Continuing with full intermediate precision gives $x_3 = 30.54431108\ldots$, $x_4 = 31.09139065\ldots$, $x_5 = 30.75292479\ldots$ and $x_6 = 30.96205562\ldots$. Continued iteration converges to $30.88212\ldots$, which lies in $(0, d)$. This is the horizontal distance at maximum height, not the maximum height itself.

$$\boxed{x_2 = 31.43, \quad x_{\text{maximum}} = 30.88 \text{ m}}$$

FINAL ANSWER

(a) $d = 54.93$ m (b) $x = 50\ln\left(\frac{150}{x+50}\right)$ (c) $x_2 = 31.43, \quad x_{\text{maximum}} = 30.88$ m

Question 8

12 marks | Numerical Methods

Main topic: Log Graphs & Exponential Models

8.

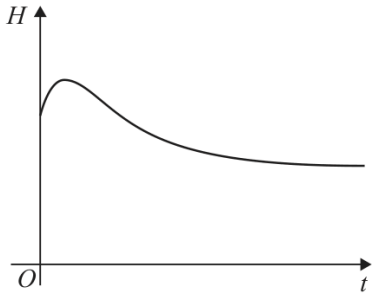


Figure 4

The heart rate of a horse is being monitored.
The heart rate H, measured in beats per minute (bpm), is modelled by the equation

$$H = 32 + 40e^{-0.2t} - 20e^{-0.9t}$$

where t minutes is the time after monitoring began.

Figure 4 is a sketch of H against t.

Use the equation of the model to answer parts (a) to (e).

(a) State the initial heart rate of the horse.

(1)

In the long term, the heart rate of the horse approaches L bpm.

(b) State the value of L.

(1)

The heart rate of the horse reaches its maximum value after T minutes.

(c) Find the value of T, giving your answer to 3 decimal places.

(*Solutions based entirely on calculator technology are not acceptable.*)

(5)

The heart rate of the horse is 37 bpm after M minutes.

(d) Show that M is a solution of the equation

$$t = 5\ln\left(\frac{8}{1 + 4e^{-0.9t}}\right)$$

(2)

Using the iteration formula

$$t_{n+1} = 5\ln\left(\frac{8}{1 + 4e^{-0.9t_n}}\right) \quad \text{with} \quad t_1 = 10$$

(e) (i) find, to 4 decimal places, the value of t_2

(ii) find, to 4 decimal places, the value of M

(3)

1 Part (a)

At $t = 0$, $H = 32 + 40 - 20 = 52$.

$$\boxed{H(0) = 52 \text{ bpm}}$$

2 Part (b)

As $t \to \infty$, both decaying exponential terms tend to zero.

$$\boxed{L = 32}$$

3 Part (c)

$$\frac{dH}{dt} = -8e^{-0.2t} + 18e^{-0.9t}.$$

At the maximum,

$$18e^{-0.9T} = 8e^{-0.2T} \quad \Longrightarrow \quad e^{0.7T} = \frac{9}{4} \quad \Longrightarrow \quad T = \frac{10}{7}\ln\frac{9}{4} = 1.1584717\ldots.$$

Also $H''(t) = 1.6e^{-0.2t} - 16.2e^{-0.9t}$. At the stationary point, $e^{-0.9T} = (4/9)e^{-0.2T}$, so $H''(T) = -5.6e^{-0.2T} < 0$, confirming a maximum.

$$\boxed{T = 1.158 \text{ minutes}}$$

4 Part (d)

For $H = 37$,

$$5 = 40e^{-0.2t} - 20e^{-0.9t} \quad \Longrightarrow \quad 8e^{-0.2t} = 1 + 4e^{-0.9t}.$$

Taking logarithms gives

$$-0.2t = \ln\left(\frac{1 + 4e^{-0.9t}}{8}\right) \quad \Longrightarrow \quad t = -5\ln\left(\frac{1 + 4e^{-0.9t}}{8}\right).$$

Using $-\ln u = \ln(1/u)$ for $u > 0$ then gives

$$t = 5\ln\left(\frac{8}{1 + 4e^{-0.9t}}\right).$$

$$\boxed{t = 5\ln\left(\frac{8}{1 + 4e^{-0.9t}}\right)}$$

5

Part (e)

Starting with $t_1 = 10$,

$$t_2 = 5\ln\left(\frac{8}{1+4e^{-9}}\right) = 10.3947401213\ldots.$$

Continuing without rounding intermediate values gives $t_3 = 10.3954778346\ldots$, $t_4 = 10.3954789826\ldots$ and $t_5 = 10.3954789844\ldots$. Thus (i) $t_2 = 10.3947$, and (ii) $M = 10.3955$ minutes, both to 4 decimal places.

$$\boxed{t_2 = 10.3947, \quad M = 10.3955 \text{ minutes}}$$

✓ **FINAL ANSWER**

(a) $H(0) = 52$ bpm (b) $L = 32$ (c) $T = 1.158$ minutes (d) $t = 5\ln\left(\frac{8}{1+4e^{-0.9t}}\right)$ (e) $t_2 = 10.3947, \quad M = 10.3955$ minutes