Elite Mathematics
Edexcel International A Level

IAL PURE 3 · WMA13

WMA13 Pure 3 Classified Questions

180 unique questions grouped by primary topic, with cross-topic placements where a question also belongs in another chapter.

564	2020–2026	Questions
topic placements	19 exam sessions	checked by Dr Eslam



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TOPIC

Rational Algebra

Every past-paper question on this topic, January 2020 to June 2026.

Question 3

8 marks | Rational Algebra

3.

$$f(x) = 3 - \frac{x-2}{x+1} + \frac{5x+26}{2x^2-3x-5} \qquad x > 4$$

(a) Show that

$$f(x) = \frac{ax+b}{cx+d} \qquad x > 4$$

where a, b, c and d are integers to be found.

(4)

(b) Hence find $f^{-1}(x)$

(2)

(c) Find the domain of f^{-1}

(2)

Question 1

9 marks | Rational Algebra

1. The function f is defined by

$$\mathrm{f}(x) = \frac{5x}{x^2 + 7x + 12} + \frac{5x}{x + 4} \qquad x > 0$$

(a) Show that $\mathrm{f}(x) = \dfrac{5x}{x + 3}$

(3)

(b) Find f^{-1}

(3)

(c) (i) Find, in simplest form, $\mathrm{f}'(x)$.

(ii) Hence, state whether f is an increasing or a decreasing function, giving a reason for your answer.

(3)

Question 4

7 marks | Rational Algebra

4. $$f(x) = \frac{2x^4 + 15x^3 + 35x^2 + 21x - 4}{(x+3)^2} \qquad x \in \mathbb{R} \quad x > -3$$

(a) Find the values of the constants A, B, C and D such that

$$f(x) = Ax^2 + Bx + C + \frac{D}{(x+3)^2}$$

(4)

(b) Hence find,

$$\int f(x)\,dx$$

(3)

Question 4

13 marks | Rational Algebra

4. The function f is defined by

$$f(x) = \frac{2x^2 - 32}{3x^2 + 7x - 20} + \frac{8}{3x - 5} \qquad x \in \mathbb{R} \quad x > 2$$

(a) Show that $f(x) = \frac{2x}{3x-5}$

(3)

(b) Show, using calculus, that f is a decreasing function.
You must make your reasoning clear.

(3)

The function g is defined by

$$g(x) = 3 + 2\ln x \qquad x \geqslant 1$$

(c) Find g^{-1}

(3)

(d) Find the exact value of a for which

$$gf(a) = 5$$

(4)

Question 4

5 marks | Rational Algebra

4: The function f is defined by

$$f(x) = \frac{49x}{x^2 + x - 12} + \frac{7x}{x + 4} \qquad x > 3$$

(a) Show that

$$f(x) = \frac{7x}{x - 3} \qquad x > 3$$

(3)

(b) Hence find $f'(x)$ giving your answer in simplest form.

(2)

WMA13/01 JANUARY 2020

Question 1

6 marks | Rational Algebra

Main topic: Log Graphs & Exponential Models

1. A population of a rare species of toad is being studied.

The number of toads, N, in the population, t years after the start of the study, is modelled by the equation

$$N = \frac{900e^{0.12t}}{2e^{0.12t} + 1} \qquad t \geqslant 0, t \in \mathbb{R}$$

According to this model,

(a) calculate the number of toads in the population at the start of the study,

(1)

(b) find the value of t when there are 420 toads in the population, giving your answer to 2 decimal places.

(4)

(c) Explain why, according to this model, the number of toads in the population can never reach 500

(1)

WMA13/01 JANUARY 2020

Question 2

8 marks | Rational Algebra

Main topic: Functions, Composition & Inverses

2. The function f and the function g are defined by

$$f(x) = \frac{12}{x+1} \qquad x > 0, x \in \mathbb{R}$$

$$g(x) = \frac{5}{2}\ln x \qquad x > 0, x \in \mathbb{R}$$

(a) Find, in simplest form, the value of $fg(e^2)$

(2)

(b) Find f^{-1}

(3)

(c) Hence, or otherwise, find all real solutions of the equation

$$f^{-1}(x) = f(x)$$

(3)

Question 4

11 marks | Rational Algebra

Main topic: Differentiation Techniques

4. (i) $$f(x) = \frac{(2x+5)^2}{x-3} \qquad x \neq 3$$

(a) Find $f'(x)$ in the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are fully factorised quadratic expressions.

(b) Hence find the range of values of x for which $f(x)$ is increasing.

(6)

(ii) $$g(x) = x\sqrt{\sin 4x} \qquad 0 \leqslant x < \frac{\pi}{4}$$

The curve with equation $y = g(x)$ has a maximum at the point M.

Show that the x coordinate of M satisfies the equation

$$\tan 4x + kx = 0$$

where k is a constant to be found.

(5)

Question 5

8 marks | Rational Algebra

Main topic: Trig Identities & Exact Values

5. (a) Use the substitution $t = \tan x$ to show that the equation

$$12\tan 2x + 5\cot x \sec^2 x = 0$$

can be written in the form

$$5t^4 - 24t^2 - 5 = 0$$

(4)

(b) Hence solve, for $0 \leqslant x < 360°$, the equation

$$12\tan 2x + 5\cot x \sec^2 x = 0$$

Show each stage of your working and give your answers to one decimal place.

(4)

Question 6

9 marks | Rational Algebra | Main topic: Modulus

6.

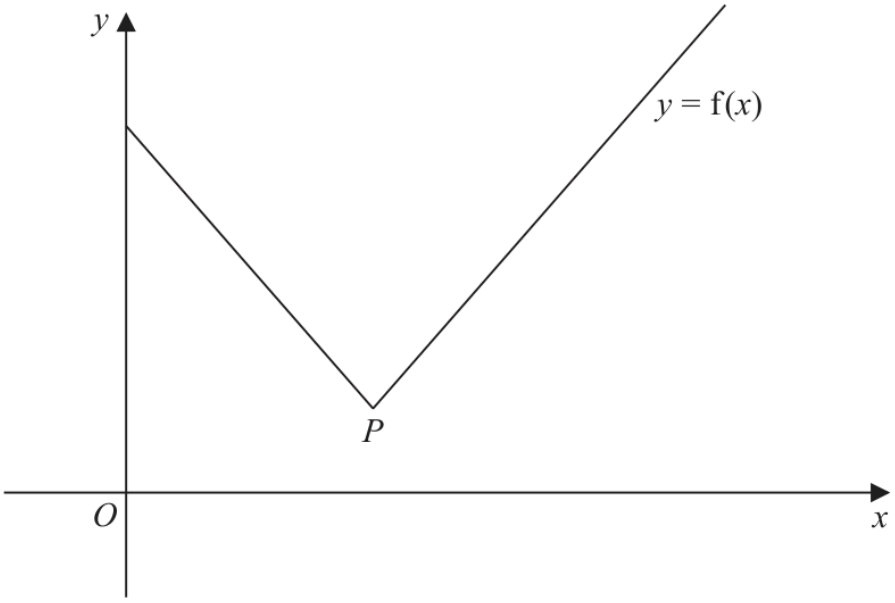


Figure 2

Figure 2 shows part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 2|2x - 5| + 3 \qquad x \geqslant 0$$

The vertex of the graph is at point P as shown.

(a) State the coordinates of P.

(2)

(b) Solve the equation $\mathrm{f}(x) = 3x - 2$

(4)

Given that the equation

$$\mathrm{f}(x) = kx + 2$$

where k is a constant, has exactly two roots,

(c) find the range of values of k.

(3)

Question 8

10 marks | Rational Algebra | Main topic: Integration

8. (i) Find, using algebraic integration, the exact value of

$$\int_3^{42} \frac{2}{3x-1}\,\mathrm{d}x$$

giving your answer in simplest form.

(4)

(ii) $$\mathrm{h}(x) = \frac{2x^3 - 7x^2 + 8x + 1}{(x-1)^2} \qquad x > 1$$

Given $\mathrm{h}(x) = Ax + B + \frac{C}{(x-1)^2}$ where A, B and C are constants to be found, find

$$\int \mathrm{h}(x)\,\mathrm{d}x$$

(6)

Question 3

7 marks | Rational Algebra

Main topic: Differentiation Techniques

3.

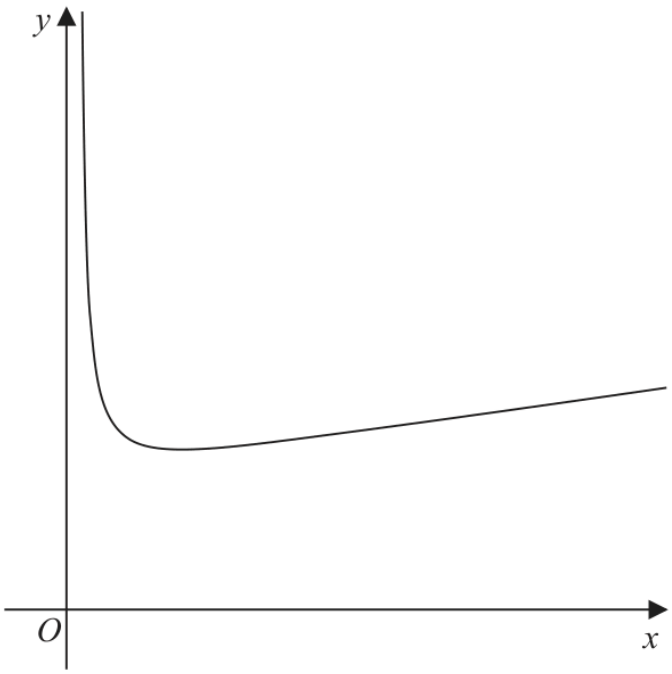


Figure 1

Figure 1 shows a sketch of a curve with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{2x + 3}{\sqrt{4x - 1}} \qquad x > \frac{1}{4}$$

(a) Find, in simplest form, $\mathrm{f}'(x)$.

(4)

(b) Hence find the range of f.

(3)

Question 7

9 marks | Rational Algebra

Main topic: Trig Equations & R-Form

7. (a) Express $\cos x + 4\sin x$ in the form $R\cos(x - \alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

A scientist is studying the behaviour of seabirds in a colony.

She models the height above sea level, H metres, of one of the birds in the colony by the equation

$$H = \frac{24}{3 + \cos\left(\frac{1}{2}t\right) + 4\sin\left(\frac{1}{2}t\right)} \qquad 0 \leqslant t \leqslant 6.5$$

where t seconds is the time after it leaves the nest.

Find, according to the model,

(b) the minimum height of the seabird above sea level, giving your answer to the nearest cm,

(2)

(c) the value of t, to 2 decimal places, when $H = 10$

(4)

Question 9

14 marks | Rational Algebra

Main topic: Applications of Differentiation

9. (a) Given that

$$\frac{x^4 - x^3 - 10x^2 + 3x - 9}{x^2 - x - 12} \equiv x^2 + P + \frac{Q}{x - 4} \qquad x > -3$$

find the value of the constant P and show that $Q = 5$

(4)

The curve C has equation $y = g(x)$, where

$$g(x) = \frac{x^4 - x^3 - 10x^2 + 3x - 9}{x^2 - x - 12} \qquad -3 < x < 3.5 \qquad x \in \mathbb{R}$$

(b) Find the equation of the tangent to C at the point where $x = 2$
Give your answer in the form $y = mx + c$, where m and c are constants to be found.

(5)

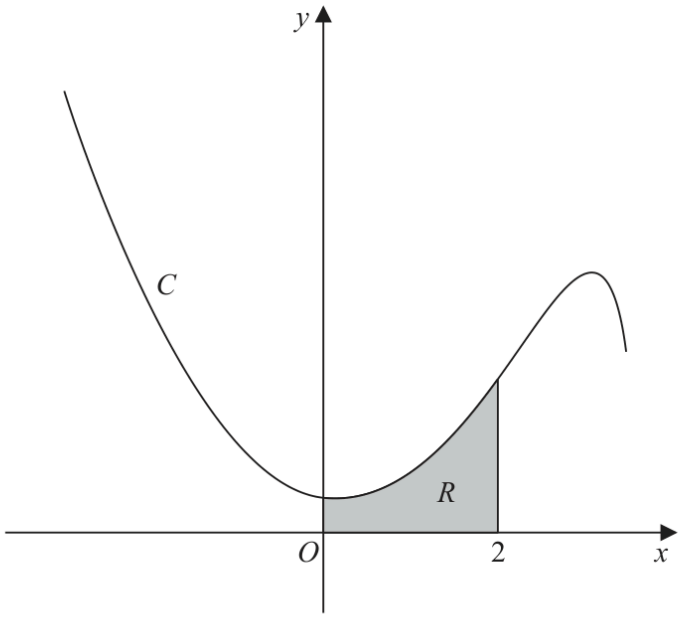


Figure 4

Figure 4 shows a sketch of the curve C.
The region R, shown shaded in Figure 4, is bounded by C, the y-axis, the x-axis and the line with equation $x = 2$

(c) Find the exact area of R, writing your answer in the form $a + b\ln 2$, where a and b are constants to be found.

(5)

WMA13/01 JANUARY 2021

Question 1

3 marks | Rational Algebra | Main topic: Integration

1. Find

$$\int \frac{x^2 - 5}{2x^3} \mathrm{d}x \qquad x > 0$$

giving your answer in simplest form.

(3)

Question 3

8 marks | Rational Algebra | Main topic: Integration

3. (i) Find

$$\int \frac{12}{(2x-1)^2}\,\mathrm{d}x$$

giving your answer in simplest form.

(2)

(ii) (a) Write $\dfrac{4x+3}{x+2}$ in the form

$$A + \frac{B}{x+2}$$ where A and B are constants to be found

(b) Hence find, using algebraic integration, the exact value of

$$\int_{-8}^{-5} \frac{4x+3}{x+2}\,\mathrm{d}x$$

giving your answer in simplest form.

(6)

Question 4

10 marks | Rational Algebra

Main topic: Functions, Composition & Inverses

4. The functions f and g are defined by

$$\mathrm{f}(x) = \frac{4x + 6}{x - 5} \qquad x \in \mathbb{R},\ x \neq 5$$

$$\mathrm{g}(x) = 5 - 2x^2 \qquad x \in \mathbb{R},\ x \leqslant 0$$

(a) Solve the equation

$$\mathrm{fg}(x) = 3$$

(4)

(b) Find f^{-1}

(3)

(c) Sketch and label, on the same axes, the curve with equation $y = \mathrm{g}(x)$ and the curve with equation $y = \mathrm{g}^{-1}(x)$. Show on your sketch the coordinates of the points where each curve meets or cuts the coordinate axes.

(3)

Question 8

13 marks | Rational Algebra

Main topic: Log Graphs & Exponential Models

8. A scientist is studying a population of fish in a lake. The number of fish, N, in the population, t years after the start of the study, is modelled by the equation

$$N = \frac{600e^{0.3t}}{2 + e^{0.3t}} \qquad t \geqslant 0$$

Use the equation of the model to answer parts (a), (b), (c), (d) and (e).

(a) Find the number of fish in the lake at the start of the study.

(1)

(b) Find the upper limit to the number of fish in the lake.

(1)

(c) Find the time, after the start of the study, when there are predicted to be 500 fish in the lake. Give your answer in years and months to the nearest month.

(4)

(d) Show that

$$\frac{dN}{dt} = \frac{Ae^{0.3t}}{(2 + e^{0.3t})^2}$$

where A is a constant to be found.

(3)

Given that when $t = T$, $\frac{dN}{dt} = 8$

(e) find the value of T to one decimal place.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(4)

Question 6

8 marks | Rational Algebra

Main topic: Differentiation Techniques

6. (i) The curve C_1 has equation

$$y = 3\ln(x^2 - 5) - 4x^2 + 15 \qquad x > \sqrt{5}$$

Show that C_1 has a stationary point at $x = \dfrac{\sqrt{p}}{2}$ where p is a constant to be found.

(4)

(ii) A different curve C_2 has equation

$$y = 4x - 12\sin^2 x$$

(a) Show that, for this curve,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = A + B\sin 2x$$

where A and B are constants to be found.

(b) Hence, state the maximum gradient of this curve.

(4)

Question 9

9 marks | Rational Algebra | Main topic: Numerical Methods

9.

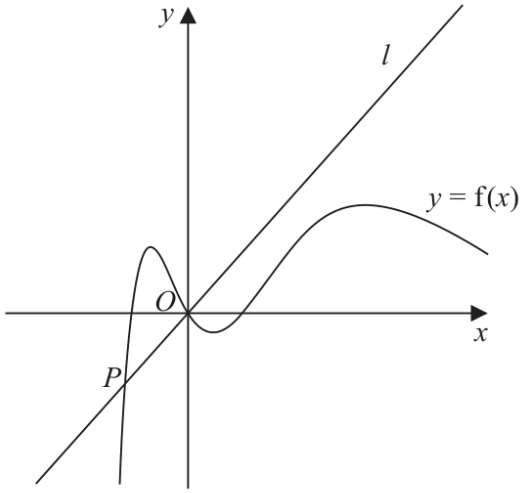


Figure 3

Figure 3 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x(x^2 - 4)\mathrm{e}^{-\frac{1}{2}x}$$

(a) Find $\mathrm{f}'(x)$.

(2)

The line l is the normal to the curve at O and meets the curve again at the point P.

The point P lies in the 3rd quadrant, as shown in Figure 3.

(b) Show that the x coordinate of P is a solution of the equation

$$x = -\frac{1}{2}\sqrt{16 + \mathrm{e}^{\frac{1}{2}x}}$$

(4)

(c) Using the iterative formula

$$x_{n+1} = -\frac{1}{2}\sqrt{16 + \mathrm{e}^{\frac{1}{2}x_n}} \quad \text{with } x_1 = -2$$

find, to 4 decimal places,

(i) the value of x_2

(ii) the x coordinate of P.

(3)

Question 4

7 marks | Rational Algebra

Main topic: Log Graphs & Exponential Models

4. The growth of a weed on the surface of a pond is being studied.

The surface area of the pond covered by the weed, $A\,\text{m}^2$, is modelled by the equation

$$A = \frac{80pe^{0.15t}}{pe^{0.15t} + 4}$$

where p is a positive constant and t is the number of days after the start of the study.

Given that

- $30\,\text{m}^2$ of the surface of the pond was covered by the weed at the start of the study
- $50\,\text{m}^2$ of the surface of the pond was covered by the weed T days after the start of the study

(a) show that $p = 2.4$

(2)

(b) find the value of T, giving your answer to one decimal place.

(*Solutions relying entirely on graphical or numerical methods are not acceptable.*)

(4)

The weed grows until it covers the surface of the pond.

(c) Find, according to the model, the maximum possible surface area of the pond.

(1)

Question 5

9 marks | Rational Algebra

Main topic: Applications of Differentiation

5.

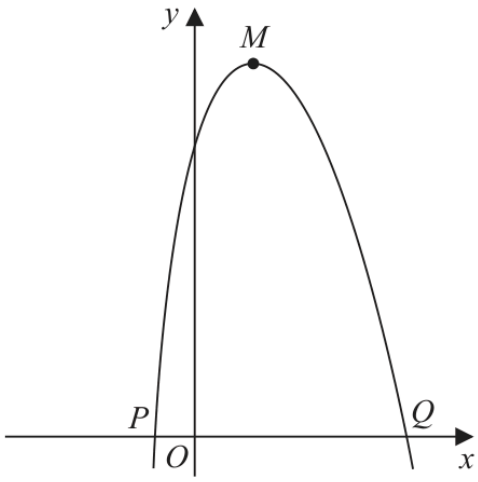


Figure 1

Figure 1 shows a sketch of part of the curve with equation

$$y = 6\ln(2x + 3) - \frac{1}{2}x^2 + 4 \qquad x > -\frac{3}{2}$$

The curve cuts the negative x-axis at the point P, as shown in Figure 1.

(a) Show that the x coordinate of P lies in the interval $[-1.25, -1.2]$

(2)

The curve cuts the positive x-axis at the point Q, also shown in Figure 1.

Using the iterative formula

$$x_{n+1} = \sqrt{12\ln(2x_n + 3) + 8} \quad \text{with} \quad x_1 = 6$$

(b) (i) find, to 4 decimal places, the value of x_2

(ii) find, by continued iteration, the x coordinate of Q. Give your answer to 4 decimal places.

(3)

The curve has a maximum turning point at M, as shown in Figure 1.

(c) Using calculus and showing each stage of your working, find the x coordinate of M.

(4)

Question 6

11 marks | Rational Algebra

Main topic: Functions, Composition & Inverses

6. The function f is defined by

$$f(x) = \frac{5x - 3}{x - 4} \qquad x > 4$$

(a) Show, by using calculus, that f is a decreasing function.

(3)

(b) Find f^{-1}

(3)

(c) (i) Show that $ff(x) = \frac{ax + b}{x + c}$ where a, b and c are constants to be found.

(ii) Deduce the range of ff.

(5)

Question 2

9 marks | Rational Algebra

Main topic: Functions, Composition & Inverses

2. The functions f and g are defined by

$$\mathrm{f}(x) = \frac{5 - x}{3x + 2} \qquad x \in \mathbb{R}, x \neq -\frac{2}{3}$$

$$\mathrm{g}(x) = 2x - 7 \qquad x \in \mathbb{R}$$

(a) Find the value of $\mathrm{fg}(5)$

(2)

(b) Find f^{-1}

(3)

(c) Solve the equation

$$\mathrm{f}\left(\frac{1}{a}\right) = \mathrm{g}(a + 3)$$

(4)

Question 3

6 marks | Rational Algebra | Main topic: Integration

3. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

Given that k is a positive constant,

(a) find

$$\int \frac{9x}{3x^2 + k} \, dx$$

(2)

Given also that

$$\int_2^5 \frac{9x}{3x^2 + k} \, dx = \ln 8$$

(b) find the value of k

(4)

Question 6

10 marks | Rational Algebra

Main topic: Applications of Differentiation

6.

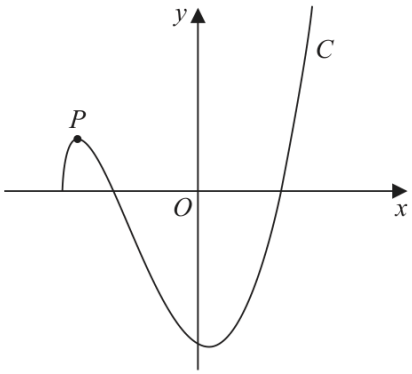


Figure 3

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

The function f is defined by

$$f(x) = 5(x^2 - 2)(4x + 9)^{\frac{1}{2}} \qquad x \geqslant -\frac{9}{4}$$

(a) Show that

$$f'(x) = \frac{k(5x^2 + 9x - 2)}{(4x + 9)^{\frac{1}{2}}}$$

where k is an integer to be found.

(4)

(b) Hence, find the values of x for which $f'(x) = 0$

(1)

Figure 3 shows a sketch of the curve C with equation $y = f(x)$.

The curve has a local maximum at the point P

(c) Find the exact coordinates of P

(2)

The function g is defined by

$$g(x) = 2f(x) + 4 \qquad -\frac{9}{4} \leqslant x \leqslant 0$$

(d) Find the range of g

(3)

Question 1

9 marks | Rational Algebra | Main topic: Integration

1. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

$$\mathrm{f}(x) = \frac{2x^3 - 4x - 15}{x^2 + 3x + 4}$$

(a) Show that

$$\mathrm{f}(x) \equiv Ax + B + \frac{C(2x+3)}{x^2 + 3x + 4}$$

where A, B and C are integers to be found.

(4)

(b) Hence, find

$$\int_3^5 \mathrm{f}(x)\,\mathrm{d}x$$

giving your answer in the form $p + \ln q$, where p and q are integers.

(5)

Question 2

7 marks | Rational Algebra

Main topic: Functions, Composition & Inverses

2. The functions f and g are defined by

$$f(x) = 5 - \frac{4}{3x + 2} \qquad x \geqslant 0$$

$$g(x) = \left| 4\sin\left(\frac{x}{3} + \frac{\pi}{6}\right) \right| \qquad x \in \mathbb{R}$$

(a) Find the range of f

(2)

(b) (i) Find $f^{-1}(x)$

(ii) Write down the domain of f^{-1}

(3)

(c) Find $fg(-\pi)$

(2)

Question 9

9 marks | Rational Algebra

Main topic: Trig Equations & R-Form

9. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

Given that $\cos 2\theta - \sin 3\theta \neq 0$

(a) prove that

$$\frac{\cos^2\theta}{\cos 2\theta - \sin 3\theta} \equiv \frac{1+\sin\theta}{1-2\sin\theta-4\sin^2\theta}$$

(4)

(b) Hence solve, for $0 < \theta \leqslant 360°$

$$\frac{\cos^2\theta}{\cos 2\theta - \sin 3\theta} = 2\operatorname{cosec}\theta$$

Give your answers to one decimal place.

(5)

WMA13/01 JANUARY 2023

Question 1

6 marks | Rational Algebra

Main topic: Functions, Composition & Inverses

1. The functions f and g are defined by

$$\mathrm{f}(x) = 9 - x^2 \qquad x \in \mathbb{R} \quad x \geqslant 0$$

$$\mathrm{g}(x) = \frac{3}{2x+1} \qquad x \in \mathbb{R} \quad x \geqslant 0$$

(a) Write down the range of f

(1)

(b) Find the value of $\mathrm{fg}(1.5)$

(2)

(c) Find g^{-1}

(3)

Question 5

9 marks | Rational Algebra

Main topic: Trig Equations & R-Form

5. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Prove that

$$\cot^2 x - \tan^2 x \equiv 4\cot 2x \operatorname{cosec} 2x \qquad x \neq \frac{n\pi}{2} \quad n \in \mathbb{Z}$$

(4)

(b) Hence solve, for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$4\cot 2\theta \operatorname{cosec} 2\theta = 2\tan^2 \theta$$

giving your answers to 2 decimal places.

(5)

Question 9

11 marks | Rational Algebra | Main topic: Numerical Methods

9.

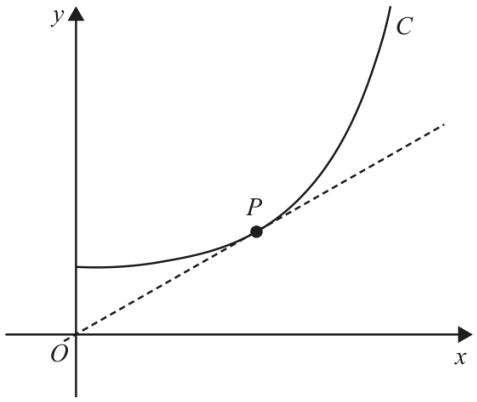


Figure 3

Figure 3 shows a sketch of part of the curve C with equation

$$y = \sqrt{3 + 4e^{x^2}} \qquad x \geqslant 0$$

(a) Find $\frac{dy}{dx}$, giving your answer in simplest form.

(2)

The point P with x coordinate α lies on C.

Given that the tangent to C at P passes through the origin, as shown in Figure 3,

(b) show that $x = \alpha$ is a solution of the equation

$$4x^2e^{x^2} - 4e^{x^2} - 3 = 0$$

(3)

(c) Hence show that α lies between 1 and 2

(2)

(d) Show that the equation in part (b) can be written in the form

$$x = \frac{1}{2}\sqrt{4 + 3e^{-x^2}}$$

(1)

The iteration formula

$$x_{n+1} = \frac{1}{2}\sqrt{4 + 3e^{-x_n^2}}$$

with $x_1 = 1$ is used to find an approximation for α.

(e) Use the iteration formula to find, to 4 decimal places, the value of

(i) x_3

(ii) α

(3)

Question 10

9 marks | Rational Algebra

Main topic: Log Graphs & Exponential Models

10. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

A population of fruit flies is being studied.

The number of fruit flies, F, in the population, t days after the start of the study, is modelled by the equation

$$F = \frac{350e^{kt}}{9 + e^{kt}}$$

where k is a constant.

Use the equation of the model to answer parts (a), (b) and (c).

(a) Find the number of fruit flies in the population at the start of the study.

(1)

Given that there are 200 fruit flies in the population 15 days after the start of the study,

(b) show that $k = \frac{1}{15}\ln 12$

(3)

Given also that, when $t = T$, the number of fruit flies in the population is increasing at a rate of 10 per day,

(c) find the possible values of T, giving your answers to one decimal place.

(5)

Question 8

9 marks | Rational Algebra

Main topic: Applications of Differentiation

8.

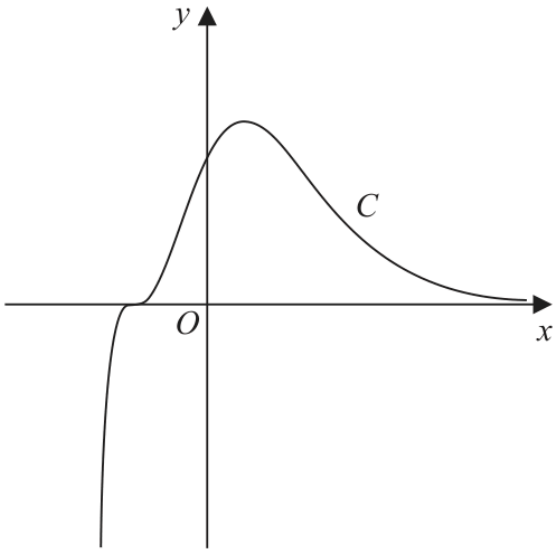


Figure 3

Figure 3 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = (2x + 1)^3 \mathrm{e}^{-4x}$$

(a) Show that

$$\mathrm{f}'(x) = A(2x + 1)^2 (1 - 4x)\, \mathrm{e}^{-4x}$$

where A is a constant to be found.

(4)

(b) Hence find the exact coordinates of the two stationary points on C.

(3)

The function g is defined by

$$\mathrm{g}(x) = 8\,\mathrm{f}(x - 2)$$

(c) Find the coordinates of the maximum stationary point on the curve with equation $y = \mathrm{g}(x)$.

(2)

Question 10

8 marks | Rational Algebra

Main topic: Differentiation Techniques

10.

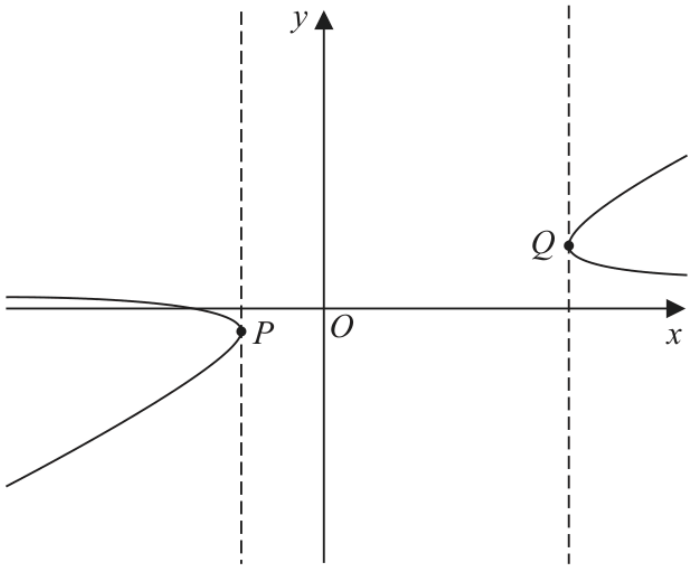


Figure 4

Figure 4 shows a sketch of the curve with equation

$$x = \frac{2y^2 + 6}{3y - 3}$$

(a) Find $\frac{dx}{dy}$ giving your answer as a fully simplified fraction.

(4)

The tangents at points P and Q on the curve are parallel to the y-axis, as shown in Figure 4.

(b) Use the answer to part (a) to find the equations of these two tangents.

(4)

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Question 2

8 marks | Rational Algebra

Main topic: Functions, Composition & Inverses

2. The function f is defined by

$$\mathrm{f}(x) = \frac{x+3}{x-4} \qquad x \in \mathbb{R}, x \neq 4$$

(a) Find $\mathrm{ff}(6)$

(2)

(b) Find f^{-1}

(3)

The function g is defined by

$$\mathrm{g}(x) = x^2 + 5 \qquad x \in \mathbb{R}, x > 0$$

(c) Find the exact value of a for which

$$\mathrm{gf}(a) = 7$$

(3)

Question 7

10 marks | Rational Algebra

Main topic: Functions, Composition & Inverses

7.

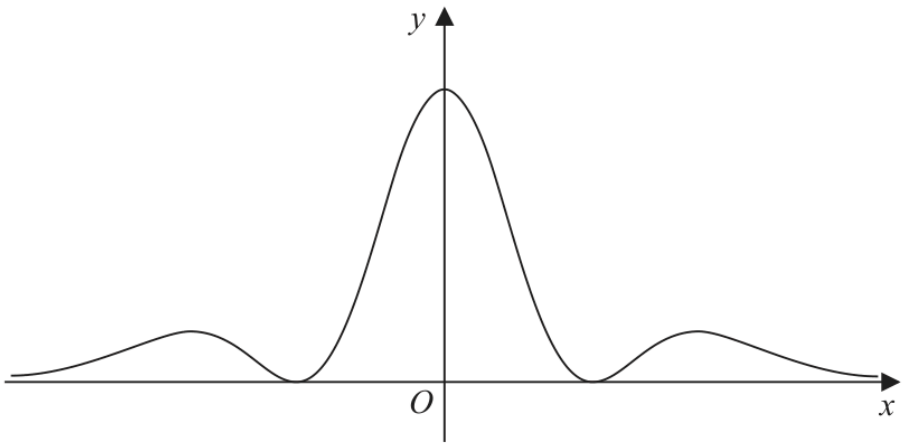


Figure 1

Figure 1 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \mathrm{e}^{-x^2}(2x^2 - 3)^2$$

(a) Find the range of f

(2)

(b) Show that

$$\mathrm{f}'(x) = 2x(2x^2 - 3)\mathrm{e}^{-x^2}(A - Bx^2)$$

where A and B are constants to be found.

(4)

Given that the line $y = k$, where k is a constant, $k > 0$, intersects the curve at exactly two distinct points,

(c) find the exact range of values of k

(4)

Question 10

9 marks | Rational Algebra

Main topic: Differentiation Techniques

10. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

A curve C has equation

$$x = \sin^2 4y \qquad 0 \leqslant y \leqslant \frac{\pi}{8} \qquad 0 \leqslant x \leqslant 1$$

The point P with x coordinate $\frac{1}{4}$ lies on C

(a) Find the exact y coordinate of P

(2)

(b) Find $\frac{\mathrm{d}x}{\mathrm{d}y}$

(2)

(c) Hence show that $\frac{\mathrm{d}y}{\mathrm{d}x}$ can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{q + r(x + s)^2}}$$

where q, r and s are constants to be found.

(3)

Using the answer to part (c),

(d) (i) state the x coordinate of the point where the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ is a minimum,

(ii) state the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at this point.

(2)

Question 2

6 marks | Rational Algebra | Main topic: Numerical Methods

2. A curve has equation $y = f(x)$ where

$$f(x) = x^4 - 5x^2 + 4x - 7 \qquad x \in \mathbb{R}$$

(a) Show that the equation $f(x) = 0$ has a root, α, in the interval $[2, 3]$

(2)

(b) Show that the equation $f(x) = 0$ can be written as

$$x = \sqrt[3]{\frac{5x^2 - 4x + 7}{x}}$$

(1)

The iterative formula

$$x_{n+1} = \sqrt[3]{\frac{5x_n^{\,2} - 4x_n + 7}{x_n}}$$

is used to find α

(c) Starting with $x_1 = 2$ and using the iterative formula,

(i) find, to 4 decimal places, the value of x_2

(ii) find, to 4 decimal places, the value of α

(3)

Question 7

12 marks | Rational Algebra

Main topic: Applications of Differentiation

7. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

The curve C has equation

$$y = \frac{16}{9(3x - k)} \qquad x \neq \frac{k}{3}$$

where k is a positive constant not equal to 3

(a) Find $\frac{dy}{dx}$ giving your answer in simplest form in terms of k.

(2)

The point P with x coordinate 1 lies on C.

Given that the gradient of the curve at P is -12

(b) find the two possible values of k

(3)

Given also that $k < 3$

(c) find the equation of the normal to C at P, writing your answer in the form $ax + by + c = 0$, where a, b and c are integers to be found.

(3)

(d) show, using algebraic integration that,

$$\int_1^3 \frac{16}{9(3x - k)} dx = \lambda \ln 10$$

where λ is a constant to be found.

(4)

Question 9

8 marks | Rational Algebra

Main topic: Trig Equations & R-Form

9. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that the equation

$$\frac{3\sin\theta\cos\theta}{\cos\theta+\sin\theta} = (2+\sec 2\theta)(\cos\theta-\sin\theta)$$

can be written in the form

$$3\sin 2\theta - 4\cos 2\theta = 2$$

(3)

(b) Hence solve for $\pi < x < \frac{3\pi}{2}$

$$\frac{3\sin x\cos x}{\cos x+\sin x} = (2+\sec 2x)(\cos x-\sin x)$$

giving the answer to 3 significant figures.

(5)

Question 2

7 marks | Rational Algebra | Main topic: Integration

2. $$g(x) = \frac{2x^2 - 5x + 8}{x - 2}$$

(a) Write $g(x)$ in the form

$$Ax + B + \frac{C}{x - 2}$$

where A, B and C are integers to be found.

(3)

(b) Hence use algebraic integration to show that

$$\int_4^8 g(x)\,dx = \alpha + \beta \ln 3$$

where α and β are integers to be found.

(4)

Question 5

10 marks | Rational Algebra

Main topic: Functions, Composition & Inverses

5. The functions f and g are defined by

$$f(x) = 2 + 5\ln x \qquad x > 0$$

$$g(x) = \frac{6x - 2}{2x + 1} \qquad x > \frac{1}{3}$$

(a) Find $f^{-1}(22)$

(2)

(b) Use differentiation to prove that g is an increasing function.

(3)

(c) Find g^{-1}

(3)

(d) Find the range of fg

(2)

Question 7

8 marks | Rational Algebra

Main topic: Trig Identities & Exact Values

7. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Given that

$$\sqrt{2}\sin(x + 45°) = \cos(x - 60°)$$

show that

$$\tan x = -2 - \sqrt{3}$$

(4)

(b) Hence or otherwise, solve, for $0 \leqslant \theta < 180°$

$$\sqrt{2}\sin(2\theta) = \cos(2\theta - 105°)$$

(4)

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Question 6

8 marks | Rational Algebra

Main topic: Functions, Composition & Inverses

6. The functions f and g are defined by

$$\mathrm{f}(x) = 6 - \frac{21}{2x+3} \qquad x \geqslant 0$$

$$\mathrm{g}(x) = x^2 + 5 \qquad x \in \mathbb{R}$$

(a) Find $\mathrm{gf}(2)$

(2)

(b) Find f^{-1}

(3)

(c) Solve the equation

$$\mathrm{gg}(x) = 126$$

(3)

Question 7

10 marks | Rational Algebra

Main topic: Graph Transformations

7.

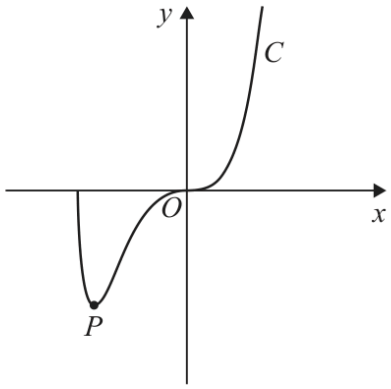


Figure 3

The curve C has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x^3\sqrt{4x+7} \qquad x \geqslant -\frac{7}{4}$$

(a) Show that

$$\mathrm{f}'(x) = \frac{kx^2(2x+3)}{\sqrt{4x+7}}$$

where k is a constant to be found.

(4)

The point P, shown in Figure 3, is the minimum turning point on C.

(b) Find the coordinates of P.

(2)

(c) Hence find the range of the function g defined by

$$\mathrm{g}(x) = -4\,\mathrm{f}(x) \qquad x \geqslant -\frac{7}{4}$$

(2)

The point Q with coordinates $\left(\frac{1}{2}, \frac{3}{8}\right)$ lies on C.

(d) Find the coordinates of the point to which Q is mapped when C is transformed to the curve with equation

$$y = 40\,\mathrm{f}\left(x - \frac{3}{2}\right) - 8$$

(2)

Question 9

14 marks | Rational Algebra | Main topic: Integration

9.

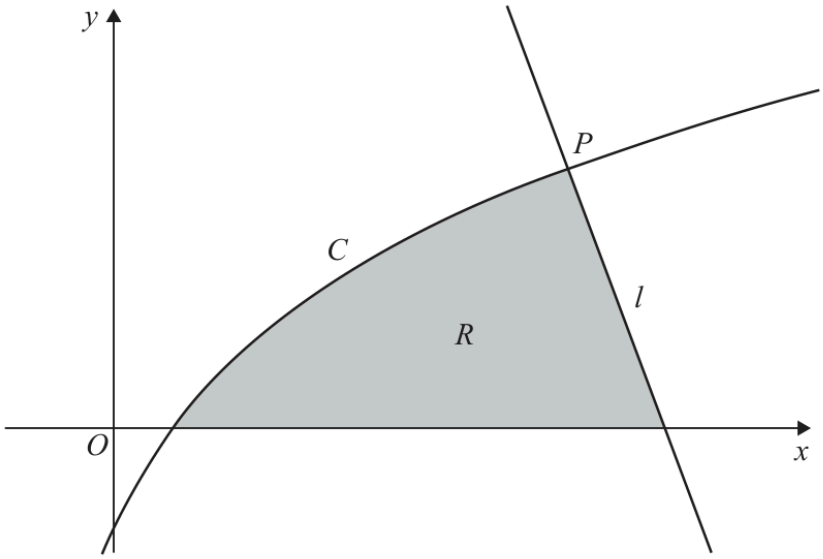


Figure 5

Figure 5 shows a sketch of part of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{6x^2 + 4x - 2}{2x + 1} \qquad x > -\frac{1}{2}$$

(a) Find $\mathrm{f}'(x)$, giving the answer in simplest form.

(3)

The line l is the normal to C at the point $P(2, 6)$

(b) Show that an equation for l is

$$16y + 5x = 106$$

(3)

(c) Write $\mathrm{f}(x)$ in the form $Ax + B + \frac{D}{2x + 1}$ where A, B and D are constants.

(3)

The region R, shown shaded in Figure 5, is bounded by C, l and the x-axis.

(d) Use algebraic integration to find the exact area of R, giving your answer in the form $P + Q\ln 3$, where P and Q are rational constants.

(*Solutions based entirely on calculator technology are not acceptable.*)

(5)

Question 3

6 marks | Rational Algebra

Main topic: Applications of Differentiation

3. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

A curve has equation

$$y = \frac{4x + 1}{(x + 3)^2} \qquad x \neq -3 \qquad x \in \mathbb{R}$$

Use calculus to find the range of values of x for which y is increasing.

(6)

Question 4

9 marks | Rational Algebra | Main topic: Integration

4. Given that

$$\frac{4x^3 + 2x^2 + 3x + 8}{x^2 + 4} \equiv Ax + B + \frac{Cx + D}{x^2 + 4}$$

(a) (i) find the values of the constants A, B and C

(ii) show that $D = 0$

(4)

(b) Hence, using algebraic integration, find

$$\int_1^4 \frac{4x^3 + 2x^2 + 3x + 8}{x^2 + 4} \, \mathrm{d}x$$

giving your answer in the form $p + q \ln 2$, where p and q are integers.

(5)

Question 6

8 marks | Rational Algebra

Main topic: Functions, Composition & Inverses

6. The function f is defined by

$$\mathrm{f}(x) = \frac{4x + 3}{x - 2} \qquad x \neq 2$$

(a) Find f^{-1}

(3)

(b) Show that

$$\mathrm{ff}(x) = \frac{ax + b}{cx + d}$$

where a, b, c and d are integers to be found.

(3)

The point $P(3, 15)$ lies on the curve with equation $y = \mathrm{f}(x)$.

(c) Find the point to which P is mapped when $y = \mathrm{f}(x)$ is transformed to the curve with equation $y = 2\mathrm{f}(3x) + 8$

(2)

Question 1

8 marks | Rational Algebra

Main topic: Functions, Composition & Inverses

1: **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The functions f and g are defined by

$$f(x) = \frac{2x}{3x+1} \qquad x \in \mathbb{R} \quad x \geqslant 0$$

$$g(x) = 4 - x^2 \qquad x \in \mathbb{R} \quad x \geqslant 0$$

(a) Find the value of $gf(1)$

(2)

(b) Find the range of f

(2)

(c) Find $f^{-1}(x)$

(2)

(d) Solve $f^{-1}(x) = f(x)$

(2)

Question 1

9 marks | Rational Algebra

Main topic: Functions, Composition & Inverses

1. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The functions f and g are defined by

$$\mathrm{f}(x) = \ln(x^2 + 3) \qquad x \in \mathbb{R}$$

$$\mathrm{g}(x) = \frac{3 + 5x}{x + 2} \qquad x \in \mathbb{R} \quad x > -2$$

(a) State the range of f

(1)

(b) Find g^{-1}

(3)

(c) Find $\mathrm{fg}(0)$

(2)

(d) Find the exact value of a for which

$$\mathrm{g}\left(\mathrm{e}^{2a}\right) = \mathrm{f}\left(\sqrt{\mathrm{e}^4 - 3}\right)$$

(3)

Question 2

7 marks | Rational Algebra | Main topic: Numerical Methods

2.

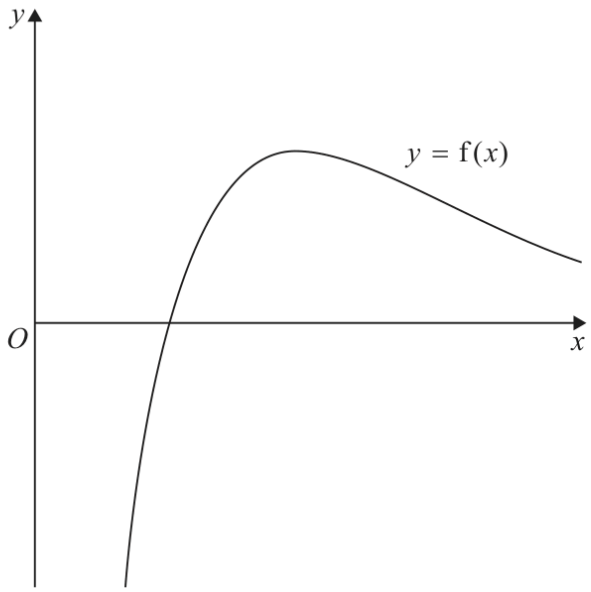


Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = f(x)$ where

$$f(x) = \frac{2x^2 + 3x - 4}{e^x} - \frac{1}{x^2} \qquad x \in \mathbb{R} \quad x \neq 0$$

(a) Show that $f(x) = 0$ has a root α in the interval $[1, 2]$

(2)

(b) Show that the equation $f(x) = 0$ can be written in the form

$$x = \sqrt[3]{\frac{e^x + 4x^2}{2x + 3}}$$

(2)

Using the iteration formula

$$x_{n+1} = \sqrt[3]{\frac{e^{x_n} + 4x_n^{\,2}}{2x_n + 3}} \qquad \text{with } x_1 = 1$$

find, to 4 decimal places,

(c) (i) the value of x_3

(ii) the value of α

(3)

Question 5

8 marks | Rational Algebra

Main topic: Log Graphs & Exponential Models

5. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The number of squirrels in a forest is being studied.

The number of squirrels, N, in the forest, t years after the start of the study, is modelled by the equation

$$N = \frac{4000e^{0.1t}}{19 + e^{0.2t}} \qquad t \geqslant 0$$

Use the equation of the model to answer parts (a), (b), (c) and (d).

(a) Find the number of squirrels in the forest at the start of the study.

(1)

(b) Find $\frac{dN}{dt}$

(2)

The number of squirrels in the forest is at a maximum when $t = T$

Using the answer to part (b),

(c) show that $e^{0.2T} = A$, where A is a constant to be found.

(2)

(d) Hence find the maximum number of squirrels in the forest.

Show your working and give your answer to the nearest whole number.

(3)

Question 6

7 marks | Rational Algebra | Main topic: Integration

6. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(i) Given $y = \ln(2x^2 + 5)$ find $\dfrac{dy}{dx}$

(2)

(ii) A curve C has equation $y = f(x)$ where

$$f(x) = \frac{21x}{3x^2 + k} \qquad x \in \mathbb{R}$$

where k is a positive integer, $k > 1$

Given that

$$\int_1^k f(x)\,dx < 7\ln 8$$

find the greatest possible value of k

(5)

Question 7

7 marks | Rational Algebra

Main topic: Applications of Differentiation

7. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

A curve C has equation

$$y = 2xe^{x^2+(3k-2)x}$$

where k is a constant.

Given that C has two distinct turning points, find the range of possible values of k.

(7)

Question 8

13 marks | Rational Algebra | Main topic: Modulus

8.

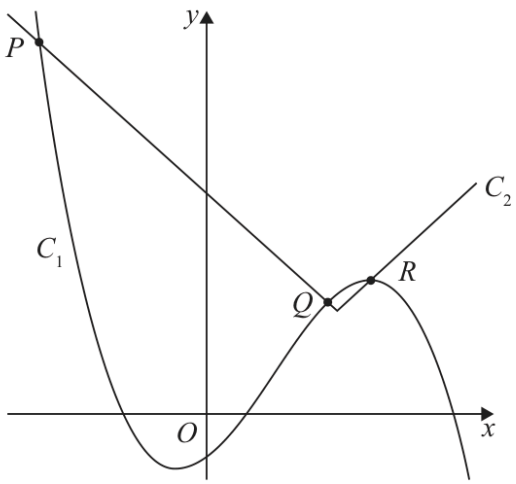


Figure 2

Figure 2 shows a sketch of the graph C_1 with equation

$$y = -2x^3 + 5x^2 + 4x - 3$$

and a sketch of the graph C_2 with equation

$$y = a + |5x + b|$$

where a and b are constants.

Graphs C_1 and C_2 intersect at point P, point Q and point R, as shown in Figure 2.

Given that P has coordinates $(-2, 25)$

(a) show that

$$a = 15 + b$$

(2)

Given also that R has coordinates $(2, 9)$

(b) find the value of a and the value of b

(3)

Using the answer to part (b),

(c) state the coordinates of the vertex of C_2

(2)

(d) Find, using algebra, the coordinates of Q. Show each stage of your working.

(*Solutions relying on calculator technology are not acceptable.*)

(6)

Question 2

6 marks | Rational Algebra

Main topic: Differentiation Techniques

2. (i) Given $y = x^2 \ln 2x$, find in simplest form, $\dfrac{\mathrm{d}y}{\mathrm{d}x}$

(2)

(ii) Given

$$\mathrm{f}(x) = \frac{\mathrm{e}^{2x}}{(4x+3)^2} \qquad x > -\frac{3}{4}$$

show that

$$\mathrm{f}'(x) = \frac{2\mathrm{e}^{2x}(ax+b)}{(4x+3)^3}$$

where a and b are constants to be found.

(4)

Question 3

10 marks | Rational Algebra | Main topic: Integration

3. (i) Find, in simplest form,

$$\int 2(3x+5)^7 \, dx$$

(2)

(ii) $$f(x) = \frac{4x^3 + x^2 - 7x + 14}{(x-1)(x+2)} \qquad x > 1$$

(a) Write $f(x)$ in the form

$$ax + b + \frac{c}{x-1}$$

where a, b and c are constants to be found.

(4)

(b) Hence, using algebraic integration, find

$$\int_3^9 f(x) \, dx$$

writing the answer in simplest form.

(4)

Question 5

8 marks | Rational Algebra

Main topic: Functions, Composition & Inverses

5. The function f is defined by

$$f : x \to \frac{2x+16}{x-4} \qquad x \neq 4$$

(a) Find $ff(x)$, giving your answer as a single fraction in its simplest form.

(3)

(b) Find $f^{-1}(12)$

(2)

The function g is defined by

$$g : x \to \ln x \qquad x > 0 \quad x \neq e^4$$

(c) Find the exact values of a such that

$$fg(a) = g(a)$$

(3)

Question 6

7 marks | Rational Algebra

Main topic: Trig Identities & Exact Values

6. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(a) Show that the equation

$$2\cos(\theta - 60°) = 3\sin\theta$$

can be written in the form

$$\tan\theta = \frac{1}{6}\left(3 + \sqrt{3}\right)$$

(4)

(b) Hence solve, for $0 \leqslant x < 180°$, the equation

$$2\cos(2x - 70°) = 3\sin(2x - 10°)$$

giving your solutions to one decimal place.

(3)

Question 3

8 marks | Rational Algebra

Main topic: Differentiation Techniques

3. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The curve C has equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{4x^2 - 14x + 15}{x - 3} \qquad x < 3$$

(a) Find $\mathrm{f}'(x)$, giving the answer in the form

$$\frac{ax^2 + bx + c}{(x-3)^2}$$

where a, b and c are constants.

(4)

(b) Hence find the coordinates of the turning point of C.

(2)

The point $P\left(\frac{5}{2}, -10\right)$ lies on C.

(c) Find the coordinates of the point to which P is mapped, when the curve with equation $y = \mathrm{f}(x)$ is transformed to the curve with equation $y = \mathrm{f}(2x) + 3$

(2)

Question 7

12 marks | Rational Algebra

Main topic: Log Graphs & Exponential Models

7. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The selling price of a car is monitored over a 15-year period.

The selling price P_C of the car, in thousands of pounds, t years after monitoring began, is modelled by the equation

$$P_C = 7 + 16e^{-0.1t} \qquad 0 \leqslant t \leqslant 15$$

Given that when $t = T$ the selling price of the car is £15 000

(a) find, according to the model, the value of T. Give the answer to 2 decimal places.

(3)

(b) Find the value of $\frac{dP_C}{dt}$ when $t = 5$. Give the answer to 2 decimal places.

(2)

The selling price of a van is monitored over the same 15-year period.

The selling price P_V of the van, in thousands of pounds, t years after monitoring began, is modelled by the equation

$$P_V = 4 + me^{-0.2t} \qquad 0 \leqslant t \leqslant 15$$

where m is a constant.

Use the equations of these models to answer parts (c) and (d).

Given that $P_V = 36$ when $t = 0$

(c) (i) state the value of m

(ii) find the difference between the initial selling price of the car and the initial selling price of the van.

(3)

When $t = \alpha$, the selling price of the car is equal to the selling price of the van.

(d) Find, using algebra and showing your working, the value of α, giving the answer to 3 significant figures.

(4)

Question 8

9 marks | Rational Algebra

Main topic: Trig Identities & Exact Values

8. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(a) Show that

$$\frac{\sec\theta\,\text{cosec}\theta}{4\text{cosec}\,4\theta} \equiv \cos 2\theta$$

(2)

(b) Hence show that the equation

$$4(5\cos^2\theta - 9)\,\text{cosec}\,4\theta = 3(1+\tan^2 2\theta)\sec\theta\,\text{cosec}\,\theta$$

can be written as

$$a\cos^2 2\theta + b\cos 2\theta + c = 0$$

where a, b and c are integers to be found.

(3)

(c) Hence, solve for $0 < x < 180°$, $x \neq (45n)°$, $n \in \mathbb{N}$, the equation

$$4(5\cos^2 x - 9)\,\text{cosec}\,4x = 3(1+\tan^2 2x)\sec x\,\text{cosec}\,x$$

giving the answers to one decimal place.

(4)

TOPIC

Functions, Composition & Inverses

Every past-paper question on this topic, January 2020 to June 2026.

Question 2

8 marks | Functions, Composition & Inverses

2. The function f and the function g are defined by

$$f(x) = \frac{12}{x+1} \qquad x > 0, x \in \mathbb{R}$$

$$g(x) = \frac{5}{2}\ln x \qquad x > 0, x \in \mathbb{R}$$

(a) Find, in simplest form, the value of $fg(e^2)$

(2)

(b) Find f^{-1}

(3)

(c) Hence, or otherwise, find all real solutions of the equation

$$f^{-1}(x) = f(x)$$

(3)

Question 4

10 marks | Functions, Composition & Inverses

4. The functions f and g are defined by

$$f(x) = \frac{4x + 6}{x - 5} \qquad x \in \mathbb{R},\ x \neq 5$$

$$g(x) = 5 - 2x^2 \qquad x \in \mathbb{R},\ x \leqslant 0$$

(a) Solve the equation

$$fg(x) = 3$$

(4)

(b) Find f^{-1}

(3)

(c) Sketch and label, on the same axes, the curve with equation $y = g(x)$ and the curve with equation $y = g^{-1}(x)$. Show on your sketch the coordinates of the points where each curve meets or cuts the coordinate axes.

(3)

Question 6

11 marks | Functions, Composition & Inverses

6. The function f is defined by

$$f(x) = \frac{5x - 3}{x - 4} \qquad x > 4$$

(a) Show, by using calculus, that f is a decreasing function.

(3)

(b) Find f^{-1}

(3)

(c) (i) Show that $ff(x) = \frac{ax + b}{x + c}$ where a, b and c are constants to be found.

(ii) Deduce the range of ff.

(5)

Question 2

9 marks | Functions, Composition & Inverses

2. The functions f and g are defined by

$$f(x) = \frac{5 - x}{3x + 2} \qquad x \in \mathbb{R}, x \neq -\frac{2}{3}$$

$$g(x) = 2x - 7 \qquad x \in \mathbb{R}$$

(a) Find the value of $fg(5)$

(2)

(b) Find f^{-1}

(3)

(c) Solve the equation

$$f\left(\frac{1}{a}\right) = g(a + 3)$$

(4)

Question 2

7 marks | Functions, Composition & Inverses

2. The functions f and g are defined by

$$f(x) = 5 - \frac{4}{3x + 2} \qquad x \geqslant 0$$

$$g(x) = \left| 4\sin\left(\frac{x}{3} + \frac{\pi}{6}\right) \right| \qquad x \in \mathbb{R}$$

(a) Find the range of f

(2)

(b) (i) Find $f^{-1}(x)$

(ii) Write down the domain of f^{-1}

(3)

(c) Find $fg(-\pi)$

(2)

WMA13/01 JANUARY 2023

Question 1

6 marks | Functions, Composition & Inverses

1. The functions f and g are defined by

$$\mathrm{f}(x) = 9 - x^2 \qquad x \in \mathbb{R} \quad x \geqslant 0$$

$$\mathrm{g}(x) = \frac{3}{2x+1} \qquad x \in \mathbb{R} \quad x \geqslant 0$$

(a) Write down the range of f

(1)

(b) Find the value of $\mathrm{fg}(1.5)$

(2)

(c) Find g^{-1}

(3)

Question 4

6 marks | Functions, Composition & Inverses

4. The function f is defined by

$$f(x) = 2x^2 - 5 \qquad x \geqslant 0 \quad x \in \mathbb{R}$$

(a) State the range of f

(1)

On the following page there is a diagram, labelled Diagram 1, which shows a sketch of the curve with equation $y = f(x)$.

(b) On Diagram 1, sketch the curve with equation $y = f^{-1}(x)$.

(2)

The curve with equation $y = f(x)$ meets the curve with equation $y = f^{-1}(x)$ at the point P

Using algebra and showing your working,

(c) find the exact x coordinate of P

(3)

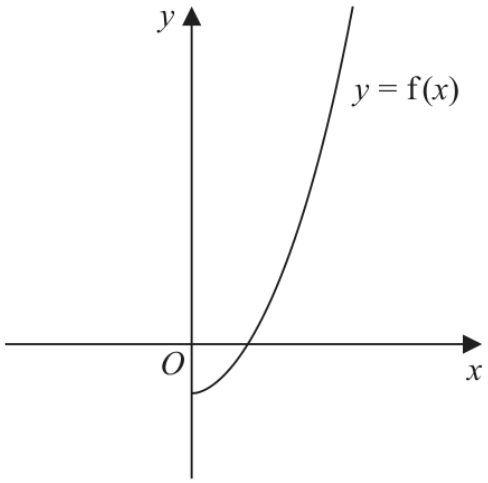


Diagram 1

Question 2

8 marks | Functions, Composition & Inverses

2. The function f is defined by

$$f(x) = \frac{x+3}{x-4} \qquad x \in \mathbb{R}, x \neq 4$$

(a) Find $ff(6)$

(2)

(b) Find f^{-1}

(3)

The function g is defined by

$$g(x) = x^2 + 5 \qquad x \in \mathbb{R}, x > 0$$

(c) Find the exact value of a for which

$$gf(a) = 7$$

(3)

Question 7

10 marks | Functions, Composition & Inverses

7.

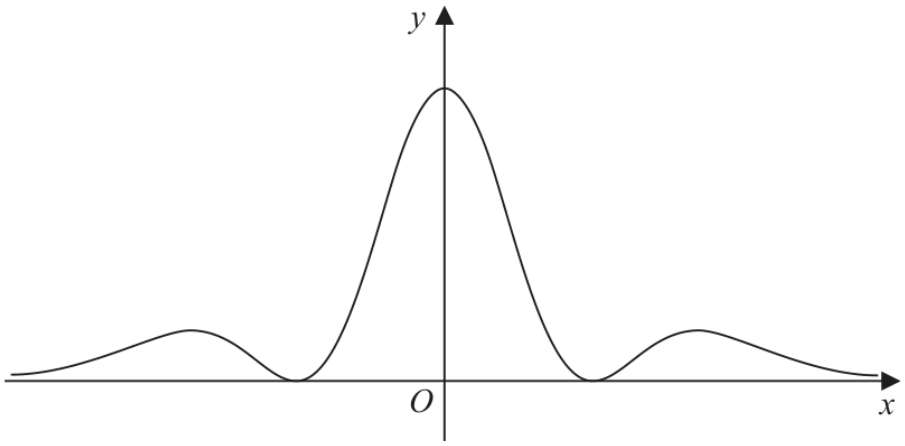


Figure 1

Figure 1 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \mathrm{e}^{-x^2}(2x^2 - 3)^2$$

(a) Find the range of f

(2)

(b) Show that

$$\mathrm{f}'(x) = 2x(2x^2 - 3)\mathrm{e}^{-x^2}(A - Bx^2)$$

where A and B are constants to be found.

(4)

Given that the line $y = k$, where k is a constant, $k > 0$, intersects the curve at exactly two distinct points,

(c) find the exact range of values of k

(4)

Question 5

10 marks | Functions, Composition & Inverses

5. The functions f and g are defined by

$$f(x) = 2 + 5\ln x \qquad x > 0$$

$$g(x) = \frac{6x - 2}{2x + 1} \qquad x > \frac{1}{3}$$

(a) Find $f^{-1}(22)$

(2)

(b) Use differentiation to prove that g is an increasing function.

(3)

(c) Find g^{-1}

(3)

(d) Find the range of fg

(2)

Question 6

8 marks | Functions, Composition & Inverses

6. The functions f and g are defined by

$$\mathrm{f}(x) = 6 - \frac{21}{2x+3} \qquad x \geqslant 0$$

$$\mathrm{g}(x) = x^2 + 5 \qquad x \in \mathbb{R}$$

(a) Find $\mathrm{gf}(2)$

(2)

(b) Find f^{-1}

(3)

(c) Solve the equation

$$\mathrm{gg}(x) = 126$$

(3)

Question 6

8 marks | Functions, Composition & Inverses

6. The function f is defined by

$$f(x) = \frac{4x + 3}{x - 2} \qquad x \neq 2$$

(a) Find f^{-1}

(3)

(b) Show that

$$ff(x) = \frac{ax + b}{cx + d}$$

where a, b, c and d are integers to be found.

(3)

The point $P(3, 15)$ lies on the curve with equation $y = f(x)$.

(c) Find the point to which P is mapped when $y = f(x)$ is transformed to the curve with equation $y = 2f(3x) + 8$

(2)

Question 1

8 marks | Functions, Composition & Inverses

1: **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The functions f and g are defined by

$$f(x) = \frac{2x}{3x+1} \qquad x \in \mathbb{R} \quad x \geqslant 0$$

$$g(x) = 4 - x^2 \qquad x \in \mathbb{R} \quad x \geqslant 0$$

(a) Find the value of $gf(1)$

(2)

(b) Find the range of f

(2)

(c) Find $f^{-1}(x)$

(2)

(d) Solve $f^{-1}(x) = f(x)$

(2)

Question 1

9 marks | Functions, Composition & Inverses

1. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The functions f and g are defined by

$$\mathrm{f}(x) = \ln(x^2 + 3) \qquad x \in \mathbb{R}$$

$$\mathrm{g}(x) = \frac{3 + 5x}{x + 2} \qquad x \in \mathbb{R} \quad x > -2$$

(a) State the range of f

(1)

(b) Find g^{-1}

(3)

(c) Find $\mathrm{fg}(0)$

(2)

(d) Find the exact value of a for which

$$\mathrm{g}\left(\mathrm{e}^{2a}\right) = \mathrm{f}\left(\sqrt{\mathrm{e}^4 - 3}\right)$$

(3)

Question 5

8 marks | Functions, Composition & Inverses

5. The function f is defined by

$$\mathrm{f} : x \to \frac{2x+16}{x-4} \qquad x \neq 4$$

(a) Find $\mathrm{ff}(x)$, giving your answer as a single fraction in its simplest form.

(3)

(b) Find $\mathrm{f}^{-1}(12)$

(2)

The function g is defined by

$$\mathrm{g} : x \to \ln x \qquad x > 0 \quad x \neq \mathrm{e}^4$$

(c) Find the exact values of a such that

$$\mathrm{fg}(a) = \mathrm{g}(a)$$

(3)

Question 2

9 marks | Functions, Composition & Inverses

2. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The functions f and g are defined by

$$f(x) = 3 + 4e^{2x} \qquad x \in \mathbb{R}$$

$$g(x) = \sqrt{x + 5} \qquad x \in \mathbb{R} \quad x > -5$$

(a) State the range of f

(1)

(b) Find g^{-1}

(3)

(c) Find $gf\left(\ln\sqrt{7}\right)$

(2)

(d) Find the value of a, $a > 0$, for which

$$fg(a^2 - 5) = g(59)$$

giving the answer in the form $\ln p$ where p is an irrational constant.

(3)

Question 9

7 marks | Functions, Composition & Inverses | Main topic: Trig Equations & R-Form

9. $$f(\theta) = 5\cos\theta - 4\sin\theta \qquad \theta \in \mathbb{R}$$

(a) Express $f(\theta)$ in the form $R\cos(\theta + \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

The curve with equation $y = \cos\theta$ is transformed onto the curve with equation $y = f(\theta)$ by a sequence of two transformations.

Given that the first transformation is a stretch and the second a translation,

(b) (i) describe fully the transformation that is a stretch,

(ii) describe fully the transformation that is a translation.

(2)

Given

$$g(\theta) = \frac{90}{4 + (f(\theta))^2} \qquad \theta \in \mathbb{R}$$

(c) find the range of g.

(2)

Question 3

7 marks | Functions, Composition & Inverses | Main topic: Differentiation Techniques

3.

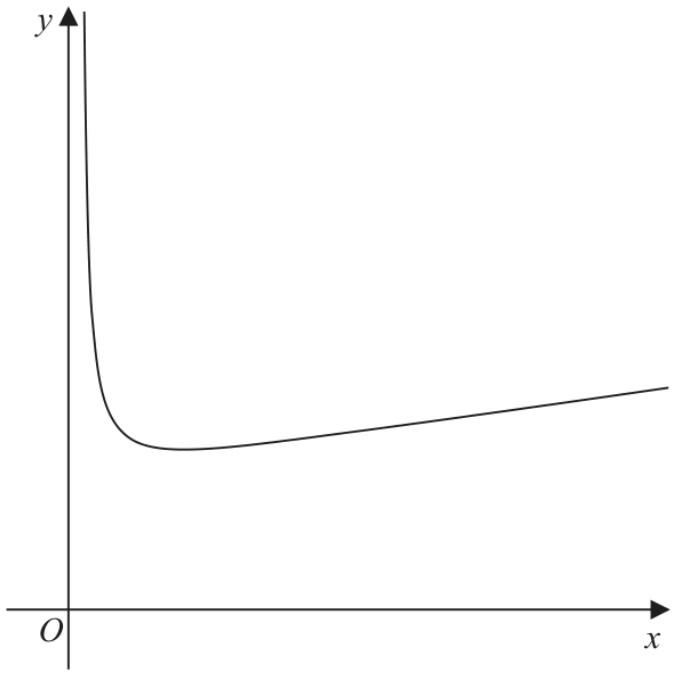


Figure 1

Figure 1 shows a sketch of a curve with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{2x + 3}{\sqrt{4x - 1}} \qquad x > \frac{1}{4}$$

(a) Find, in simplest form, $\mathrm{f}'(x)$.

(4)

(b) Hence find the range of f.

(3)

Question 4

8 marks | Functions, Composition & Inverses | Main topic: Modulus

4.

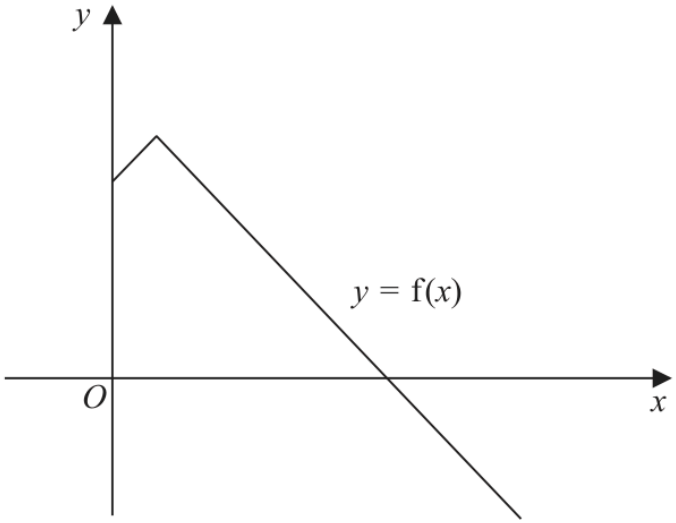


Figure 2

Figure 2 shows a sketch of part of the graph with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 21 - 2|2 - x| \qquad x \geqslant 0$$

(a) Find $\mathrm{ff}(6)$

(2)

(b) Solve the equation $\mathrm{f}(x) = 5x$

(2)

Given that the equation $\mathrm{f}(x) = k$, where k is a constant, has exactly two roots,

(c) state the set of possible values of k.

(2)

The graph with equation $y = \mathrm{f}(x)$ is transformed onto the graph with equation $y = a\mathrm{f}(x - b)$

The vertex of the graph with equation $y = a\mathrm{f}(x - b)$ is $(6, 3)$.

Given that a and b are constants,

(d) find the value of a and the value of b.

(2)

Question 7

9 marks | Functions, Composition & Inverses | Main topic: Trig Equations & R-Form

7. (a) Express $\cos x + 4\sin x$ in the form $R\cos(x - \alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

A scientist is studying the behaviour of seabirds in a colony.

She models the height above sea level, H metres, of one of the birds in the colony by the equation

$$H = \frac{24}{3 + \cos\left(\frac{1}{2}t\right) + 4\sin\left(\frac{1}{2}t\right)} \qquad 0 \leqslant t \leqslant 6.5$$

where t seconds is the time after it leaves the nest.

Find, according to the model,

(b) the minimum height of the seabird above sea level, giving your answer to the nearest cm,

(2)

(c) the value of t, to 2 decimal places, when $H = 10$

(4)

Question 3

8 marks | Functions, Composition & Inverses | Main topic: Rational Algebra

3.

$$f(x) = 3 - \frac{x-2}{x+1} + \frac{5x+26}{2x^2-3x-5} \qquad x > 4$$

(a) Show that

$$f(x) = \frac{ax+b}{cx+d} \qquad x > 4$$

where a, b, c and d are integers to be found.

(4)

(b) Hence find $f^{-1}(x)$

(2)

(c) Find the domain of f^{-1}

(2)

Question 9

8 marks | Functions, Composition & Inverses | Main topic: Trig Equations & R-Form

9. (a) Express $12\sin x - 5\cos x$ in the form $R\sin(x - \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the exact value of R and give the value of α in radians, to 3 decimal places.

(3)

The function g is defined by

$$g(\theta) = 10 + 12\sin\left(2\theta - \frac{\pi}{6}\right) - 5\cos\left(2\theta - \frac{\pi}{6}\right) \qquad \theta > 0$$

Find

(b) (i) the minimum value of $g(\theta)$

(ii) the smallest value of θ at which the minimum value occurs.

(3)

The function h is defined by

$$h(\beta) = 10 - (12\sin\beta - 5\cos\beta)^2$$

(c) Find the range of h.

(2)

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Question 1

9 marks | Functions, Composition & Inverses | Main topic: Rational Algebra

1. The function f is defined by

$$\mathrm{f}(x) = \frac{5x}{x^2 + 7x + 12} + \frac{5x}{x + 4} \qquad x > 0$$

(a) Show that $\mathrm{f}(x) = \dfrac{5x}{x + 3}$

(3)

(b) Find f^{-1}

(3)

(c) (i) Find, in simplest form, $\mathrm{f}'(x)$.

(ii) Hence, state whether f is an increasing or a decreasing function, giving a reason for your answer.

(3)

Question 2

10 marks | Functions, Composition & Inverses | Main topic: Modulus

2.

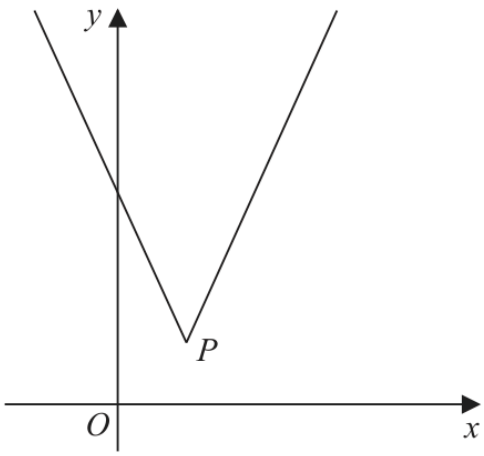


Figure 1

Figure 1 shows a sketch of part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = |3x - 13| + 5 \qquad x \in \mathbb{R}$$

The vertex of the graph is at point P, as shown in Figure 1.

(a) State the coordinates of P.

(2)

(b) (i) State the range of f.

(ii) Find the value of ff(4)

(2)

(c) Solve, using algebra and showing your working,

$$16 - 2x > |3x - 13| + 5$$

(4)

The graph with equation $y = \mathrm{f}(x)$ is transformed onto the graph with equation $y = a\mathrm{f}(x + b)$

The vertex of the graph with equation $y = a\mathrm{f}(x + b)$ is (4, 20)

Given that a and b are constants,

(d) find the value of a and the value of b.

(2)

Question 8

7 marks | Functions, Composition & Inverses | Main topic: Differentiation Techniques

8. A curve C has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \arcsin\left(\frac{1}{2}x\right) \qquad -2 \leqslant x \leqslant 2 \qquad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

(a) Sketch C.

(1)

(b) Given $x = 2\sin y$, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{A - x^2}}$$

where A is a constant to be found.

(3)

The point P lies on C and has y coordinate $\frac{\pi}{4}$

(c) Find the equation of the tangent to C at P. Write your answer in the form $y = mx + c$, where m and c are constants to be found.

(3)

Question 10

7 marks | Functions, Composition & Inverses | Main topic: Differentiation Techniques

10.

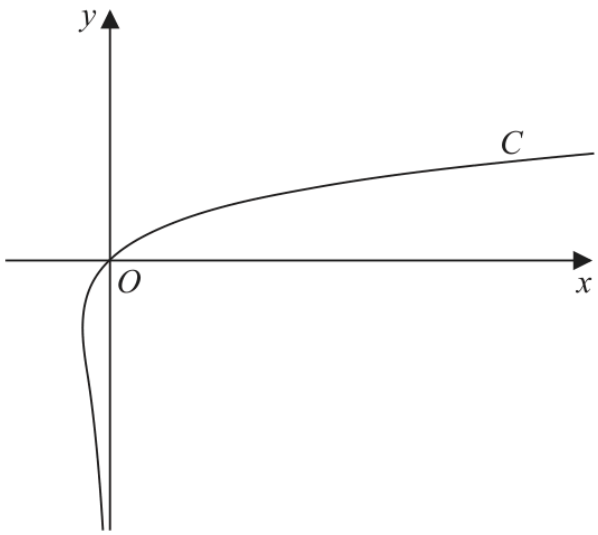


Figure 3

Figure 3 shows a sketch of the curve C with equation

$$x = ye^{2y} \qquad y \in \mathbb{R}$$

(a) Show that

$$\frac{dy}{dx} = \frac{y}{x(1+2y)}$$

(4)

Given that the straight line with equation $x = k$, where k is a constant, cuts C at exactly two points,

(b) find the range of possible values for k.

(3)

Question 5

8 marks | Functions, Composition & Inverses | Main topic: Modulus

5.

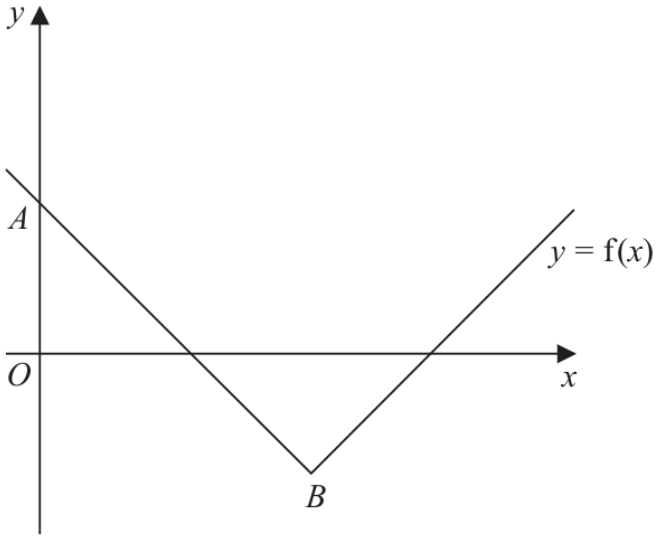


Figure 2

Figure 2 shows part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = |kx - 9| - 2 \qquad x \in \mathbb{R}$$

and k is a positive constant.

The graph intersects the y-axis at the point A and has a minimum point at B as shown.

(a) (i) Find the y coordinate of A

(ii) Find, in terms of k, the x coordinate of B

(2)

(b) Find, in terms of k, the range of values of x that satisfy the inequality

$$|kx - 9| - 2 < 0$$

(3)

Given that the line $y = 3 - 2x$ intersects the graph $y = \mathrm{f}(x)$ at two distinct points,

(c) find the range of possible values of k

(3)

Question 6

10 marks | Functions, Composition & Inverses | Main topic: Applications of Differentiation

6.

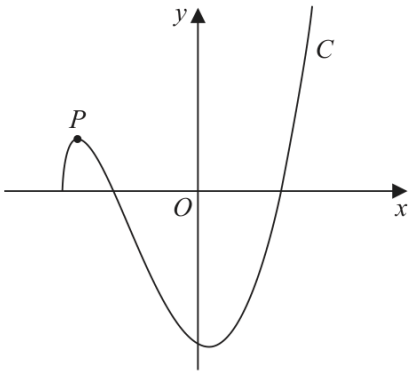


Figure 3

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

The function f is defined by

$$f(x) = 5(x^2 - 2)(4x + 9)^{\frac{1}{2}} \qquad x \geqslant -\frac{9}{4}$$

(a) Show that

$$f'(x) = \frac{k(5x^2 + 9x - 2)}{(4x + 9)^{\frac{1}{2}}}$$

where k is an integer to be found.

(4)

(b) Hence, find the values of x for which $f'(x) = 0$

(1)

Figure 3 shows a sketch of the curve C with equation $y = f(x)$.

The curve has a local maximum at the point P

(c) Find the exact coordinates of P

(2)

The function g is defined by

$$g(x) = 2f(x) + 4 \qquad -\frac{9}{4} \leqslant x \leqslant 0$$

(d) Find the range of g

(3)

Question 3

7 marks | Functions, Composition & Inverses | Main topic: Applications of Differentiation

3. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

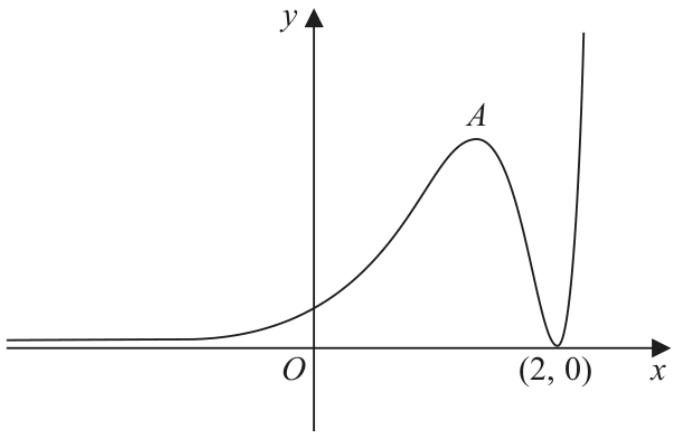


Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = (x - 2)^2 \mathrm{e}^{3x} \qquad x \in \mathbb{R}$$

The curve has a maximum turning point at A and a minimum turning point at $(2, 0)$

(a) Use calculus to find the exact coordinates of A.

(5)

Given that the equation $\mathrm{f}(x) = k$, where k is a constant, has **at least** two distinct roots,

(b) state the range of possible values for k.

(2)

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Question 4

5 marks | Functions, Composition & Inverses | Main topic: Differentiation Techniques

4. $y = \log_{10}(2x + 1)$

(a) Express x in terms of y.

(2)

(b) Hence, giving your answer in terms of x, find $\frac{dy}{dx}$

(3)

Question 7

12 marks | Functions, Composition & Inverses | Main topic: Modulus

7.

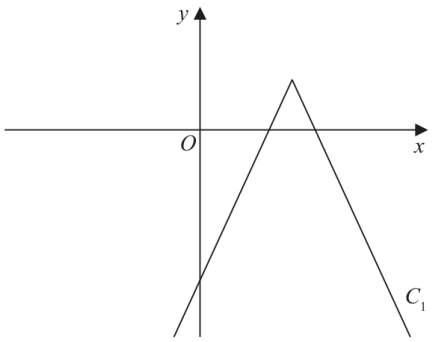


Figure 3

Figure 3 shows a sketch of the graph of C_1 with equation

$$y = 5 - |3x - 22|$$

(a) Write down the coordinates of

(i) the vertex of C_1

(ii) the intersection of C_1 with the y-axis.

(2)

(b) Find the x coordinates of the intersections of C_1 with the x-axis.

(2)

Diagram 1, shown on page 21, is a copy of Figure 3.

(c) On Diagram 1, sketch the curve C_2 with equation

$$y = \frac{1}{9}x^2 - 9$$

Identify clearly the coordinates of any points of intersection of C_2 with the coordinate axes.

(3)

(d) Find the coordinates of the points of intersection of C_1 and C_2
(*Solutions relying entirely on calculator technology are not acceptable.*)

(5)

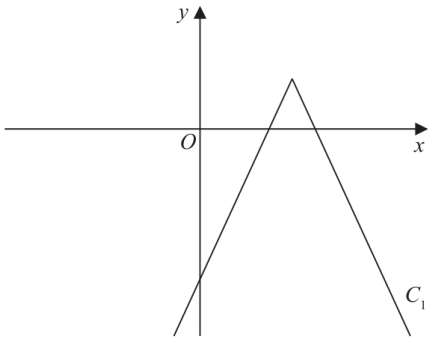


Diagram 1

Question 8

9 marks | Functions, Composition & Inverses | Main topic: Trig Equations & R-Form

8. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Express $8\sin x - 15\cos x$ in the form $R\sin(x-\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R, and give the value of α, in radians, to 4 significant figures.

(3)

$$f(x) = \frac{15}{41 + 16\sin x - 30\cos x} \qquad x > 0$$

(b) Find

(i) the minimum value of $f(x)$

(ii) the smallest value of x at which this minimum value occurs.

(4)

(c) State the y coordinate of the minimum points on the curve with equation

$$y = 2f(x) - 5 \qquad x > 0$$

(1)

(d) State the smallest value of x at which a maximum point occurs for the curve with equation

$$y = -f(2x) \qquad x > 0$$

(1)

Question 6

9 marks | Functions, Composition & Inverses | Main topic: Modulus

6.

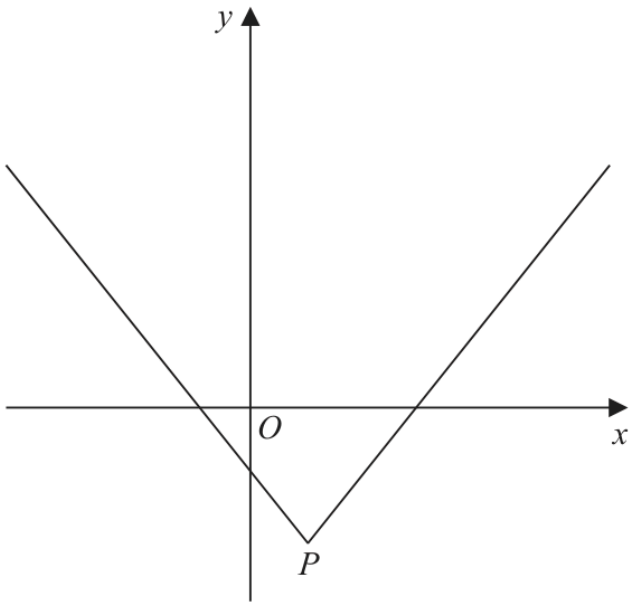


Figure 2

Figure 2 shows a sketch of the graph $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 3|x - 2| - 10$$

The vertex of the graph is at point P, shown in Figure 2.

(a) Find the coordinates of P

(2)

(b) Find $\mathrm{ff}(0)$

(2)

(c) Solve the inequality

$$3|x - 2| - 10 < 5x + 10$$

(2)

(d) Solve the equation

$$\mathrm{f}(|x|) = 0$$

(3)

Question 5

7 marks | Functions, Composition & Inverses | Main topic: Applications of Differentiation

5. The curve C has equation

$$y = \frac{\ln(x^2 + k)}{x^2 + k} \qquad x \in \mathbb{R}$$

where k is a positive constant.

(a) Show that

$$\frac{dy}{dx} = \frac{Ax(B - \ln(x^2 + k))}{(x^2 + k)^2}$$

where A and B are constants to be found.

(3)

Given that C has exactly three turning points,

(b) find the x coordinate of each of these points. Give your answer in terms of k where appropriate.

(3)

(c) find the upper limit to the value for k.

(1)

Question 9

9 marks | Functions, Composition & Inverses | Main topic: Exponentials & Logarithms

9. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

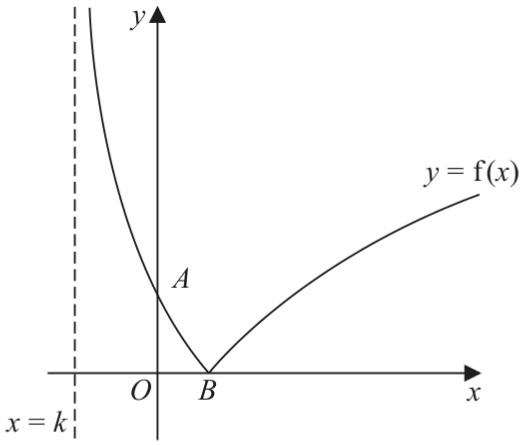


Figure 2

Figure 2 shows a sketch of the curve with equation

$$y = |2 - 4\ln(x + 1)| \qquad x > k$$

where k is a constant.

Given that the curve

- has an asymptote at $x = k$
- cuts the y-axis at point A
- meets the x-axis at point B

as shown in Figure 2,

(a) state the value of k

(1)

(b) (i) find the y coordinate of A

(ii) find the exact x coordinate of B

(3)

(c) Using algebra and showing your working, find the set of values of x such that

$$|2 - 4\ln(x + 1)| > 3$$

(5)

Question 10

9 marks | Functions, Composition & Inverses | Main topic: Differentiation Techniques

10. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

A curve C has equation

$$x = \sin^2 4y \qquad 0 \leqslant y \leqslant \frac{\pi}{8} \qquad 0 \leqslant x \leqslant 1$$

The point P with x coordinate $\frac{1}{4}$ lies on C

(a) Find the exact y coordinate of P

(2)

(b) Find $\frac{\mathrm{d}x}{\mathrm{d}y}$

(2)

(c) Hence show that $\frac{\mathrm{d}y}{\mathrm{d}x}$ can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{q + r(x + s)^2}}$$

where q, r and s are constants to be found.

(3)

Using the answer to part (c),

(d) (i) state the x coordinate of the point where the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ is a minimum,

(ii) state the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at this point.

(2)

Question 4

13 marks | Functions, Composition & Inverses | Main topic: Rational Algebra

4. The function f is defined by

$$f(x) = \frac{2x^2 - 32}{3x^2 + 7x - 20} + \frac{8}{3x - 5} \qquad x \in \mathbb{R} \quad x > 2$$

(a) Show that $f(x) = \dfrac{2x}{3x-5}$

(3)

(b) Show, using calculus, that f is a decreasing function.
You must make your reasoning clear.

(3)

The function g is defined by

$$g(x) = 3 + 2\ln x \qquad x \geqslant 1$$

(c) Find g^{-1}

(3)

(d) Find the exact value of a for which

$$gf(a) = 5$$

(4)

Question 8

11 marks | Functions, Composition & Inverses | Main topic: Modulus

8.

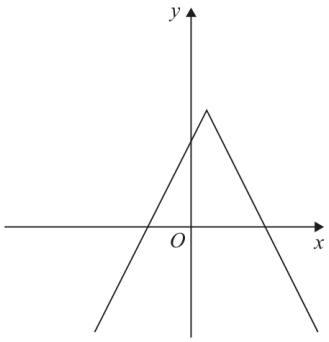


Figure 2

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

The graph shown in Figure 2 has equation

$$y = a - |2x - b|$$

where a and b are positive constants, $a > b$

(a) Find, giving your answer in terms of a and b,

(i) the coordinates of the maximum point of the graph,

(ii) the coordinates of the point of intersection of the graph with the y-axis,

(iii) the coordinates of the points of intersection of the graph with the x-axis.

(5)

On page 24 there is a copy of Figure 2 called Diagram 1.

(b) On Diagram 1, sketch the graph with equation

$$y = |x| - 1$$

(2)

Given that the graphs $y = |x| - 1$ and $y = a - |2x - b|$ intersect at $x = -3$ and $x = 5$

(c) find the value of a and the value of b

(4)

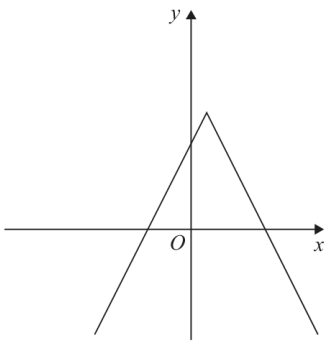


Diagram 1

Question 7

10 marks | Functions, Composition & Inverses | Main topic: Graph Transformations

7.

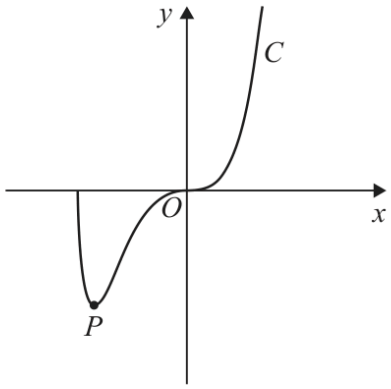


Figure 3

The curve C has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x^3\sqrt{4x+7} \qquad x \geqslant -\frac{7}{4}$$

(a) Show that

$$\mathrm{f}'(x) = \frac{kx^2(2x+3)}{\sqrt{4x+7}}$$

where k is a constant to be found.

(4)

The point P, shown in Figure 3, is the minimum turning point on C.

(b) Find the coordinates of P.

(2)

(c) Hence find the range of the function g defined by

$$\mathrm{g}(x) = -4\,\mathrm{f}(x) \qquad x \geqslant -\frac{7}{4}$$

(2)

The point Q with coordinates $\left(\frac{1}{2}, \frac{3}{8}\right)$ lies on C.

(d) Find the coordinates of the point to which Q is mapped when C is transformed to the curve with equation

$$y = 40\,\mathrm{f}\left(x - \frac{3}{2}\right) - 8$$

(2)

Question 9

8 marks | Functions, Composition & Inverses | Main topic: Applications of Differentiation

9.

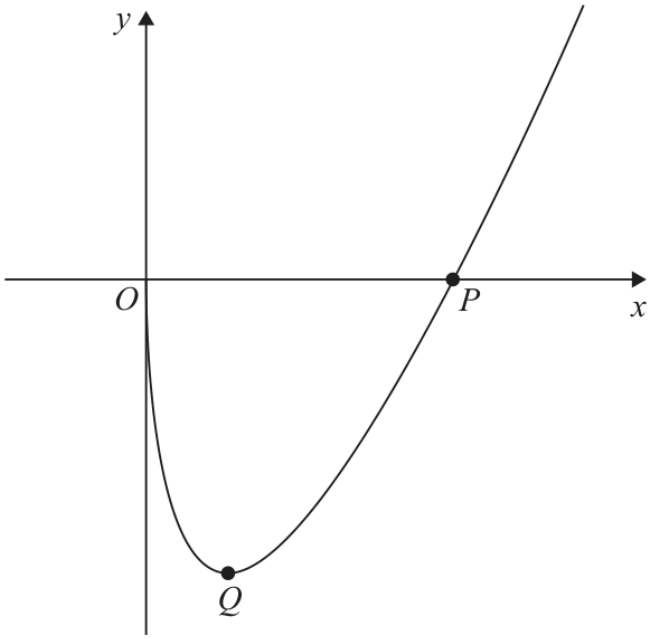


Figure 1

Figure 1 shows a sketch of part of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 6\sqrt{x}\ln(4x) \qquad x > 0$$

The curve cuts the x-axis at point P

(a) State the x coordinate of P

(1)

The point Q, shown in Figure 1, is the stationary point on C

(b) Use calculus to find the exact coordinates of Q

(5)

(c) Hence find the range of the function $\mathrm{g}(x)$ where

$$\mathrm{g}(x) = -2\,\mathrm{f}(x)$$

(2)

Question 2

7 marks | Functions, Composition & Inverses | Main topic: Trig Equations & R-Form

2: **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

$$f(x) = 7\cos x - 24\sin x$$

(a) Express $f(x)$ in the form $R\cos(x + \alpha)$ where R and α are constants, $R > 0$

and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

$$g(x) = \frac{5}{90 - 3f(2x)}$$

(b) Using the answer to part (a), find

(i) the minimum value of $g(x)$, giving your answer as a fully simplified fraction,

(ii) the smallest positive value of x for which this minimum value occurs, giving your answer to 3 decimal places.

(4)

Question 10

10 marks | Functions, Composition & Inverses | Main topic: Modulus

10:

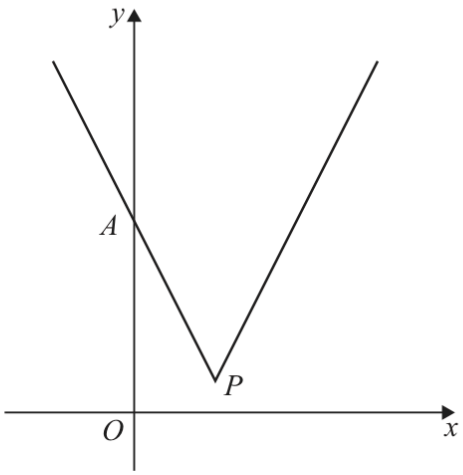


Figure 3

Figure 3 shows a sketch of part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = |kx - 10| + k \qquad x \in \mathbb{R}$$

and k is a positive constant.

The graph

- cuts the y-axis at the point A
- has a vertex at the point P

(a) Find, in simplest form in terms of k,

(i) the y coordinate of A

(ii) the coordinates of P

(3)

(b) Find, in terms of k, the range of values of x which satisfy

$$|kx - 10| + k \geqslant 2k$$

(3)

Given that the line with equation $y = 3x + 1$ intersects the graph of $y = \mathrm{f}(x)$ at 2 distinct points,

(c) find the range of values of k.

(4)

Question 4

6 marks | Functions, Composition & Inverses | Main topic: Trig Equations & R-Form

4. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(a) $$f(x) = \sqrt{3}\sin 2x - 3\cos 2x$$

Express $f(x)$ in the form $R\sin(2x - \alpha)$, where R and α are constants,

$$R > 0 \text{ and } 0 < \alpha < \frac{\pi}{2}$$

Give the exact value of R and the exact value of α.

(3)

(b) $$g(x) = \frac{18}{f(3x) + 4\sqrt{3}} \qquad x > 0$$

Using the answer to part (a),

(i) write down the exact minimum value of $g(x)$,

(ii) find the smallest value of x for which this minimum value occurs.

You must make your method clear.

(3)

Question 8

13 marks | Functions, Composition & Inverses | Main topic: Modulus

8.

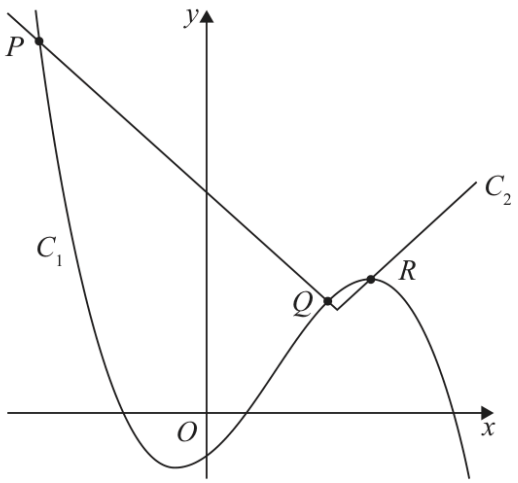


Figure 2

Figure 2 shows a sketch of the graph C_1 with equation

$$y = -2x^3 + 5x^2 + 4x - 3$$

and a sketch of the graph C_2 with equation

$$y = a + |5x + b|$$

where a and b are constants.

Graphs C_1 and C_2 intersect at point P, point Q and point R, as shown in Figure 2.

Given that P has coordinates $(-2, 25)$

(a) show that

$$a = 15 + b$$

(2)

Given also that R has coordinates $(2, 9)$

(b) find the value of a and the value of b

(3)

Using the answer to part (b),

(c) state the coordinates of the vertex of C_2

(2)

(d) Find, using algebra, the coordinates of Q. Show each stage of your working.

(*Solutions relying on calculator technology are not acceptable.*)

(6)

Question 7

10 marks | Functions, Composition & Inverses | Main topic: Modulus

7.

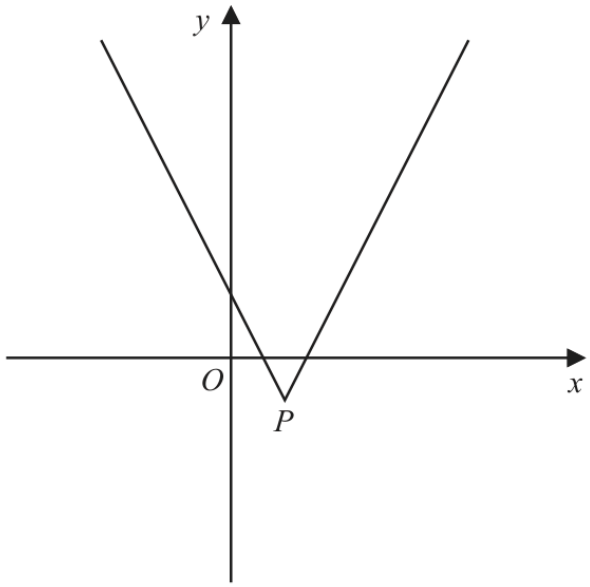


Figure 1

Figure 1 shows a sketch of part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = |3x - 11| - 4$$

The vertex of the graph is at the point P, as shown in Figure 1.

(a) State the coordinates of P.

(2)

(b) Solve the equation $\mathrm{f}(x) = 8$

(3)

The line l has equation $y = mx$, where m is a constant.

Given that l intersects the graph of $y = \mathrm{f}(x)$ at exactly one point,

(c) find the possible values of m.

(3)

The graph with equation $y = \mathrm{f}(x)$ is transformed onto the graph with equation $y = a\,\mathrm{f}(x - b)$, where a and b are constants.

Given that the vertex of the graph with equation $y = a\,\mathrm{f}(x - b)$ is $(5, 16)$

(d) find the value of a and the value of b.

(2)

Question 9

11 marks | Functions, Composition & Inverses | Main topic: Modulus

9.

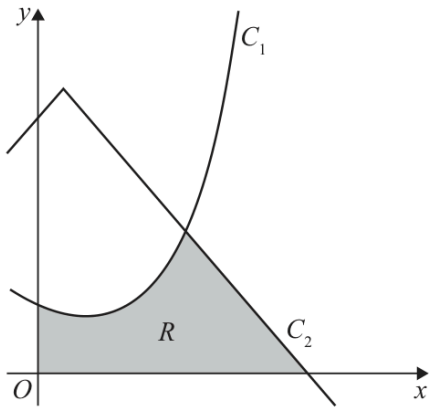


Figure 2

In this question you must show all stages of your working.
Solutions relying entirely on calculator technology are not acceptable.

(a) Find $\int \left(\frac{11}{2x+5} + 4e^{x-3} \right) dx$

(2)

Figure 2 shows a sketch of part of the curve C_1 and part of the graph C_2
The curve C_1 has equation

$$y = \frac{11}{2x+5} + 4e^{x-3}$$

The graph C_2 has equation

$$y = a - |bx - 1|$$

where a and b are positive integers.

(b) State, in terms of a and/or b, the coordinates of the maximum point of C_2

(2)

Given that

- C_1 intersects C_2 at $x = 3$
- C_2 crosses the y-axis at $(0, 9)$

(c) find the value of a and the value of b

(3)

The region R, shown shaded in Figure 2, is bounded by C_1, C_2, the x-axis and the y-axis.

(d) Find the area of R, giving the answer to 3 significant figures.

(4)

TOPIC

Graph Transformations

Every past-paper question on this topic, January 2020 to June 2026.

Question 2

6 marks | Graph Transformations

2.

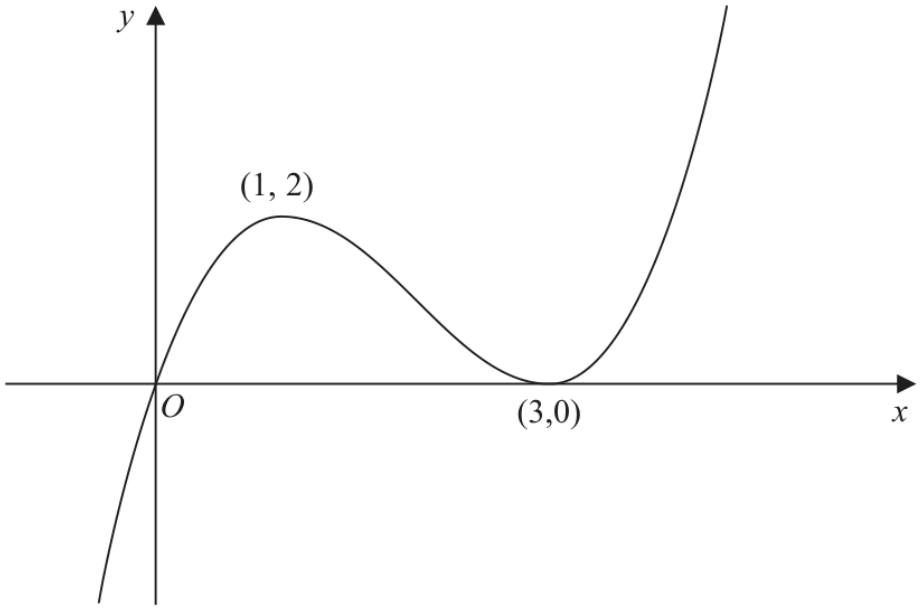


Figure 1

Figure 1 shows a sketch of the curve with equation $y = f(x)$, where $x \in \mathbb{R}$ and $f(x)$ is a polynomial.

The curve passes through the origin and touches the x-axis at the point (3, 0)

There is a maximum turning point at (1, 2) and a minimum turning point at (3, 0)

On separate diagrams, sketch the curve with equation

(i) $y = 3f(2x)$

(3)

(ii) $y = f(-x) - 1$

(3)

On each sketch, show clearly the coordinates of

- the point where the curve crosses the y-axis
- any maximum or minimum turning points

Question 1

4 marks | Graph Transformations

1. The point $P(-4, -3)$ lies on the curve with equation $y = f(x)$, $x \in \mathbb{R}$

Find the point to which P is mapped when the curve with equation $y = f(x)$ is transformed to the curve with equation

(a) $y = f(2x)$

(1)

(b) $y = 3f(x - 1)$

(2)

(c) $y = |f(x)|$

(1)

Question 7

10 marks | Graph Transformations

7.

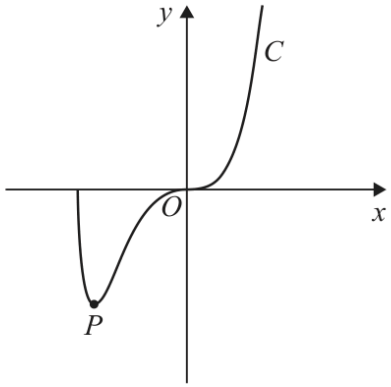


Figure 3

The curve C has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x^3\sqrt{4x+7} \qquad x \geqslant -\frac{7}{4}$$

(a) Show that

$$\mathrm{f}'(x) = \frac{kx^2(2x+3)}{\sqrt{4x+7}}$$

where k is a constant to be found.

(4)

The point P, shown in Figure 3, is the minimum turning point on C.

(b) Find the coordinates of P.

(2)

(c) Hence find the range of the function g defined by

$$\mathrm{g}(x) = -4\,\mathrm{f}(x) \qquad x \geqslant -\frac{7}{4}$$

(2)

The point Q with coordinates $\left(\frac{1}{2}, \frac{3}{8}\right)$ lies on C.

(d) Find the coordinates of the point to which Q is mapped when C is transformed to the curve with equation

$$y = 40\,\mathrm{f}\left(x - \frac{3}{2}\right) - 8$$

(2)

Question 6

9 marks | Graph Transformations | Main topic: Modulus

6.

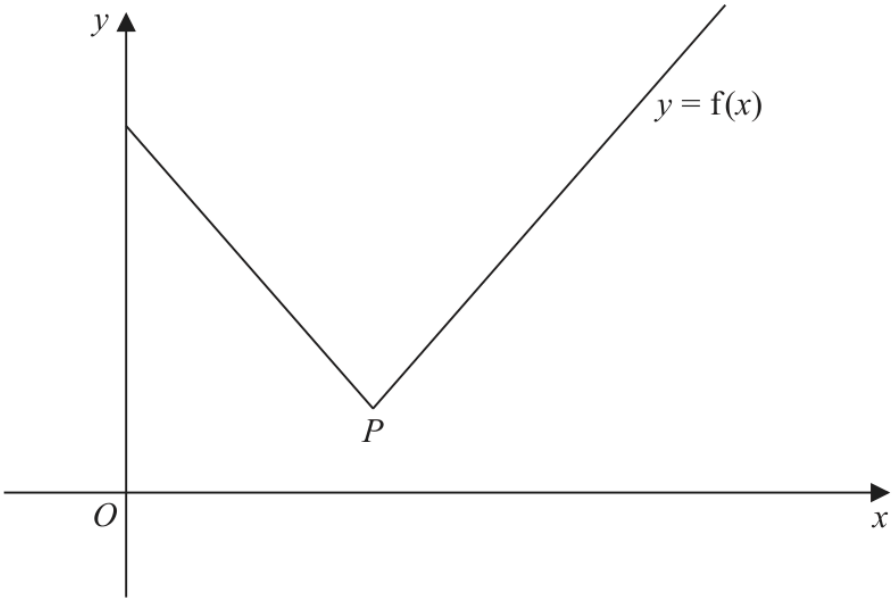


Figure 2

Figure 2 shows part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 2|2x - 5| + 3 \qquad x \geqslant 0$$

The vertex of the graph is at point P as shown.

(a) State the coordinates of P.

(2)

(b) Solve the equation $\mathrm{f}(x) = 3x - 2$

(4)

Given that the equation

$$\mathrm{f}(x) = kx + 2$$

where k is a constant, has exactly two roots,

(c) find the range of values of k.

(3)

Question 9

7 marks | Graph Transformations | Main topic: Trig Equations & R-Form

9. $\mathrm{f}(\theta) = 5\cos\theta - 4\sin\theta \qquad \theta \in \mathbb{R}$

(a) Express $\mathrm{f}(\theta)$ in the form $R\cos(\theta + \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

The curve with equation $y = \cos\theta$ is transformed onto the curve with equation $y = \mathrm{f}(\theta)$ by a sequence of two transformations.

Given that the first transformation is a stretch and the second a translation,

(b) (i) describe fully the transformation that is a stretch,

(ii) describe fully the transformation that is a translation.

(2)

Given

$$\mathrm{g}(\theta) = \frac{90}{4 + (\mathrm{f}(\theta))^2} \qquad \theta \in \mathbb{R}$$

(c) find the range of g.

(2)

Question 4

8 marks | Graph Transformations | Main topic: Modulus

4.

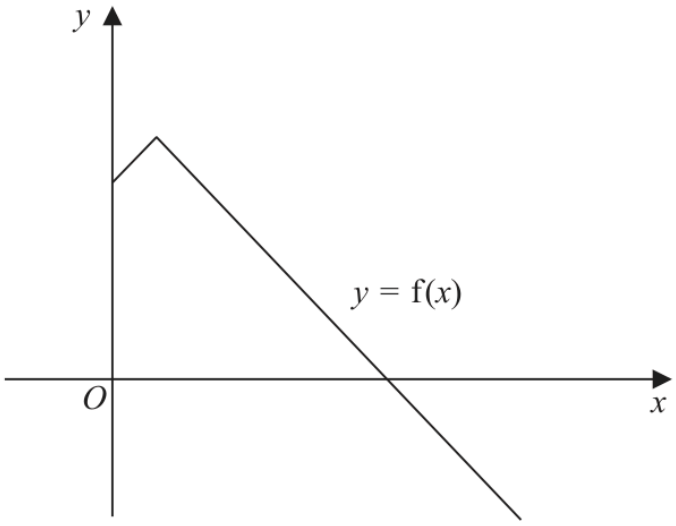


Figure 2

Figure 2 shows a sketch of part of the graph with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 21 - 2|2 - x| \qquad x \geqslant 0$$

(a) Find $\mathrm{ff}(6)$

(2)

(b) Solve the equation $\mathrm{f}(x) = 5x$

(2)

Given that the equation $\mathrm{f}(x) = k$, where k is a constant, has exactly two roots,

(c) state the set of possible values of k.

(2)

The graph with equation $y = \mathrm{f}(x)$ is transformed onto the graph with equation $y = a\mathrm{f}(x - b)$

The vertex of the graph with equation $y = a\mathrm{f}(x - b)$ is $(6, 3)$.

Given that a and b are constants,

(d) find the value of a and the value of b.

(2)

Question 6

9 marks | Graph Transformations | Main topic: Numerical Methods

6.

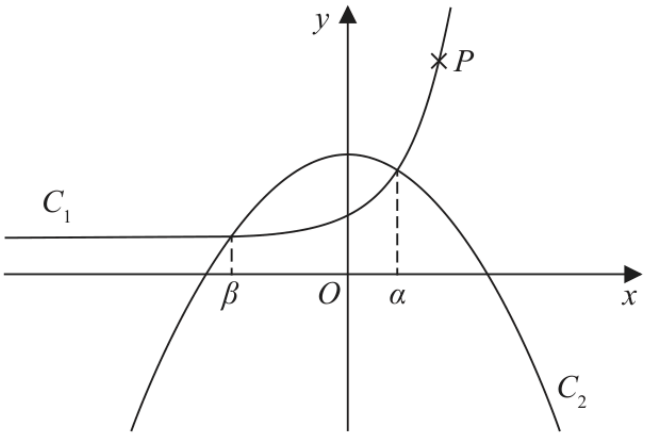


Figure 3

Figure 3 shows a sketch of curve C_1 with equation $y = 5e^{x-1} + 3$

and curve C_2 with equation $y = 10 - x^2$

The point P lies on C_1 and has y coordinate 18

(a) Find the x coordinate of P, writing your answer in the form $\ln k$, where k is a constant to be found.

(3)

The curve C_1 meets the curve C_2 at $x = \alpha$ and at $x = \beta$, as shown in Figure 3.

(b) Using a suitable interval and a suitable function that should be stated, show that to 3 decimal places $\alpha = 1.134$

(3)

The iterative equation

$$x_{n+1} = -\sqrt{7 - 5e^{x_n - 1}}$$

is used to find an approximation to β.

Using this iterative formula with $x_1 = -3$

(c) find the value of x_2 and the value of β, giving each answer to 6 decimal places.

(3)

Question 7

9 marks | Graph Transformations | Main topic: Trig Equations & R-Form

7. (a) Express $\cos x + 4\sin x$ in the form $R\cos(x - \alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

A scientist is studying the behaviour of seabirds in a colony.

She models the height above sea level, H metres, of one of the birds in the colony by the equation

$$H = \frac{24}{3 + \cos\left(\frac{1}{2}t\right) + 4\sin\left(\frac{1}{2}t\right)} \qquad 0 \leqslant t \leqslant 6.5$$

where t seconds is the time after it leaves the nest.

Find, according to the model,

(b) the minimum height of the seabird above sea level, giving your answer to the nearest cm,

(2)

(c) the value of t, to 2 decimal places, when $H = 10$

(4)

Question 4

9 marks | Graph Transformations | Main topic: Modulus

4.

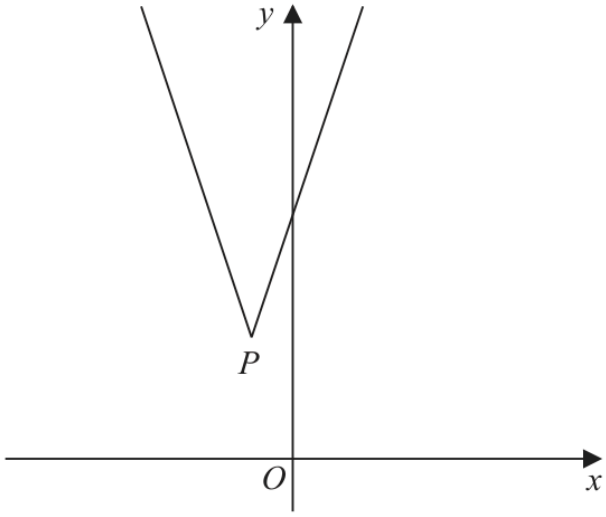


Figure 2

Figure 2 shows a sketch of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = |3x + a| + a$$

and where a is a positive constant.

The graph has a vertex at the point P, as shown in Figure 2.

(a) Find, in terms of a, the coordinates of P.

(2)

(b) Sketch the graph with equation $y = \mathrm{g}(x)$, where

$$\mathrm{g}(x) = |x + 5a|$$

On your sketch, show the coordinates, in terms of a, of each point where the graph cuts or meets the coordinate axes.

(2)

The graph with equation $y = \mathrm{g}(x)$ intersects the graph with equation $y = \mathrm{f}(x)$ at two points.

(c) Find, in terms of a, the coordinates of the two points.

(5)

Question 5

11 marks | Graph Transformations | Main topic: Log Graphs & Exponential Models

5. The temperature, θ°C, inside an oven, t minutes after the oven is switched on, is given by

$$\theta = A - 180e^{-kt}$$

where A and k are positive constants.

Given that the temperature inside the oven is initially 18 °C,

(a) find the value of A.

(2)

The temperature inside the oven, 5 minutes after the oven is switched on, is 90 °C.

(b) Show that $k = p \ln q$ where p and q are rational numbers to be found.

(4)

Hence find

(c) the temperature inside the oven 9 minutes after the oven is switched on, giving your answer to 3 significant figures,

(2)

(d) the rate of increase of the temperature inside the oven 9 minutes after the oven is switched on. Give your answer in °C min^{-1} to 3 significant figures.

(3)

Question 4

10 marks | Graph Transformations

Main topic: Functions, Composition & Inverses

4. The functions f and g are defined by

$$f(x) = \frac{4x + 6}{x - 5} \qquad x \in \mathbb{R},\ x \neq 5$$

$$g(x) = 5 - 2x^2 \qquad x \in \mathbb{R},\ x \leqslant 0$$

(a) Solve the equation

$$fg(x) = 3$$

(4)

(b) Find f^{-1}

(3)

(c) Sketch and label, on the same axes, the curve with equation $y = g(x)$ and the curve with equation $y = g^{-1}(x)$. Show on your sketch the coordinates of the points where each curve meets or cuts the coordinate axes.

(3)

Question 6

8 marks | Graph Transformations | Main topic: Modulus

6. Given that k is a positive constant,

(a) on separate diagrams, sketch the graph with equation

(i) $y = k - 2|x|$

(ii) $y = \left|2x - \frac{k}{3}\right|$

Show on each sketch the coordinates, in terms of k, of each point where the graph meets or cuts the axes.

(4)

(b) Hence find, in terms of k, the values of x for which

$$\left|2x - \frac{k}{3}\right| = k - 2|x|$$

giving your answers in simplest form.

(4)

Question 9

8 marks | Graph Transformations | Main topic: Trig Equations & R-Form

9. (a) Express $12\sin x - 5\cos x$ in the form $R\sin(x - \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the exact value of R and give the value of α in radians, to 3 decimal places.

(3)

The function g is defined by

$$g(\theta) = 10 + 12\sin\left(2\theta - \frac{\pi}{6}\right) - 5\cos\left(2\theta - \frac{\pi}{6}\right) \qquad \theta > 0$$

Find

(b) (i) the minimum value of $g(\theta)$

(ii) the smallest value of θ at which the minimum value occurs.

(3)

The function h is defined by

$$h(\beta) = 10 - (12\sin\beta - 5\cos\beta)^2$$

(c) Find the range of h.

(2)

Question 2

10 marks | Graph Transformations | Main topic: Modulus

2.

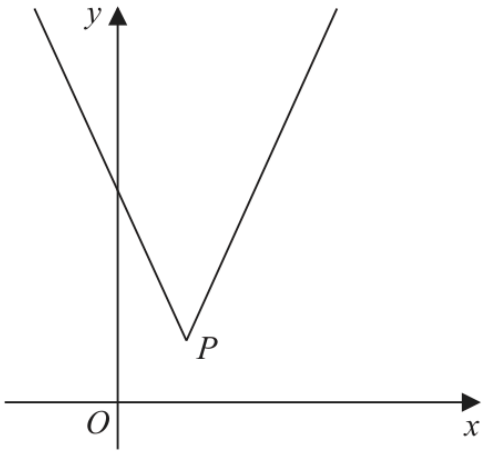


Figure 1

Figure 1 shows a sketch of part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = |3x - 13| + 5 \qquad x \in \mathbb{R}$$

The vertex of the graph is at point P, as shown in Figure 1.

(a) State the coordinates of P.

(2)

(b) (i) State the range of f.

(ii) Find the value of ff(4)

(2)

(c) Solve, using algebra and showing your working,

$$16 - 2x > |3x - 13| + 5$$

(4)

The graph with equation $y = \mathrm{f}(x)$ is transformed onto the graph with equation $y = a\mathrm{f}(x + b)$

The vertex of the graph with equation $y = a\mathrm{f}(x + b)$ is (4, 20)

Given that a and b are constants,

(d) find the value of a and the value of b.

(2)

Question 3

6 marks | Graph Transformations

Main topic: Log Graphs & Exponential Models

3.

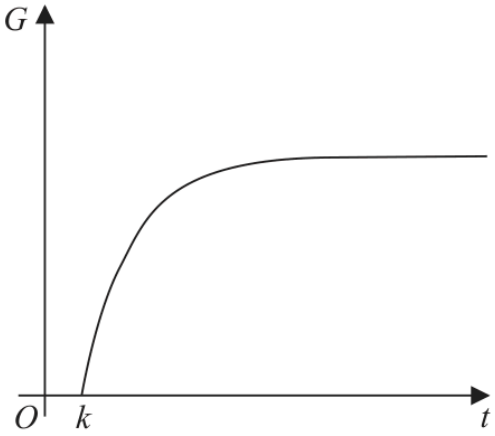


Figure 2

The total mass of gold, G tonnes, extracted from a mine is modelled by the equation

$$G = 40 - 30e^{1-0.05t} \qquad t \geqslant k \qquad G \geqslant 0$$

where t is the number of years after 1st January 1800.

Figure 2 shows a sketch of G against t.

Use the equation of the model to answer parts (a), (b) and (c).

(a) (i) Find the value of k.

(ii) Hence find the year and month in which gold started being extracted from the mine.

(3)

(b) Find the total mass of gold extracted from the mine up to 1st January 1870.

(2)

There is a limit to the mass of gold that can be extracted from the mine.

(c) State the value of this limit.

(1)

Question 8

7 marks | Graph Transformations | Main topic: Differentiation Techniques

8. A curve C has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \arcsin\left(\frac{1}{2}x\right) \qquad -2 \leqslant x \leqslant 2 \qquad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

(a) Sketch C.

(1)

(b) Given $x = 2\sin y$, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{A - x^2}}$$

where A is a constant to be found.

(3)

The point P lies on C and has y coordinate $\frac{\pi}{4}$

(c) Find the equation of the tangent to C at P. Write your answer in the form $y = mx + c$, where m and c are constants to be found.

(3)

Question 7

10 marks | Graph Transformations | Main topic: Modulus

7.

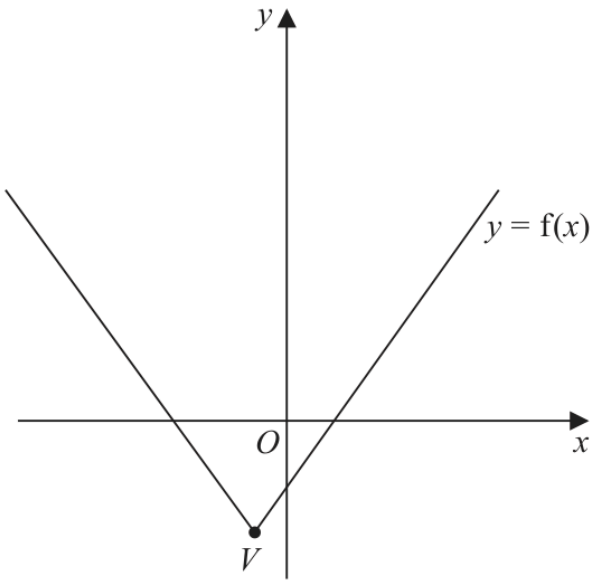


Figure 2

Figure 2 shows a sketch of part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \frac{1}{2}|2x + 7| - 10$$

(a) State the coordinates of the vertex, V, of the graph.

(2)

(b) Solve, using algebra,

$$\frac{1}{2}|2x + 7| - 10 \geqslant \frac{1}{3}x + 1$$

(4)

(c) Sketch the graph with equation

$$y = |\mathrm{f}(x)|$$

stating the coordinates of the local maximum point and each local minimum point.

(4)

Question 6

10 marks | Graph Transformations | Main topic: Applications of Differentiation

6.

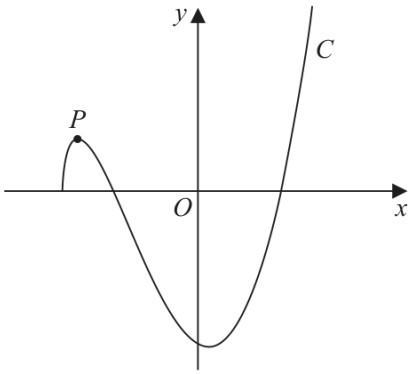


Figure 3

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

The function f is defined by

$$f(x) = 5(x^2 - 2)(4x + 9)^{\frac{1}{2}} \qquad x \geqslant -\frac{9}{4}$$

(a) Show that

$$f'(x) = \frac{k(5x^2 + 9x - 2)}{(4x + 9)^{\frac{1}{2}}}$$

where k is an integer to be found.

(4)

(b) Hence, find the values of x for which $f'(x) = 0$

(1)

Figure 3 shows a sketch of the curve C with equation $y = f(x)$.

The curve has a local maximum at the point P

(c) Find the exact coordinates of P

(2)

The function g is defined by

$$g(x) = 2f(x) + 4 \qquad -\frac{9}{4} \leqslant x \leqslant 0$$

(d) Find the range of g

(3)

Question 2

7 marks | Graph Transformations

Main topic: Functions, Composition & Inverses

2. The functions f and g are defined by

$$\mathrm{f}(x) = 5 - \frac{4}{3x+2} \qquad x \geqslant 0$$

$$\mathrm{g}(x) = \left| 4\sin\left(\frac{x}{3} + \frac{\pi}{6}\right) \right| \qquad x \in \mathbb{R}$$

(a) Find the range of f

(2)

(b) (i) Find $\mathrm{f}^{-1}(x)$

(ii) Write down the domain of f^{-1}

(3)

(c) Find $\mathrm{fg}(-\pi)$

(2)

Question 7

12 marks | Graph Transformations | Main topic: Modulus

7.

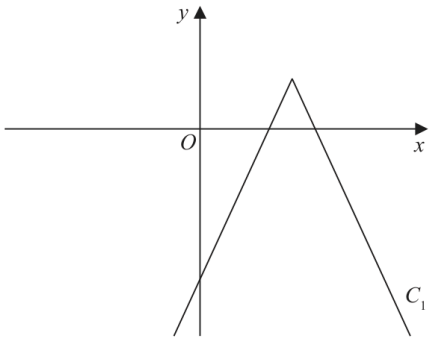


Figure 3

Figure 3 shows a sketch of the graph of C_1 with equation

$$y = 5 - |3x - 22|$$

(a) Write down the coordinates of

(i) the vertex of C_1

(ii) the intersection of C_1 with the y-axis.

(2)

(b) Find the x coordinates of the intersections of C_1 with the x-axis.

(2)

Diagram 1, shown on page 21, is a copy of Figure 3.

(c) On Diagram 1, sketch the curve C_2 with equation

$$y = \frac{1}{9}x^2 - 9$$

Identify clearly the coordinates of any points of intersection of C_2 with the coordinate axes.

(3)

(d) Find the coordinates of the points of intersection of C_1 and C_2
(*Solutions relying entirely on calculator technology are not acceptable.*)

(5)

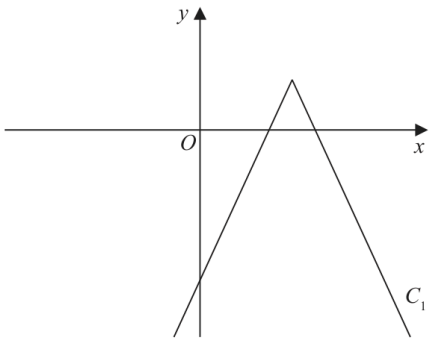


Diagram 1

Question 8

9 marks | Graph Transformations | Main topic: Trig Equations & R-Form

8. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Express $8\sin x - 15\cos x$ in the form $R\sin(x-\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R, and give the value of α, in radians, to 4 significant figures.

(3)

$$f(x) = \frac{15}{41 + 16\sin x - 30\cos x} \qquad x > 0$$

(b) Find

(i) the minimum value of $f(x)$

(ii) the smallest value of x at which this minimum value occurs.

(4)

(c) State the y coordinate of the minimum points on the curve with equation

$$y = 2f(x) - 5 \qquad x > 0$$

(1)

(d) State the smallest value of x at which a maximum point occurs for the curve with equation

$$y = -f(2x) \qquad x > 0$$

(1)

Question 2

6 marks | Graph Transformations | Main topic: Trig Equations & R-Form

2. $$f(x) = \cos x + 2\sin x$$

(a) Express $f(x)$ in the form $R\cos(x - \alpha)$, where R and α are constants,

$R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

$$g(x) = 3 - 7f(2x)$$

(b) Using the answer to part (a),

(i) write down the exact maximum value of $g(x)$,

(ii) find the smallest positive value of x for which this maximum value occurs, giving your answer to 2 decimal places.

(3)

Question 6

8 marks | Graph Transformations | Main topic: Modulus

6.

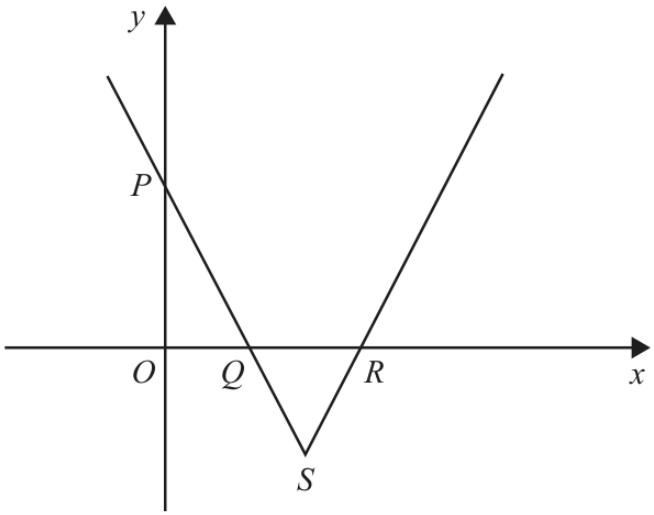


Figure 2

Figure 2 shows a sketch of the graph with equation

$$y = |3x - 5a| - 2a$$

where a is a positive constant.

The graph

- cuts the y-axis at the point P
- cuts the x-axis at the points Q and R
- has a minimum point at S

(a) Find, in simplest form in terms of a, the coordinates of

(i) point P

(ii) points Q and R

(iii) point S

(4)

(b) Find, in simplest form in terms of a, the values of x for which

$$|3x - 5a| - 2a = |x - 2a|$$

(4)

Question 4

6 marks | Graph Transformations

Main topic: Functions, Composition & Inverses

4. The function f is defined by

$$f(x) = 2x^2 - 5 \qquad x \geqslant 0 \quad x \in \mathbb{R}$$

(a) State the range of f

(1)

On the following page there is a diagram, labelled Diagram 1, which shows a sketch of the curve with equation $y = f(x)$.

(b) On Diagram 1, sketch the curve with equation $y = f^{-1}(x)$.

(2)

The curve with equation $y = f(x)$ meets the curve with equation $y = f^{-1}(x)$ at the point P

Using algebra and showing your working,

(c) find the exact x coordinate of P

(3)

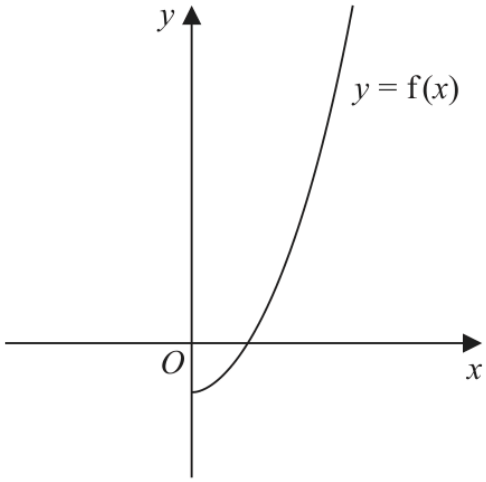


Diagram 1

Question 6

9 marks | Graph Transformations | Main topic: Modulus

6.

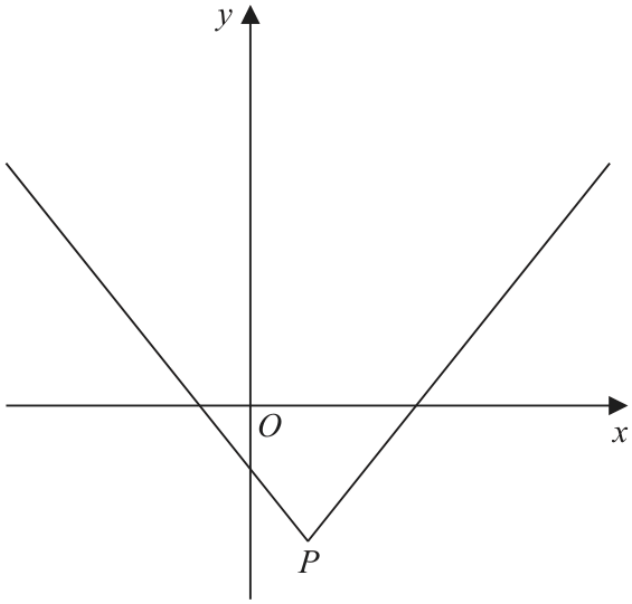


Figure 2

Figure 2 shows a sketch of the graph $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 3|x - 2| - 10$$

The vertex of the graph is at point P, shown in Figure 2.

(a) Find the coordinates of P

(2)

(b) Find $\mathrm{ff}(0)$

(2)

(c) Solve the inequality

$$3|x - 2| - 10 < 5x + 10$$

(2)

(d) Solve the equation

$$\mathrm{f}(|x|) = 0$$

(3)

Question 8

9 marks | Graph Transformations

Main topic: Applications of Differentiation

8.

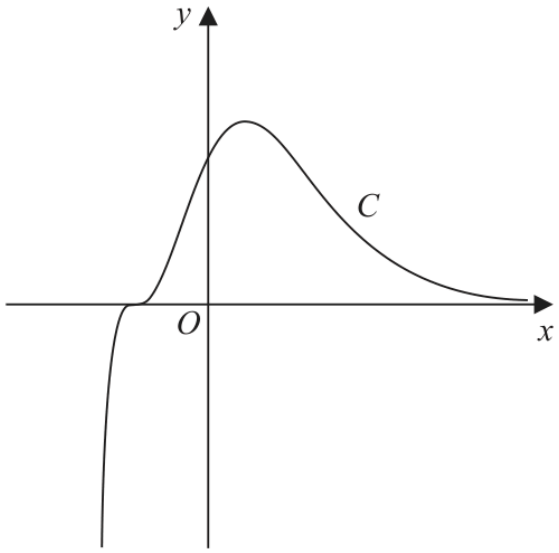


Figure 3

Figure 3 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = (2x + 1)^3 \mathrm{e}^{-4x}$$

(a) Show that

$$\mathrm{f}'(x) = A(2x + 1)^2 (1 - 4x)\, \mathrm{e}^{-4x}$$

where A is a constant to be found.

(4)

(b) Hence find the exact coordinates of the two stationary points on C.

(3)

The function g is defined by

$$\mathrm{g}(x) = 8\,\mathrm{f}(x - 2)$$

(c) Find the coordinates of the maximum stationary point on the curve with equation $y = \mathrm{g}(x)$.

(2)

Question 9

9 marks | Graph Transformations | Main topic: Exponentials & Logarithms

9. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

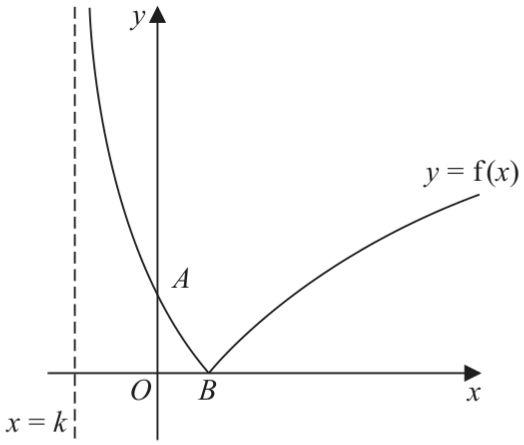


Figure 2

Figure 2 shows a sketch of the curve with equation

$$y = |2 - 4\ln(x + 1)| \qquad x > k$$

where k is a constant.

Given that the curve

- has an asymptote at $x = k$
- cuts the y-axis at point A
- meets the x-axis at point B

as shown in Figure 2,

(a) state the value of k

(1)

(b) (i) find the y coordinate of A

(ii) find the exact x coordinate of B

(3)

(c) Using algebra and showing your working, find the set of values of x such that

$$|2 - 4\ln(x + 1)| > 3$$

(5)

Question 8

11 marks | Graph Transformations | Main topic: Modulus

8.

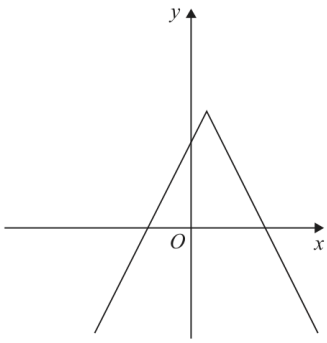


Figure 2

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

The graph shown in Figure 2 has equation

$$y = a - |2x - b|$$

where a and b are positive constants, $a > b$

(a) Find, giving your answer in terms of a and b,

(i) the coordinates of the maximum point of the graph,

(ii) the coordinates of the point of intersection of the graph with the y-axis,

(iii) the coordinates of the points of intersection of the graph with the x-axis.

(5)

On page 24 there is a copy of Figure 2 called Diagram 1.

(b) On Diagram 1, sketch the graph with equation

$$y = |x| - 1$$

(2)

Given that the graphs $y = |x| - 1$ and $y = a - |2x - b|$ intersect at $x = -3$ and $x = 5$

(c) find the value of a and the value of b

(4)

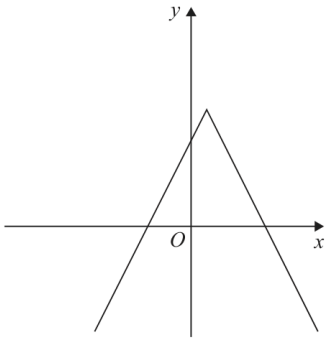


Diagram 1

Question 1

6 marks | Graph Transformations | Main topic: Modulus

1.

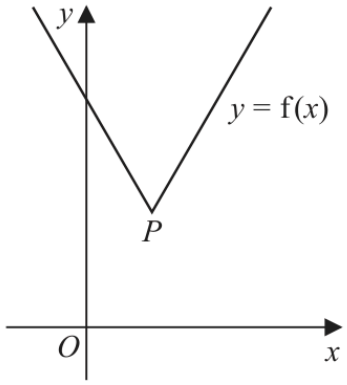


Figure 1

Figure 1 shows a sketch of the graph with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 2|x - 5| + 10$$

The point P, shown in Figure 1, is the vertex of the graph.

(a) State the coordinates of P

(2)

(b) Use algebra to solve

$$2|x - 5| + 10 > 6x$$

(*Solutions relying on calculator technology are not acceptable.*)

(2)

(c) Find the point to which P is mapped, when the graph with equation $y = \mathrm{f}(x)$ is transformed to the graph with equation $y = 3\mathrm{f}(x - 2)$

(2)

Question 4

9 marks | Graph Transformations

Main topic: Trig Identities & Exact Values

4. $$f(x) = 8\sin x\cos x + 4\cos^2 x - 3$$

(a) Write $f(x)$ in the form

$$a\sin 2x + b\cos 2x + c$$

where a, b and c are integers to be found.

(3)

(b) Use the answer to part (a) to write $f(x)$ in the form

$$R\sin(2x + \alpha) + c$$

where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α in radians to 3 significant figures.

(3)

(c) Hence, or otherwise,

(i) state the maximum value of $f(x)$

(ii) find the **second** smallest positive value of x at which a maximum value of $f(x)$ occurs. Give your answer to 3 significant figures.

(3)

Question 3

7 marks | Graph Transformations | Main topic: Modulus

3.

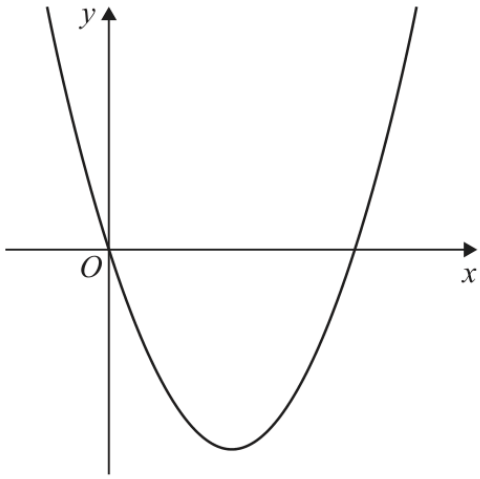


Figure 2

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 2 shows a sketch of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 2x^2 - 10x \qquad x \in \mathbb{R}$$

(a) Solve the equation

$$\mathrm{f}(|x|) = 48$$

(3)

(b) Find the set of values of x for which

$$|\mathrm{f}(x)| \geqslant \frac{5}{2}x$$

(4)

Question 5

9 marks | Graph Transformations | Main topic: Log Graphs & Exponential Models

5. A hot piece of metal is cooled by dropping it into water. The temperature, H°C, of the metal, t minutes after it is dropped into the water, is modelled by the equation

$$H = 280e^{-0.05t} + 24 \qquad t \geqslant 0$$

Use the equation of the model to answer parts (a) to (d).

(a) Find the initial temperature of the piece of metal.

(1)

(b) On Diagram 1, sketch the graph of H against t. On your sketch, state the equation of the asymptote to the curve.

(2)

(c) Find the value of t for which $H = 144$, giving your answer to 2 decimal places.

(*Solutions based entirely on calculator technology are not acceptable.*)

(3)

(d) Show by differentiation that

$$\frac{dH}{dt} = a + bH$$

where a and b are constants to be found.

(3)

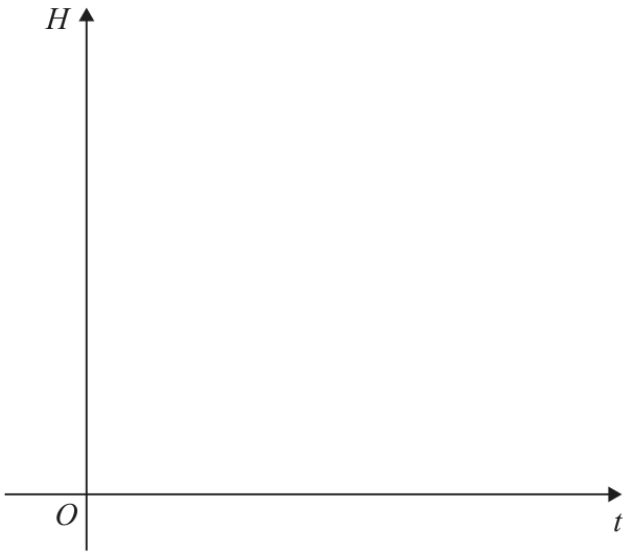


Diagram 1

Question 6

8 marks | Graph Transformations

Main topic: Functions, Composition & Inverses

6. The function f is defined by

$$f(x) = \frac{4x + 3}{x - 2} \qquad x \neq 2$$

(a) Find f^{-1}

(3)

(b) Show that

$$ff(x) = \frac{ax + b}{cx + d}$$

where a, b, c and d are integers to be found.

(3)

The point $P(3, 15)$ lies on the curve with equation $y = f(x)$.

(c) Find the point to which P is mapped when $y = f(x)$ is transformed to the curve with equation $y = 2f(3x) + 8$

(2)

Question 7

8 marks | Graph Transformations | Main topic: Modulus

7. Given that a and b are positive constants with $a > b$,

(a) sketch, on **separate** diagrams, the graph with equation

(i) $y = |3x - a|$

(ii) $y = |3x - a| - b$

Show on each sketch

- the coordinates of the minimum point on the graph
- the coordinates of the point at which the graph crosses the y-axis

(6)

(b) Solve the equation

$$|3x - a| - b = 5x$$

giving any solution for x in terms of a and b.

(2)

Question 9

8 marks | Graph Transformations

Main topic: Applications of Differentiation

9.

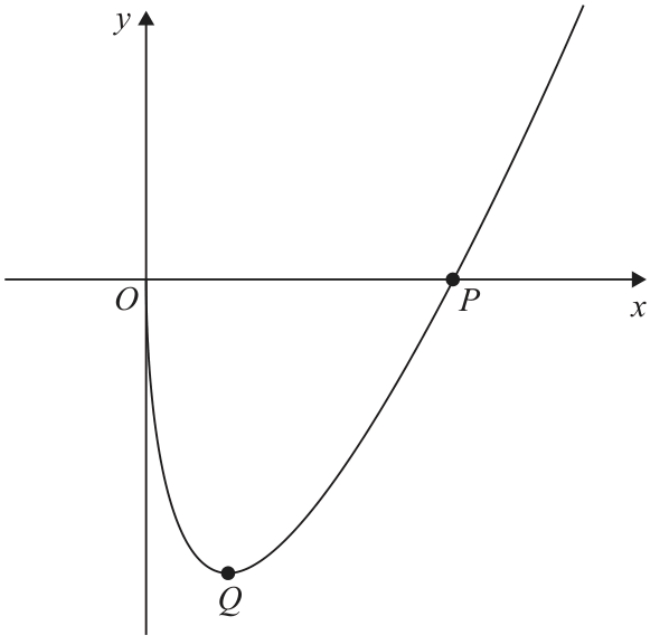


Figure 1

Figure 1 shows a sketch of part of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 6\sqrt{x}\ \ln(4x) \qquad x > 0$$

The curve cuts the x-axis at point P

(a) State the x coordinate of P

(1)

The point Q, shown in Figure 1, is the stationary point on C

(b) Use calculus to find the exact coordinates of Q

(5)

(c) Hence find the range of the function $\mathrm{g}(x)$ where

$$\mathrm{g}(x) = -2\,\mathrm{f}(x)$$

(2)

Question 7

10 marks | Graph Transformations | Main topic: Modulus

7.

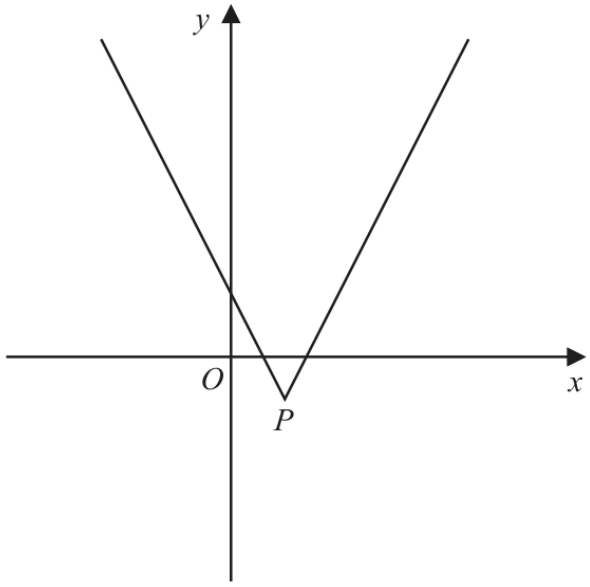


Figure 1

Figure 1 shows a sketch of part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = |3x - 11| - 4$$

The vertex of the graph is at the point P, as shown in Figure 1.

(a) State the coordinates of P.

(2)

(b) Solve the equation $\mathrm{f}(x) = 8$

(3)

The line l has equation $y = mx$, where m is a constant.

Given that l intersects the graph of $y = \mathrm{f}(x)$ at exactly one point,

(c) find the possible values of m.

(3)

The graph with equation $y = \mathrm{f}(x)$ is transformed onto the graph with equation $y = a\,\mathrm{f}(x - b)$, where a and b are constants.

Given that the vertex of the graph with equation $y = a\,\mathrm{f}(x - b)$ is $(5, 16)$

(d) find the value of a and the value of b.

(2)

Question 9

9 marks | Graph Transformations | Main topic: Trig Equations & R-Form

9. $$\mathrm{h}(x) = 8 + 3\sin x(\cos x - 2\sin x) \quad 0 \leqslant x \leqslant 2\pi$$

(a) Write $\mathrm{h}(x)$ in the form

$$P + R\sin(2x + \alpha)$$

where P, R and α are positive constants, $0 < \alpha < \frac{\pi}{2}$

Give the exact values of P and R and give the value of α, in radians, to 3 decimal places.

(6)

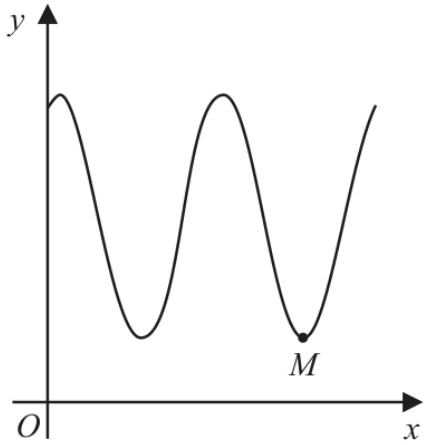


Figure 2

Figure 2 is a sketch of the curve with equation $y = \mathrm{h}(x)$

(b) State the range of h.

(1)

The point M, shown in Figure 2, is the minimum point on the curve with the largest x coordinate.

(c) Find the x coordinate of M.

(Solutions based entirely on calculator technology are not acceptable.)

(2)

Question 3

8 marks | Graph Transformations | Main topic: Differentiation Techniques

3. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The curve C has equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{4x^2 - 14x + 15}{x - 3} \qquad x < 3$$

(a) Find $\mathrm{f}'(x)$, giving the answer in the form

$$\frac{ax^2 + bx + c}{(x-3)^2}$$

where a, b and c are constants.

(4)

(b) Hence find the coordinates of the turning point of C.

(2)

The point $P\left(\frac{5}{2}, -10\right)$ lies on C.

(c) Find the coordinates of the point to which P is mapped, when the curve with equation $y = \mathrm{f}(x)$ is transformed to the curve with equation $y = \mathrm{f}(2x) + 3$

(2)

TOPIC

Modulus

Every past-paper question on this topic, January 2020 to June 2026.

Question 6

9 marks | Modulus

6.

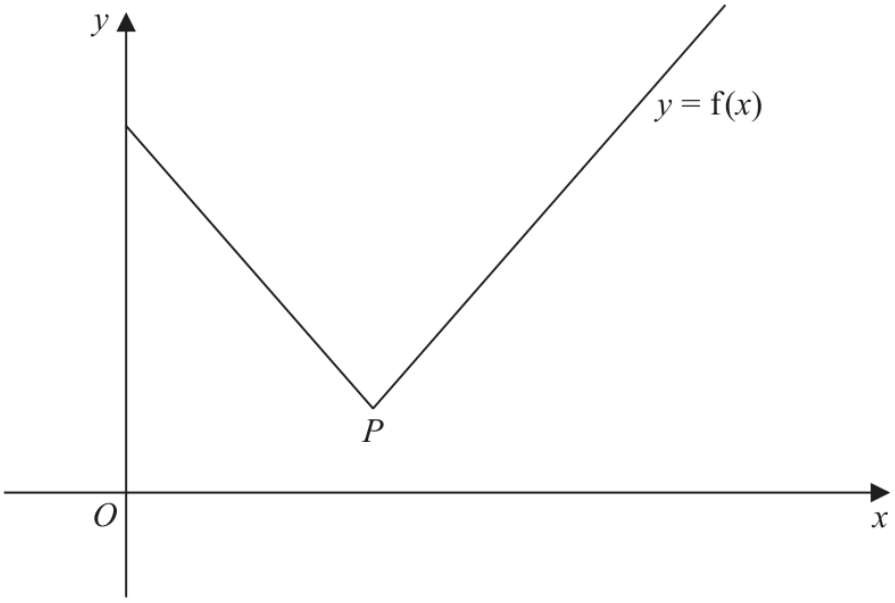


Figure 2

Figure 2 shows part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 2|2x - 5| + 3 \qquad x \geqslant 0$$

The vertex of the graph is at point P as shown.

(a) State the coordinates of P.

(2)

(b) Solve the equation $\mathrm{f}(x) = 3x - 2$

(4)

Given that the equation

$$\mathrm{f}(x) = kx + 2$$

where k is a constant, has exactly two roots,

(c) find the range of values of k.

(3)

Question 4

8 marks | Modulus

4.

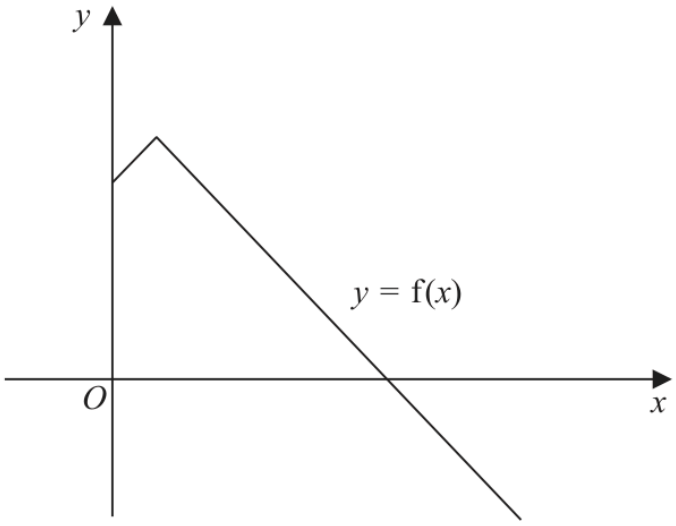


Figure 2

Figure 2 shows a sketch of part of the graph with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 21 - 2|2 - x| \qquad x \geqslant 0$$

(a) Find $\mathrm{ff}(6)$

(2)

(b) Solve the equation $\mathrm{f}(x) = 5x$

(2)

Given that the equation $\mathrm{f}(x) = k$, where k is a constant, has exactly two roots,

(c) state the set of possible values of k.

(2)

The graph with equation $y = \mathrm{f}(x)$ is transformed onto the graph with equation $y = a\mathrm{f}(x - b)$

The vertex of the graph with equation $y = a\mathrm{f}(x - b)$ is $(6, 3)$.

Given that a and b are constants,

(d) find the value of a and the value of b.

(2)

Question 4

9 marks | Modulus

4.

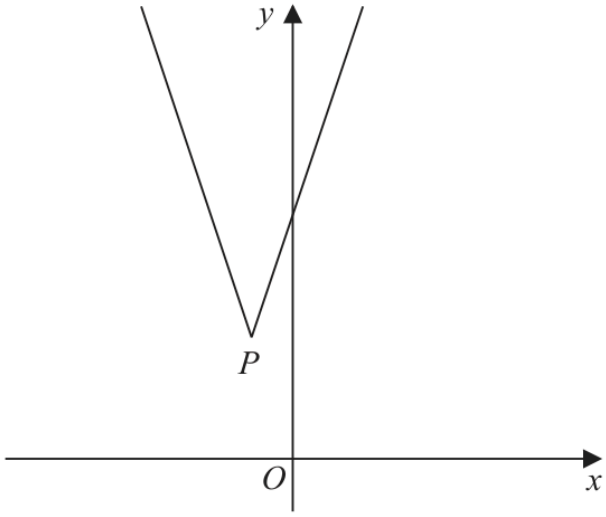


Figure 2

Figure 2 shows a sketch of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = |3x + a| + a$$

and where a is a positive constant.

The graph has a vertex at the point P, as shown in Figure 2.

(a) Find, in terms of a, the coordinates of P.

(2)

(b) Sketch the graph with equation $y = \mathrm{g}(x)$, where

$$\mathrm{g}(x) = |x + 5a|$$

On your sketch, show the coordinates, in terms of a, of each point where the graph cuts or meets the coordinate axes.

(2)

The graph with equation $y = \mathrm{g}(x)$ intersects the graph with equation $y = \mathrm{f}(x)$ at two points.

(c) Find, in terms of a, the coordinates of the two points.

(5)

Question 6

8 marks | Modulus

6. Given that k is a positive constant,

(a) on separate diagrams, sketch the graph with equation

(i) $y = k - 2|x|$

(ii) $y = \left|2x - \frac{k}{3}\right|$

Show on each sketch the coordinates, in terms of k, of each point where the graph meets or cuts the axes.

(4)

(b) Hence find, in terms of k, the values of x for which

$$\left|2x - \frac{k}{3}\right| = k - 2|x|$$

giving your answers in simplest form.

(4)

Question 2

10 marks | Modulus

2.

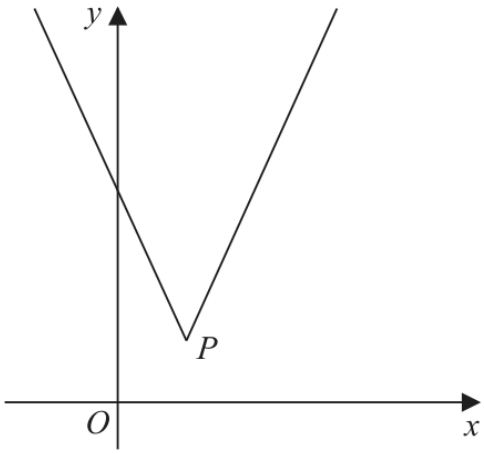


Figure 1

Figure 1 shows a sketch of part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = |3x - 13| + 5 \qquad x \in \mathbb{R}$$

The vertex of the graph is at point P, as shown in Figure 1.

(a) State the coordinates of P.

(2)

(b) (i) State the range of f.

(ii) Find the value of ff(4)

(2)

(c) Solve, using algebra and showing your working,

$$16 - 2x > |3x - 13| + 5$$

(4)

The graph with equation $y = \mathrm{f}(x)$ is transformed onto the graph with equation $y = a\mathrm{f}(x + b)$

The vertex of the graph with equation $y = a\mathrm{f}(x + b)$ is (4, 20)

Given that a and b are constants,

(d) find the value of a and the value of b.

(2)

Question 7

10 marks | Modulus

7.

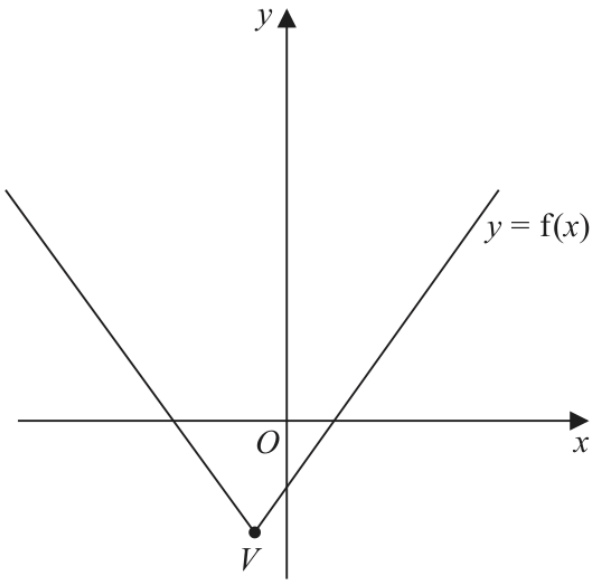


Figure 2

Figure 2 shows a sketch of part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \frac{1}{2}|2x + 7| - 10$$

(a) State the coordinates of the vertex, V, of the graph.

(2)

(b) Solve, using algebra,

$$\frac{1}{2}|2x + 7| - 10 \geqslant \frac{1}{3}x + 1$$

(4)

(c) Sketch the graph with equation

$$y = |\mathrm{f}(x)|$$

stating the coordinates of the local maximum point and each local minimum point.

(4)

Question 5

8 marks | Modulus

5.

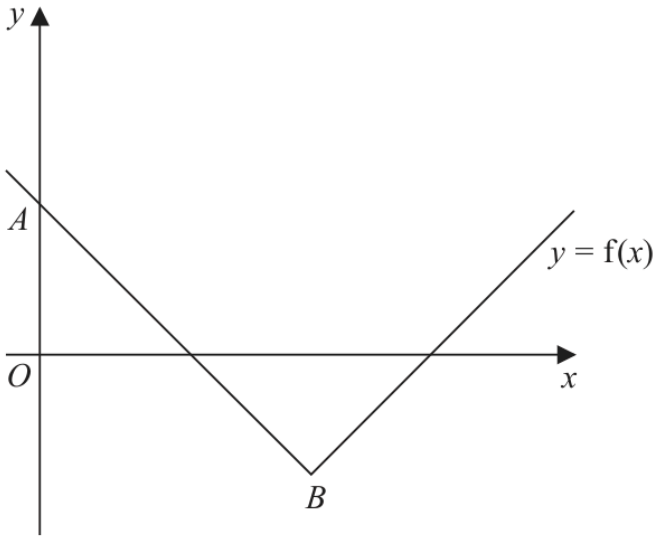


Figure 2

Figure 2 shows part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = |kx - 9| - 2 \qquad x \in \mathbb{R}$$

and k is a positive constant.

The graph intersects the y-axis at the point A and has a minimum point at B as shown.

(a) (i) Find the y coordinate of A

(ii) Find, in terms of k, the x coordinate of B

(2)

(b) Find, in terms of k, the range of values of x that satisfy the inequality

$$|kx - 9| - 2 < 0$$

(3)

Given that the line $y = 3 - 2x$ intersects the graph $y = \mathrm{f}(x)$ at two distinct points,

(c) find the range of possible values of k

(3)

Question 7

12 marks | Modulus

7.

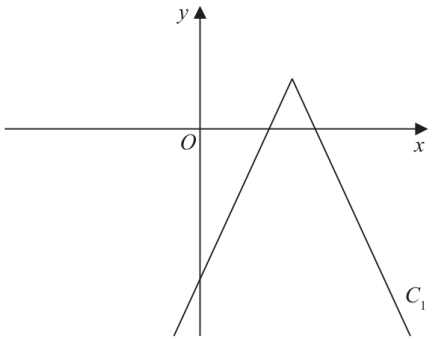


Figure 3

Figure 3 shows a sketch of the graph of C_1 with equation

$$y = 5 - |3x - 22|$$

(a) Write down the coordinates of

(i) the vertex of C_1

(ii) the intersection of C_1 with the y-axis.

(2)

(b) Find the x coordinates of the intersections of C_1 with the x-axis.

(2)

Diagram 1, shown on page 21, is a copy of Figure 3.

(c) On Diagram 1, sketch the curve C_2 with equation

$$y = \frac{1}{9}x^2 - 9$$

Identify clearly the coordinates of any points of intersection of C_2 with the coordinate axes.

(3)

(d) Find the coordinates of the points of intersection of C_1 and C_2
(*Solutions relying entirely on calculator technology are not acceptable.*)

(5)

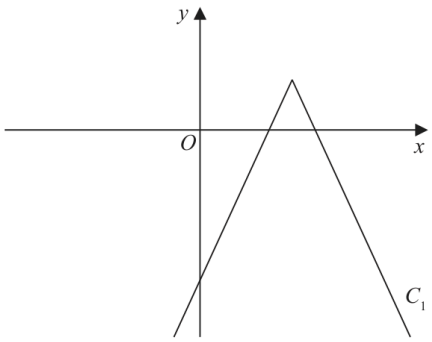


Diagram 1

Question 6

8 marks | Modulus

6.

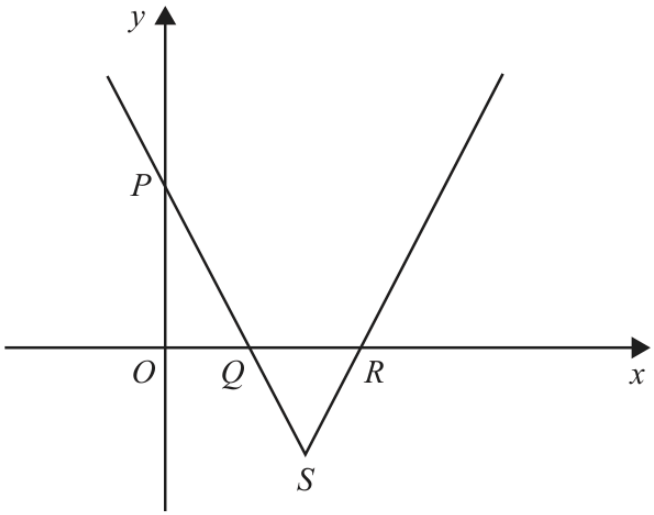


Figure 2

Figure 2 shows a sketch of the graph with equation

$$y = |3x - 5a| - 2a$$

where a is a positive constant.

The graph

- cuts the y-axis at the point P
- cuts the x-axis at the points Q and R
- has a minimum point at S

(a) Find, in simplest form in terms of a, the coordinates of

(i) point P

(ii) points Q and R

(iii) point S

(4)

(b) Find, in simplest form in terms of a, the values of x for which

$$|3x - 5a| - 2a = |x - 2a|$$

(4)

Question 6

9 marks | Modulus

6.

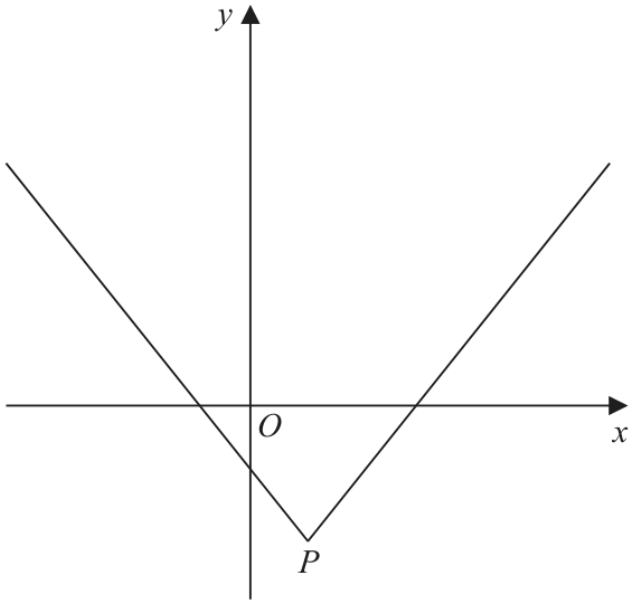


Figure 2

Figure 2 shows a sketch of the graph $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 3|x - 2| - 10$$

The vertex of the graph is at point P, shown in Figure 2.

(a) Find the coordinates of P

(2)

(b) Find $\mathrm{ff}(0)$

(2)

(c) Solve the inequality

$$3|x - 2| - 10 < 5x + 10$$

(2)

(d) Solve the equation

$$\mathrm{f}(|x|) = 0$$

(3)

Question 8

11 marks | Modulus

8.

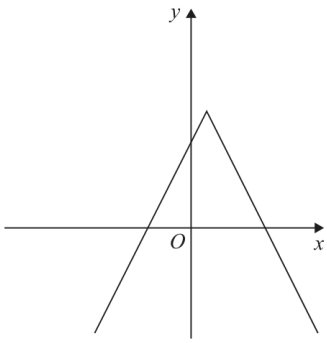


Figure 2

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

The graph shown in Figure 2 has equation

$$y = a - |2x - b|$$

where a and b are positive constants, $a > b$

(a) Find, giving your answer in terms of a and b,

(i) the coordinates of the maximum point of the graph,

(ii) the coordinates of the point of intersection of the graph with the y-axis,

(iii) the coordinates of the points of intersection of the graph with the x-axis.

(5)

On page 24 there is a copy of Figure 2 called Diagram 1.

(b) On Diagram 1, sketch the graph with equation

$$y = |x| - 1$$

(2)

Given that the graphs $y = |x| - 1$ and $y = a - |2x - b|$ intersect at $x = -3$ and $x = 5$

(c) find the value of a and the value of b

(4)

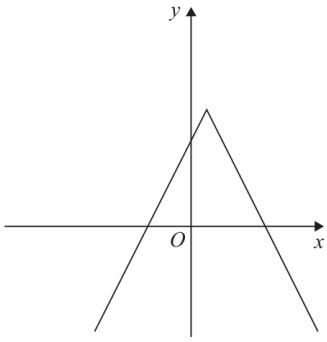


Diagram 1

Question 1

6 marks | Modulus

1.

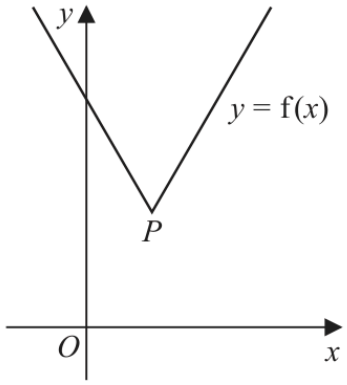


Figure 1

Figure 1 shows a sketch of the graph with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 2|x - 5| + 10$$

The point P, shown in Figure 1, is the vertex of the graph.

(a) State the coordinates of P

(2)

(b) Use algebra to solve

$$2|x - 5| + 10 > 6x$$

(*Solutions relying on calculator technology are not acceptable.*)

(2)

(c) Find the point to which P is mapped, when the graph with equation $y = \mathrm{f}(x)$ is transformed to the graph with equation $y = 3\mathrm{f}(x - 2)$

(2)

Question 3

7 marks | Modulus

3.

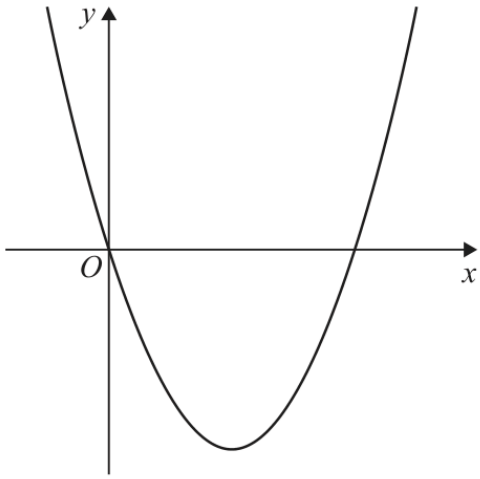


Figure 2

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 2 shows a sketch of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 2x^2 - 10x \qquad x \in \mathbb{R}$$

(a) Solve the equation

$$\mathrm{f}(|x|) = 48$$

(3)

(b) Find the set of values of x for which

$$|\mathrm{f}(x)| \geqslant \frac{5}{2}x$$

(4)

Question 7

8 marks | Modulus

7. Given that a and b are positive constants with $a > b$,

(a) sketch, on **separate** diagrams, the graph with equation

(i) $y = |3x - a|$

(ii) $y = |3x - a| - b$

Show on each sketch

- the coordinates of the minimum point on the graph
- the coordinates of the point at which the graph crosses the y-axis

(6)

(b) Solve the equation

$$|3x - a| - b = 5x$$

giving any solution for x in terms of a and b.

(2)

Question 10

10 marks | Modulus

10:

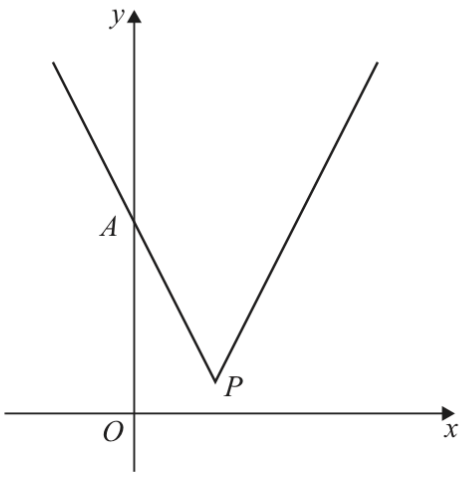


Figure 3

Figure 3 shows a sketch of part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = |kx - 10| + k \qquad x \in \mathbb{R}$$

and k is a positive constant.

The graph

- cuts the y-axis at the point A
- has a vertex at the point P

(a) Find, in simplest form in terms of k,

(i) the y coordinate of A

(ii) the coordinates of P

(3)

(b) Find, in terms of k, the range of values of x which satisfy

$$|kx - 10| + k \geqslant 2k$$

(3)

Given that the line with equation $y = 3x + 1$ intersects the graph of $y = \mathrm{f}(x)$ at 2 distinct points,

(c) find the range of values of k.

(4)

Question 8

13 marks | Modulus

8.

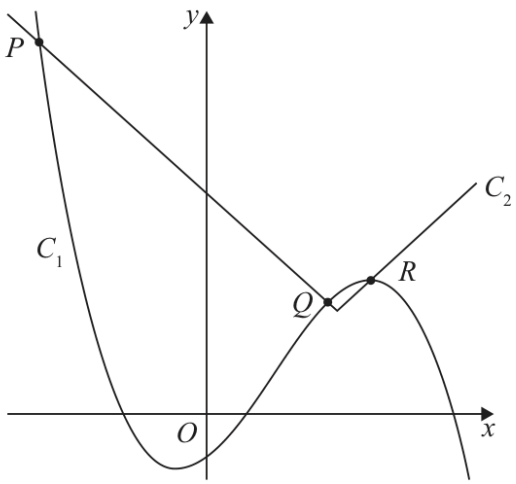


Figure 2

Figure 2 shows a sketch of the graph C_1 with equation

$$y = -2x^3 + 5x^2 + 4x - 3$$

and a sketch of the graph C_2 with equation

$$y = a + |5x + b|$$

where a and b are constants.

Graphs C_1 and C_2 intersect at point P, point Q and point R, as shown in Figure 2.

Given that P has coordinates $(-2, 25)$

(a) show that

$$a = 15 + b$$

(2)

Given also that R has coordinates $(2, 9)$

(b) find the value of a and the value of b

(3)

Using the answer to part (b),

(c) state the coordinates of the vertex of C_2

(2)

(d) Find, using algebra, the coordinates of Q. Show each stage of your working.

(*Solutions relying on calculator technology are not acceptable.*)

(6)

Question 7

10 marks | Modulus

7.

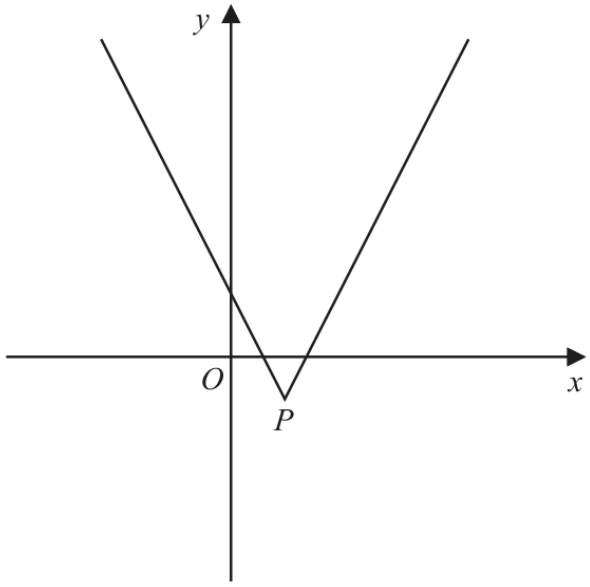


Figure 1

Figure 1 shows a sketch of part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = |3x - 11| - 4$$

The vertex of the graph is at the point P, as shown in Figure 1.

(a) State the coordinates of P.

(2)

(b) Solve the equation $\mathrm{f}(x) = 8$

(3)

The line l has equation $y = mx$, where m is a constant.

Given that l intersects the graph of $y = \mathrm{f}(x)$ at exactly one point,

(c) find the possible values of m.

(3)

The graph with equation $y = \mathrm{f}(x)$ is transformed onto the graph with equation $y = a\,\mathrm{f}(x - b)$, where a and b are constants.

Given that the vertex of the graph with equation $y = a\,\mathrm{f}(x - b)$ is $(5, 16)$

(d) find the value of a and the value of b.

(2)

Question 9

11 marks | Modulus

9.

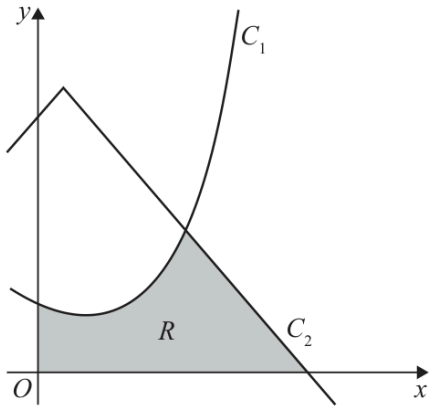


Figure 2

In this question you must show all stages of your working.
Solutions relying entirely on calculator technology are not acceptable.

(a) Find $\int \left(\frac{11}{2x+5} + 4e^{x-3} \right) dx$

(2)

Figure 2 shows a sketch of part of the curve C_1 and part of the graph C_2
The curve C_1 has equation

$$y = \frac{11}{2x+5} + 4e^{x-3}$$

The graph C_2 has equation

$$y = a - |bx - 1|$$

where a and b are positive integers.

(b) State, in terms of a and/or b, the coordinates of the maximum point of C_2

(2)

Given that

- C_1 intersects C_2 at $x = 3$
- C_2 crosses the y-axis at $(0, 9)$

(c) find the value of a and the value of b

(3)

The region R, shown shaded in Figure 2, is bounded by C_1, C_2, the x-axis and the y-axis.

(d) Find the area of R, giving the answer to 3 significant figures.

(4)

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Question 2

7 marks | Modulus

Main topic: Functions, Composition & Inverses

2. The functions f and g are defined by

$$\mathrm{f}(x) = 5 - \frac{4}{3x+2} \qquad x \geqslant 0$$

$$\mathrm{g}(x) = \left| 4\sin\left(\frac{x}{3} + \frac{\pi}{6}\right) \right| \qquad x \in \mathbb{R}$$

(a) Find the range of f

(2)

(b) (i) Find $\mathrm{f}^{-1}(x)$

(ii) Write down the domain of f^{-1}

(3)

(c) Find $\mathrm{fg}(-\pi)$

(2)

Question 9

9 marks | Modulus | Main topic: Exponentials & Logarithms

9. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

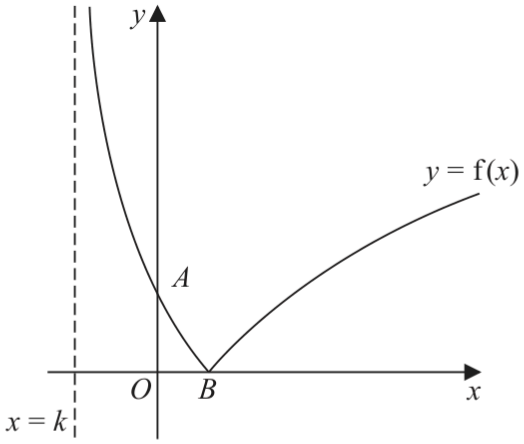


Figure 2

Figure 2 shows a sketch of the curve with equation

$$y = |2 - 4\ln(x + 1)| \qquad x > k$$

where k is a constant.

Given that the curve

- has an asymptote at $x = k$
- cuts the y-axis at point A
- meets the x-axis at point B

as shown in Figure 2,

(a) state the value of k

(1)

(b) (i) find the y coordinate of A

(ii) find the exact x coordinate of B

(3)

(c) Using algebra and showing your working, find the set of values of x such that

$$|2 - 4\ln(x + 1)| > 3$$

(5)

Question 1

4 marks | Modulus | Main topic: Graph Transformations

1. The point $P(-4, -3)$ lies on the curve with equation $y = f(x)$, $x \in \mathbb{R}$

Find the point to which P is mapped when the curve with equation $y = f(x)$ is transformed to the curve with equation

(a) $y = f(2x)$

(1)

(b) $y = 3f(x - 1)$

(2)

(c) $y = |f(x)|$

(1)

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Question 7

7 marks | Modulus

Main topic: Applications of Differentiation

7. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

A curve C has equation

$$y = 2xe^{x^2+(3k-2)x}$$

where k is a constant.

Given that C has two distinct turning points, find the range of possible values of k.

(7)

Dr Eslam Ahmed · eliteigcse.com

TOPIC

Trigonometric Functions & Graphs

Every past-paper question on this topic, January 2020 to June 2026.

Question 9

7 marks | Trigonometric Functions & Graphs | Main topic: Trig Equations & R-Form

9. $\mathrm{f}(\theta) = 5\cos\theta - 4\sin\theta \qquad \theta \in \mathbb{R}$

(a) Express $\mathrm{f}(\theta)$ in the form $R\cos(\theta + \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

The curve with equation $y = \cos\theta$ is transformed onto the curve with equation $y = \mathrm{f}(\theta)$ by a sequence of two transformations.

Given that the first transformation is a stretch and the second a translation,

(b) (i) describe fully the transformation that is a stretch,

(ii) describe fully the transformation that is a translation.

(2)

Given

$$\mathrm{g}(\theta) = \frac{90}{4 + (\mathrm{f}(\theta))^2} \qquad \theta \in \mathbb{R}$$

(c) find the range of g.

(2)

Question 7

9 marks | Trigonometric Functions & Graphs | Main topic: Trig Equations & R-Form

7. (a) Express $\cos x + 4\sin x$ in the form $R\cos(x - \alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

A scientist is studying the behaviour of seabirds in a colony.

She models the height above sea level, H metres, of one of the birds in the colony by the equation

$$H = \frac{24}{3 + \cos\left(\frac{1}{2}t\right) + 4\sin\left(\frac{1}{2}t\right)} \qquad 0 \leqslant t \leqslant 6.5$$

where t seconds is the time after it leaves the nest.

Find, according to the model,

(b) the minimum height of the seabird above sea level, giving your answer to the nearest cm,

(2)

(c) the value of t, to 2 decimal places, when $H = 10$

(4)

Question 9

8 marks | Trigonometric Functions & Graphs | Main topic: Trig Equations & R-Form

9. (a) Express $12\sin x - 5\cos x$ in the form $R\sin(x - \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the exact value of R and give the value of α in radians, to 3 decimal places.

(3)

The function g is defined by

$$g(\theta) = 10 + 12\sin\left(2\theta - \frac{\pi}{6}\right) - 5\cos\left(2\theta - \frac{\pi}{6}\right) \qquad \theta > 0$$

Find

(b) (i) the minimum value of $g(\theta)$

(ii) the smallest value of θ at which the minimum value occurs.

(3)

The function h is defined by

$$h(\beta) = 10 - (12\sin\beta - 5\cos\beta)^2$$

(c) Find the range of h.

(2)

Question 6

8 marks | Trigonometric Functions & Graphs | Main topic: Differentiation Techniques

6. (i) The curve C_1 has equation

$$y = 3\ln(x^2 - 5) - 4x^2 + 15 \qquad x > \sqrt{5}$$

Show that C_1 has a stationary point at $x = \frac{\sqrt{p}}{2}$ where p is a constant to be found.

(4)

(ii) A different curve C_2 has equation

$$y = 4x - 12\sin^2 x$$

(a) Show that, for this curve,

$$\frac{dy}{dx} = A + B\sin 2x$$

where A and B are constants to be found.

(b) Hence, state the maximum gradient of this curve.

(4)

Question 8

7 marks | Trigonometric Functions & Graphs | Main topic: Differentiation Techniques

8. A curve C has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \arcsin\left(\frac{1}{2}x\right) \qquad -2 \leqslant x \leqslant 2 \qquad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

(a) Sketch C.

(1)

(b) Given $x = 2\sin y$, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{A - x^2}}$$

where A is a constant to be found.

(3)

The point P lies on C and has y coordinate $\frac{\pi}{4}$

(c) Find the equation of the tangent to C at P. Write your answer in the form $y = mx + c$, where m and c are constants to be found.

(3)

WMA13/01 OCTOBER 2022

Question 2

7 marks | Trigonometric Functions & Graphs

Main topic: Functions, Composition & Inverses

2. The functions f and g are defined by

$$\mathrm{f}(x) = 5 - \frac{4}{3x + 2} \qquad x \geqslant 0$$

$$\mathrm{g}(x) = \left| 4\sin\left(\frac{x}{3} + \frac{\pi}{6}\right) \right| \qquad x \in \mathbb{R}$$

(a) Find the range of f

(2)

(b) (i) Find $\mathrm{f}^{-1}(x)$

(ii) Write down the domain of f^{-1}

(3)

(c) Find $\mathrm{fg}(-\pi)$

(2)

Question 8

9 marks | Trigonometric Functions & Graphs | Main topic: Trig Equations & R-Form

8. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Express $8\sin x - 15\cos x$ in the form $R\sin(x-\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R, and give the value of α, in radians, to 4 significant figures.

(3)

$$f(x) = \frac{15}{41 + 16\sin x - 30\cos x} \qquad x > 0$$

(b) Find

(i) the minimum value of $f(x)$

(ii) the smallest value of x at which this minimum value occurs.

(4)

(c) State the y coordinate of the minimum points on the curve with equation

$$y = 2f(x) - 5 \qquad x > 0$$

(1)

(d) State the smallest value of x at which a maximum point occurs for the curve with equation

$$y = -f(2x) \qquad x > 0$$

(1)

Question 2

6 marks | Trigonometric Functions & Graphs | Main topic: Trig Equations & R-Form

2. $$f(x) = \cos x + 2\sin x$$

(a) Express $f(x)$ in the form $R\cos(x - \alpha)$, where R and α are constants,

$R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

$$g(x) = 3 - 7f(2x)$$

(b) Using the answer to part (a),

(i) write down the exact maximum value of $g(x)$,

(ii) find the smallest positive value of x for which this maximum value occurs, giving your answer to 2 decimal places.

(3)

Question 4

9 marks | Trigonometric Functions & Graphs | Main topic: Trig Identities & Exact Values

4. $f(x) = 8\sin x\cos x + 4\cos^2 x - 3$

(a) Write $f(x)$ in the form

$$a\sin 2x + b\cos 2x + c$$

where a, b and c are integers to be found.

(3)

(b) Use the answer to part (a) to write $f(x)$ in the form

$$R\sin(2x + \alpha) + c$$

where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α in radians to 3 significant figures.

(3)

(c) Hence, or otherwise,

(i) state the maximum value of $f(x)$

(ii) find the **second** smallest positive value of x at which a maximum value of $f(x)$ occurs. Give your answer to 3 significant figures.

(3)

Question 9

10 marks | Trigonometric Functions & Graphs | Main topic: Applications of Differentiation

9.

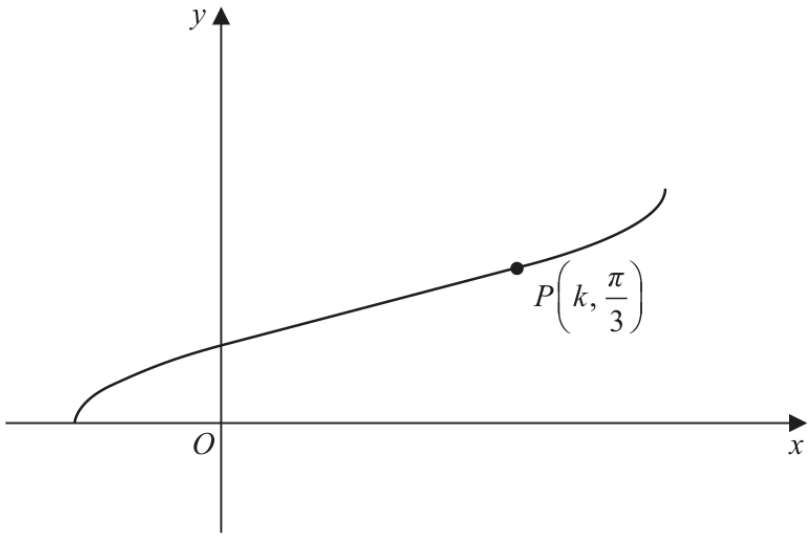


Figure 5

The curve shown in Figure 5 has equation

$$x = 4\sin^2 y - 1 \qquad 0 \leqslant y \leqslant \frac{\pi}{2}$$

The point $P\left(k, \frac{\pi}{3}\right)$ lies on the curve.

(a) Verify that $k = 2$

(1)

(b) (i) Find $\frac{\mathrm{d}x}{\mathrm{d}y}$ in terms of y

(ii) Hence show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\sqrt{x+1}\sqrt{3-x}}$

(6)

The normal to the curve at P cuts the x-axis at the point N.

(c) Find the exact area of triangle OPN, where O is the origin.

Give your answer in the form $a\pi + b\pi^2$ where a and b are constants.

(3)

Question 2

7 marks | Trigonometric Functions & Graphs | Main topic: Trig Equations & R-Form

2: **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

$$f(x) = 7\cos x - 24\sin x$$

(a) Express $f(x)$ in the form $R\cos(x + \alpha)$ where R and α are constants, $R > 0$

and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

$$g(x) = \frac{5}{90 - 3f(2x)}$$

(b) Using the answer to part (a), find

(i) the minimum value of $g(x)$, giving your answer as a fully simplified fraction,

(ii) the smallest positive value of x for which this minimum value occurs, giving your answer to 3 decimal places.

(4)

Question 4

6 marks | Trigonometric Functions & Graphs | Main topic: Trig Equations & R-Form

4. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(a) $$f(x) = \sqrt{3}\sin 2x - 3\cos 2x$$

Express $f(x)$ in the form $R\sin(2x - \alpha)$, where R and α are constants,

$$R > 0 \text{ and } 0 < \alpha < \frac{\pi}{2}$$

Give the exact value of R and the exact value of α.

(3)

(b) $$g(x) = \frac{18}{f(3x) + 4\sqrt{3}} \qquad x > 0$$

Using the answer to part (a),

(i) write down the exact minimum value of $g(x)$,

(ii) find the smallest value of x for which this minimum value occurs.

You must make your method clear.

(3)

Question 9

9 marks | Trigonometric Functions & Graphs | Main topic: Trig Equations & R-Form

9. $$\mathrm{h}(x) = 8 + 3\sin x(\cos x - 2\sin x) \quad 0 \leqslant x \leqslant 2\pi$$

(a) Write $\mathrm{h}(x)$ in the form

$$P + R\sin(2x + \alpha)$$

where P, R and α are positive constants, $0 < \alpha < \frac{\pi}{2}$

Give the exact values of P and R and give the value of α, in radians, to 3 decimal places.

(6)

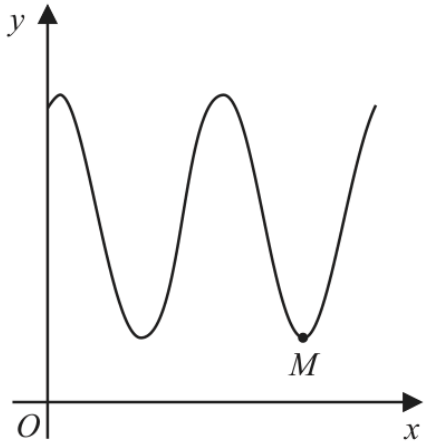


Figure 2

Figure 2 is a sketch of the curve with equation $y = \mathrm{h}(x)$

(b) State the range of h.

(1)

The point M, shown in Figure 2, is the minimum point on the curve with the largest x coordinate.

(c) Find the x coordinate of M.

(Solutions based entirely on calculator technology are not acceptable.)

(2)

TOPIC

Trig Identities & Exact Values

Every past-paper question on this topic, January 2020 to June 2026.

Question 5

8 marks | Trig Identities & Exact Values

5. (a) Use the substitution $t = \tan x$ to show that the equation

$$12\tan 2x + 5\cot x \sec^2 x = 0$$

can be written in the form

$$5t^4 - 24t^2 - 5 = 0$$

(4)

(b) Hence solve, for $0 \leqslant x < 360°$, the equation

$$12\tan 2x + 5\cot x \sec^2 x = 0$$

Show each stage of your working and give your answers to one decimal place.

(4)

Question 5

8 marks | Trig Identities & Exact Values

5. (a) Show that

$$\sin 3x \equiv 3\sin x - 4\sin^3 x$$

(4)

(b) Hence find, using algebraic integration,

$$\int_0^{\frac{\pi}{3}} \sin^3 x \, dx$$

(4)

Question 7

9 marks | Trig Identities & Exact Values

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

7. (a) Prove that

$$\frac{\sin 2x}{\cos x} + \frac{\cos 2x}{\sin x} \equiv \operatorname{cosec} x \qquad x \neq \frac{n\pi}{2} \quad n \in \mathbb{Z}$$

(3)

(b) Hence solve, for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$7 + \frac{\sin 4\theta}{\cos 2\theta} + \frac{\cos 4\theta}{\sin 2\theta} = 3\cot^2 2\theta$$

giving your answers in radians to 3 significant figures where appropriate.

(6)

Question 2

9 marks | Trig Identities & Exact Values

2. (a) Show that

$$\frac{1-\cos 2x}{2\sin 2x} \equiv k\tan x \qquad x \neq (90n)^\circ \qquad n \in \mathbb{Z}$$

where k is a constant to be found.

(3)

(b) Hence solve, for $0 < \theta < 90^\circ$

$$\frac{9(1-\cos 2\theta)}{2\sin 2\theta} = 2\sec^2\theta$$

giving your answers to one decimal place.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(6)

Question 4

7 marks | Trig Identities & Exact Values

4. **In this question you should show detailed reasoning.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that the equation

$$2\sin(\theta - 30°) = 5\cos\theta$$

can be written in the form

$$\tan\theta = 2\sqrt{3}$$

(4)

(b) Hence, or otherwise, solve for $0 \leqslant x \leqslant 360°$

$$2\sin(x - 10°) = 5\cos(x + 20°)$$

giving your answers to one decimal place.

(3)

Question 2

5 marks | Trig Identities & Exact Values

2. (a) Show that the equation

$$8\cos\theta = 3\operatorname{cosec}\theta$$

can be written in the form

$$\sin 2\theta = k$$

where k is a constant to be found.

(3)

(b) Hence find the smallest positive solution of the equation

$$8\cos\theta = 3\operatorname{cosec}\theta$$

giving your answer, in degrees, to one decimal place.

(2)

Question 9

8 marks | Trig Identities & Exact Values

9. **In this question you must show detailed reasoning.**

Solutions relying entirely on calculator technology are not acceptable.

(i) Solve, for $0 < x \leqslant \pi$, the equation

$$2\sec^2 x - 3\tan x = 2$$

giving the answers, as appropriate, to 3 significant figures.

(4)

(ii) Prove that

$$\frac{\sin 3\theta}{\sin \theta} - \frac{\cos 3\theta}{\cos \theta} \equiv 2$$

(4)

Question 7

8 marks | Trig Identities & Exact Values

7. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that the equation

$$2\sin\theta(3\cot^2 2\theta - 7) = 13\sec\theta$$

can be written as

$$3\operatorname{cosec}^2 2\theta - 13\operatorname{cosec} 2\theta - 10 = 0$$

(4)

(b) Hence solve, for $0 < \theta < \frac{\pi}{2}$, the equation

$$2\sin\theta(3\cot^2 2\theta - 7) = 13\sec\theta$$

giving your answers to 3 significant figures.

(4)

Question 9

10 marks | Trig Identities & Exact Values

9. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that

$$\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} \equiv \operatorname{cosec} x \qquad x \neq \frac{n\pi}{2} \qquad n \in \mathbb{Z}$$

(3)

(b) Hence solve, for $0 < \theta < \frac{\pi}{2}$

$$\left(\frac{\cos 2\theta}{\sin \theta} + \frac{\sin 2\theta}{\cos \theta}\right)^2 = 6\cot\theta - 4$$

giving your answers to 3 significant figures as appropriate.

(5)

(c) Using the result from part (a), or otherwise, find the exact value of

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x}\right) \cot x \, dx$$

(2)

Question 3

6 marks | Trig Identities & Exact Values

3. (a) Using the identity for $\cos(A + B)$, prove that

$$\cos 2A \equiv 2\cos^2 A - 1$$

(2)

(b) Hence, using algebraic integration, find the exact value of

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{8}} (5 - 4\cos^2 3x)\,\mathrm{d}x$$

(4)

Question 8

8 marks | Trig Identities & Exact Values

8. (a) Prove that

$$2\operatorname{cosec}^2 2\theta(1 - \cos 2\theta) \equiv 1 + \tan^2\theta$$

(4)

(b) Hence solve for $0 < x < 360°$, where $x \neq (90n)°$, $n \in \mathbb{N}$, the equation

$$2\operatorname{cosec}^2 2x(1 - \cos 2x) = 4 + 3\sec x$$

giving your answers to one decimal place.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(4)

Question 4

9 marks | Trig Identities & Exact Values

4. $$f(x) = 8\sin x \cos x + 4\cos^2 x - 3$$

(a) Write $f(x)$ in the form

$$a\sin 2x + b\cos 2x + c$$

where a, b and c are integers to be found.

(3)

(b) Use the answer to part (a) to write $f(x)$ in the form

$$R\sin(2x + \alpha) + c$$

where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α in radians to 3 significant figures.

(3)

(c) Hence, or otherwise,

(i) state the maximum value of $f(x)$

(ii) find the **second** smallest positive value of x at which a maximum value of $f(x)$ occurs. Give your answer to 3 significant figures.

(3)

Question 7

8 marks | Trig Identities & Exact Values

7. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Given that

$$\sqrt{2}\sin(x + 45°) = \cos(x - 60°)$$

show that

$$\tan x = -2 - \sqrt{3}$$

(4)

(b) Hence or otherwise, solve, for $0 \leqslant \theta < 180°$

$$\sqrt{2}\sin(2\theta) = \cos(2\theta - 105°)$$

(4)

Question 5

8 marks | Trig Identities & Exact Values

5. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that $\sin 3x$ can be written in the form

$$P\sin x + Q\sin^3 x$$

where P and Q are constants to be found.

(4)

(b) Hence or otherwise, solve, for $0 < \theta \leqslant 360°$, the equation

$$2\sin 3\theta = 5\sin 2\theta$$

giving your answers, in degrees, to one decimal place as appropriate.

(4)

Question 8

8 marks | Trig Identities & Exact Values

8: **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(a) Prove that

$$\tan 3x \equiv \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x} \qquad x \neq (2n+1)\frac{\pi}{6} \qquad n \in \mathbb{Z}$$

(3)

(b) Hence solve, for $0 < \theta < \frac{\pi}{2}$

$$\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = 2\sec^2 3\theta - 8$$

giving your answers to 2 decimal places.

(5)

Question 9

9 marks | Trig Identities & Exact Values

9. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(a) Express

$$6\sin^2\theta\cot 2\theta + 4\sin\theta\cos\theta$$

in terms of $\sin 2\theta$ and $\cos 2\theta$ only.

(3)

(b) Hence show that the equation

$$3\cot 2\theta - 14 = 6\sin^2\theta\cot 2\theta + 4\sin\theta\cos\theta$$

can be written in the form

$$5\sin^2 2\theta + 14\sin 2\theta - 3 = 0$$

(3)

(c) Hence solve, for $0 < x < 90°$, the equation

$$3\cot 2x - 14 = 6\sin^2 x\cot 2x + 4\sin x\cos x$$

giving your answers to one decimal place.

(3)

Question 6

7 marks | Trig Identities & Exact Values

6. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(a) Show that the equation

$$2\cos(\theta - 60°) = 3\sin\theta$$

can be written in the form

$$\tan\theta = \frac{1}{6}\left(3 + \sqrt{3}\right)$$

(4)

(b) Hence solve, for $0 \leqslant x < 180°$, the equation

$$2\cos(2x - 70°) = 3\sin(2x - 10°)$$

giving your solutions to one decimal place.

(3)

Question 8

9 marks | Trig Identities & Exact Values

8. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(a) Show that

$$\frac{\sec\theta\,\text{cosec}\theta}{4\text{cosec}\,4\theta} \equiv \cos 2\theta$$

(2)

(b) Hence show that the equation

$$4(5\cos^2\theta - 9)\,\text{cosec}\,4\theta = 3(1+\tan^2 2\theta)\sec\theta\,\text{cosec}\,\theta$$

can be written as

$$a\cos^2 2\theta + b\cos 2\theta + c = 0$$

where a, b and c are integers to be found.

(3)

(c) Hence, solve for $0 < x < 180°$, $x \neq (45n)°$, $n \in \mathbb{N}$, the equation

$$4(5\cos^2 x - 9)\,\text{cosec}\,4x = 3(1+\tan^2 2x)\sec x\,\text{cosec}\,x$$

giving the answers to one decimal place.

(4)

Question 4

11 marks | Trig Identities & Exact Values | Main topic: Differentiation Techniques

4. (i) $$f(x) = \frac{(2x+5)^2}{x-3} \qquad x \neq 3$$

(a) Find $f'(x)$ in the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are fully factorised quadratic expressions.

(b) Hence find the range of values of x for which $f(x)$ is increasing.

(6)

(ii) $$g(x) = x\sqrt{\sin 4x} \qquad 0 \leqslant x < \frac{\pi}{4}$$

The curve with equation $y = g(x)$ has a maximum at the point M.

Show that the x coordinate of M satisfies the equation

$$\tan 4x + kx = 0$$

where k is a constant to be found.

(5)

Question 9

7 marks | Trig Identities & Exact Values | Main topic: Trig Equations & R-Form

9. $\mathrm{f}(\theta) = 5\cos\theta - 4\sin\theta \qquad \theta \in \mathbb{R}$

(a) Express $\mathrm{f}(\theta)$ in the form $R\cos(\theta + \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

The curve with equation $y = \cos\theta$ is transformed onto the curve with equation $y = \mathrm{f}(\theta)$ by a sequence of two transformations.

Given that the first transformation is a stretch and the second a translation,

(b) (i) describe fully the transformation that is a stretch,

(ii) describe fully the transformation that is a translation.

(2)

Given

$$\mathrm{g}(\theta) = \frac{90}{4 + (\mathrm{f}(\theta))^2} \qquad \theta \in \mathbb{R}$$

(c) find the range of g.

(2)

WMA13/01 OCTOBER 2020

Question 1

5 marks | Trig Identities & Exact Values | Main topic: Trig Equations & R-Form

1. Solve, for $0 \leqslant x < 360°$, the equation

$$2\cos 2x = 7\cos x$$

giving your solutions to one decimal place.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(5)

Question 7

9 marks | Trig Identities & Exact Values | Main topic: Trig Equations & R-Form

7. (a) Express $\cos x + 4\sin x$ in the form $R\cos(x - \alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

A scientist is studying the behaviour of seabirds in a colony.

She models the height above sea level, H metres, of one of the birds in the colony by the equation

$$H = \frac{24}{3 + \cos\left(\frac{1}{2}t\right) + 4\sin\left(\frac{1}{2}t\right)} \qquad 0 \leqslant t \leqslant 6.5$$

where t seconds is the time after it leaves the nest.

Find, according to the model,

(b) the minimum height of the seabird above sea level, giving your answer to the nearest cm,

(2)

(c) the value of t, to 2 decimal places, when $H = 10$

(4)

Question 8

9 marks | Trig Identities & Exact Values | Main topic: Differentiation Techniques

8. (i) The curve C has equation $y = g(x)$ where

$$g(x) = e^{3x} \sec 2x \qquad -\frac{\pi}{4} < x < \frac{\pi}{4}$$

(a) Find $g'(x)$

(2)

(b) Hence find the x coordinate of the stationary point of C.

(3)

(ii) A different curve has equation

$$x = \ln(\sin y) \qquad 0 < y < \frac{\pi}{2}$$

Show that

$$\frac{dy}{dx} = \frac{e^x}{f(x)}$$

where $f(x)$ is a function of e^x that should be found.

(4)

Question 6

8 marks | Trig Identities & Exact Values | Main topic: Numerical Methods

6.

$$f(x) = x\cos\left(\frac{x}{3}\right) \qquad x > 0$$

(a) Find $f'(x)$

(2)

(b) Show that the equation $f'(x) = 0$ can be written as

$$x = k\arctan\left(\frac{k}{x}\right)$$

where k is an integer to be found.

(2)

(c) Starting with $x_1 = 2.5$ use the iteration formula

$$x_{n+1} = k\arctan\left(\frac{k}{x_n}\right)$$

with the value of k found in part (b), to calculate the values of x_2 and x_6 giving your answers to 3 decimal places.

(2)

(d) Using a suitable interval and a suitable function that should be stated, show that a root of $f'(x) = 0$ is 2.581 correct to 3 decimal places.

(2)

Question 10

10 marks | Trig Identities & Exact Values | Main topic: Differentiation Techniques

10. The curve C has equation

$$x = 3\sec^2 2y \qquad x > 3 \qquad 0 < y < \frac{\pi}{4}$$

(a) Find $\frac{dx}{dy}$ in terms of y.

(2)

(b) Hence show that

$$\frac{dy}{dx} = \frac{p}{qx\sqrt{x-3}}$$

where p is irrational and q is an integer, stating the values of p and q.

(3)

(c) Find the equation of the normal to C at the point where $y = \frac{\pi}{12}$, giving your answer in the form $y = mx + c$, giving m and c as exact irrational numbers.

(5)

Question 1

7 marks | Trig Identities & Exact Values | Main topic: Numerical Methods

1. The curve C has equation

$$y = x^2 \cos\left(\frac{1}{2}x\right) \qquad 0 < x \leqslant \pi$$

The curve has a stationary point at the point P.

(a) Show, using calculus, that the x coordinate of P is a solution of the equation

$$x = 2\arctan\left(\frac{4}{x}\right)$$

(4)

Using the iteration formula

$$x_{n+1} = 2\arctan\left(\frac{4}{x_n}\right) \qquad x_1 = 2$$

(b) find the value of x_2 and the value of x_6, giving your answers to 3 decimal places.

(3)

Question 7

5 marks | Trig Identities & Exact Values | Main topic: Differentiation Techniques

7. Given that

$$x = 6\sin^2 2y \qquad 0 < y < \frac{\pi}{4}$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{A\sqrt{(Bx - x^2)}}$$

where A and B are integers to be found.

(5)

Question 9

8 marks | Trig Identities & Exact Values | Main topic: Trig Equations & R-Form

9. (a) Express $12\sin x - 5\cos x$ in the form $R\sin(x - \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the exact value of R and give the value of α in radians, to 3 decimal places.

(3)

The function g is defined by

$$g(\theta) = 10 + 12\sin\left(2\theta - \frac{\pi}{6}\right) - 5\cos\left(2\theta - \frac{\pi}{6}\right) \qquad \theta > 0$$

Find

(b) (i) the minimum value of $g(\theta)$

(ii) the smallest value of θ at which the minimum value occurs.

(3)

The function h is defined by

$$h(\beta) = 10 - (12\sin\beta - 5\cos\beta)^2$$

(c) Find the range of h.

(2)

Question 6

8 marks | Trig Identities & Exact Values | Main topic: Differentiation Techniques

6. (i) The curve C_1 has equation

$$y = 3\ln(x^2 - 5) - 4x^2 + 15 \qquad x > \sqrt{5}$$

Show that C_1 has a stationary point at $x = \frac{\sqrt{p}}{2}$ where p is a constant to be found.

(4)

(ii) A different curve C_2 has equation

$$y = 4x - 12\sin^2 x$$

(a) Show that, for this curve,

$$\frac{dy}{dx} = A + B\sin 2x$$

where A and B are constants to be found.

(b) Hence, state the maximum gradient of this curve.

(4)

Question 8

7 marks | Trig Identities & Exact Values | Main topic: Differentiation Techniques

8. A curve C has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \arcsin\left(\frac{1}{2}x\right) \qquad -2 \leqslant x \leqslant 2 \qquad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

(a) Sketch C.

(1)

(b) Given $x = 2\sin y$, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{A - x^2}}$$

where A is a constant to be found.

(3)

The point P lies on C and has y coordinate $\frac{\pi}{4}$

(c) Find the equation of the tangent to C at P. Write your answer in the form $y = mx + c$, where m and c are constants to be found.

(3)

Question 10

7 marks | Trig Identities & Exact Values | Main topic: Integration

10.

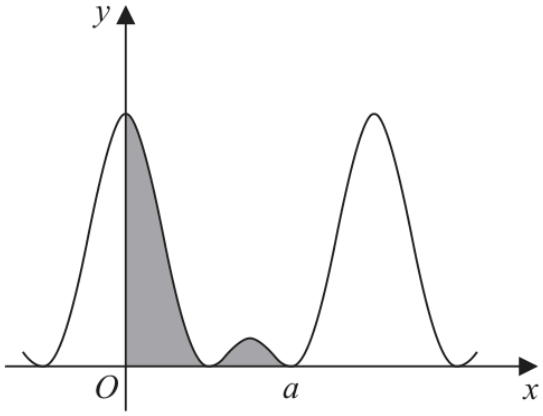


Figure 4

Figure 4 shows a sketch of part of the curve with equation

$$y = (1 + 2\cos 2x)^2$$

(a) Show that

$$(1 + 2\cos 2x)^2 \equiv p + q\cos 2x + r\cos 4x$$

where p, q and r are constants to be found.

(2)

The curve touches the positive x-axis for the second time when $x = a$, as shown in Figure 4.

The regions bounded by the curve, the y-axis and the x-axis up to $x = a$ are shown shaded in Figure 4.

(b) Find, using algebraic integration and making your method clear, the exact total area of the shaded regions. Write your answer in simplest form.

(5)

Question 9

9 marks | Trig Identities & Exact Values | Main topic: Trig Equations & R-Form

9.

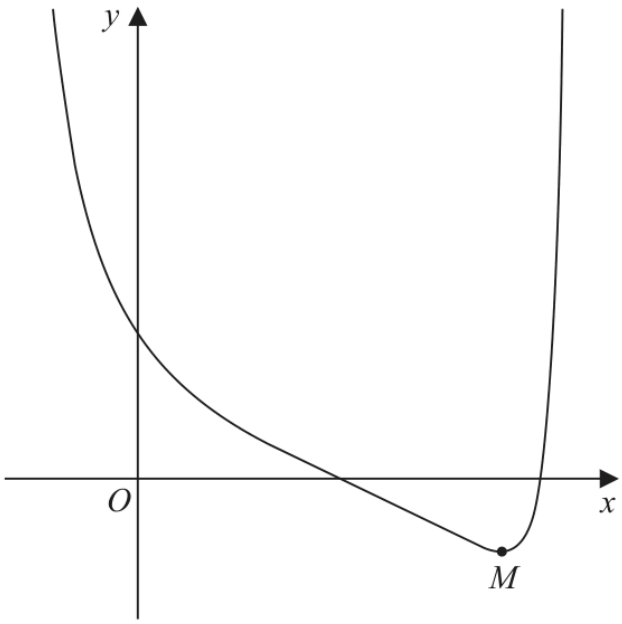


Figure 5

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 5 shows the curve with equation

$$y = \frac{1 + 2\cos x}{1 + \sin x} \qquad -\frac{\pi}{2} < x < \frac{3\pi}{2}$$

The point M, shown in Figure 5, is the minimum point on the curve.

(a) Show that the x coordinate of M is a solution of the equation

$$2\sin x + \cos x = -2$$

(4)

(b) Hence find, to 3 significant figures, the x coordinate of M.

(5)

WMA13/01 OCTOBER 2022

Question 6

6 marks | Trig Identities & Exact Values | Main topic: Differentiation Techniques

6.

$$y = \frac{2 + 3\sin x}{\cos x + \sin x}$$

Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a\tan x + b\sec x + c}{\sec x + 2\sin x}$$

where a, b and c are integers to be found.

(6)

Question 8

9 marks | Trig Identities & Exact Values | Main topic: Trig Equations & R-Form

8. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Express $8\sin x - 15\cos x$ in the form $R\sin(x-\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R, and give the value of α, in radians, to 4 significant figures.

(3)

$$f(x) = \frac{15}{41 + 16\sin x - 30\cos x} \qquad x > 0$$

(b) Find

(i) the minimum value of $f(x)$

(ii) the smallest value of x at which this minimum value occurs.

(4)

(c) State the y coordinate of the minimum points on the curve with equation

$$y = 2f(x) - 5 \qquad x > 0$$

(1)

(d) State the smallest value of x at which a maximum point occurs for the curve with equation

$$y = -f(2x) \qquad x > 0$$

(1)

Question 9

9 marks | Trig Identities & Exact Values | Main topic: Trig Equations & R-Form

9. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

Given that $\cos 2\theta - \sin 3\theta \neq 0$

(a) prove that

$$\frac{\cos^2\theta}{\cos 2\theta - \sin 3\theta} \equiv \frac{1 + \sin\theta}{1 - 2\sin\theta - 4\sin^2\theta}$$

(4)

(b) Hence solve, for $0 < \theta \leqslant 360°$

$$\frac{\cos^2\theta}{\cos 2\theta - \sin 3\theta} = 2\operatorname{cosec}\theta$$

Give your answers to one decimal place.

(5)

Question 2

6 marks | Trig Identities & Exact Values | Main topic: Trig Equations & R-Form

2. $$\mathrm{f}(x) = \cos x + 2\sin x$$

(a) Express $\mathrm{f}(x)$ in the form $R\cos(x - \alpha)$, where R and α are constants,

$R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

$$\mathrm{g}(x) = 3 - 7\mathrm{f}(2x)$$

(b) Using the answer to part (a),

(i) write down the exact maximum value of $\mathrm{g}(x)$,

(ii) find the smallest positive value of x for which this maximum value occurs, giving your answer to 2 decimal places.

(3)

Question 5

9 marks | Trig Identities & Exact Values | Main topic: Trig Equations & R-Form

5. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Prove that

$$\cot^2 x - \tan^2 x \equiv 4\cot 2x \operatorname{cosec} 2x \qquad x \neq \frac{n\pi}{2} \quad n \in \mathbb{Z}$$

(4)

(b) Hence solve, for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$4\cot 2\theta \operatorname{cosec} 2\theta = 2\tan^2 \theta$$

giving your answers to 2 decimal places.

(5)

WMA13/01 JANUARY 2023

Question 7

9 marks | Trig Identities & Exact Values | Main topic: Applications of Differentiation

7. The curve C has equation

$$x = 3\tan\left(y - \frac{\pi}{6}\right) \qquad x \in \mathbb{R} \quad -\frac{\pi}{3} < y < \frac{2\pi}{3}$$

(a) Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a}{x^2 + b}$$

where a and b are integers to be found.

(4)

The point P with y coordinate $\frac{\pi}{3}$ lies on C.

Given that the tangent to C at P crosses the x-axis at the point Q.

(b) find, in simplest form, the exact x coordinate of Q.

(5)

Question 8

5 marks | Trig Identities & Exact Values | Main topic: Integration

8. Find, in simplest form,

$$\int (2\cos x - \sin x)^2 \, \mathrm{d}x$$

(5)

Question 3

7 marks | Trig Identities & Exact Values | Main topic: Differentiation Techniques

3. (i) Find $\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\sin^2 3x\right)$ writing your answer in simplest form.

(2)

(ii)(a) Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(3x^2 - 4\right)^6$

(2)

(b) Hence show that

$$\int_0^{\sqrt{2}} x\left(3x^2 - 4\right)^5 \mathrm{d}x = R$$

where R is an integer to be found.

(*Solutions relying on calculator technology are not acceptable.*)

(3)

WMA13/01 MAY/JUNE 2023

Question 5

7 marks | Trig Identities & Exact Values | Main topic: Trig Equations & R-Form

5. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(i) Solve, for $0 < x < \pi$

$$(x - 2)\left(\sqrt{3}\sec x + 2\right) = 0$$

(3)

(ii) Solve, for $0 < \theta < 360°$

$$10\sin\theta = 3\cos 2\theta$$

(4)

Question 10

9 marks | Trig Identities & Exact Values | Main topic: Differentiation Techniques

10. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

A curve C has equation

$$x = \sin^2 4y \qquad 0 \leqslant y \leqslant \frac{\pi}{8} \qquad 0 \leqslant x \leqslant 1$$

The point P with x coordinate $\frac{1}{4}$ lies on C

(a) Find the exact y coordinate of P

(2)

(b) Find $\frac{\mathrm{d}x}{\mathrm{d}y}$

(2)

(c) Hence show that $\frac{\mathrm{d}y}{\mathrm{d}x}$ can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{q + r(x + s)^2}}$$

where q, r and s are constants to be found.

(3)

Using the answer to part (c),

(d) (i) state the x coordinate of the point where the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ is a minimum,

(ii) state the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at this point.

(2)

Question 6

7 marks | Trig Identities & Exact Values | Main topic: Applications of Differentiation

6.

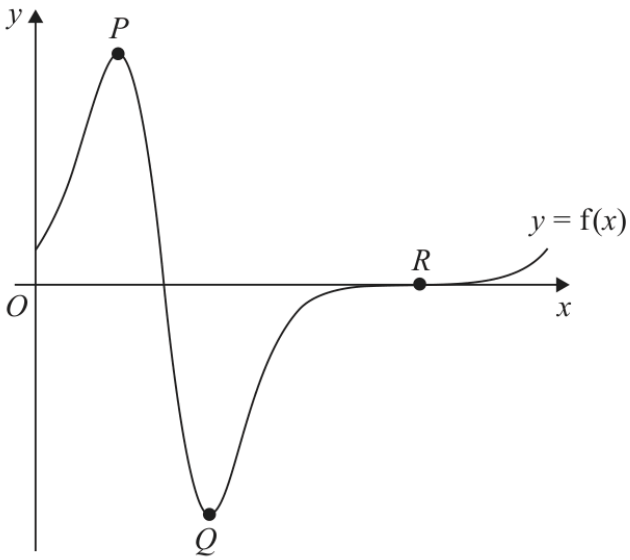


Figure 1

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 1 shows a sketch of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 2\mathrm{e}^{3\sin x}\cos x \qquad 0 \leqslant x \leqslant 2\pi$$

The curve intersects the x-axis at point R, as shown in Figure 1.

(a) State the coordinates of R

(1)

The curve has two turning points, at point P and point Q, also shown in Figure 1.

(b) Show that, at points P and Q,

$$a\sin^2 x + b\sin x + c = 0$$

where a, b and c are integers to be found.

(4)

(c) Hence find the x coordinate of point Q, giving your answer to 3 decimal places.

(2)

Question 9

8 marks | Trig Identities & Exact Values | Main topic: Trig Equations & R-Form

9. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that the equation

$$\frac{3\sin\theta\cos\theta}{\cos\theta+\sin\theta} = (2+\sec 2\theta)(\cos\theta-\sin\theta)$$

can be written in the form

$$3\sin 2\theta - 4\cos 2\theta = 2$$

(3)

(b) Hence solve for $\pi < x < \frac{3\pi}{2}$

$$\frac{3\sin x\cos x}{\cos x+\sin x} = (2+\sec 2x)(\cos x-\sin x)$$

giving the answer to 3 significant figures.

(5)

Question 9

10 marks | Trig Identities & Exact Values | Main topic: Applications of Differentiation

9.

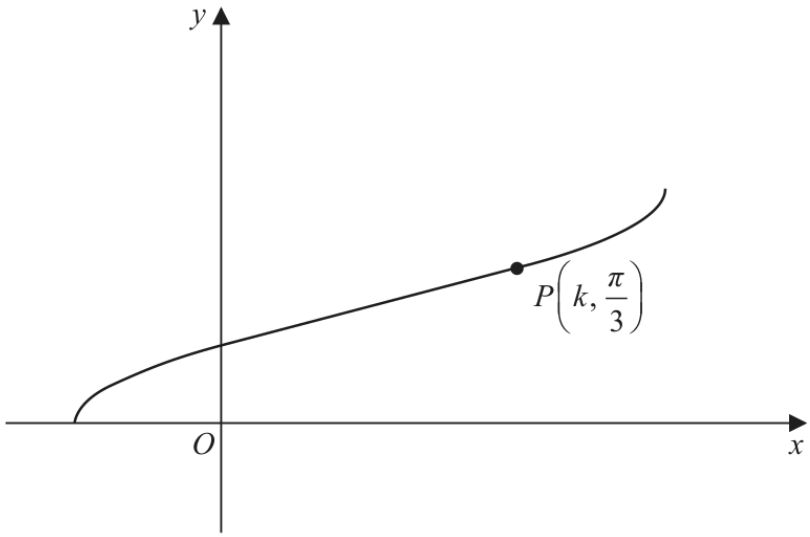


Figure 5

The curve shown in Figure 5 has equation

$$x = 4\sin^2 y - 1 \qquad 0 \leqslant y \leqslant \frac{\pi}{2}$$

The point $P\left(k, \frac{\pi}{3}\right)$ lies on the curve.

(a) Verify that $k = 2$

(1)

(b) (i) Find $\frac{\mathrm{d}x}{\mathrm{d}y}$ in terms of y

(ii) Hence show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\sqrt{x+1}\sqrt{3-x}}$

(6)

The normal to the curve at P cuts the x-axis at the point N.

(c) Find the exact area of triangle OPN, where O is the origin.

Give your answer in the form $a\pi + b\pi^2$ where a and b are constants.

(3)

WMA13/01 OCTOBER 2024

Question 1

5 marks | Trig Identities & Exact Values | Main topic: Trig Equations & R-Form

1. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

Solve, for $0 < \theta \leqslant 360°$, the equation

$$3\tan^2\theta + 7\sec\theta - 3 = 0$$

giving your answers to one decimal place.

(5)

WMA13/01 JANUARY 2025

Question 1

6 marks | Trig Identities & Exact Values | Main topic: Numerical Methods

1. $f(x) = 2\sec x + 6x - 3 \qquad 0 < x < \frac{\pi}{2}$

The equation $f(x) = 0$ has a single root α

(a) Show that $0.1 < \alpha < 0.2$

(2)

(b) Show that α is a solution of

$$x = \frac{1}{2} - \frac{1}{3\cos x}$$

(1)

The iterative formula

$$x_{n+1} = \frac{1}{2} - \frac{1}{3\cos x_n}$$

is used to find α

(c) Starting with $x_1 = 0.15$ and using the iterative formula,

(i) find, to 4 decimal places, the value of x_2

(ii) find, to 4 decimal places, the value of α

(3)

Question 8

9 marks | Trig Identities & Exact Values | Main topic: Trig Equations & R-Form

8. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(i) Solve, for $0 < \theta < \pi$

$$3\,\text{cosec}\,\theta = 8\cos\theta$$

giving your answers, in radians, to 3 significant figures.

(5)

(ii) Solve, for $0 < x < 180°$

$$\frac{\tan 2x - \tan 70°}{1 + \tan 2x \tan 70°} = -\frac{3}{8}$$

giving your answers, in degrees, to one decimal place.

(4)

Question 10

9 marks | Trig Identities & Exact Values | Main topic: Differentiation Techniques

10.

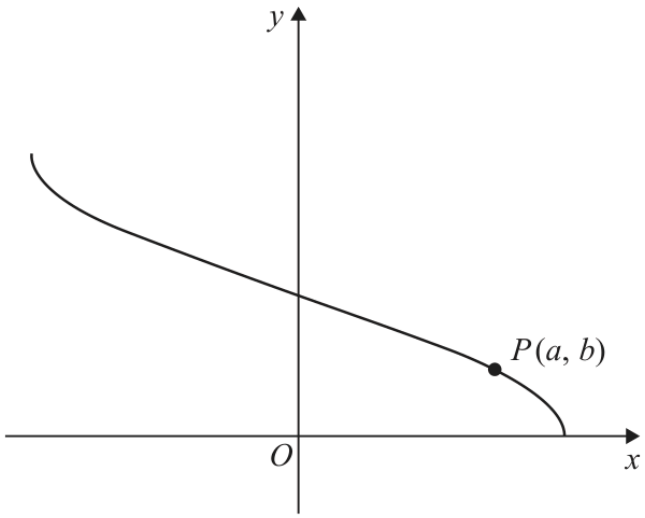


Figure 2

Figure 2 shows a sketch of the curve with equation

$$x = 3\cos 2y \qquad -3 \leqslant x \leqslant 3 \qquad 0 \leqslant y \leqslant \frac{\pi}{2}$$

(a) Find $\frac{\mathrm{d}x}{\mathrm{d}y}$ in terms of y.

(2)

(b) Hence show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{k}{\sqrt{9 - x^2}}$$

where k is a constant to be found.

(3)

The point $P(a, b)$ lies on the curve and is shown in Figure 2.

Given that

- the gradient of the curve at P is $-\frac{1}{4}$
- both a and b are positive

(c) find the exact values of a and b.

(4)

Question 2

7 marks | Trig Identities & Exact Values | Main topic: Trig Equations & R-Form

2: **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

$$f(x) = 7\cos x - 24\sin x$$

(a) Express $f(x)$ in the form $R\cos(x + \alpha)$ where R and α are constants, $R > 0$

and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

$$g(x) = \frac{5}{90 - 3f(2x)}$$

(b) Using the answer to part (a), find

(i) the minimum value of $g(x)$, giving your answer as a fully simplified fraction,

(ii) the smallest positive value of x for which this minimum value occurs, giving your answer to 3 decimal places.

(4)

WMA13/01 MAY/JUNE 2025

Question 5

4 marks | Trig Identities & Exact Values | Main topic: Integration

5: Find

(i) $\int \sin^2 3x \, dx$

(2)

(ii) $\int x(x^2 + 4)^{\frac{3}{2}} dx$

(2)

Question 7

9 marks | Trig Identities & Exact Values | Main topic: Numerical Methods

7: A continuous curve has equation

$$y = e^{-x^2} \sin 3x \qquad 0 \leqslant x \leqslant \frac{\pi}{3}$$

The curve has a stationary point at the point P.

(a) Show, using calculus, that the x coordinate of P is a solution of the equation

$$x = \frac{1}{3} \arctan\left(\frac{3}{2x}\right)$$

(4)

Using the iteration formula

$$x_{n+1} = \frac{1}{3} \arctan\left(\frac{3}{2x_n}\right) \qquad x_1 = 0.4$$

(b) find the value of

(i) x_2

(ii) x_4

giving your answers to 4 decimal places.

(3)

(c) Using a suitable interval and a suitable function which should be stated, show that the x coordinate of P is 0.430 correct to 3 decimal places.

(2)

Question 9

11 marks | Trig Identities & Exact Values | Main topic: Differentiation Techniques

9:

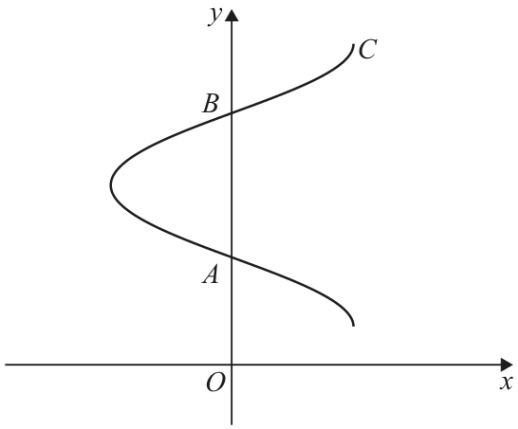


Figure 2

Figure 2 shows a sketch of the curve C with equation

$$x = \frac{2}{3}\sin\left(3y + \frac{\pi}{4}\right) \qquad \frac{\pi}{12} < y < \frac{3\pi}{4}$$

The curve intersects the y-axis at the points A and B as shown.

(a) Find the exact value of the y coordinate of

- point A
- point B

(3)

(b) Show that

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{p - qx^2}$$

where p and q are integers to be found.

(4)

The **normal** to C at A and the **tangent** to C at B intersect at the point D.

Using

- the answer to part (b)
- the sketch of curve C in Figure 2

(c) find, in simplest form, the exact x coordinate of D.

(4)

Question 4

6 marks | Trig Identities & Exact Values | Main topic: Trig Equations & R-Form

4. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(a) $$f(x) = \sqrt{3}\sin 2x - 3\cos 2x$$

Express $f(x)$ in the form $R\sin(2x - \alpha)$, where R and α are constants,

$$R > 0 \text{ and } 0 < \alpha < \frac{\pi}{2}$$

Give the exact value of R and the exact value of α.

(3)

(b) $$g(x) = \frac{18}{f(3x) + 4\sqrt{3}} \qquad x > 0$$

Using the answer to part (a),

(i) write down the exact minimum value of $g(x)$,

(ii) find the smallest value of x for which this minimum value occurs.

You must make your method clear.

(3)

Question 8

4 marks | Trig Identities & Exact Values | Main topic: Differentiation Techniques

8. A curve has equation

$$x = e^{2\tan y} \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Show that

$$\frac{dy}{dx} = \frac{a}{x\left(b + (\ln x)^n\right)}$$

where a, b and n are integers to be found.

(4)

Question 9

9 marks | Trig Identities & Exact Values | Main topic: Trig Equations & R-Form

9. $$\mathrm{h}(x) = 8 + 3\sin x(\cos x - 2\sin x) \quad 0 \leqslant x \leqslant 2\pi$$

(a) Write $\mathrm{h}(x)$ in the form

$$P + R\sin(2x + \alpha)$$

where P, R and α are positive constants, $0 < \alpha < \frac{\pi}{2}$

Give the exact values of P and R and give the value of α, in radians, to 3 decimal places.

(6)

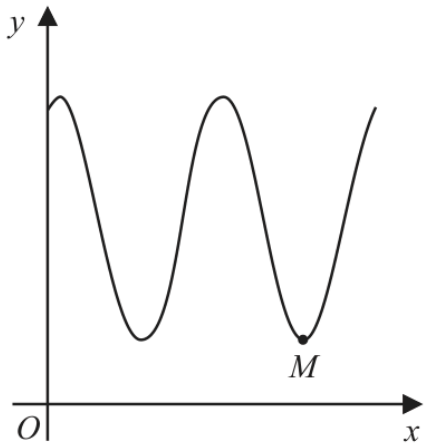


Figure 2

Figure 2 is a sketch of the curve with equation $y = \mathrm{h}(x)$

(b) State the range of h.

(1)

The point M, shown in Figure 2, is the minimum point on the curve with the largest x coordinate.

(c) Find the x coordinate of M.

(Solutions based entirely on calculator technology are not acceptable.)

(2)

WMA13/01 MAY/JUNE 2026

Question 4

5 marks | Trig Identities & Exact Values | Main topic: Integration

4. **In this question you must show all stages of your working.**
Solutions relying on calculator technology are not acceptable.

Using algebraic integration, find the exact value of

$$\int_{\frac{\pi}{18}}^{\frac{\pi}{12}} (\sin 3x + \cos 3x)^2 \, \mathrm{d}x$$

(5)

Dr Eslam Ahmed · eliteigcse.com

TOPIC

Trig Equations & R-Form

Every past-paper question on this topic, January 2020 to June 2026.

Question 9

7 marks | Trig Equations & R-Form

9. $$f(\theta) = 5\cos\theta - 4\sin\theta \qquad \theta \in \mathbb{R}$$

(a) Express $f(\theta)$ in the form $R\cos(\theta + \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

The curve with equation $y = \cos\theta$ is transformed onto the curve with equation $y = f(\theta)$ by a sequence of two transformations.

Given that the first transformation is a stretch and the second a translation,

(b) (i) describe fully the transformation that is a stretch,

(ii) describe fully the transformation that is a translation.

(2)

Given

$$g(\theta) = \frac{90}{4 + (f(\theta))^2} \qquad \theta \in \mathbb{R}$$

(c) find the range of g.

(2)

WMA13/01 OCTOBER 2020

Question 1

5 marks | Trig Equations & R-Form

1. Solve, for $0 \leqslant x < 360°$, the equation

$$2\cos 2x = 7\cos x$$

giving your solutions to one decimal place.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(5)

Question 7

9 marks | Trig Equations & R-Form

7. (a) Express $\cos x + 4\sin x$ in the form $R\cos(x - \alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

A scientist is studying the behaviour of seabirds in a colony.

She models the height above sea level, H metres, of one of the birds in the colony by the equation

$$H = \frac{24}{3 + \cos\left(\frac{1}{2}t\right) + 4\sin\left(\frac{1}{2}t\right)} \qquad 0 \leqslant t \leqslant 6.5$$

where t seconds is the time after it leaves the nest.

Find, according to the model,

(b) the minimum height of the seabird above sea level, giving your answer to the nearest cm,

(2)

(c) the value of t, to 2 decimal places, when $H = 10$

(4)

Question 9

8 marks | Trig Equations & R-Form

9. (a) Express $12\sin x - 5\cos x$ in the form $R\sin(x - \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the exact value of R and give the value of α in radians, to 3 decimal places.

(3)

The function g is defined by

$$g(\theta) = 10 + 12\sin\left(2\theta - \frac{\pi}{6}\right) - 5\cos\left(2\theta - \frac{\pi}{6}\right) \qquad \theta > 0$$

Find

(b) (i) the minimum value of $g(\theta)$

(ii) the smallest value of θ at which the minimum value occurs.

(3)

The function h is defined by

$$h(\beta) = 10 - (12\sin\beta - 5\cos\beta)^2$$

(c) Find the range of h.

(2)

Question 9

9 marks | Trig Equations & R-Form

9.

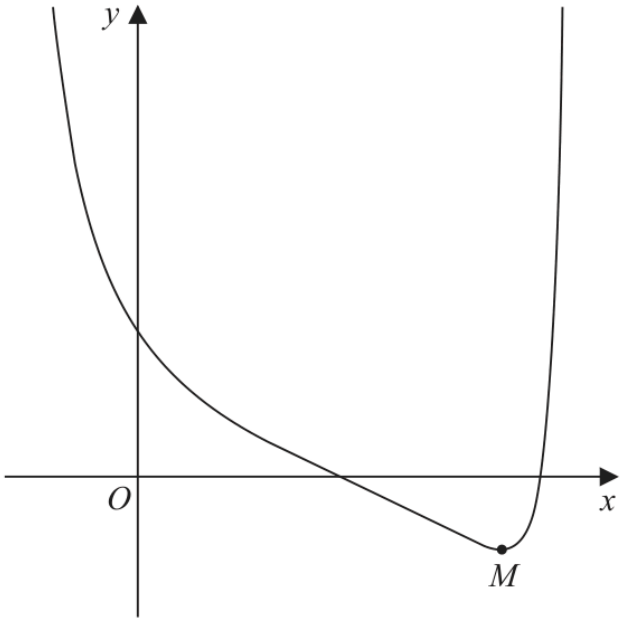


Figure 5

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 5 shows the curve with equation

$$y = \frac{1 + 2\cos x}{1 + \sin x} \qquad -\frac{\pi}{2} < x < \frac{3\pi}{2}$$

The point M, shown in Figure 5, is the minimum point on the curve.

(a) Show that the x coordinate of M is a solution of the equation

$$2\sin x + \cos x = -2$$

(4)

(b) Hence find, to 3 significant figures, the x coordinate of M.

(5)

Question 8

9 marks | Trig Equations & R-Form

8. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Express $8\sin x - 15\cos x$ in the form $R\sin(x-\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R, and give the value of α, in radians, to 4 significant figures.

(3)

$$f(x) = \frac{15}{41 + 16\sin x - 30\cos x} \qquad x > 0$$

(b) Find

(i) the minimum value of $f(x)$

(ii) the smallest value of x at which this minimum value occurs.

(4)

(c) State the y coordinate of the minimum points on the curve with equation

$$y = 2f(x) - 5 \qquad x > 0$$

(1)

(d) State the smallest value of x at which a maximum point occurs for the curve with equation

$$y = -f(2x) \qquad x > 0$$

(1)

Question 9

9 marks | Trig Equations & R-Form

9. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

Given that $\cos 2\theta - \sin 3\theta \neq 0$

(a) prove that

$$\frac{\cos^2\theta}{\cos 2\theta - \sin 3\theta} \equiv \frac{1 + \sin\theta}{1 - 2\sin\theta - 4\sin^2\theta}$$

(4)

(b) Hence solve, for $0 < \theta \leqslant 360°$

$$\frac{\cos^2\theta}{\cos 2\theta - \sin 3\theta} = 2\operatorname{cosec}\theta$$

Give your answers to one decimal place.

(5)

Question 2

6 marks | Trig Equations & R-Form

2. $$f(x) = \cos x + 2\sin x$$

(a) Express $f(x)$ in the form $R\cos(x - \alpha)$, where R and α are constants,

$R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

$$g(x) = 3 - 7f(2x)$$

(b) Using the answer to part (a),

(i) write down the exact maximum value of $g(x)$,

(ii) find the smallest positive value of x for which this maximum value occurs, giving your answer to 2 decimal places.

(3)

Question 5

9 marks | Trig Equations & R-Form

5. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Prove that

$$\cot^2 x - \tan^2 x \equiv 4\cot 2x \operatorname{cosec} 2x \qquad x \neq \frac{n\pi}{2} \quad n \in \mathbb{Z}$$

(4)

(b) Hence solve, for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$4\cot 2\theta \operatorname{cosec} 2\theta = 2\tan^2 \theta$$

giving your answers to 2 decimal places.

(5)

Question 5

7 marks | Trig Equations & R-Form

5. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(i) Solve, for $0 < x < \pi$

$$(x - 2)\left(\sqrt{3}\sec x + 2\right) = 0$$

(3)

(ii) Solve, for $0 < \theta < 360°$

$$10\sin\theta = 3\cos 2\theta$$

(4)

Question 9

8 marks | Trig Equations & R-Form

9. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that the equation

$$\frac{3\sin\theta\cos\theta}{\cos\theta+\sin\theta} = (2+\sec 2\theta)(\cos\theta-\sin\theta)$$

can be written in the form

$$3\sin 2\theta - 4\cos 2\theta = 2$$

(3)

(b) Hence solve for $\pi < x < \frac{3\pi}{2}$

$$\frac{3\sin x\cos x}{\cos x+\sin x} = (2+\sec 2x)(\cos x-\sin x)$$

giving the answer to 3 significant figures.

(5)

WMA13/01 OCTOBER 2024

Question 1

5 marks | Trig Equations & R-Form

1. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

Solve, for $0 < \theta \leqslant 360°$, the equation

$$3\tan^2\theta + 7\sec\theta - 3 = 0$$

giving your answers to one decimal place.

(5)

Question 8

9 marks | Trig Equations & R-Form

8. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(i) Solve, for $0 < \theta < \pi$

$$3\,\text{cosec}\,\theta = 8\cos\theta$$

giving your answers, in radians, to 3 significant figures.

(5)

(ii) Solve, for $0 < x < 180°$

$$\frac{\tan 2x - \tan 70°}{1 + \tan 2x \tan 70°} = -\frac{3}{8}$$

giving your answers, in degrees, to one decimal place.

(4)

Question 2

7 marks | Trig Equations & R-Form

2: **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

$$f(x) = 7\cos x - 24\sin x$$

(a) Express $f(x)$ in the form $R\cos(x + \alpha)$ where R and α are constants, $R > 0$

and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

$$g(x) = \frac{5}{90 - 3f(2x)}$$

(b) Using the answer to part (a), find

(i) the minimum value of $g(x)$, giving your answer as a fully simplified fraction,

(ii) the smallest positive value of x for which this minimum value occurs, giving your answer to 3 decimal places.

(4)

Question 4

6 marks | Trig Equations & R-Form

4. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(a) $$f(x) = \sqrt{3}\sin 2x - 3\cos 2x$$

Express $f(x)$ in the form $R\sin(2x - \alpha)$, where R and α are constants,

$$R > 0 \text{ and } 0 < \alpha < \frac{\pi}{2}$$

Give the exact value of R and the exact value of α.

(3)

(b) $$g(x) = \frac{18}{f(3x) + 4\sqrt{3}} \qquad x > 0$$

Using the answer to part (a),

(i) write down the exact minimum value of $g(x)$,

(ii) find the smallest value of x for which this minimum value occurs.

You must make your method clear.

(3)

Question 9

9 marks | Trig Equations & R-Form

9. $$\mathrm{h}(x) = 8 + 3\sin x(\cos x - 2\sin x) \quad 0 \leqslant x \leqslant 2\pi$$

(a) Write $\mathrm{h}(x)$ in the form

$$P + R\sin(2x + \alpha)$$

where P, R and α are positive constants, $0 < \alpha < \frac{\pi}{2}$

Give the exact values of P and R and give the value of α, in radians, to 3 decimal places.

(6)

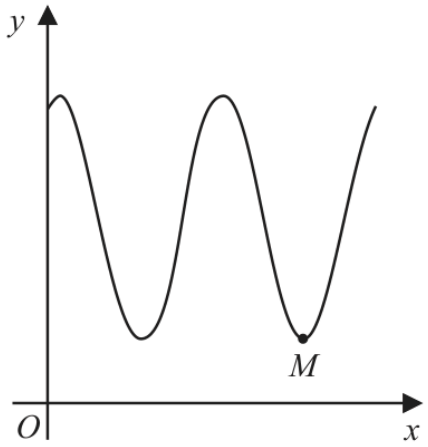


Figure 2

Figure 2 is a sketch of the curve with equation $y = \mathrm{h}(x)$

(b) State the range of h.

(1)

The point M, shown in Figure 2, is the minimum point on the curve with the largest x coordinate.

(c) Find the x coordinate of M.

(Solutions based entirely on calculator technology are not acceptable.)

(2)

Question 5

8 marks | Trig Equations & R-Form

Main topic: Trig Identities & Exact Values

5. (a) Use the substitution $t = \tan x$ to show that the equation

$$12\tan 2x + 5\cot x \sec^2 x = 0$$

can be written in the form

$$5t^4 - 24t^2 - 5 = 0$$

(4)

(b) Hence solve, for $0 \leqslant x < 360°$, the equation

$$12\tan 2x + 5\cot x \sec^2 x = 0$$

Show each stage of your working and give your answers to one decimal place.

(4)

Question 7

11 marks | Trig Equations & R-Form

Main topic: Applications of Differentiation

7.

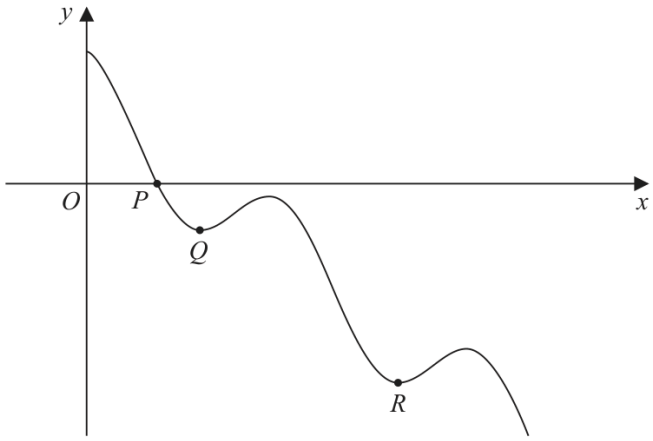


Figure 3

Figure 3 shows a sketch of part of the curve with equation

$$y = 2\cos 3x - 3x + 4 \qquad x > 0$$

where x is measured in radians.

The curve crosses the x-axis at the point P, as shown in Figure 3.

Given that the x coordinate of P is α,

(a) show that α lies between 0.8 and 0.9

(2)

The iteration formula

$$x_{n+1} = \frac{1}{3}\arccos(1.5x_n - 2)$$

can be used to find an approximate value for α.

(b) Using this iteration formula with $x_1 = 0.8$ find, to 4 decimal places, the value of

(i) x_2

(ii) x_5

(3)

The point Q and the point R are local minimum points on the curve, as shown in Figure 3.

Given that the x coordinates of Q and R are β and λ respectively, and that they are the two smallest values of x at which local minima occur,

(c) find, using calculus, the exact value of β and the exact value of λ.

(6)

Question 8

9 marks | Trig Equations & R-Form

Main topic: Differentiation Techniques

8. (i) The curve C has equation $y = \mathrm{g}(x)$ where

$$\mathrm{g}(x) = \mathrm{e}^{3x} \sec 2x \qquad -\frac{\pi}{4} < x < \frac{\pi}{4}$$

(a) Find $\mathrm{g}'(x)$

(2)

(b) Hence find the x coordinate of the stationary point of C.

(3)

(ii) A different curve has equation

$$x = \ln(\sin y) \qquad 0 < y < \frac{\pi}{2}$$

Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^x}{\mathrm{f}(x)}$$

where $\mathrm{f}(x)$ is a function of e^x that should be found.

(4)

Question 6

8 marks | Trig Equations & R-Form | Main topic: Numerical Methods

6.

$$f(x) = x\cos\left(\frac{x}{3}\right) \qquad x > 0$$

(a) Find $f'(x)$

(2)

(b) Show that the equation $f'(x) = 0$ can be written as

$$x = k\arctan\left(\frac{k}{x}\right)$$

where k is an integer to be found.

(2)

(c) Starting with $x_1 = 2.5$ use the iteration formula

$$x_{n+1} = k\arctan\left(\frac{k}{x_n}\right)$$

with the value of k found in part (b), to calculate the values of x_2 and x_6 giving your answers to 3 decimal places.

(2)

(d) Using a suitable interval and a suitable function that should be stated, show that a root of $f'(x) = 0$ is 2.581 correct to 3 decimal places.

(2)

Question 7

9 marks | Trig Equations & R-Form

Main topic: Trig Identities & Exact Values

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

7. (a) Prove that

$$\frac{\sin 2x}{\cos x} + \frac{\cos 2x}{\sin x} \equiv \operatorname{cosec} x \qquad x \neq \frac{n\pi}{2} \quad n \in \mathbb{Z}$$

(3)

(b) Hence solve, for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$7 + \frac{\sin 4\theta}{\cos 2\theta} + \frac{\cos 4\theta}{\sin 2\theta} = 3\cot^2 2\theta$$

giving your answers in radians to 3 significant figures where appropriate.

(6)

Question 1

7 marks | Trig Equations & R-Form | Main topic: Numerical Methods

1. The curve C has equation

$$y = x^2 \cos\left(\frac{1}{2}x\right) \qquad 0 < x \leqslant \pi$$

The curve has a stationary point at the point P.

(a) Show, using calculus, that the x coordinate of P is a solution of the equation

$$x = 2\arctan\left(\frac{4}{x}\right)$$

(4)

Using the iteration formula

$$x_{n+1} = 2\arctan\left(\frac{4}{x_n}\right) \qquad x_1 = 2$$

(b) find the value of x_2 and the value of x_6, giving your answers to 3 decimal places.

(3)

Question 2

9 marks | Trig Equations & R-Form

Main topic: Trig Identities & Exact Values

2. (a) Show that

$$\frac{1-\cos 2x}{2\sin 2x} \equiv k\tan x \qquad x \neq (90n)^\circ \qquad n \in \mathbb{Z}$$

where k is a constant to be found.

(3)

(b) Hence solve, for $0 < \theta < 90^\circ$

$$\frac{9(1-\cos 2\theta)}{2\sin 2\theta} = 2\sec^2\theta$$

giving your answers to one decimal place.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(6)

Question 4

7 marks | Trig Equations & R-Form

Main topic: Trig Identities & Exact Values

4. **In this question you should show detailed reasoning.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that the equation

$$2\sin(\theta - 30°) = 5\cos\theta$$

can be written in the form

$$\tan\theta = 2\sqrt{3}$$

(4)

(b) Hence, or otherwise, solve for $0 \leqslant x \leqslant 360°$

$$2\sin(x - 10°) = 5\cos(x + 20°)$$

giving your answers to one decimal place.

(3)

Question 10

7 marks | Trig Equations & R-Form | Main topic: Integration

10.

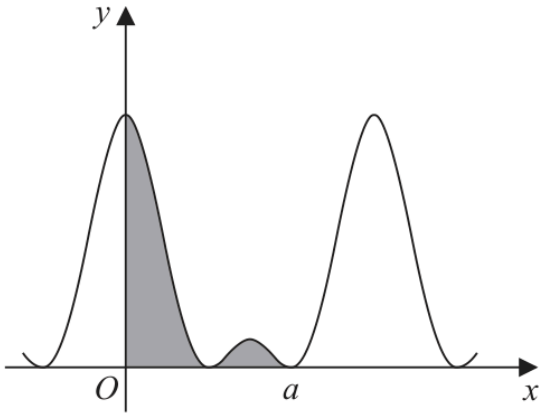


Figure 4

Figure 4 shows a sketch of part of the curve with equation

$$y = (1 + 2\cos 2x)^2$$

(a) Show that

$$(1 + 2\cos 2x)^2 \equiv p + q\cos 2x + r\cos 4x$$

where p, q and r are constants to be found.

(2)

The curve touches the positive x-axis for the second time when $x = a$, as shown in Figure 4.

The regions bounded by the curve, the y-axis and the x-axis up to $x = a$ are shown shaded in Figure 4.

(b) Find, using algebraic integration and making your method clear, the exact total area of the shaded regions. Write your answer in simplest form.

(5)

Question 2

5 marks | Trig Equations & R-Form

Main topic: Trig Identities & Exact Values

2. (a) Show that the equation

$$8\cos\theta = 3\operatorname{cosec}\theta$$

can be written in the form

$$\sin 2\theta = k$$

where k is a constant to be found.

(3)

(b) Hence find the smallest positive solution of the equation

$$8\cos\theta = 3\operatorname{cosec}\theta$$

giving your answer, in degrees, to one decimal place.

(2)

Question 9

8 marks | Trig Equations & R-Form

Main topic: Trig Identities & Exact Values

9. **In this question you must show detailed reasoning.**

Solutions relying entirely on calculator technology are not acceptable.

(i) Solve, for $0 < x \leqslant \pi$, the equation

$$2\sec^2 x - 3\tan x = 2$$

giving the answers, as appropriate, to 3 significant figures.

(4)

(ii) Prove that

$$\frac{\sin 3\theta}{\sin \theta} - \frac{\cos 3\theta}{\cos \theta} \equiv 2$$

(4)

Question 7

8 marks | Trig Equations & R-Form

Main topic: Trig Identities & Exact Values

7. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that the equation

$$2\sin\theta\,(3\cot^2 2\theta - 7) = 13\sec\theta$$

can be written as

$$3\operatorname{cosec}^2 2\theta - 13\operatorname{cosec} 2\theta - 10 = 0$$

(4)

(b) Hence solve, for $0 < \theta < \frac{\pi}{2}$, the equation

$$2\sin\theta\,(3\cot^2 2\theta - 7) = 13\sec\theta$$

giving your answers to 3 significant figures.

(4)

Question 9

10 marks | Trig Equations & R-Form

Main topic: Trig Identities & Exact Values

9. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that

$$\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} \equiv \operatorname{cosec} x \qquad x \neq \frac{n\pi}{2} \qquad n \in \mathbb{Z}$$

(3)

(b) Hence solve, for $0 < \theta < \frac{\pi}{2}$

$$\left(\frac{\cos 2\theta}{\sin \theta} + \frac{\sin 2\theta}{\cos \theta}\right)^2 = 6\cot\theta - 4$$

giving your answers to 3 significant figures as appropriate.

(5)

(c) Using the result from part (a), or otherwise, find the exact value of

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x}\right) \cot x \, dx$$

(2)

Question 8

8 marks | Trig Equations & R-Form

Main topic: Trig Identities & Exact Values

8. (a) Prove that

$$2\operatorname{cosec}^2 2\theta(1-\cos 2\theta) \equiv 1+\tan^2\theta$$

(4)

(b) Hence solve for $0 < x < 360°$, where $x \neq (90n)°$, $n \in \mathbb{N}$, the equation

$$2\operatorname{cosec}^2 2x(1-\cos 2x) = 4+3\sec x$$

giving your answers to one decimal place.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(4)

Question 10

9 marks | Trig Equations & R-Form

Main topic: Differentiation Techniques

10. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

A curve C has equation

$$x = \sin^2 4y \qquad 0 \leqslant y \leqslant \frac{\pi}{8} \qquad 0 \leqslant x \leqslant 1$$

The point P with x coordinate $\frac{1}{4}$ lies on C

(a) Find the exact y coordinate of P

(2)

(b) Find $\frac{\mathrm{d}x}{\mathrm{d}y}$

(2)

(c) Hence show that $\frac{\mathrm{d}y}{\mathrm{d}x}$ can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{q + r(x + s)^2}}$$

where q, r and s are constants to be found.

(3)

Using the answer to part (c),

(d) (i) state the x coordinate of the point where the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ is a minimum,

(ii) state the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at this point.

(2)

Question 6

7 marks | Trig Equations & R-Form

Main topic: Applications of Differentiation

6.

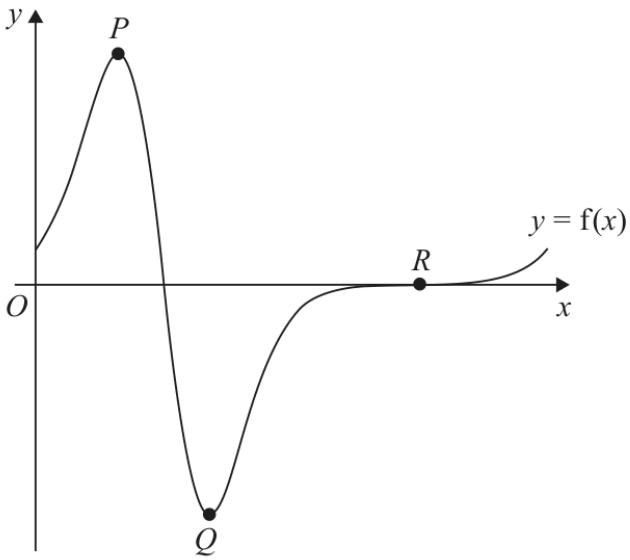


Figure 1

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 1 shows a sketch of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 2\mathrm{e}^{3\sin x}\cos x \qquad 0 \leqslant x \leqslant 2\pi$$

The curve intersects the x-axis at point R, as shown in Figure 1.

(a) State the coordinates of R

(1)

The curve has two turning points, at point P and point Q, also shown in Figure 1.

(b) Show that, at points P and Q,

$$a\sin^2 x + b\sin x + c = 0$$

where a, b and c are integers to be found.

(4)

(c) Hence find the x coordinate of point Q, giving your answer to 3 decimal places.

(2)

Question 4

9 marks | Trig Equations & R-Form

Main topic: Trig Identities & Exact Values

4. $$f(x) = 8\sin x\cos x + 4\cos^2 x - 3$$

(a) Write $f(x)$ in the form

$$a\sin 2x + b\cos 2x + c$$

where a, b and c are integers to be found.

(3)

(b) Use the answer to part (a) to write $f(x)$ in the form

$$R\sin(2x + \alpha) + c$$

where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α in radians to 3 significant figures.

(3)

(c) Hence, or otherwise,

(i) state the maximum value of $f(x)$

(ii) find the **second** smallest positive value of x at which a maximum value of $f(x)$ occurs. Give your answer to 3 significant figures.

(3)

Question 7

8 marks | Trig Equations & R-Form

Main topic: Trig Identities & Exact Values

7. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Given that

$$\sqrt{2}\sin(x + 45°) = \cos(x - 60°)$$

show that

$$\tan x = -2 - \sqrt{3}$$

(4)

(b) Hence or otherwise, solve, for $0 \leqslant \theta < 180°$

$$\sqrt{2}\sin(2\theta) = \cos(2\theta - 105°)$$

(4)

Question 5

8 marks | Trig Equations & R-Form

Main topic: Trig Identities & Exact Values

5. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that $\sin 3x$ can be written in the form

$$P\sin x + Q\sin^3 x$$

where P and Q are constants to be found.

(4)

(b) Hence or otherwise, solve, for $0 < \theta \leqslant 360°$, the equation

$$2\sin 3\theta = 5\sin 2\theta$$

giving your answers, in degrees, to one decimal place as appropriate.

(4)

Question 10

9 marks | Trig Equations & R-Form

Main topic: Differentiation Techniques

10.

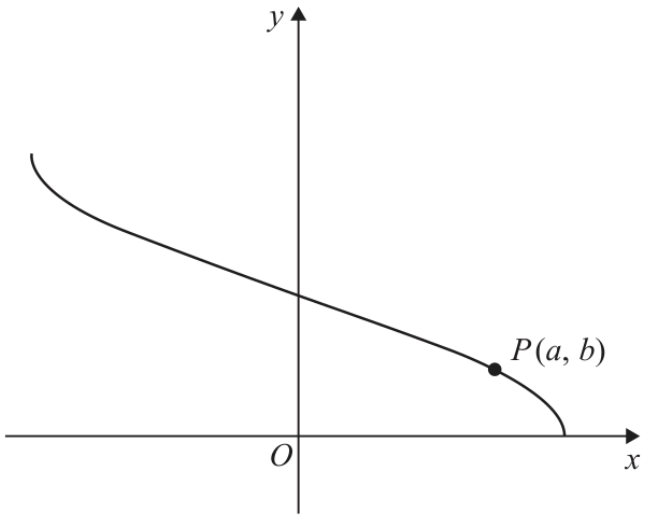


Figure 2

Figure 2 shows a sketch of the curve with equation

$$x = 3\cos 2y \qquad -3 \leqslant x \leqslant 3 \qquad 0 \leqslant y \leqslant \frac{\pi}{2}$$

(a) Find $\frac{\mathrm{d}x}{\mathrm{d}y}$ in terms of y.

(2)

(b) Hence show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{k}{\sqrt{9 - x^2}}$$

where k is a constant to be found.

(3)

The point $P(a, b)$ lies on the curve and is shown in Figure 2.

Given that

- the gradient of the curve at P is $-\frac{1}{4}$
- both a and b are positive

(c) find the exact values of a and b.

(4)

Question 7

9 marks | Trig Equations & R-Form | Main topic: Numerical Methods

7: A continuous curve has equation

$$y = e^{-x^2}\sin 3x \qquad 0 \leqslant x \leqslant \frac{\pi}{3}$$

The curve has a stationary point at the point P.

(a) Show, using calculus, that the x coordinate of P is a solution of the equation

$$x = \frac{1}{3}\arctan\left(\frac{3}{2x}\right)$$

(4)

Using the iteration formula

$$x_{n+1} = \frac{1}{3}\arctan\left(\frac{3}{2x_n}\right) \qquad x_1 = 0.4$$

(b) find the value of

(i) x_2

(ii) x_4

giving your answers to 4 decimal places.

(3)

(c) Using a suitable interval and a suitable function which should be stated, show that the x coordinate of P is 0.430 correct to 3 decimal places.

(2)

Question 8

8 marks | Trig Equations & R-Form

Main topic: Trig Identities & Exact Values

8: **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(a) Prove that

$$\tan 3x \equiv \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x} \qquad x \neq (2n+1)\frac{\pi}{6} \qquad n \in \mathbb{Z}$$

(3)

(b) Hence solve, for $0 < \theta < \frac{\pi}{2}$

$$\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = 2\sec^2 3\theta - 8$$

giving your answers to 2 decimal places.

(5)

Question 9

11 marks | Trig Equations & R-Form

Main topic: Differentiation Techniques

9:

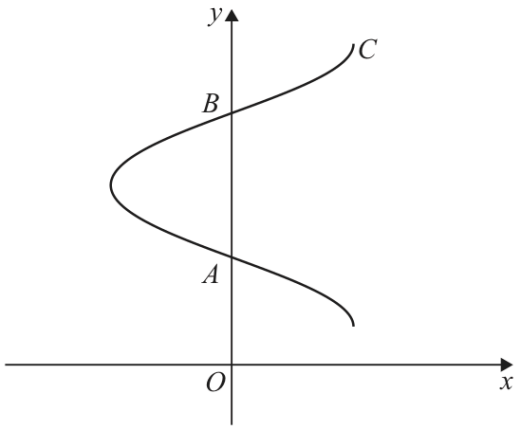


Figure 2

Figure 2 shows a sketch of the curve C with equation

$$x = \frac{2}{3}\sin\left(3y + \frac{\pi}{4}\right) \qquad \frac{\pi}{12} < y < \frac{3\pi}{4}$$

The curve intersects the y-axis at the points A and B as shown.

(a) Find the exact value of the y coordinate of

- point A
- point B

(3)

(b) Show that

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{p - qx^2}$$

where p and q are integers to be found.

(4)

The **normal** to C at A and the **tangent** to C at B intersect at the point D.

Using

- the answer to part (b)
- the sketch of curve C in Figure 2

(c) find, in simplest form, the exact x coordinate of D.

(4)

Question 9

9 marks | Trig Equations & R-Form

Main topic: Trig Identities & Exact Values

9. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(a) Express

$$6\sin^2\theta\cot 2\theta + 4\sin\theta\cos\theta$$

in terms of $\sin 2\theta$ and $\cos 2\theta$ only.

(3)

(b) Hence show that the equation

$$3\cot 2\theta - 14 = 6\sin^2\theta\cot 2\theta + 4\sin\theta\cos\theta$$

can be written in the form

$$5\sin^2 2\theta + 14\sin 2\theta - 3 = 0$$

(3)

(c) Hence solve, for $0 < x < 90°$, the equation

$$3\cot 2x - 14 = 6\sin^2 x\cot 2x + 4\sin x\cos x$$

giving your answers to one decimal place.

(3)

Question 1

6 marks | Trig Equations & R-Form | Main topic: Numerical Methods

1. $$g(x) = 3x - 5 + 5\sin\left(x^2\right)$$

where x is measured in radians.

(a) Show that $g(x) = 0$ has a root α in the interval $[0.7, 0.8]$

(2)

(b) Show that the equation $g(x) = 0$ can be written in the form

$$x = \left[\arcsin\left(1 - px\right)\right]^{\frac{1}{2}}$$

where p is a constant to be found.

(1)

Using the iteration formula

$$x_{n+1} = \left[\arcsin\left(1 - px_n\right)\right]^{\frac{1}{2}}$$

with the value of p found in part (b) and starting with $x_1 = 0.7$

(c) (i) find, to 4 decimal places, the value of x_2

(ii) find, by repeated iteration, the value of α, giving your answer to 4 decimal places.

(3)

Question 6

7 marks | Trig Equations & R-Form

Main topic: Trig Identities & Exact Values

6. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(a) Show that the equation

$$2\cos(\theta - 60°) = 3\sin\theta$$

can be written in the form

$$\tan\theta = \frac{1}{6}\left(3 + \sqrt{3}\right)$$

(4)

(b) Hence solve, for $0 \leqslant x < 180°$, the equation

$$2\cos(2x - 70°) = 3\sin(2x - 10°)$$

giving your solutions to one decimal place.

(3)

Question 8

9 marks | Trig Equations & R-Form

Main topic: Trig Identities & Exact Values

8. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(a) Show that

$$\frac{\sec\theta\,\text{cosec}\theta}{4\text{cosec}\,4\theta} \equiv \cos 2\theta$$

(2)

(b) Hence show that the equation

$$4(5\cos^2\theta - 9)\,\text{cosec}\,4\theta = 3(1+\tan^2 2\theta)\sec\theta\,\text{cosec}\,\theta$$

can be written as

$$a\cos^2 2\theta + b\cos 2\theta + c = 0$$

where a, b and c are integers to be found.

(3)

(c) Hence, solve for $0 < x < 180°$, $x \neq (45n)°$, $n \in \mathbb{N}$, the equation

$$4(5\cos^2 x - 9)\,\text{cosec}\,4x = 3(1+\tan^2 2x)\sec x\,\text{cosec}\,x$$

giving the answers to one decimal place.

(4)

TOPIC

Exponentials & Logarithms

Every past-paper question on this topic, January 2020 to June 2026.

Question 9

9 marks | Exponentials & Logarithms

9. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

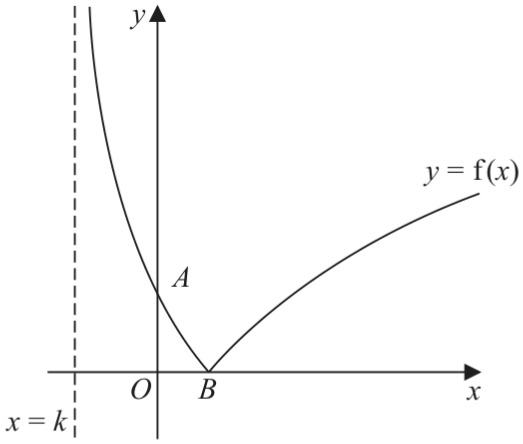


Figure 2

Figure 2 shows a sketch of the curve with equation

$$y = |2 - 4\ln(x + 1)| \qquad x > k$$

where k is a constant.

Given that the curve

- has an asymptote at $x = k$
- cuts the y-axis at point A
- meets the x-axis at point B

as shown in Figure 2,

(a) state the value of k

(1)

(b) (i) find the y coordinate of A

(ii) find the exact x coordinate of B

(3)

(c) Using algebra and showing your working, find the set of values of x such that

$$|2 - 4\ln(x + 1)| > 3$$

(5)

WMA13/01 JANUARY 2020

Question 1

6 marks | Exponentials & Logarithms | Main topic: Log Graphs & Exponential Models

1. A population of a rare species of toad is being studied.

The number of toads, N, in the population, t years after the start of the study, is modelled by the equation

$$N = \frac{900e^{0.12t}}{2e^{0.12t} + 1} \qquad t \geqslant 0, t \in \mathbb{R}$$

According to this model,

(a) calculate the number of toads in the population at the start of the study,

(1)

(b) find the value of t when there are 420 toads in the population, giving your answer to 2 decimal places.

(4)

(c) Explain why, according to this model, the number of toads in the population can never reach 500

(1)

Question 2

8 marks | Exponentials & Logarithms | Main topic: Functions, Composition & Inverses

2. The function f and the function g are defined by

$$f(x) = \frac{12}{x+1} \qquad x > 0, x \in \mathbb{R}$$

$$g(x) = \frac{5}{2}\ln x \qquad x > 0, x \in \mathbb{R}$$

(a) Find, in simplest form, the value of $fg(e^2)$

(2)

(b) Find f^{-1}

(3)

(c) Hence, or otherwise, find all real solutions of the equation

$$f^{-1}(x) = f(x)$$

(3)

Question 3

5 marks | Exponentials & Logarithms

Main topic: Log Graphs & Exponential Models

3.

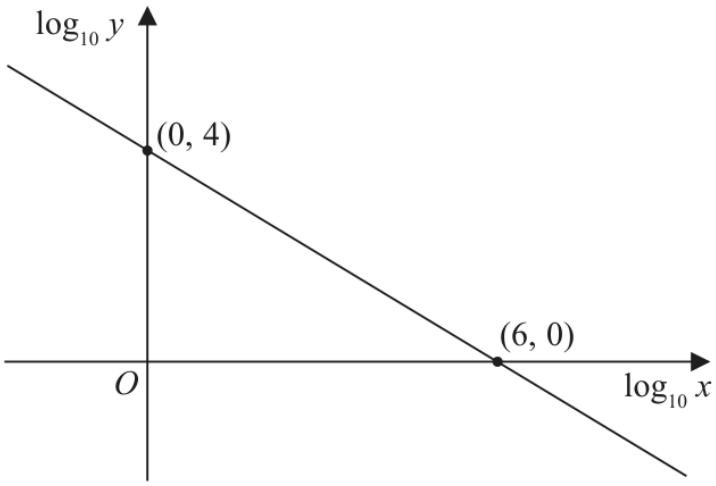


Figure 1

Figure 1 shows a linear relationship between $\log_{10} y$ and $\log_{10} x$

The line passes through the points (0, 4) and (6, 0) as shown.

(a) Find an equation linking $\log_{10} y$ with $\log_{10} x$

(2)

(b) Hence, or otherwise, express y in the form px^q, where p and q are constants to be found.

(3)

Question 8

10 marks | Exponentials & Logarithms | Main topic: Integration

8. (i) Find, using algebraic integration, the exact value of

$$\int_3^{42} \frac{2}{3x-1}\,\mathrm{d}x$$

giving your answer in simplest form.

(4)

(ii) $$\mathrm{h}(x) = \frac{2x^3 - 7x^2 + 8x + 1}{(x-1)^2} \qquad x > 1$$

Given $\mathrm{h}(x) = Ax + B + \frac{C}{(x-1)^2}$ where A, B and C are constants to be found, find

$$\int \mathrm{h}(x)\,\mathrm{d}x$$

(6)

Question 2

6 marks | Exponentials & Logarithms

Main topic: Log Graphs & Exponential Models

2. A scientist monitored the growth of bacteria on a dish over a 30-day period.

The area, $N\text{mm}^2$, of the dish covered by bacteria, t days after monitoring began, is modelled by the equation

$$\log_{10} N = 0.0646t + 1.478 \qquad 0 \leqslant t \leqslant 30$$

(a) Show that this equation may be written in the form

$$N = a\,b^t$$

where a and b are constants to be found. Give the value of a to the nearest integer and give the value of b to 3 significant figures.

(4)

(b) Use the model to find the area of the dish covered by bacteria 30 days after monitoring began. Give your answer, in mm^2, to 2 significant figures.

(2)

Question 6

9 marks | Exponentials & Logarithms | Main topic: Numerical Methods

6.

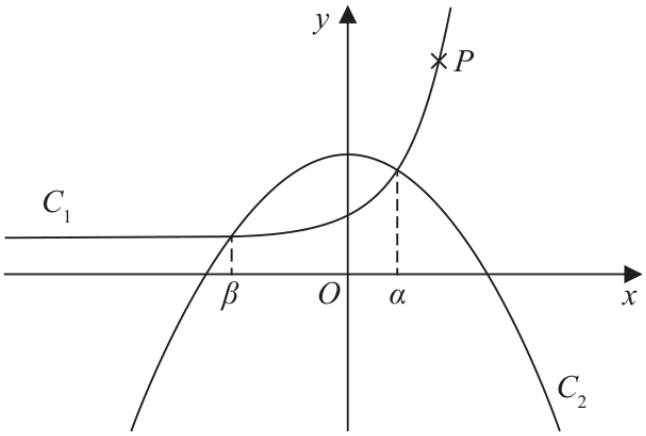


Figure 3

Figure 3 shows a sketch of curve C_1 with equation $y = 5e^{x-1} + 3$

and curve C_2 with equation $y = 10 - x^2$

The point P lies on C_1 and has y coordinate 18

(a) Find the x coordinate of P, writing your answer in the form $\ln k$, where k is a constant to be found.

(3)

The curve C_1 meets the curve C_2 at $x = \alpha$ and at $x = \beta$, as shown in Figure 3.

(b) Using a suitable interval and a suitable function that should be stated, show that to 3 decimal places $\alpha = 1.134$

(3)

The iterative equation

$$x_{n+1} = -\sqrt{7 - 5e^{x_n - 1}}$$

is used to find an approximation to β.

Using this iterative formula with $x_1 = -3$

(c) find the value of x_2 and the value of β, giving each answer to 6 decimal places.

(3)

Question 8

9 marks | Exponentials & Logarithms | Main topic: Differentiation Techniques

8. (i) The curve C has equation $y = \mathrm{g}(x)$ where

$$\mathrm{g}(x) = \mathrm{e}^{3x} \sec 2x \qquad -\frac{\pi}{4} < x < \frac{\pi}{4}$$

(a) Find $\mathrm{g}'(x)$

(2)

(b) Hence find the x coordinate of the stationary point of C.

(3)

(ii) A different curve has equation

$$x = \ln(\sin y) \qquad 0 < y < \frac{\pi}{2}$$

Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^x}{\mathrm{f}(x)}$$

where $\mathrm{f}(x)$ is a function of e^x that should be found.

(4)

Question 9

14 marks | Exponentials & Logarithms | Main topic: Applications of Differentiation

9. (a) Given that

$$\frac{x^4 - x^3 - 10x^2 + 3x - 9}{x^2 - x - 12} \equiv x^2 + P + \frac{Q}{x - 4} \qquad x > -3$$

find the value of the constant P and show that $Q = 5$

(4)

The curve C has equation $y = \mathrm{g}(x)$, where

$$\mathrm{g}(x) = \frac{x^4 - x^3 - 10x^2 + 3x - 9}{x^2 - x - 12} \qquad -3 < x < 3.5 \qquad x \in \mathbb{R}$$

(b) Find the equation of the tangent to C at the point where $x = 2$
Give your answer in the form $y = mx + c$, where m and c are constants to be found.

(5)

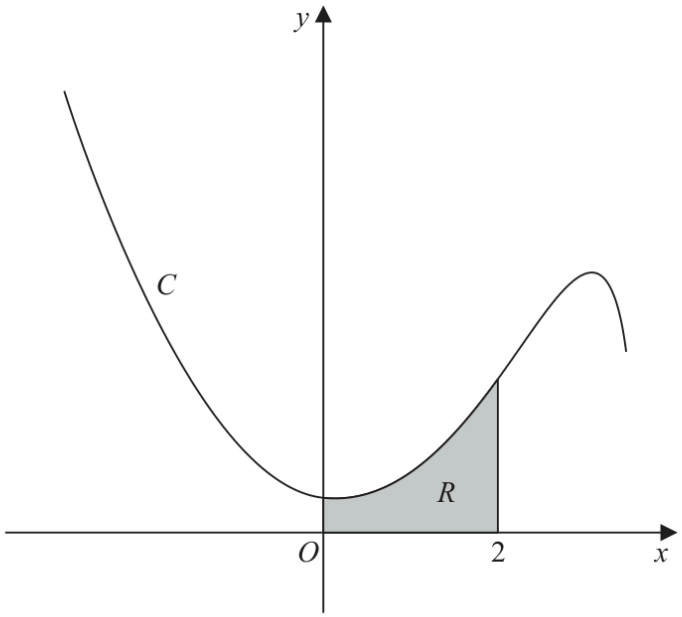


Figure 4

Figure 4 shows a sketch of the curve C.
The region R, shown shaded in Figure 4, is bounded by C, the y-axis, the x-axis and the line with equation $x = 2$

(c) Find the exact area of R, writing your answer in the form $a + b\ln 2$, where a and b are constants to be found.

(5)

Question 5

11 marks | Exponentials & Logarithms | Main topic: Log Graphs & Exponential Models

5. The temperature, θ°C, inside an oven, t minutes after the oven is switched on, is given by

$$\theta = A - 180e^{-kt}$$

where A and k are positive constants.

Given that the temperature inside the oven is initially 18 °C,

(a) find the value of A.

(2)

The temperature inside the oven, 5 minutes after the oven is switched on, is 90 °C.

(b) Show that $k = p \ln q$ where p and q are rational numbers to be found.

(4)

Hence find

(c) the temperature inside the oven 9 minutes after the oven is switched on, giving your answer to 3 significant figures,

(2)

(d) the rate of increase of the temperature inside the oven 9 minutes after the oven is switched on. Give your answer in °C min^{-1} to 3 significant figures.

(3)

Question 8

7 marks | Exponentials & Logarithms

Main topic: Log Graphs & Exponential Models

8. The percentage, P, of the population of a small country who have access to the internet, is modelled by the equation

$$P = ab^t$$

where a and b are constants and t is the number of years after the start of 2005

Using the data for the years between the start of 2005 and the start of 2010, a graph is plotted of $\log_{10} P$ against t.

The points are found to lie approximately on a straight line with gradient 0.09 and intercept 0.68 on the $\log_{10} P$ axis.

(a) Find, according to the model, the value of a and the value of b, giving your answers to 2 decimal places.

(4)

(b) In the context of the model, give a practical interpretation of the constant a.

(1)

(c) Use the model to estimate the percentage of the population who had access to the internet at the start of 2015

(2)

Question 3

8 marks | Exponentials & Logarithms | Main topic: Integration

3. (i) Find

$$\int \frac{12}{(2x-1)^2}\,\mathrm{d}x$$

giving your answer in simplest form.

(2)

(ii) (a) Write $\dfrac{4x+3}{x+2}$ in the form

$$A + \frac{B}{x+2}$$ where A and B are constants to be found

(b) Hence find, using algebraic integration, the exact value of

$$\int_{-8}^{-5} \frac{4x+3}{x+2}\,\mathrm{d}x$$

giving your answer in simplest form.

(6)

Question 5

7 marks | Exponentials & Logarithms

Main topic: Log Graphs & Exponential Models

5.

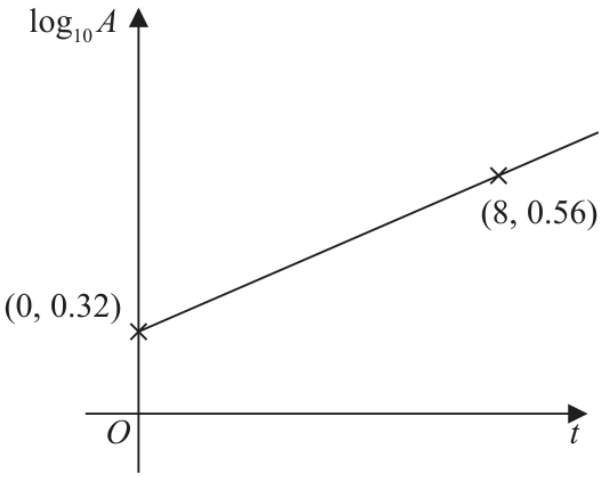


Figure 1

The growth of duckweed on a pond is being studied.

The surface area of the pond covered by duckweed, $A\text{m}^2$, at a time t days after the start of the study is modelled by the equation

$$A = pq^t \qquad \text{where } p \text{ and } q \text{ are positive constants}$$

Figure 1 shows the linear relationship between $\log_{10} A$ and t.

The points (0, 0.32) and (8, 0.56) lie on the line as shown.

(a) Find, to 3 decimal places, the value of p and the value of q.

(4)

Using the model with the values of p and q found in part (a),

(b) find the rate of increase of the surface area of the pond covered by duckweed, in m^2/day, exactly 6 days after the start of the study.
Give your answer to 2 decimal places.

(3)

Question 8

13 marks | Exponentials & Logarithms | Main topic: Log Graphs & Exponential Models

8. A scientist is studying a population of fish in a lake. The number of fish, N, in the population, t years after the start of the study, is modelled by the equation

$$N = \frac{600e^{0.3t}}{2 + e^{0.3t}} \qquad t \geqslant 0$$

Use the equation of the model to answer parts (a), (b), (c), (d) and (e).

(a) Find the number of fish in the lake at the start of the study.

(1)

(b) Find the upper limit to the number of fish in the lake.

(1)

(c) Find the time, after the start of the study, when there are predicted to be 500 fish in the lake. Give your answer in years and months to the nearest month.

(4)

(d) Show that

$$\frac{dN}{dt} = \frac{Ae^{0.3t}}{(2 + e^{0.3t})^2}$$

where A is a constant to be found.

(3)

Given that when $t = T$, $\frac{dN}{dt} = 8$

(e) find the value of T to one decimal place.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(4)

Question 3

6 marks | Exponentials & Logarithms

Main topic: Log Graphs & Exponential Models

3.

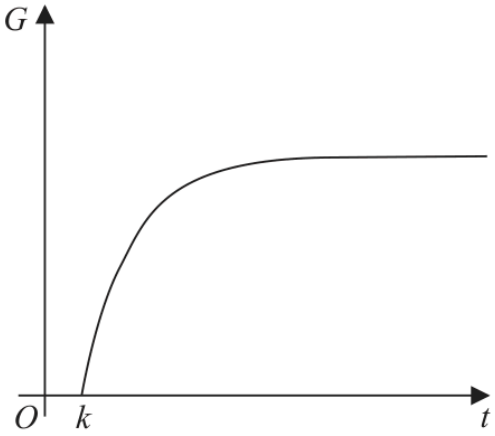


Figure 2

The total mass of gold, G tonnes, extracted from a mine is modelled by the equation

$$G = 40 - 30e^{1-0.05t} \qquad t \geqslant k \qquad G \geqslant 0$$

where t is the number of years after 1st January 1800.

Figure 2 shows a sketch of G against t.

Use the equation of the model to answer parts (a), (b) and (c).

(a) (i) Find the value of k.

(ii) Hence find the year and month in which gold started being extracted from the mine.

(3)

(b) Find the total mass of gold extracted from the mine up to 1st January 1870.

(2)

There is a limit to the mass of gold that can be extracted from the mine.

(c) State the value of this limit.

(1)

Question 7

6 marks | Exponentials & Logarithms | Main topic: Log Graphs & Exponential Models

7 The mass, M kg, of a species of tree can be modelled by the equation

$$\log_{10} M = 1.93 \log_{10} r + 0.684$$

where r cm is the base radius of the tree.

The base radius of a particular tree of this species is 45 cm.

According to the model,

(a) find the mass of this tree, giving your answer to 2 significant figures.

(2)

(b) Show that the equation of the model can be written in the form

$$M = pr^q$$

giving the values of the constants p and q to 3 significant figures.

(3)

(c) With reference to the model, interpret the value of the constant p.

(1)

Question 3

6 marks | Exponentials & Logarithms | Main topic: Integration

3. (i) Find, in simplest form,

$$\int (2x - 5)^7 \, dx$$

(2)

(ii) Show, by algebraic integration, that

$$\int_0^{\frac{\pi}{3}} \frac{4 \sin x}{1 + 2 \cos x} dx = \ln a$$

where a is a rational constant to be found.

(4)

Question 4

7 marks | Exponentials & Logarithms | Main topic: Log Graphs & Exponential Models

4. The growth of a weed on the surface of a pond is being studied.

The surface area of the pond covered by the weed, $A\,\text{m}^2$, is modelled by the equation

$$A = \frac{80pe^{0.15t}}{pe^{0.15t} + 4}$$

where p is a positive constant and t is the number of days after the start of the study.

Given that

- $30\,\text{m}^2$ of the surface of the pond was covered by the weed at the start of the study
- $50\,\text{m}^2$ of the surface of the pond was covered by the weed T days after the start of the study

(a) show that $p = 2.4$

(2)

(b) find the value of T, giving your answer to one decimal place.

(*Solutions relying entirely on graphical or numerical methods are not acceptable.*)

(4)

The weed grows until it covers the surface of the pond.

(c) Find, according to the model, the maximum possible surface area of the pond.

(1)

Question 8

8 marks | Exponentials & Logarithms | Main topic: Log Graphs & Exponential Models

8. A dose of antibiotics is given to a patient.

The amount of the antibiotic, x milligrams, in the patient's bloodstream t hours after the dose was given, is found to satisfy the equation

$$\log_{10} x = 2.74 - 0.079t$$

(a) Show that this equation can be written in the form

$$x = pq^{-t}$$

where p and q are constants to be found. Give the value of p to the nearest whole number and the value of q to 2 significant figures.

(4)

(b) With reference to the equation in part (a), interpret the value of the constant p.

(1)

When a different dose of the antibiotic is given to another patient, the values of x and t satisfy the equation

$$x = 400 \times 1.4^{-t}$$

(c) Use calculus to find, to 2 significant figures, the value of $\dfrac{dx}{dt}$ when $t = 5$

(3)

Question 3

6 marks | Exponentials & Logarithms | Main topic: Integration

3. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

Given that k is a positive constant,

(a) find

$$\int \frac{9x}{3x^2 + k} \, \mathrm{d}x$$

(2)

Given also that

$$\int_2^5 \frac{9x}{3x^2 + k} \, \mathrm{d}x = \ln 8$$

(b) find the value of k

(4)

Question 4

7 marks | Exponentials & Logarithms

Main topic: Log Graphs & Exponential Models

4.

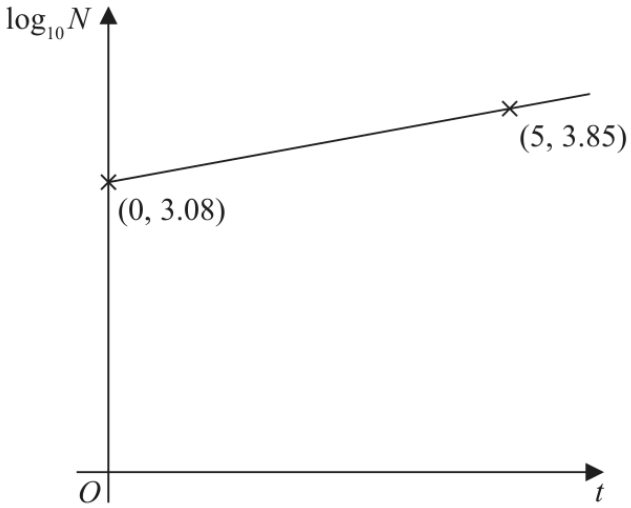


Figure 1

The number of subscribers to an online video streaming service, N, is modelled by the equation

$$N = ab^t$$

where a and b are constants and t is the number of years since monitoring began.

The line in Figure 1 shows the linear relationship between t and $\log_{10} N$

The line passes through the points (0, 3.08) and (5, 3.85)

Using this information,

(a) find an equation for this line.

(2)

(b) Find the value of a and the value of b, giving your answers to 3 significant figures.

(3)

When $t = T$ the number of subscribers is 500 000

According to the model,

(c) find the value of T

(2)

Question 8

12 marks | Exponentials & Logarithms | Main topic: Numerical Methods

8.

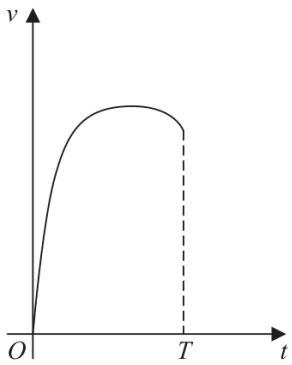


Figure 4

Figure 4 is a graph showing the velocity of a sprinter during a 100 m race.

The sprinter's velocity during the race, $v\,\text{m s}^{-1}$, is modelled by the equation

$$v = 12 - \text{e}^{t-10} - 12\text{e}^{-0.75t} \qquad t \geqslant 0$$

where t seconds is the time after the sprinter begins to run.

According to the model,

(a) find, using calculus, the sprinter's maximum velocity during the race.

(5)

Given that the sprinter runs 100 m in T seconds, such that

$$\int_0^T v\,\text{d}t = 100$$

(b) show that T is a solution of the equation

$$T = \frac{1}{12}\left(116 - 16\text{e}^{-0.75T} + \text{e}^{T-10} - \text{e}^{-10}\right)$$

(4)

The iteration formula

$$T_{n+1} = \frac{1}{12}\left(116 - 16\text{e}^{-0.75T_n} + \text{e}^{T_n-10} - \text{e}^{-10}\right)$$

is used to find an approximate value for T

Using this iteration formula with $T_1 = 10$

(c) find, to 4 decimal places,

(i) the value of T_2

(ii) the time taken by the sprinter to run the race, according to the model.

(3)

Question 1

9 marks | Exponentials & Logarithms | Main topic: Integration

1. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

$$\mathrm{f}(x) = \frac{2x^3 - 4x - 15}{x^2 + 3x + 4}$$

(a) Show that

$$\mathrm{f}(x) \equiv Ax + B + \frac{C(2x + 3)}{x^2 + 3x + 4}$$

where A, B and C are integers to be found.

(4)

(b) Hence, find

$$\int_3^5 \mathrm{f}(x)\,\mathrm{d}x$$

giving your answer in the form $p + \ln q$, where p and q are integers.

(5)

WMA13/01 OCTOBER 2022

Question 4

5 marks | Exponentials & Logarithms | Main topic: Differentiation Techniques

4. $y = \log_{10}(2x + 1)$

(a) Express x in terms of y.

(2)

(b) Hence, giving your answer in terms of x, find $\frac{dy}{dx}$

(3)

Question 5

11 marks | Exponentials & Logarithms | Main topic: Log Graphs & Exponential Models

5.

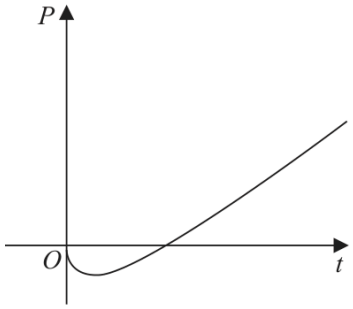


Figure 2

The profit made by a company, £P million, t years after the company started trading, is modelled by the equation

$$P = \frac{4t-1}{10} + \frac{3}{4}\ln\left[\frac{t+1}{(2t+1)^2}\right]$$

The graph of P against t is shown in Figure 2.

According to the model,

(a) show that exactly one year after it started trading, the company had made a loss of approximately £830 000

(2)

A manager of the company wants to know the value of t for which $P = 0$

(b) Show that this value of t occurs in the interval $[6, 7]$

(2)

(c) Show that the equation $P = 0$ can be expressed in the form

$$t = \frac{1}{4} + \frac{15}{8}\ln\left[\frac{(2t+1)^2}{t+1}\right]$$

(2)

(d) Using the iteration formula

$$t_{n+1} = \frac{1}{4} + \frac{15}{8}\ln\left[\frac{(2t_n+1)^2}{t_n+1}\right] \quad \text{with} \quad t_1 = 6$$

find the value of t_2 and the value of t_6, giving your answers to 3 decimal places.

(3)

(e) Hence find, according to the model, how many months it takes in total, from when the company started trading, for it to make a profit.

(2)

Question 3

5 marks | Exponentials & Logarithms | Main topic: Log Graphs & Exponential Models

3.

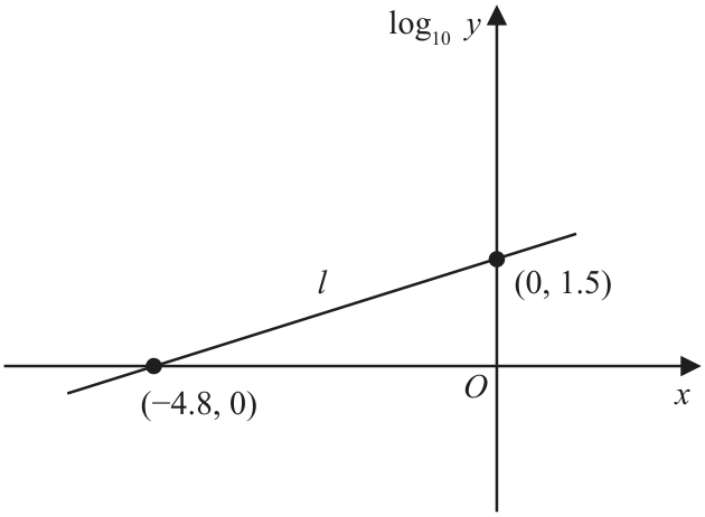


Figure 1

The line l in Figure 1 shows a linear relationship between $\log_{10} y$ and x.

The line passes through the points (0, 1.5) and (−4.8, 0) as shown.

(a) Write down an equation for l.

(2)

(b) Hence, or otherwise, express y in the form kb^x, giving the values of the constants k and b to 3 significant figures.

(3)

Question 10

9 marks | Exponentials & Logarithms

Main topic: Log Graphs & Exponential Models

10. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

A population of fruit flies is being studied.

The number of fruit flies, F, in the population, t days after the start of the study, is modelled by the equation

$$F = \frac{350e^{kt}}{9 + e^{kt}}$$

where k is a constant.

Use the equation of the model to answer parts (a), (b) and (c).

(a) Find the number of fruit flies in the population at the start of the study.

(1)

Given that there are 200 fruit flies in the population 15 days after the start of the study,

(b) show that $k = \frac{1}{15}\ln 12$

(3)

Given also that, when $t = T$, the number of fruit flies in the population is increasing at a rate of 10 per day,

(c) find the possible values of T, giving your answers to one decimal place.

(5)

Question 2

6 marks | Exponentials & Logarithms

Main topic: Log Graphs & Exponential Models

2.

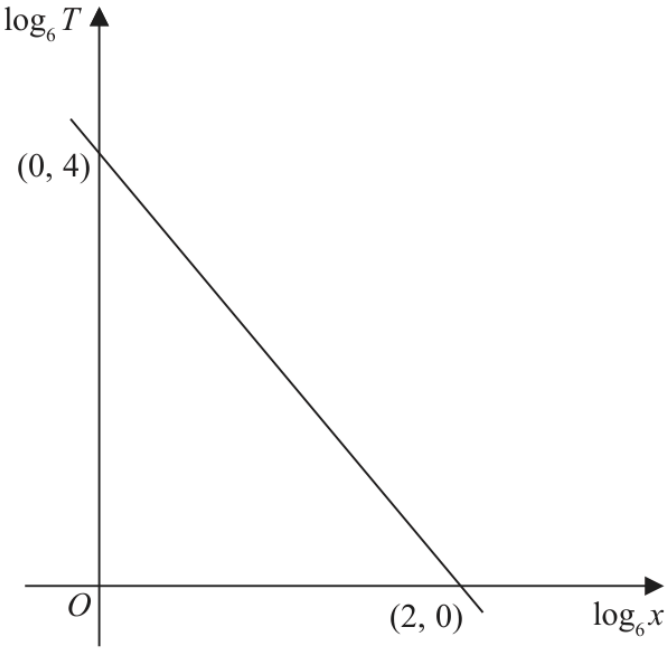


Figure 1

Figure 1 shows the linear relationship between $\log_6 T$ and $\log_6 x$

The line passes through the points (0, 4) and (2, 0) as shown.

(a) (i) Find an equation linking $\log_6 T$ and $\log_6 x$

(ii) Hence find the exact value of T when $x = 216$

(3)

(b) Find an equation, not involving logs, linking T with x

(3)

Question 7

8 marks | Exponentials & Logarithms

Main topic: Log Graphs & Exponential Models

7. A scientist is studying two different populations of bacteria.

The number of bacteria N in the first population is modelled by the equation

$$N = Ae^{kt} \qquad t \geqslant 0$$

where A and k are positive constants and t is the time in hours from the start of the study.

Given that

- there were 2500 bacteria in this population at the start of the study
- there were 10 000 bacteria 8 hours later

(a) find the exact value of A and the value of k to 4 significant figures.

(3)

The number of bacteria N in the second population is modelled by the equation

$$N = 60\,000e^{-0.6t} \qquad t \geqslant 0$$

where t is the time in hours from the start of the study.

(b) Find the rate of decrease of bacteria in this population exactly 5 hours from the start of the study. Give your answer to 3 significant figures.

(2)

When $t = T$, the number of bacteria in the two different populations was the same.

(c) Find the value of T, giving your answer to 3 significant figures.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(3)

Question 4

7 marks | Exponentials & Logarithms

Main topic: Log Graphs & Exponential Models

4. A new mobile phone is released for sale.

The total sales N of this phone, in **thousands**, is modelled by the equation

$$N = 125 - Ae^{-0.109t} \qquad t \geqslant 0$$

where A is a constant and t is the time in months after the phone was released for sale.

Given that when $t = 0$, $N = 32$

(a) state the value of A.

(1)

Given that when $t = T$ the total sales of the phone was 100 000

(b) find, according to the model, the value of T. Give your answer to 2 decimal places.

(3)

(c) Find, according to the model, the rate of increase in total sales when $t = 7$, giving your answer to 3 significant figures.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(2)

The total sales of the mobile phone is expected to reach 150 000

Using this information,

(d) give a reason why the given equation is not suitable for modelling the total sales of the phone.

(1)

Question 5

7 marks | Exponentials & Logarithms | Main topic: Applications of Differentiation

5. The curve C has equation

$$y = \frac{\ln(x^2 + k)}{x^2 + k} \qquad x \in \mathbb{R}$$

where k is a positive constant.

(a) Show that

$$\frac{dy}{dx} = \frac{Ax(B - \ln(x^2 + k))}{(x^2 + k)^2}$$

where A and B are constants to be found.

(3)

Given that C has exactly three turning points,

(b) find the x coordinate of each of these points. Give your answer in terms of k where appropriate.

(3)

(c) find the upper limit to the value for k.

(1)

Question 6

6 marks | Exponentials & Logarithms | Main topic: Log Graphs & Exponential Models

6. An area of sea floor is being monitored.

The area of the sea floor, $S\,\text{km}^2$, covered by coral reefs is modelled by the equation

$$S = pq^t$$

where p and q are constants and t is the number of years after monitoring began.

Given that

$$\log_{10} S = 4.5 - 0.006t$$

(a) find, according to the model, the area of sea floor covered by coral reefs when $t = 2$

(2)

(b) find a complete equation for the model in the form

$$S = pq^t$$

giving the value of p and the value of q each to 3 significant figures.

(3)

(c) With reference to the model, interpret the value of the constant q

(1)

Question 3

7 marks | Exponentials & Logarithms | Main topic: Log Graphs & Exponential Models

3. The amount of money raised for a charity is being monitored.

The total amount raised in the t months after monitoring began, £D, is modelled by the equation

$$\log_{10} D = 1.04 + 0.38t$$

(a) Write this equation in the form

$$D = ab^t$$

where a and b are constants to be found. Give each value to 4 significant figures.

(3)

When $t = T$, the total amount of money raised is £45 000

According to the model,

(b) find the value of T, giving your answer to 3 significant figures.

(2)

The charity aims to raise a total of £350 000 within the first 12 months of monitoring.

According to the model,

(c) determine whether or not the charity will achieve its aim.

(2)

Question 4

13 marks | Exponentials & Logarithms | Main topic: Rational Algebra

4. The function f is defined by

$$\mathrm{f}(x) = \frac{2x^2 - 32}{3x^2 + 7x - 20} + \frac{8}{3x - 5} \qquad x \in \mathbb{R} \quad x > 2$$

(a) Show that $\mathrm{f}(x) = \dfrac{2x}{3x-5}$

(3)

(b) Show, using calculus, that f is a decreasing function.
You must make your reasoning clear.

(3)

The function g is defined by

$$\mathrm{g}(x) = 3 + 2\ln x \qquad x \geqslant 1$$

(c) Find g^{-1}

(3)

(d) Find the exact value of a for which

$$\mathrm{gf}(a) = 5$$

(4)

Question 5

7 marks | Exponentials & Logarithms

Main topic: Log Graphs & Exponential Models

5. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

The temperature, T °C, of the air in a room t minutes after a heat source is switched off, is modelled by the equation

$$T = 10 + A\mathrm{e}^{-Bt}$$

where A and B are constants.

Given that the temperature of the air in the room at the instant the heat source was switched off was 18 °C,

(a) find the value of A

(1)

Given also that, exactly 45 minutes after the heat source was switched off, the temperature of the air in the room was 16 °C,

(b) find the value of B to 3 significant figures.

(3)

Using the values for A and B,

(c) find, according to the model, the rate of change of the temperature of the air in the room exactly two minutes after the heat source was switched off.
Give your answer in °C min^{-1} to 3 significant figures.

(2)

(d) Explain why, according to the model, the temperature of the air in the room cannot fall to 5 °C

(1)

Question 7

12 marks | Exponentials & Logarithms | Main topic: Applications of Differentiation

7. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

The curve C has equation

$$y = \frac{16}{9(3x - k)} \qquad x \neq \frac{k}{3}$$

where k is a positive constant not equal to 3

(a) Find $\frac{dy}{dx}$ giving your answer in simplest form in terms of k.

(2)

The point P with x coordinate 1 lies on C.

Given that the gradient of the curve at P is -12

(b) find the two possible values of k

(3)

Given also that $k < 3$

(c) find the equation of the normal to C at P, writing your answer in the form $ax + by + c = 0$, where a, b and c are integers to be found.

(3)

(d) show, using algebraic integration that,

$$\int_1^3 \frac{16}{9(3x - k)} dx = \lambda \ln 10$$

where λ is a constant to be found.

(4)

Question 2

7 marks | Exponentials & Logarithms | Main topic: Integration

2. $$g(x) = \frac{2x^2 - 5x + 8}{x - 2}$$

(a) Write $g(x)$ in the form

$$Ax + B + \frac{C}{x - 2}$$

where A, B and C are integers to be found.

(3)

(b) Hence use algebraic integration to show that

$$\int_4^8 g(x)\,dx = \alpha + \beta \ln 3$$

where α and β are integers to be found.

(4)

Question 3

6 marks | Exponentials & Logarithms

Main topic: Log Graphs & Exponential Models

3. (i) The variables x and y are connected by the equation

$$y = \frac{10^6}{x^3} \qquad x > 0$$

Sketch the graph of $\log_{10} y$ against $\log_{10} x$

Show on your sketch the coordinates of the points of intersection of the graph with the axes.

(3)

(ii)

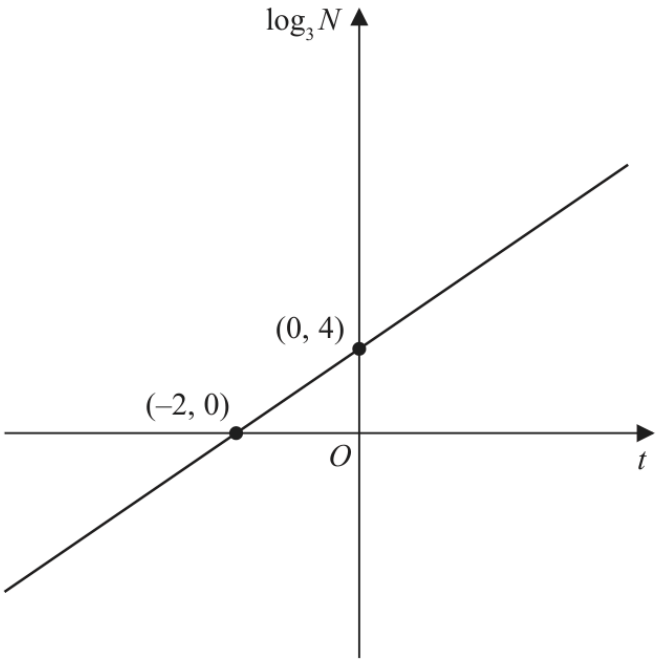


Figure 2

Figure 2 shows the linear relationship between $\log_3 N$ and t.

Show that $N = ab^t$ where a and b are constants to be found.

(3)

Question 5

10 marks | Exponentials & Logarithms | Main topic: Functions, Composition & Inverses

5. The functions f and g are defined by

$$\mathrm{f}(x) = 2 + 5\ln x \qquad x > 0$$

$$\mathrm{g}(x) = \frac{6x - 2}{2x + 1} \qquad x > \frac{1}{3}$$

(a) Find $\mathrm{f}^{-1}(22)$

(2)

(b) Use differentiation to prove that g is an increasing function.

(3)

(c) Find g^{-1}

(3)

(d) Find the range of fg

(2)

Question 8

10 marks | Exponentials & Logarithms | Main topic: Log Graphs & Exponential Models

8.

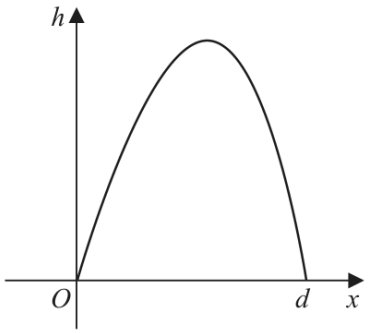


Figure 4

Figure 4 is a graph showing the path of a golf ball after the ball has been hit until it first hits the ground.

The vertical height, h metres, of the ball above the ground has been plotted against the horizontal distance travelled, x metres, measured from where the ball was hit.

The ball travels a horizontal distance of d metres before it first hits the ground.

The ball is modelled as a particle travelling in a vertical plane above horizontal ground.

The path of the ball is modelled by the equation

$$h = 1.5x - 0.5x\,e^{0.02x} \qquad 0 \leqslant x \leqslant d$$

Use the model to answer parts (a), (b) and (c).

(a) Find the value of d, giving your answer to 2 decimal places.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(3)

(b) Show that the maximum value of h occurs when

$$x = 50\ln\left(\frac{150}{x + 50}\right)$$

(4)

Using the iteration formula

$$x_{n+1} = 50\ln\left(\frac{150}{x_n + 50}\right) \qquad \text{with } x_1 = 30$$

(c) (i) find the value of x_2 to 2 decimal places,

(ii) find, by repeated iteration, the horizontal distance travelled by the golf ball before it reaches its maximum height. Give your answer to 2 decimal places.

(3)

Question 4

6 marks | Exponentials & Logarithms

Main topic: Log Graphs & Exponential Models

4. The number of bacteria on a surface is being monitored.

The number of bacteria, N, on the surface, t hours after monitoring began is modelled by the equation

$$\log_{10} N = 0.35t + 2$$

Use the equation of the model to answer parts (a) to (c).

(a) Find the initial number of bacteria on the surface.

(1)

(b) Show that the equation of the model can be written in the form

$$N = ab^t$$

where a and b are constants to be found. Give the value of b to 2 decimal places.

(3)

(c) Hence find the rate of growth of bacteria on the surface exactly 5 hours after monitoring began.

(2)

Question 8

12 marks | Exponentials & Logarithms | Main topic: Log Graphs & Exponential Models

8.

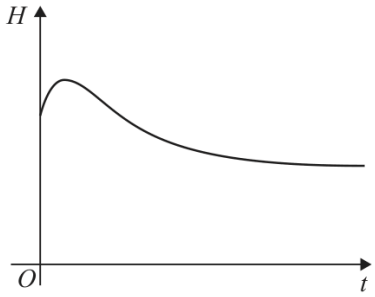


Figure 4

The heart rate of a horse is being monitored.
The heart rate H, measured in beats per minute (bpm), is modelled by the equation

$$H = 32 + 40e^{-0.2t} - 20e^{-0.9t}$$

where t minutes is the time after monitoring began.

Figure 4 is a sketch of H against t.

Use the equation of the model to answer parts (a) to (e).

(a) State the initial heart rate of the horse.

(1)

In the long term, the heart rate of the horse approaches L bpm.

(b) State the value of L.

(1)

The heart rate of the horse reaches its maximum value after T minutes.

(c) Find the value of T, giving your answer to 3 decimal places.

(*Solutions based entirely on calculator technology are not acceptable.*)

(5)

The heart rate of the horse is 37 bpm after M minutes.

(d) Show that M is a solution of the equation

$$t = 5\ln\left(\frac{8}{1 + 4e^{-0.9t}}\right)$$

(2)

Using the iteration formula

$$t_{n+1} = 5\ln\left(\frac{8}{1 + 4e^{-0.9t_n}}\right) \quad \text{with} \quad t_1 = 10$$

(e) (i) find, to 4 decimal places, the value of t_2

(ii) find, to 4 decimal places, the value of M

(3)

Question 2

3 marks | Exponentials & Logarithms

Main topic: Log Graphs & Exponential Models

2. The weed on the surface of a pond is being monitored.

The surface area of the pond covered by the weed, $A\,\text{m}^2$, is modelled by the equation

$$\log_{10} A = 1 + 0.03t$$

where t is the number of weeks after monitoring began.

Use the equation of the model to answer parts (a) and (b).

(a) Find the surface area of the pond initially covered by the weed.

(1)

After T weeks, $25\,\text{m}^2$ of the pond is covered by the weed.

(b) Find the value of T, giving your answer to 2 decimal places.

(2)

Question 4

9 marks | Exponentials & Logarithms | Main topic: Integration

4. Given that

$$\frac{4x^3 + 2x^2 + 3x + 8}{x^2 + 4} \equiv Ax + B + \frac{Cx + D}{x^2 + 4}$$

(a) (i) find the values of the constants A, B and C

(ii) show that $D = 0$

(4)

(b) Hence, using algebraic integration, find

$$\int_1^4 \frac{4x^3 + 2x^2 + 3x + 8}{x^2 + 4} \, \mathrm{d}x$$

giving your answer in the form $p + q \ln 2$, where p and q are integers.

(5)

Question 5

9 marks | Exponentials & Logarithms

Main topic: Log Graphs & Exponential Models

5. A hot piece of metal is cooled by dropping it into water. The temperature, H°C, of the metal, t minutes after it is dropped into the water, is modelled by the equation

$$H = 280e^{-0.05t} + 24 \qquad t \geqslant 0$$

Use the equation of the model to answer parts (a) to (d).

(a) Find the initial temperature of the piece of metal.

(1)

(b) On Diagram 1, sketch the graph of H against t. On your sketch, state the equation of the asymptote to the curve.

(2)

(c) Find the value of t for which $H = 144$, giving your answer to 2 decimal places.

(*Solutions based entirely on calculator technology are not acceptable.*)

(3)

(d) Show by differentiation that

$$\frac{dH}{dt} = a + bH$$

where a and b are constants to be found.

(3)

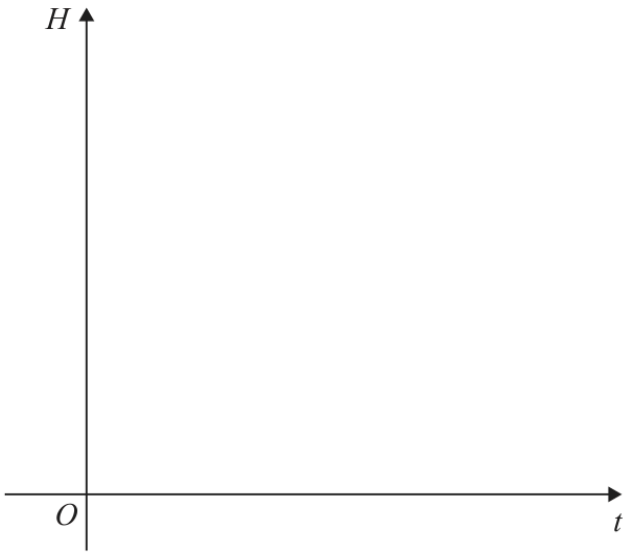


Diagram 1

Question 9

8 marks | Exponentials & Logarithms

Main topic: Applications of Differentiation

9.

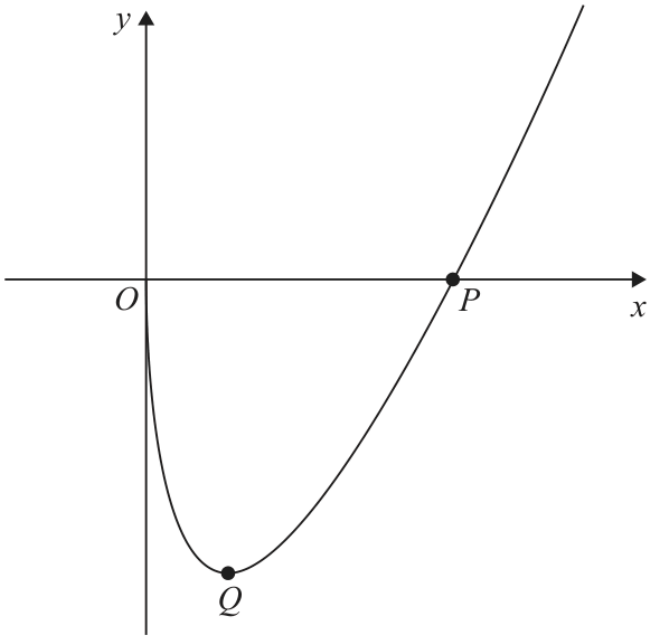


Figure 1

Figure 1 shows a sketch of part of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 6\sqrt{x}\ln(4x) \qquad x > 0$$

The curve cuts the x-axis at point P

(a) State the x coordinate of P

(1)

The point Q, shown in Figure 1, is the stationary point on C

(b) Use calculus to find the exact coordinates of Q

(5)

(c) Hence find the range of the function $\mathrm{g}(x)$ where

$$\mathrm{g}(x) = -2\,\mathrm{f}(x)$$

(2)

Question 3

5 marks | Exponentials & Logarithms

Main topic: Log Graphs & Exponential Models

3:

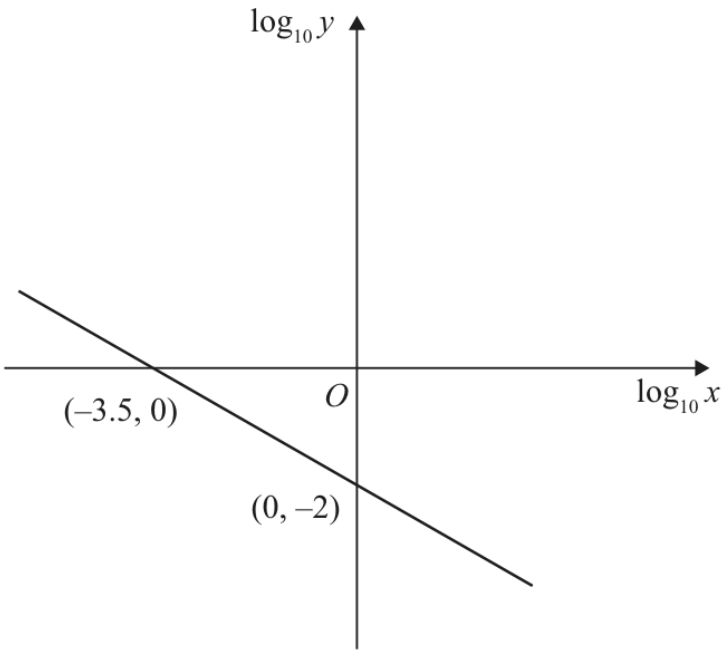


Figure 1

Figure 1 shows a linear relationship between $\log_{10} y$ and $\log_{10} x$

The line passes through the points (–3.5, 0) and (0, –2) as shown.

(a) Find an equation linking $\log_{10} y$ with $\log_{10} x$

(2)

(b) Hence, or otherwise, express y in the form px^q where p and q are rational constants.

(3)

Question 6

8 marks | Exponentials & Logarithms

Main topic: Log Graphs & Exponential Models

6: **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The temperature, θ°C, of a computer processor, t minutes after the computer is switched off, is modelled by the equation

$$\theta = 21 + Ae^{-kt}$$

where A and k are positive constants.

Given that the temperature of the processor was 75°C when the computer was switched off,

(a) find the value of A.

(2)

Given also that it takes 5 minutes for the temperature of the processor to decrease from 75°C to 25°C,

(b) find the value of k, giving your answer to 3 significant figures.

(3)

At time T minutes, the temperature of the processor is decreasing at a rate of 9°C per minute.

(c) Find the value of T according to the model, giving your answer to 2 decimal places.

(3)

WMA13/01 OCTOBER 2025

Question 1

9 marks | Exponentials & Logarithms | Main topic: Functions, Composition & Inverses

1. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The functions f and g are defined by

$$\mathrm{f}(x) = \ln(x^2 + 3) \qquad x \in \mathbb{R}$$

$$\mathrm{g}(x) = \frac{3 + 5x}{x + 2} \qquad x \in \mathbb{R} \quad x > -2$$

(a) State the range of f

(1)

(b) Find g^{-1}

(3)

(c) Find $\mathrm{fg}(0)$

(2)

(d) Find the exact value of a for which

$$\mathrm{g}\left(\mathrm{e}^{2a}\right) = \mathrm{f}\left(\sqrt{\mathrm{e}^4 - 3}\right)$$

(3)

Question 3

9 marks | Exponentials & Logarithms | Main topic: Log Graphs & Exponential Models

3. The share price, £V, of a company is being monitored.

A graph is drawn of $\log_{10} V$ against t, where t is the number of years after monitoring began.

The graph is a straight line passing through the points $(0, 2)$ and $(5, 2.25)$

Using this information,

(a) find an equation for the line in the form

$$\log_{10} V = mt + c$$

where m and c are constants.

(2)

(b) Write the answer to part (a) in the form

$$V = ab^t$$

where a and b are constants to be found.

Give the exact value of a and the value of b to 3 significant figures.

(3)

When $t = T$, the rate of increase in the share price of the company was £50 per year.

(c) Find the value of T, giving your answer to the nearest integer.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(4)

Question 5

8 marks | Exponentials & Logarithms

Main topic: Log Graphs & Exponential Models

5. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The number of squirrels in a forest is being studied.

The number of squirrels, N, in the forest, t years after the start of the study, is modelled by the equation

$$N = \frac{4000e^{0.1t}}{19 + e^{0.2t}} \qquad t \geqslant 0$$

Use the equation of the model to answer parts (a), (b), (c) and (d).

(a) Find the number of squirrels in the forest at the start of the study.

(1)

(b) Find $\frac{dN}{dt}$

(2)

The number of squirrels in the forest is at a maximum when $t = T$

Using the answer to part (b),

(c) show that $e^{0.2T} = A$, where A is a constant to be found.

(2)

(d) Hence find the maximum number of squirrels in the forest.

Show your working and give your answer to the nearest whole number.

(3)

Question 6

7 marks | Exponentials & Logarithms | Main topic: Integration

6. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(i) Given $y = \ln(2x^2 + 5)$ find $\dfrac{dy}{dx}$

(2)

(ii) A curve C has equation $y = f(x)$ where

$$f(x) = \frac{21x}{3x^2 + k} \qquad x \in \mathbb{R}$$

where k is a positive integer, $k > 1$

Given that

$$\int_1^k f(x)\,dx < 7\ln 8$$

find the greatest possible value of k

(5)

Question 3

10 marks | Exponentials & Logarithms | Main topic: Integration

3. (i) Find, in simplest form,

$$\int 2(3x+5)^7 \, dx$$

(2)

(ii) $$f(x) = \frac{4x^3 + x^2 - 7x + 14}{(x-1)(x+2)} \qquad x > 1$$

(a) Write $f(x)$ in the form

$$ax + b + \frac{c}{x-1}$$

where a, b and c are constants to be found.

(4)

(b) Hence, using algebraic integration, find

$$\int_3^9 f(x) \, dx$$

writing the answer in simplest form.

(4)

Question 4

6 marks | Exponentials & Logarithms

Main topic: Log Graphs & Exponential Models

4. The resting heart rate, h beats per minute, of a particular group of mammals, can be modelled by the equation

$$\log_{10} h = 2.4 - 0.25 \log_{10} m$$

where m kg is the mass of the mammal.

The mass of one of these mammals is 3 kg.

According to the model,

(a) find the resting heart rate of this mammal. Give your answer to 3 significant figures.

(2)

(b) Show that the equation of the model can be written in the form

$$h = \frac{p}{m^q}$$

giving each of the values of the constants p and q to 3 significant figures where appropriate.

(3)

(c) With reference to the model, interpret the value of the constant p.

(1)

Question 5

8 marks | Exponentials & Logarithms

Main topic: Functions, Composition & Inverses

5. The function f is defined by

$$\mathrm{f} : x \to \frac{2x+16}{x-4} \qquad x \neq 4$$

(a) Find $\mathrm{ff}(x)$, giving your answer as a single fraction in its simplest form.

(3)

(b) Find $\mathrm{f}^{-1}(12)$

(2)

The function g is defined by

$$\mathrm{g} : x \to \ln x \qquad x > 0 \quad x \neq \mathrm{e}^4$$

(c) Find the exact values of a such that

$$\mathrm{fg}(a) = \mathrm{g}(a)$$

(3)

Question 8

4 marks | Exponentials & Logarithms | Main topic: Differentiation Techniques

8. A curve has equation

$$x = e^{2\tan y} \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Show that

$$\frac{dy}{dx} = \frac{a}{x\left(b + (\ln x)^n\right)}$$

where a, b and n are integers to be found.

(4)

Question 10

9 marks | Exponentials & Logarithms

Main topic: Log Graphs & Exponential Models

10. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

Two populations of insects are being studied over the same period of time.

The number of insects N_A in population A, t years after the start of the study, is modelled by the equation

$$N_A = 8000 + 900e^{0.2t}$$

According to the model,

(a) find the rate of growth in the number of insects in population A exactly 5 years after the start of the study.

(2)

The number of insects N_B in population B, t years after the start of the study, is modelled by the equation

$$N_B = 8000 + Pe^{kt}$$

where P and k are positive constants.

Given that

- there are 10 000 insects in population B at the start of the study
- there are 11 570 insects in population B exactly 4 years after the start of the study
- there are the same number of insects in population A and population B after T years

find, according to the models,

(b) the exact value of P and the value of k to 3 decimal places,

(4)

(c) the value of T, giving your answer to one decimal place.

(3)

Question 2

9 marks | Exponentials & Logarithms | Main topic: Functions, Composition & Inverses

2. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The functions f and g are defined by

$$f(x) = 3 + 4e^{2x} \qquad x \in \mathbb{R}$$

$$g(x) = \sqrt{x + 5} \qquad x \in \mathbb{R} \quad x > -5$$

(a) State the range of f

(1)

(b) Find g^{-1}

(3)

(c) Find $gf\left(\ln\sqrt{7}\right)$

(2)

(d) Find the value of a, $a > 0$, for which

$$fg(a^2 - 5) = g(59)$$

giving the answer in the form $\ln p$ where p is an irrational constant.

(3)

Question 5

7 marks | Exponentials & Logarithms | Main topic: Log Graphs & Exponential Models

5.

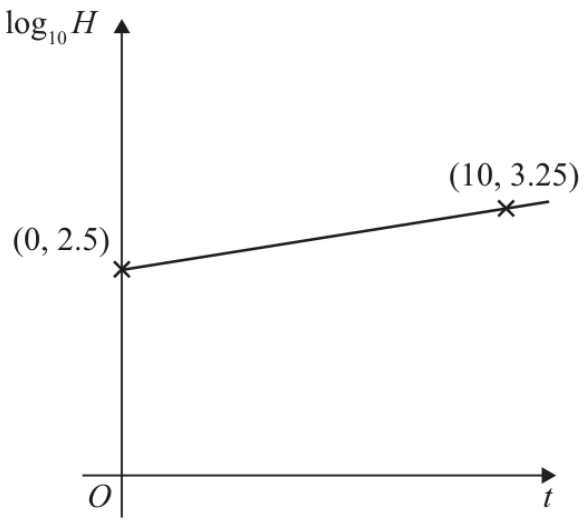


Figure 1

The recycling of household waste in a particular town is being monitored.

The number of households in the town, H, recycling their household waste, t years after monitoring began, is modelled by the equation

$$H = ab^t$$

where a and b are constants.

The line in Figure 1 shows the linear relationship between t and $\log_{10} H$

The line passes through the points (0, 2.5) and (10, 3.25)

Using this information,

(a) find, to 3 significant figures, the value of a and the value of b

(4)

(b) With reference to the model, interpret the value of the constant b

(1)

(c) Find the value of $\frac{dH}{dt}$ when $t = 5$, giving the answer to 2 significant figures.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(2)

Question 7

12 marks | Exponentials & Logarithms | Main topic: Log Graphs & Exponential Models

7. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The selling price of a car is monitored over a 15-year period.

The selling price P_C of the car, in thousands of pounds, t years after monitoring began, is modelled by the equation

$$P_C = 7 + 16e^{-0.1t} \qquad 0 \leqslant t \leqslant 15$$

Given that when $t = T$ the selling price of the car is £15 000

(a) find, according to the model, the value of T. Give the answer to 2 decimal places.

(3)

(b) Find the value of $\frac{dP_C}{dt}$ when $t = 5$. Give the answer to 2 decimal places.

(2)

The selling price of a van is monitored over the same 15-year period.

The selling price P_V of the van, in thousands of pounds, t years after monitoring began, is modelled by the equation

$$P_V = 4 + me^{-0.2t} \qquad 0 \leqslant t \leqslant 15$$

where m is a constant.

Use the equations of these models to answer parts (c) and (d).

Given that $P_V = 36$ when $t = 0$

(c) (i) state the value of m

(ii) find the difference between the initial selling price of the car and the initial selling price of the van.

(3)

When $t = \alpha$, the selling price of the car is equal to the selling price of the van.

(d) Find, using algebra and showing your working, the value of α, giving the answer to 3 significant figures.

(4)

TOPIC

Log Graphs & Exponential Models

Every past-paper question on this topic, January 2020 to June 2026.

WMA13/01 JANUARY 2020

Question 1

6 marks | Log Graphs & Exponential Models

1. A population of a rare species of toad is being studied.

The number of toads, N, in the population, t years after the start of the study, is modelled by the equation

$$N = \frac{900e^{0.12t}}{2e^{0.12t} + 1} \qquad t \geqslant 0, t \in \mathbb{R}$$

According to this model,

(a) calculate the number of toads in the population at the start of the study,

(1)

(b) find the value of t when there are 420 toads in the population, giving your answer to 2 decimal places.

(4)

(c) Explain why, according to this model, the number of toads in the population can never reach 500

(1)

Question 3

5 marks | Log Graphs & Exponential Models

3.

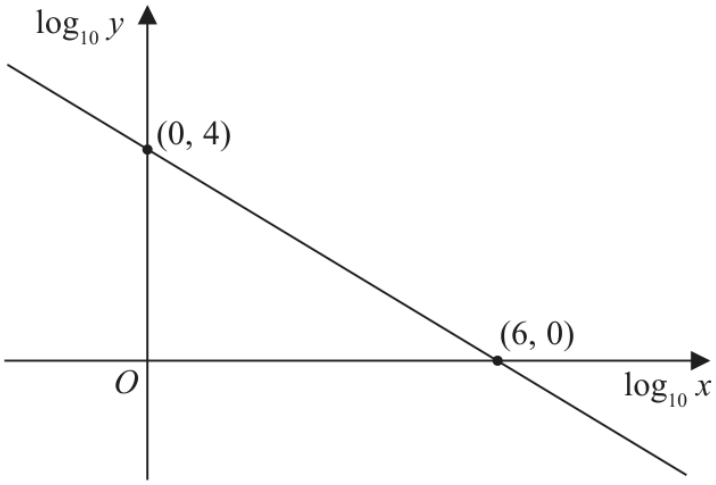


Figure 1

Figure 1 shows a linear relationship between $\log_{10} y$ and $\log_{10} x$

The line passes through the points (0, 4) and (6, 0) as shown.

(a) Find an equation linking $\log_{10} y$ with $\log_{10} x$

(2)

(b) Hence, or otherwise, express y in the form px^q, where p and q are constants to be found.

(3)

Question 2

6 marks | Log Graphs & Exponential Models

2. A scientist monitored the growth of bacteria on a dish over a 30-day period.

The area, $N\text{mm}^2$, of the dish covered by bacteria, t days after monitoring began, is modelled by the equation

$$\log_{10} N = 0.0646t + 1.478 \qquad 0 \leqslant t \leqslant 30$$

(a) Show that this equation may be written in the form

$$N = a\,b^t$$

where a and b are constants to be found. Give the value of a to the nearest integer and give the value of b to 3 significant figures.

(4)

(b) Use the model to find the area of the dish covered by bacteria 30 days after monitoring began. Give your answer, in mm^2, to 2 significant figures.

(2)

Question 5

11 marks | Log Graphs & Exponential Models

5. The temperature, θ°C, inside an oven, t minutes after the oven is switched on, is given by

$$\theta = A - 180e^{-kt}$$

where A and k are positive constants.

Given that the temperature inside the oven is initially 18 °C,

(a) find the value of A.

(2)

The temperature inside the oven, 5 minutes after the oven is switched on, is 90 °C.

(b) Show that $k = p \ln q$ where p and q are rational numbers to be found.

(4)

Hence find

(c) the temperature inside the oven 9 minutes after the oven is switched on, giving your answer to 3 significant figures,

(2)

(d) the rate of increase of the temperature inside the oven 9 minutes after the oven is switched on. Give your answer in °C min^{-1} to 3 significant figures.

(3)

Question 8

7 marks | Log Graphs & Exponential Models

8. The percentage, P, of the population of a small country who have access to the internet, is modelled by the equation

$$P = ab^t$$

where a and b are constants and t is the number of years after the start of 2005

Using the data for the years between the start of 2005 and the start of 2010, a graph is plotted of $\log_{10} P$ against t.

The points are found to lie approximately on a straight line with gradient 0.09 and intercept 0.68 on the $\log_{10} P$ axis.

(a) Find, according to the model, the value of a and the value of b, giving your answers to 2 decimal places.

(4)

(b) In the context of the model, give a practical interpretation of the constant a.

(1)

(c) Use the model to estimate the percentage of the population who had access to the internet at the start of 2015

(2)

Question 5

7 marks | Log Graphs & Exponential Models

5.

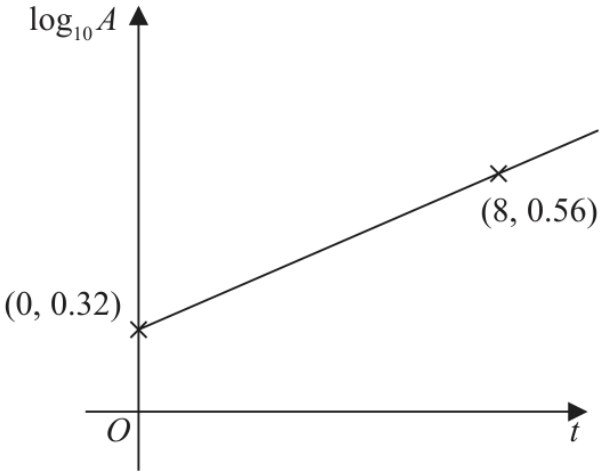


Figure 1

The growth of duckweed on a pond is being studied.

The surface area of the pond covered by duckweed, $A\text{m}^2$, at a time t days after the start of the study is modelled by the equation

$$A = pq^t \qquad \text{where } p \text{ and } q \text{ are positive constants}$$

Figure 1 shows the linear relationship between $\log_{10} A$ and t.

The points (0, 0.32) and (8, 0.56) lie on the line as shown.

(a) Find, to 3 decimal places, the value of p and the value of q.

(4)

Using the model with the values of p and q found in part (a),

(b) find the rate of increase of the surface area of the pond covered by duckweed, in m^2/day, exactly 6 days after the start of the study.
Give your answer to 2 decimal places.

(3)

Question 8

13 marks | Log Graphs & Exponential Models

8. A scientist is studying a population of fish in a lake. The number of fish, N, in the population, t years after the start of the study, is modelled by the equation

$$N = \frac{600e^{0.3t}}{2 + e^{0.3t}} \qquad t \geqslant 0$$

Use the equation of the model to answer parts (a), (b), (c), (d) and (e).

(a) Find the number of fish in the lake at the start of the study.

(1)

(b) Find the upper limit to the number of fish in the lake.

(1)

(c) Find the time, after the start of the study, when there are predicted to be 500 fish in the lake. Give your answer in years and months to the nearest month.

(4)

(d) Show that

$$\frac{dN}{dt} = \frac{Ae^{0.3t}}{(2 + e^{0.3t})^2}$$

where A is a constant to be found.

(3)

Given that when $t = T$, $\frac{dN}{dt} = 8$

(e) find the value of T to one decimal place.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(4)

Question 3

6 marks | Log Graphs & Exponential Models

3.

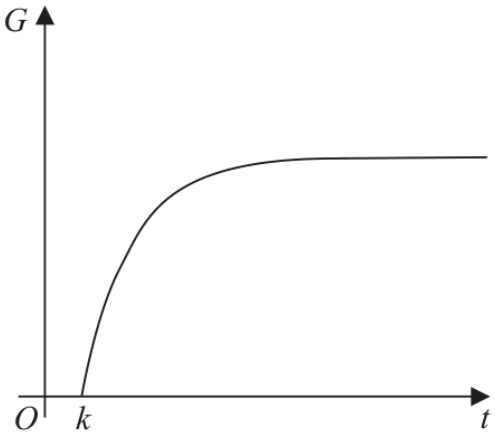


Figure 2

The total mass of gold, G tonnes, extracted from a mine is modelled by the equation

$$G = 40 - 30e^{1-0.05t} \qquad t \geqslant k \qquad G \geqslant 0$$

where t is the number of years after 1st January 1800.

Figure 2 shows a sketch of G against t.

Use the equation of the model to answer parts (a), (b) and (c).

(a) (i) Find the value of k.

(ii) Hence find the year and month in which gold started being extracted from the mine.

(3)

(b) Find the total mass of gold extracted from the mine up to 1st January 1870.

(2)

There is a limit to the mass of gold that can be extracted from the mine.

(c) State the value of this limit.

(1)

Question 7

6 marks | Log Graphs & Exponential Models

7 The mass, M kg, of a species of tree can be modelled by the equation

$$\log_{10} M = 1.93 \log_{10} r + 0.684$$

where r cm is the base radius of the tree.

The base radius of a particular tree of this species is 45 cm.

According to the model,

(a) find the mass of this tree, giving your answer to 2 significant figures.

(2)

(b) Show that the equation of the model can be written in the form

$$M = pr^q$$

giving the values of the constants p and q to 3 significant figures.

(3)

(c) With reference to the model, interpret the value of the constant p.

(1)

Question 4

7 marks | Log Graphs & Exponential Models

4. The growth of a weed on the surface of a pond is being studied.

The surface area of the pond covered by the weed, $A\,\text{m}^2$, is modelled by the equation

$$A = \frac{80pe^{0.15t}}{pe^{0.15t} + 4}$$

where p is a positive constant and t is the number of days after the start of the study.

Given that

- $30\,\text{m}^2$ of the surface of the pond was covered by the weed at the start of the study
- $50\,\text{m}^2$ of the surface of the pond was covered by the weed T days after the start of the study

(a) show that $p = 2.4$

(2)

(b) find the value of T, giving your answer to one decimal place.

(*Solutions relying entirely on graphical or numerical methods are not acceptable.*)

(4)

The weed grows until it covers the surface of the pond.

(c) Find, according to the model, the maximum possible surface area of the pond.

(1)

Question 8

8 marks | Log Graphs & Exponential Models

8. A dose of antibiotics is given to a patient.

The amount of the antibiotic, x milligrams, in the patient's bloodstream t hours after the dose was given, is found to satisfy the equation

$$\log_{10} x = 2.74 - 0.079t$$

(a) Show that this equation can be written in the form

$$x = pq^{-t}$$

where p and q are constants to be found. Give the value of p to the nearest whole number and the value of q to 2 significant figures.

(4)

(b) With reference to the equation in part (a), interpret the value of the constant p.

(1)

When a different dose of the antibiotic is given to another patient, the values of x and t satisfy the equation

$$x = 400 \times 1.4^{-t}$$

(c) Use calculus to find, to 2 significant figures, the value of $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ when $t = 5$

(3)

Question 4

7 marks | Log Graphs & Exponential Models

4.

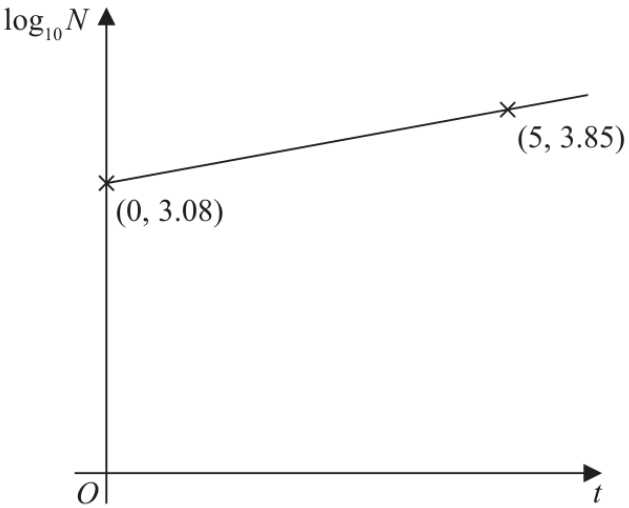


Figure 1

The number of subscribers to an online video streaming service, N, is modelled by the equation

$$N = ab^t$$

where a and b are constants and t is the number of years since monitoring began.

The line in Figure 1 shows the linear relationship between t and $\log_{10} N$

The line passes through the points (0, 3.08) and (5, 3.85)

Using this information,

(a) find an equation for this line.

(2)

(b) Find the value of a and the value of b, giving your answers to 3 significant figures.

(3)

When $t = T$ the number of subscribers is 500 000

According to the model,

(c) find the value of T

(2)

Question 5

11 marks | Log Graphs & Exponential Models

5.

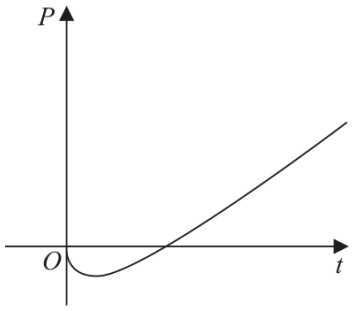


Figure 2

The profit made by a company, £P million, t years after the company started trading, is modelled by the equation

$$P = \frac{4t-1}{10} + \frac{3}{4}\ln\left[\frac{t+1}{(2t+1)^2}\right]$$

The graph of P against t is shown in Figure 2.

According to the model,

(a) show that exactly one year after it started trading, the company had made a loss of approximately £830 000

(2)

A manager of the company wants to know the value of t for which $P = 0$

(b) Show that this value of t occurs in the interval $[6, 7]$

(2)

(c) Show that the equation $P = 0$ can be expressed in the form

$$t = \frac{1}{4} + \frac{15}{8}\ln\left[\frac{(2t+1)^2}{t+1}\right]$$

(2)

(d) Using the iteration formula

$$t_{n+1} = \frac{1}{4} + \frac{15}{8}\ln\left[\frac{(2t_n+1)^2}{t_n+1}\right] \quad \text{with} \quad t_1 = 6$$

find the value of t_2 and the value of t_6, giving your answers to 3 decimal places.

(3)

(e) Hence find, according to the model, how many months it takes in total, from when the company started trading, for it to make a profit.

(2)

Question 3

5 marks | Log Graphs & Exponential Models

3.

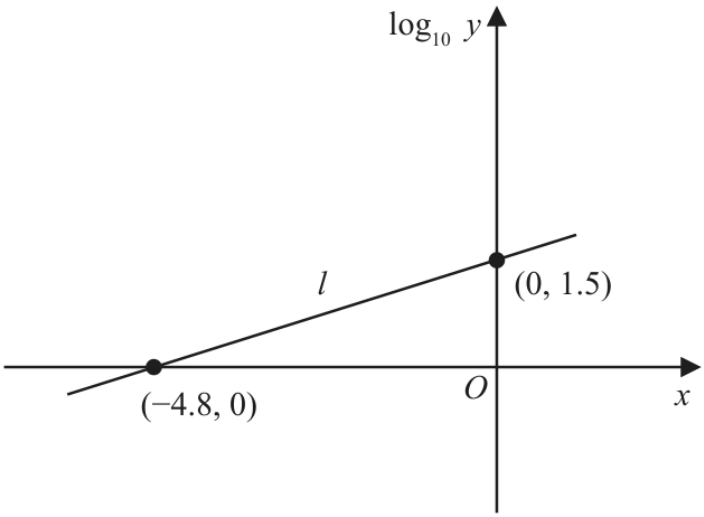


Figure 1

The line l in Figure 1 shows a linear relationship between $\log_{10} y$ and x.

The line passes through the points (0, 1.5) and (−4.8, 0) as shown.

(a) Write down an equation for l.

(2)

(b) Hence, or otherwise, express y in the form kb^x, giving the values of the constants k and b to 3 significant figures.

(3)

Question 10

9 marks | Log Graphs & Exponential Models

10. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

A population of fruit flies is being studied.

The number of fruit flies, F, in the population, t days after the start of the study, is modelled by the equation

$$F = \frac{350e^{kt}}{9 + e^{kt}}$$

where k is a constant.

Use the equation of the model to answer parts (a), (b) and (c).

(a) Find the number of fruit flies in the population at the start of the study.

(1)

Given that there are 200 fruit flies in the population 15 days after the start of the study,

(b) show that $k = \frac{1}{15}\ln 12$

(3)

Given also that, when $t = T$, the number of fruit flies in the population is increasing at a rate of 10 per day,

(c) find the possible values of T, giving your answers to one decimal place.

(5)

Question 2

6 marks | Log Graphs & Exponential Models

2.

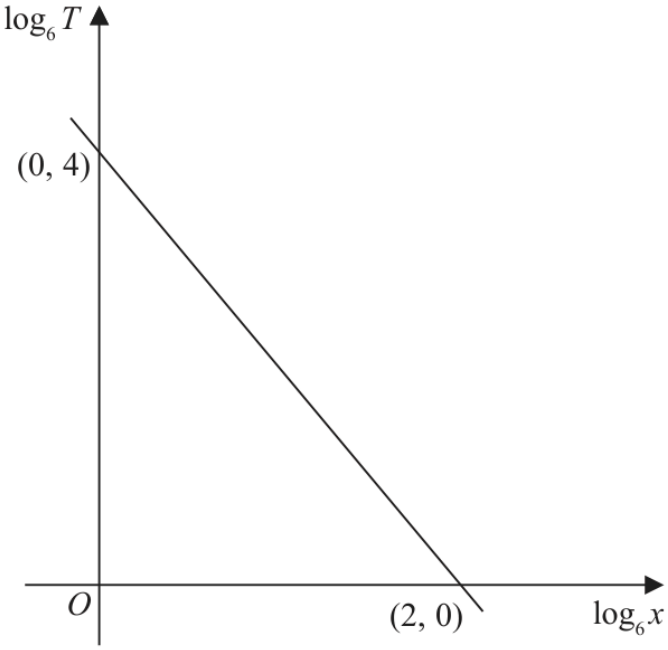


Figure 1

Figure 1 shows the linear relationship between $\log_6 T$ and $\log_6 x$

The line passes through the points (0, 4) and (2, 0) as shown.

(a) (i) Find an equation linking $\log_6 T$ and $\log_6 x$

(ii) Hence find the exact value of T when $x = 216$

(3)

(b) Find an equation, not involving logs, linking T with x

(3)

Question 7

8 marks | Log Graphs & Exponential Models

7. A scientist is studying two different populations of bacteria.

The number of bacteria N in the first population is modelled by the equation

$$N = Ae^{kt} \qquad t \geqslant 0$$

where A and k are positive constants and t is the time in hours from the start of the study.

Given that

- there were 2500 bacteria in this population at the start of the study
- there were 10 000 bacteria 8 hours later

(a) find the exact value of A and the value of k to 4 significant figures.

(3)

The number of bacteria N in the second population is modelled by the equation

$$N = 60\,000e^{-0.6t} \qquad t \geqslant 0$$

where t is the time in hours from the start of the study.

(b) Find the rate of decrease of bacteria in this population exactly 5 hours from the start of the study. Give your answer to 3 significant figures.

(2)

When $t = T$, the number of bacteria in the two different populations was the same.

(c) Find the value of T, giving your answer to 3 significant figures.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(3)

Question 4

7 marks | Log Graphs & Exponential Models

4. A new mobile phone is released for sale.

The total sales N of this phone, in **thousands**, is modelled by the equation

$$N = 125 - Ae^{-0.109t} \qquad t \geqslant 0$$

where A is a constant and t is the time in months after the phone was released for sale.

Given that when $t = 0$, $N = 32$

(a) state the value of A.

(1)

Given that when $t = T$ the total sales of the phone was 100 000

(b) find, according to the model, the value of T. Give your answer to 2 decimal places.

(3)

(c) Find, according to the model, the rate of increase in total sales when $t = 7$, giving your answer to 3 significant figures.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(2)

The total sales of the mobile phone is expected to reach 150 000

Using this information,

(d) give a reason why the given equation is not suitable for modelling the total sales of the phone.

(1)

Question 6

6 marks | Log Graphs & Exponential Models

6. An area of sea floor is being monitored.

The area of the sea floor, $S\,\text{km}^2$, covered by coral reefs is modelled by the equation

$$S = pq^t$$

where p and q are constants and t is the number of years after monitoring began.

Given that

$$\log_{10} S = 4.5 - 0.006t$$

(a) find, according to the model, the area of sea floor covered by coral reefs when $t = 2$

(2)

(b) find a complete equation for the model in the form

$$S = pq^t$$

giving the value of p and the value of q each to 3 significant figures.

(3)

(c) With reference to the model, interpret the value of the constant q

(1)

Question 3

7 marks | Log Graphs & Exponential Models

3. The amount of money raised for a charity is being monitored.

The total amount raised in the t months after monitoring began, £D, is modelled by the equation

$$\log_{10} D = 1.04 + 0.38t$$

(a) Write this equation in the form

$$D = ab^t$$

where a and b are constants to be found. Give each value to 4 significant figures.

(3)

When $t = T$, the total amount of money raised is £45 000

According to the model,

(b) find the value of T, giving your answer to 3 significant figures.

(2)

The charity aims to raise a total of £350 000 within the first 12 months of monitoring.

According to the model,

(c) determine whether or not the charity will achieve its aim.

(2)

Question 5

7 marks | Log Graphs & Exponential Models

5. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

The temperature, T °C, of the air in a room t minutes after a heat source is switched off, is modelled by the equation

$$T = 10 + Ae^{-Bt}$$

where A and B are constants.

Given that the temperature of the air in the room at the instant the heat source was switched off was 18 °C,

(a) find the value of A

(1)

Given also that, exactly 45 minutes after the heat source was switched off, the temperature of the air in the room was 16 °C,

(b) find the value of B to 3 significant figures.

(3)

Using the values for A and B,

(c) find, according to the model, the rate of change of the temperature of the air in the room exactly two minutes after the heat source was switched off.
Give your answer in °C min^{-1} to 3 significant figures.

(2)

(d) Explain why, according to the model, the temperature of the air in the room cannot fall to 5 °C

(1)

Question 3

6 marks | Log Graphs & Exponential Models

3. (i) The variables x and y are connected by the equation

$$y = \frac{10^6}{x^3} \qquad x > 0$$

Sketch the graph of $\log_{10} y$ against $\log_{10} x$

Show on your sketch the coordinates of the points of intersection of the graph with the axes.

(3)

(ii)

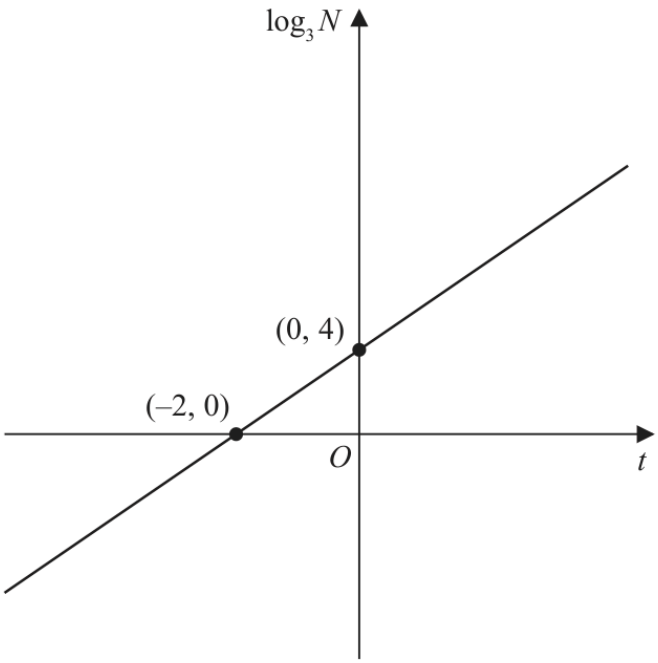


Figure 2

Figure 2 shows the linear relationship between $\log_3 N$ and t.

Show that $N = ab^t$ where a and b are constants to be found.

(3)

Question 8

10 marks | Log Graphs & Exponential Models

8.

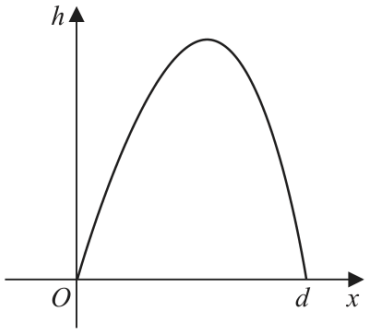


Figure 4

Figure 4 is a graph showing the path of a golf ball after the ball has been hit until it first hits the ground.

The vertical height, h metres, of the ball above the ground has been plotted against the horizontal distance travelled, x metres, measured from where the ball was hit.

The ball travels a horizontal distance of d metres before it first hits the ground.

The ball is modelled as a particle travelling in a vertical plane above horizontal ground.

The path of the ball is modelled by the equation

$$h = 1.5x - 0.5x\,e^{0.02x} \qquad 0 \leqslant x \leqslant d$$

Use the model to answer parts (a), (b) and (c).

(a) Find the value of d, giving your answer to 2 decimal places.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(3)

(b) Show that the maximum value of h occurs when

$$x = 50\ln\left(\frac{150}{x + 50}\right)$$

(4)

Using the iteration formula

$$x_{n+1} = 50\ln\left(\frac{150}{x_n + 50}\right) \qquad \text{with } x_1 = 30$$

(c) (i) find the value of x_2 to 2 decimal places,

(ii) find, by repeated iteration, the horizontal distance travelled by the golf ball before it reaches its maximum height. Give your answer to 2 decimal places.

(3)

Question 4

6 marks | Log Graphs & Exponential Models

4. The number of bacteria on a surface is being monitored.

The number of bacteria, N, on the surface, t hours after monitoring began is modelled by the equation

$$\log_{10} N = 0.35t + 2$$

Use the equation of the model to answer parts (a) to (c).

(a) Find the initial number of bacteria on the surface.

(1)

(b) Show that the equation of the model can be written in the form

$$N = ab^t$$

where a and b are constants to be found. Give the value of b to 2 decimal places.

(3)

(c) Hence find the rate of growth of bacteria on the surface exactly 5 hours after monitoring began.

(2)

Question 8

12 marks | Log Graphs & Exponential Models

8.

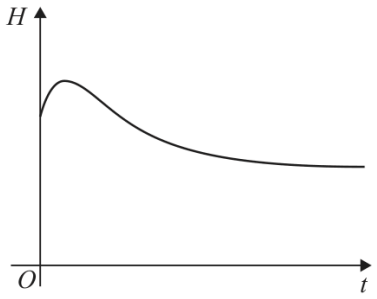


Figure 4

The heart rate of a horse is being monitored.
The heart rate H, measured in beats per minute (bpm), is modelled by the equation

$$H = 32 + 40e^{-0.2t} - 20\,e^{-0.9t}$$

where t minutes is the time after monitoring began.

Figure 4 is a sketch of H against t.

Use the equation of the model to answer parts (a) to (e).

(a) State the initial heart rate of the horse.

(1)

In the long term, the heart rate of the horse approaches L bpm.

(b) State the value of L.

(1)

The heart rate of the horse reaches its maximum value after T minutes.

(c) Find the value of T, giving your answer to 3 decimal places.

(*Solutions based entirely on calculator technology are not acceptable.*)

(5)

The heart rate of the horse is 37 bpm after M minutes.

(d) Show that M is a solution of the equation

$$t = 5\ln\left(\frac{8}{1 + 4e^{-0.9t}}\right)$$

(2)

Using the iteration formula

$$t_{n+1} = 5\ln\left(\frac{8}{1 + 4e^{-0.9t_n}}\right) \quad \text{with} \quad t_1 = 10$$

(e) (i) find, to 4 decimal places, the value of t_2

(ii) find, to 4 decimal places, the value of M

(3)

Question 2

3 marks | Log Graphs & Exponential Models

2. The weed on the surface of a pond is being monitored.

The surface area of the pond covered by the weed, $A\,\text{m}^2$, is modelled by the equation

$$\log_{10} A = 1 + 0.03t$$

where t is the number of weeks after monitoring began.

Use the equation of the model to answer parts (a) and (b).

(a) Find the surface area of the pond initially covered by the weed.

(1)

After T weeks, $25\,\text{m}^2$ of the pond is covered by the weed.

(b) Find the value of T, giving your answer to 2 decimal places.

(2)

Question 5

9 marks | Log Graphs & Exponential Models

5. A hot piece of metal is cooled by dropping it into water. The temperature, H°C, of the metal, t minutes after it is dropped into the water, is modelled by the equation

$$H = 280e^{-0.05t} + 24 \qquad t \geqslant 0$$

Use the equation of the model to answer parts (a) to (d).

(a) Find the initial temperature of the piece of metal.

(1)

(b) On Diagram 1, sketch the graph of H against t. On your sketch, state the equation of the asymptote to the curve.

(2)

(c) Find the value of t for which $H = 144$, giving your answer to 2 decimal places.

(*Solutions based entirely on calculator technology are not acceptable.*)

(3)

(d) Show by differentiation that

$$\frac{dH}{dt} = a + bH$$

where a and b are constants to be found.

(3)

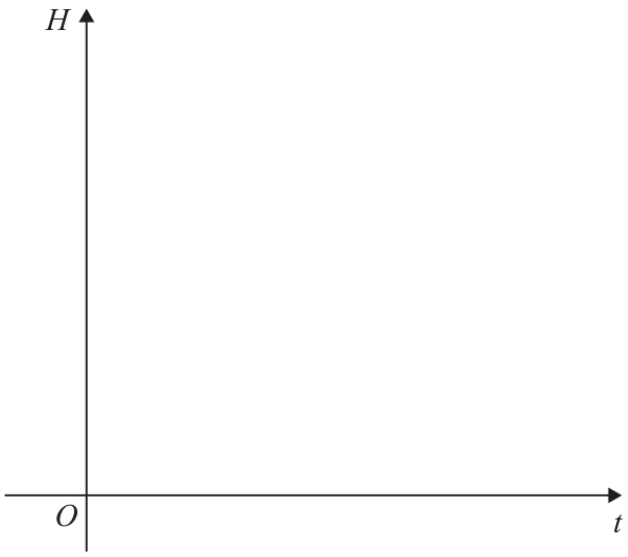


Diagram 1

Question 3

5 marks | Log Graphs & Exponential Models

3:

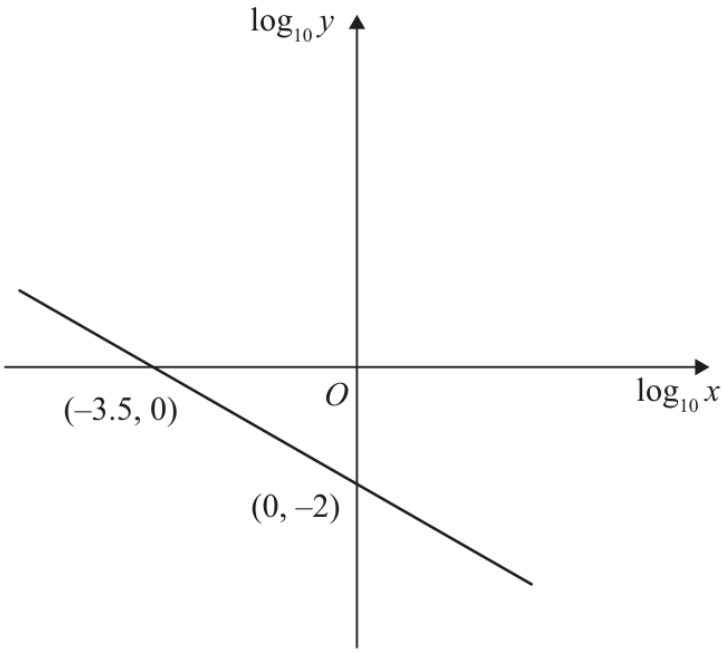


Figure 1

Figure 1 shows a linear relationship between $\log_{10} y$ and $\log_{10} x$

The line passes through the points (–3.5, 0) and (0, –2) as shown.

(a) Find an equation linking $\log_{10} y$ with $\log_{10} x$

(2)

(b) Hence, or otherwise, express y in the form px^q where p and q are rational constants.

(3)

Question 6

8 marks | Log Graphs & Exponential Models

6: **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The temperature, θ°C, of a computer processor, t minutes after the computer is switched off, is modelled by the equation

$$\theta = 21 + Ae^{-kt}$$

where A and k are positive constants.

Given that the temperature of the processor was 75°C when the computer was switched off,

(a) find the value of A.

(2)

Given also that it takes 5 minutes for the temperature of the processor to decrease from 75°C to 25°C,

(b) find the value of k, giving your answer to 3 significant figures.

(3)

At time T minutes, the temperature of the processor is decreasing at a rate of 9°C per minute.

(c) Find the value of T according to the model, giving your answer to 2 decimal places.

(3)

Question 3

9 marks | Log Graphs & Exponential Models

3. The share price, £V, of a company is being monitored.

A graph is drawn of $\log_{10} V$ against t, where t is the number of years after monitoring began.

The graph is a straight line passing through the points $(0, 2)$ and $(5, 2.25)$

Using this information,

(a) find an equation for the line in the form

$$\log_{10} V = mt + c$$

where m and c are constants.

(2)

(b) Write the answer to part (a) in the form

$$V = ab^t$$

where a and b are constants to be found.

Give the exact value of a and the value of b to 3 significant figures.

(3)

When $t = T$, the rate of increase in the share price of the company was £50 per year.

(c) Find the value of T, giving your answer to the nearest integer.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(4)

Question 5

8 marks | Log Graphs & Exponential Models

5. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The number of squirrels in a forest is being studied.

The number of squirrels, N, in the forest, t years after the start of the study, is modelled by the equation

$$N = \frac{4000e^{0.1t}}{19 + e^{0.2t}} \qquad t \geqslant 0$$

Use the equation of the model to answer parts (a), (b), (c) and (d).

(a) Find the number of squirrels in the forest at the start of the study.

(1)

(b) Find $\frac{dN}{dt}$

(2)

The number of squirrels in the forest is at a maximum when $t = T$

Using the answer to part (b),

(c) show that $e^{0.2T} = A$, where A is a constant to be found.

(2)

(d) Hence find the maximum number of squirrels in the forest.

Show your working and give your answer to the nearest whole number.

(3)

Question 4

6 marks | Log Graphs & Exponential Models

4. The resting heart rate, h beats per minute, of a particular group of mammals, can be modelled by the equation

$$\log_{10} h = 2.4 - 0.25 \log_{10} m$$

where m kg is the mass of the mammal.

The mass of one of these mammals is 3 kg.

According to the model,

(a) find the resting heart rate of this mammal. Give your answer to 3 significant figures.

(2)

(b) Show that the equation of the model can be written in the form

$$h = \frac{p}{m^q}$$

giving each of the values of the constants p and q to 3 significant figures where appropriate.

(3)

(c) With reference to the model, interpret the value of the constant p.

(1)

Question 10

9 marks | Log Graphs & Exponential Models

10. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

Two populations of insects are being studied over the same period of time.

The number of insects N_A in population A, t years after the start of the study, is modelled by the equation

$$N_A = 8000 + 900e^{0.2t}$$

According to the model,

(a) find the rate of growth in the number of insects in population A exactly 5 years after the start of the study.

(2)

The number of insects N_B in population B, t years after the start of the study, is modelled by the equation

$$N_B = 8000 + Pe^{kt}$$

where P and k are positive constants.

Given that

- there are 10 000 insects in population B at the start of the study
- there are 11 570 insects in population B exactly 4 years after the start of the study
- there are the same number of insects in population A and population B after T years

find, according to the models,

(b) the exact value of P and the value of k to 3 decimal places,

(4)

(c) the value of T, giving your answer to one decimal place.

(3)

Question 5

7 marks | Log Graphs & Exponential Models

5.

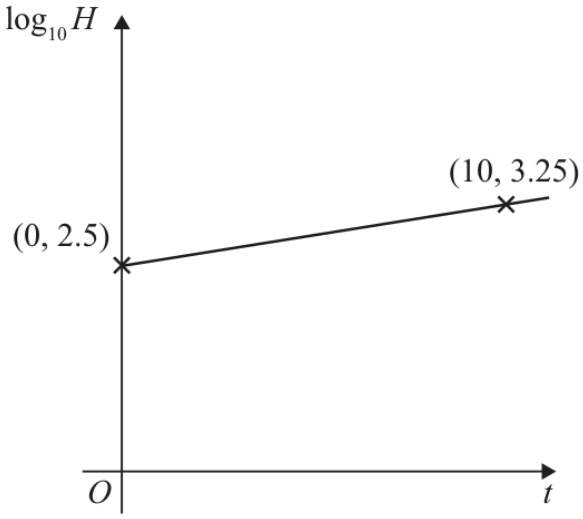


Figure 1

The recycling of household waste in a particular town is being monitored.

The number of households in the town, H, recycling their household waste, t years after monitoring began, is modelled by the equation

$$H = ab^t$$

where a and b are constants.

The line in Figure 1 shows the linear relationship between t and $\log_{10} H$

The line passes through the points (0, 2.5) and (10, 3.25)

Using this information,

(a) find, to 3 significant figures, the value of a and the value of b

(4)

(b) With reference to the model, interpret the value of the constant b

(1)

(c) Find the value of $\frac{dH}{dt}$ when $t = 5$, giving the answer to 2 significant figures.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(2)

Question 7

12 marks | Log Graphs & Exponential Models

7. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The selling price of a car is monitored over a 15-year period.

The selling price P_C of the car, in thousands of pounds, t years after monitoring began, is modelled by the equation

$$P_C = 7 + 16e^{-0.1t} \qquad 0 \leqslant t \leqslant 15$$

Given that when $t = T$ the selling price of the car is £15 000

(a) find, according to the model, the value of T. Give the answer to 2 decimal places.

(3)

(b) Find the value of $\frac{dP_C}{dt}$ when $t = 5$. Give the answer to 2 decimal places.

(2)

The selling price of a van is monitored over the same 15-year period.

The selling price P_V of the van, in thousands of pounds, t years after monitoring began, is modelled by the equation

$$P_V = 4 + me^{-0.2t} \qquad 0 \leqslant t \leqslant 15$$

where m is a constant.

Use the equations of these models to answer parts (c) and (d).

Given that $P_V = 36$ when $t = 0$

(c) (i) state the value of m

(ii) find the difference between the initial selling price of the car and the initial selling price of the van.

(3)

When $t = \alpha$, the selling price of the car is equal to the selling price of the van.

(d) Find, using algebra and showing your working, the value of α, giving the answer to 3 significant figures.

(4)

TOPIC

Differentiation Techniques

Every past-paper question on this topic, January 2020 to June 2026.

Question 4

11 marks | Differentiation Techniques

4. (i) $$f(x) = \frac{(2x+5)^2}{x-3} \qquad x \neq 3$$

(a) Find $f'(x)$ in the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are fully factorised quadratic expressions.

(b) Hence find the range of values of x for which $f(x)$ is increasing.

(6)

(ii) $$g(x) = x\sqrt{\sin 4x} \qquad 0 \leqslant x < \frac{\pi}{4}$$

The curve with equation $y = g(x)$ has a maximum at the point M.

Show that the x coordinate of M satisfies the equation

$$\tan 4x + kx = 0$$

where k is a constant to be found.

(5)

Question 3

7 marks | Differentiation Techniques

3.

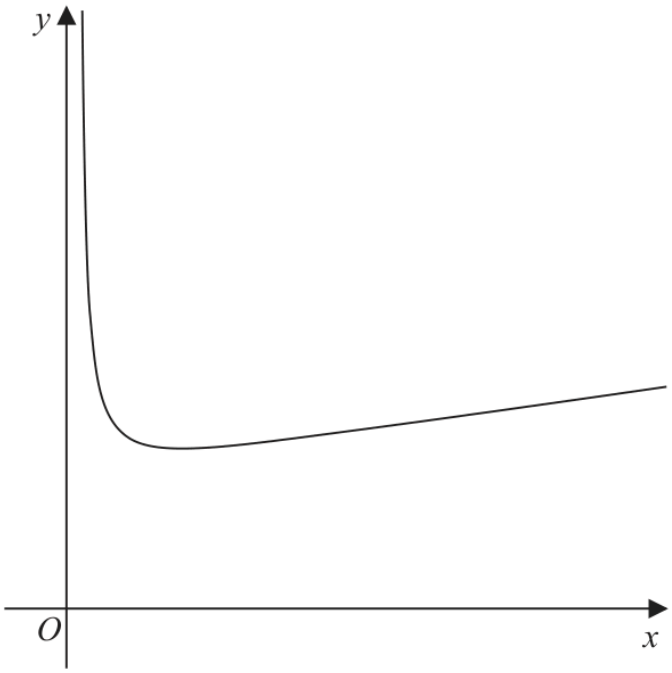


Figure 1

Figure 1 shows a sketch of a curve with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{2x + 3}{\sqrt{4x - 1}} \qquad x > \frac{1}{4}$$

(a) Find, in simplest form, $\mathrm{f}'(x)$.

(4)

(b) Hence find the range of f.

(3)

Question 8

9 marks | Differentiation Techniques

8. (i) The curve C has equation $y = \mathrm{g}(x)$ where

$$\mathrm{g}(x) = \mathrm{e}^{3x} \sec 2x \qquad -\frac{\pi}{4} < x < \frac{\pi}{4}$$

(a) Find $\mathrm{g}'(x)$

(2)

(b) Hence find the x coordinate of the stationary point of C.

(3)

(ii) A different curve has equation

$$x = \ln(\sin y) \qquad 0 < y < \frac{\pi}{2}$$

Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^x}{\mathrm{f}(x)}$$

where $\mathrm{f}(x)$ is a function of e^x that should be found.

(4)

Question 10

10 marks | Differentiation Techniques

10. The curve C has equation

$$x = 3\sec^2 2y \qquad x > 3 \qquad 0 < y < \frac{\pi}{4}$$

(a) Find $\frac{dx}{dy}$ in terms of y.

(2)

(b) Hence show that

$$\frac{dy}{dx} = \frac{p}{qx\sqrt{x-3}}$$

where p is irrational and q is an integer, stating the values of p and q.

(3)

(c) Find the equation of the normal to C at the point where $y = \frac{\pi}{12}$, giving your answer in the form $y = mx + c$, giving m and c as exact irrational numbers.

(5)

Question 7

5 marks | Differentiation Techniques

7. Given that

$$x = 6\sin^2 2y \qquad 0 < y < \frac{\pi}{4}$$

show that

$$\frac{dy}{dx} = \frac{1}{A\sqrt{(Bx - x^2)}}$$

where A and B are integers to be found.

(5)

Question 6

8 marks | Differentiation Techniques

6. (i) The curve C_1 has equation

$$y = 3\ln(x^2 - 5) - 4x^2 + 15 \qquad x > \sqrt{5}$$

Show that C_1 has a stationary point at $x = \dfrac{\sqrt{p}}{2}$ where p is a constant to be found.

(4)

(ii) A different curve C_2 has equation

$$y = 4x - 12\sin^2 x$$

(a) Show that, for this curve,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = A + B\sin 2x$$

where A and B are constants to be found.

(b) Hence, state the maximum gradient of this curve.

(4)

Question 8

7 marks | Differentiation Techniques

8. A curve C has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \arcsin\left(\frac{1}{2}x\right) \qquad -2 \leqslant x \leqslant 2 \qquad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

(a) Sketch C.

(1)

(b) Given $x = 2\sin y$, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{A - x^2}}$$

where A is a constant to be found.

(3)

The point P lies on C and has y coordinate $\frac{\pi}{4}$

(c) Find the equation of the tangent to C at P. Write your answer in the form $y = mx + c$, where m and c are constants to be found.

(3)

Question 10

7 marks | Differentiation Techniques

10.

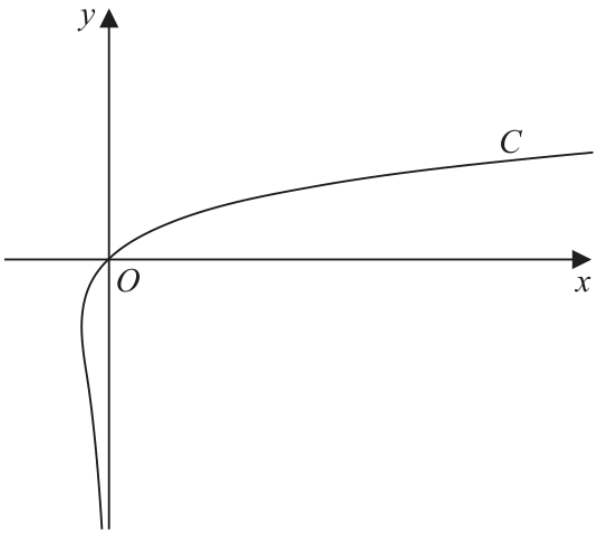


Figure 3

Figure 3 shows a sketch of the curve C with equation

$$x = ye^{2y} \qquad y \in \mathbb{R}$$

(a) Show that

$$\frac{dy}{dx} = \frac{y}{x(1+2y)}$$

(4)

Given that the straight line with equation $x = k$, where k is a constant, cuts C at exactly two points,

(b) find the range of possible values for k.

(3)

Question 4

5 marks | Differentiation Techniques

4. $y = \log_{10}(2x + 1)$

(a) Express x in terms of y.

(2)

(b) Hence, giving your answer in terms of x, find $\dfrac{dy}{dx}$

(3)

Question 6

6 marks | Differentiation Techniques

6.

$$y = \frac{2 + 3\sin x}{\cos x + \sin x}$$

Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a\tan x + b\sec x + c}{\sec x + 2\sin x}$$

where a, b and c are integers to be found.

(6)

Question 3

7 marks | Differentiation Techniques

3. (i) Find $\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\sin^2 3x\right)$ writing your answer in simplest form.

(2)

(ii)(a) Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(3x^2-4\right)^6$

(2)

(b) Hence show that

$$\int_0^{\sqrt{2}} x\left(3x^2-4\right)^5 \mathrm{d}x = R$$

where R is an integer to be found.

(*Solutions relying on calculator technology are not acceptable.*)

(3)

Question 10

8 marks | Differentiation Techniques

10.

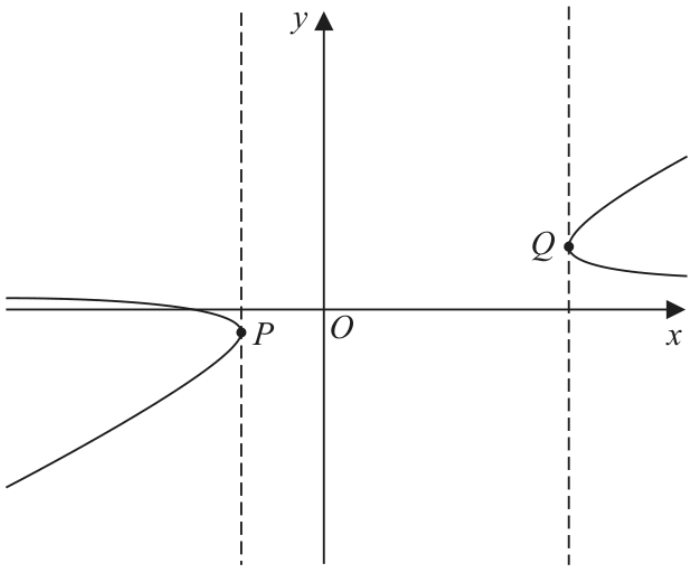


Figure 4

Figure 4 shows a sketch of the curve with equation

$$x = \frac{2y^2 + 6}{3y - 3}$$

(a) Find $\frac{dx}{dy}$ giving your answer as a fully simplified fraction.

(4)

The tangents at points P and Q on the curve are parallel to the y-axis, as shown in Figure 4.

(b) Use the answer to part (a) to find the equations of these two tangents.

(4)

Question 10

9 marks | Differentiation Techniques

10. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

A curve C has equation

$$x = \sin^2 4y \qquad 0 \leqslant y \leqslant \frac{\pi}{8} \qquad 0 \leqslant x \leqslant 1$$

The point P with x coordinate $\frac{1}{4}$ lies on C

(a) Find the exact y coordinate of P

(2)

(b) Find $\frac{dx}{dy}$

(2)

(c) Hence show that $\frac{dy}{dx}$ can be written in the form

$$\frac{dy}{dx} = \frac{1}{\sqrt{q + r(x + s)^2}}$$

where q, r and s are constants to be found.

(3)

Using the answer to part (c),

(d) (i) state the x coordinate of the point where the value of $\frac{dy}{dx}$ is a minimum,

(ii) state the value of $\frac{dy}{dx}$ at this point.

(2)

Question 10

9 marks | Differentiation Techniques

10.

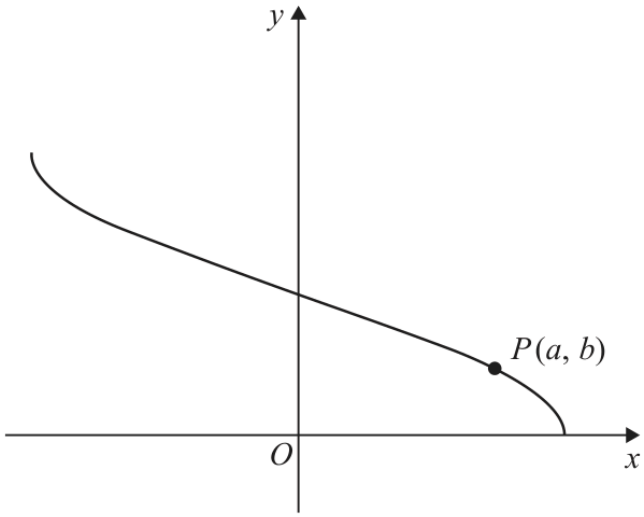


Figure 2

Figure 2 shows a sketch of the curve with equation

$$x = 3\cos 2y \qquad -3 \leqslant x \leqslant 3 \qquad 0 \leqslant y \leqslant \frac{\pi}{2}$$

(a) Find $\frac{\mathrm{d}x}{\mathrm{d}y}$ in terms of y.

(2)

(b) Hence show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{k}{\sqrt{9 - x^2}}$$

where k is a constant to be found.

(3)

The point $P(a, b)$ lies on the curve and is shown in Figure 2.

Given that

- the gradient of the curve at P is $-\frac{1}{4}$
- both a and b are positive

(c) find the exact values of a and b.

(4)

Question 9

11 marks | Differentiation Techniques

9:

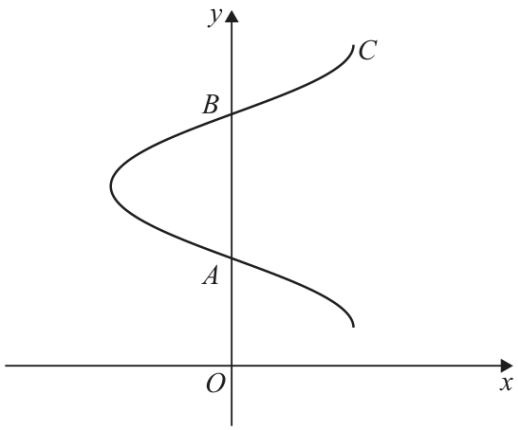


Figure 2

Figure 2 shows a sketch of the curve C with equation

$$x = \frac{2}{3}\sin\left(3y + \frac{\pi}{4}\right) \qquad \frac{\pi}{12} < y < \frac{3\pi}{4}$$

The curve intersects the y-axis at the points A and B as shown.

(a) Find the exact value of the y coordinate of

- point A
- point B

(3)

(b) Show that

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{p - qx^2}$$

where p and q are integers to be found.

(4)

The **normal** to C at A and the **tangent** to C at B intersect at the point D.

Using

- the answer to part (b)
- the sketch of curve C in Figure 2

(c) find, in simplest form, the exact x coordinate of D.

(4)

Question 2

6 marks | Differentiation Techniques

2. (i) Given $y = x^2 \ln 2x$, find in simplest form, $\dfrac{dy}{dx}$

(2)

(ii) Given

$$f(x) = \frac{e^{2x}}{(4x+3)^2} \qquad x > -\frac{3}{4}$$

show that

$$f'(x) = \frac{2e^{2x}(ax+b)}{(4x+3)^3}$$

where a and b are constants to be found.

(4)

Question 8

4 marks | Differentiation Techniques

8. A curve has equation

$$x = e^{2\tan y} \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Show that

$$\frac{dy}{dx} = \frac{a}{x\left(b + (\ln x)^n\right)}$$

where a, b and n are integers to be found.

(4)

Question 3

8 marks | Differentiation Techniques

3. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The curve C has equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{4x^2 - 14x + 15}{x - 3} \qquad x < 3$$

(a) Find $\mathrm{f}'(x)$, giving the answer in the form

$$\frac{ax^2 + bx + c}{(x-3)^2}$$

where a, b and c are constants.

(4)

(b) Hence find the coordinates of the turning point of C.

(2)

The point $P\left(\frac{5}{2}, -10\right)$ lies on C.

(c) Find the coordinates of the point to which P is mapped, when the curve with equation $y = \mathrm{f}(x)$ is transformed to the curve with equation $y = \mathrm{f}(2x) + 3$

(2)

Question 6

6 marks | Differentiation Techniques

6. The curve C has equation

$$x = \left(3\sin 2y - \frac{6y}{\pi}\right)^2$$

Given that

- the point P with y coordinate $\frac{\pi}{4}$ lies on C
- the tangent to C at P cuts the x-axis at the point Q

use calculus to find the x coordinate of Q

(6)

Question 7

11 marks | Differentiation Techniques | Main topic: Applications of Differentiation

7.

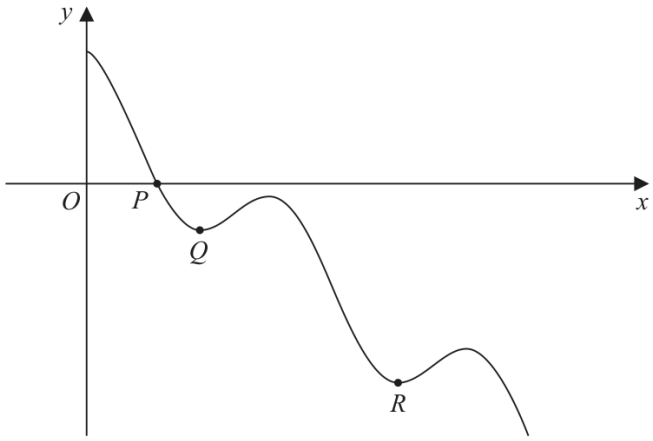


Figure 3

Figure 3 shows a sketch of part of the curve with equation

$$y = 2\cos 3x - 3x + 4 \qquad x > 0$$

where x is measured in radians.

The curve crosses the x-axis at the point P, as shown in Figure 3.

Given that the x coordinate of P is α,

(a) show that α lies between 0.8 and 0.9

(2)

The iteration formula

$$x_{n+1} = \frac{1}{3}\arccos(1.5x_n - 2)$$

can be used to find an approximate value for α.

(b) Using this iteration formula with $x_1 = 0.8$ find, to 4 decimal places, the value of

(i) x_2

(ii) x_5

(3)

The point Q and the point R are local minimum points on the curve, as shown in Figure 3.

Given that the x coordinates of Q and R are β and λ respectively, and that they are the two smallest values of x at which local minima occur,

(c) find, using calculus, the exact value of β and the exact value of λ.

(6)

Question 9

14 marks | Differentiation Techniques | Main topic: Applications of Differentiation

9. (a) Given that

$$\frac{x^4 - x^3 - 10x^2 + 3x - 9}{x^2 - x - 12} \equiv x^2 + P + \frac{Q}{x - 4} \qquad x > -3$$

find the value of the constant P and show that $Q = 5$

(4)

The curve C has equation $y = \mathrm{g}(x)$, where

$$\mathrm{g}(x) = \frac{x^4 - x^3 - 10x^2 + 3x - 9}{x^2 - x - 12} \qquad -3 < x < 3.5 \qquad x \in \mathbb{R}$$

(b) Find the equation of the tangent to C at the point where $x = 2$
Give your answer in the form $y = mx + c$, where m and c are constants to be found.

(5)

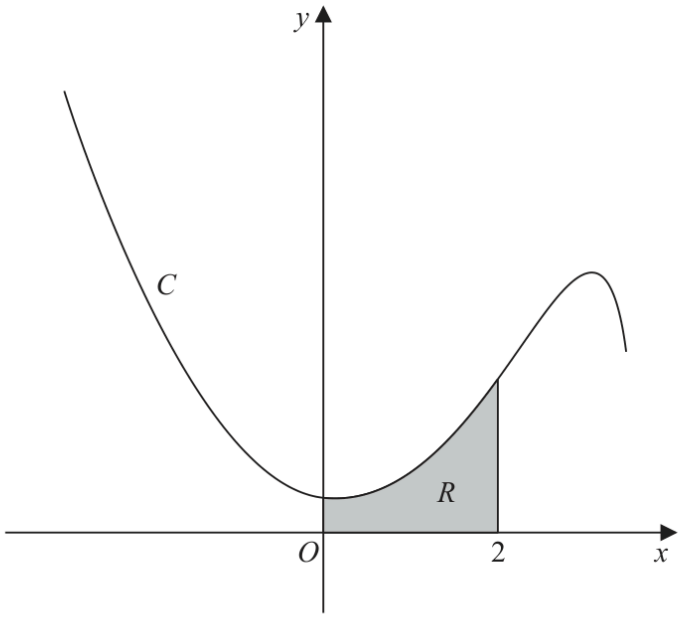


Figure 4

Figure 4 shows a sketch of the curve C.
The region R, shown shaded in Figure 4, is bounded by C, the y-axis, the x-axis and the line with equation $x = 2$

(c) Find the exact area of R, writing your answer in the form $a + b\ln 2$, where a and b are constants to be found.

(5)

Question 5

11 marks | Differentiation Techniques | Main topic: Log Graphs & Exponential Models

5. The temperature, θ°C, inside an oven, t minutes after the oven is switched on, is given by

$$\theta = A - 180e^{-kt}$$

where A and k are positive constants.

Given that the temperature inside the oven is initially 18 °C,

(a) find the value of A.

(2)

The temperature inside the oven, 5 minutes after the oven is switched on, is 90 °C.

(b) Show that $k = p \ln q$ where p and q are rational numbers to be found.

(4)

Hence find

(c) the temperature inside the oven 9 minutes after the oven is switched on, giving your answer to 3 significant figures,

(2)

(d) the rate of increase of the temperature inside the oven 9 minutes after the oven is switched on. Give your answer in °C min^{-1} to 3 significant figures.

(3)

Question 6

8 marks | Differentiation Techniques | Main topic: Numerical Methods

6.

$$f(x) = x\cos\left(\frac{x}{3}\right) \qquad x > 0$$

(a) Find $f'(x)$

(2)

(b) Show that the equation $f'(x) = 0$ can be written as

$$x = k\arctan\left(\frac{k}{x}\right)$$

where k is an integer to be found.

(2)

(c) Starting with $x_1 = 2.5$ use the iteration formula

$$x_{n+1} = k\arctan\left(\frac{k}{x_n}\right)$$

with the value of k found in part (b), to calculate the values of x_2 and x_6 giving your answers to 3 decimal places.

(2)

(d) Using a suitable interval and a suitable function that should be stated, show that a root of $f'(x) = 0$ is 2.581 correct to 3 decimal places.

(2)

Question 9

4 marks | Differentiation Techniques | Main topic: Integration

9. Find

(i) $\displaystyle\int \frac{3x-2}{3x^2-4x+5}\,\mathrm{d}x$

(2)

(ii) $\displaystyle\int \frac{\mathrm{e}^{2x}}{(\mathrm{e}^{2x}-1)^3}\,\mathrm{d}x \qquad x \neq 0$

(2)

Question 1

7 marks | Differentiation Techniques | Main topic: Numerical Methods

1. The curve C has equation

$$y = x^2 \cos\left(\frac{1}{2}x\right) \qquad 0 < x \leqslant \pi$$

The curve has a stationary point at the point P.

(a) Show, using calculus, that the x coordinate of P is a solution of the equation

$$x = 2\arctan\left(\frac{4}{x}\right)$$

(4)

Using the iteration formula

$$x_{n+1} = 2\arctan\left(\frac{4}{x_n}\right) \qquad x_1 = 2$$

(b) find the value of x_2 and the value of x_6, giving your answers to 3 decimal places.

(3)

Question 5

7 marks | Differentiation Techniques | Main topic: Log Graphs & Exponential Models

5.

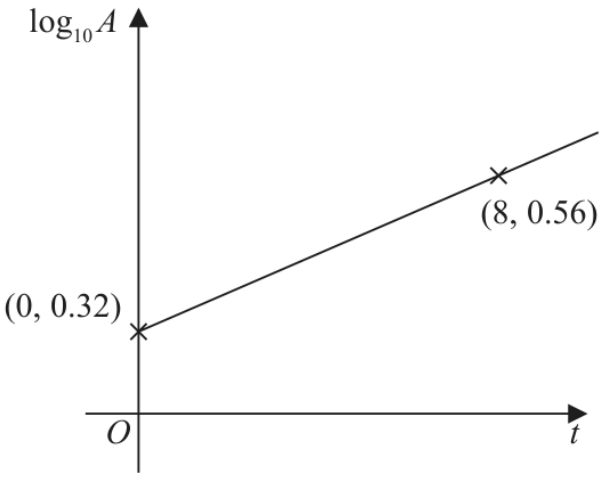


Figure 1

The growth of duckweed on a pond is being studied.

The surface area of the pond covered by duckweed, $A\text{m}^2$, at a time t days after the start of the study is modelled by the equation

$$A = pq^t \qquad \text{where } p \text{ and } q \text{ are positive constants}$$

Figure 1 shows the linear relationship between $\log_{10} A$ and t.

The points (0, 0.32) and (8, 0.56) lie on the line as shown.

(a) Find, to 3 decimal places, the value of p and the value of q.

(4)

Using the model with the values of p and q found in part (a),

(b) find the rate of increase of the surface area of the pond covered by duckweed, in m^2/day, exactly 6 days after the start of the study.
Give your answer to 2 decimal places.

(3)

Question 8

13 marks | Differentiation Techniques | Main topic: Log Graphs & Exponential Models

8. A scientist is studying a population of fish in a lake. The number of fish, N, in the population, t years after the start of the study, is modelled by the equation

$$N = \frac{600e^{0.3t}}{2 + e^{0.3t}} \qquad t \geqslant 0$$

Use the equation of the model to answer parts (a), (b), (c), (d) and (e).

(a) Find the number of fish in the lake at the start of the study.

(1)

(b) Find the upper limit to the number of fish in the lake.

(1)

(c) Find the time, after the start of the study, when there are predicted to be 500 fish in the lake. Give your answer in years and months to the nearest month.

(4)

(d) Show that

$$\frac{dN}{dt} = \frac{Ae^{0.3t}}{(2 + e^{0.3t})^2}$$

where A is a constant to be found.

(3)

Given that when $t = T$, $\frac{dN}{dt} = 8$

(e) find the value of T to one decimal place.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(4)

WMA13/01 OCTOBER 2021

Question 1

9 marks | Differentiation Techniques | Main topic: Rational Algebra

1. The function f is defined by

$$\mathrm{f}(x) = \frac{5x}{x^2 + 7x + 12} + \frac{5x}{x + 4} \qquad x > 0$$

(a) Show that $\mathrm{f}(x) = \dfrac{5x}{x + 3}$

(3)

(b) Find f^{-1}

(3)

(c) (i) Find, in simplest form, $\mathrm{f}'(x)$.

(ii) Hence, state whether f is an increasing or a decreasing function, giving a reason for your answer.

(3)

WMA13/01 OCTOBER 2021

Question 5

6 marks | Differentiation Techniques | Main topic: Integration

5. (i) Find, by algebraic integration, the exact value of

$$\int_2^4 \frac{8}{(2x-3)^3}\,\mathrm{d}x$$

(4)

(ii) Find, in simplest form,

$$\int x\left(x^2+3\right)^7\,\mathrm{d}x$$

(2)

Question 9

9 marks | Differentiation Techniques | Main topic: Numerical Methods

9.

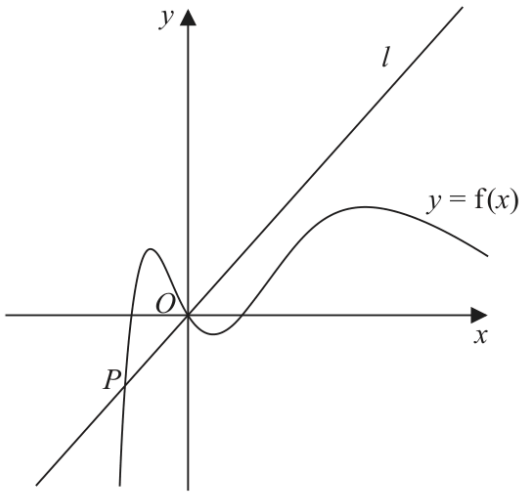


Figure 3

Figure 3 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x(x^2 - 4)\mathrm{e}^{-\frac{1}{2}x}$$

(a) Find $\mathrm{f}'(x)$.

(2)

The line l is the normal to the curve at O and meets the curve again at the point P.

The point P lies in the 3rd quadrant, as shown in Figure 3.

(b) Show that the x coordinate of P is a solution of the equation

$$x = -\frac{1}{2}\sqrt{16 + \mathrm{e}^{\frac{1}{2}x}}$$

(4)

(c) Using the iterative formula

$$x_{n+1} = -\frac{1}{2}\sqrt{16 + \mathrm{e}^{\frac{1}{2}x_n}} \quad \text{with } x_1 = -2$$

find, to 4 decimal places,

(i) the value of x_2

(ii) the x coordinate of P.

(3)

WMA13/01 JANUARY 2022

Question 1

4 marks | Differentiation Techniques | Main topic: Applications of Differentiation

1. Find, using calculus, the x coordinate of the stationary point on the curve with equation

$$y = (2x + 5)e^{3x}$$

(4)

Question 3

6 marks | Differentiation Techniques | Main topic: Integration

3. (i) Find, in simplest form,

$$\int (2x - 5)^7 \, dx$$

(2)

(ii) Show, by algebraic integration, that

$$\int_0^{\frac{\pi}{3}} \frac{4 \sin x}{1 + 2 \cos x} dx = \ln a$$

where a is a rational constant to be found.

(4)

Question 5

9 marks | Differentiation Techniques | Main topic: Applications of Differentiation

5.

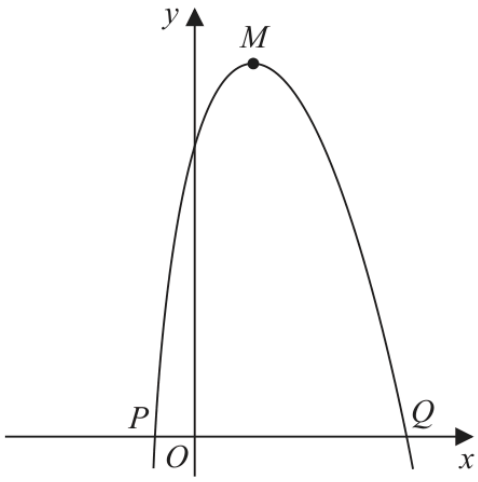


Figure 1

Figure 1 shows a sketch of part of the curve with equation

$$y = 6\ln(2x + 3) - \frac{1}{2}x^2 + 4 \qquad x > -\frac{3}{2}$$

The curve cuts the negative x-axis at the point P, as shown in Figure 1.

(a) Show that the x coordinate of P lies in the interval $[-1.25, -1.2]$

(2)

The curve cuts the positive x-axis at the point Q, also shown in Figure 1.

Using the iterative formula

$$x_{n+1} = \sqrt{12\ln(2x_n + 3) + 8} \quad \text{with} \quad x_1 = 6$$

(b) (i) find, to 4 decimal places, the value of x_2

(ii) find, by continued iteration, the x coordinate of Q. Give your answer to 4 decimal places.

(3)

The curve has a maximum turning point at M, as shown in Figure 1.

(c) Using calculus and showing each stage of your working, find the x coordinate of M.

(4)

Question 6

11 marks | Differentiation Techniques

Main topic: Functions, Composition & Inverses

6. The function f is defined by

$$\mathrm{f}(x) = \frac{5x - 3}{x - 4} \qquad x > 4$$

(a) Show, by using calculus, that f is a decreasing function.

(3)

(b) Find f^{-1}

(3)

(c) (i) Show that $\mathrm{ff}(x) = \dfrac{ax + b}{x + c}$ where a, b and c are constants to be found.

(ii) Deduce the range of ff.

(5)

Question 8

8 marks | Differentiation Techniques | Main topic: Log Graphs & Exponential Models

8. A dose of antibiotics is given to a patient.

The amount of the antibiotic, x milligrams, in the patient's bloodstream t hours after the dose was given, is found to satisfy the equation

$$\log_{10} x = 2.74 - 0.079t$$

(a) Show that this equation can be written in the form

$$x = pq^{-t}$$

where p and q are constants to be found. Give the value of p to the nearest whole number and the value of q to 2 significant figures.

(4)

(b) With reference to the equation in part (a), interpret the value of the constant p.

(1)

When a different dose of the antibiotic is given to another patient, the values of x and t satisfy the equation

$$x = 400 \times 1.4^{-t}$$

(c) Use calculus to find, to 2 significant figures, the value of $\dfrac{dx}{dt}$ when $t = 5$

(3)

WMA13/01 MAY/JUNE 2022

Question 1

6 marks | Differentiation Techniques | Main topic: Applications of Differentiation

1. The curve C has equation

$$y = (3x - 2)^6$$

(a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$

(2)

Given that the point $P\left(\frac{1}{3}, 1\right)$ lies on C,

(b) find the equation of the normal to C at P. Write your answer in the form $ax + by + c = 0$ where a, b and c are integers to be found.

(4)

Question 3

6 marks | Differentiation Techniques | Main topic: Integration

3. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

Given that k is a positive constant,

(a) find

$$\int \frac{9x}{3x^2 + k} \, dx$$

(2)

Given also that

$$\int_2^5 \frac{9x}{3x^2 + k} \, dx = \ln 8$$

(b) find the value of k

(4)

Question 6

10 marks | Differentiation Techniques | Main topic: Applications of Differentiation

6.

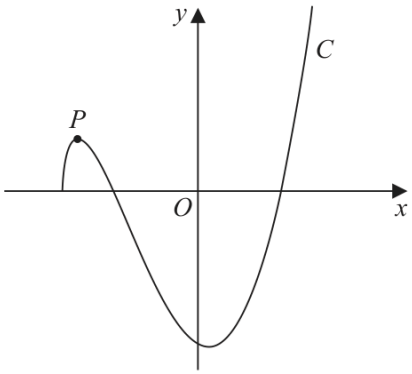


Figure 3

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

The function f is defined by

$$f(x) = 5(x^2 - 2)(4x + 9)^{\frac{1}{2}} \qquad x \geqslant -\frac{9}{4}$$

(a) Show that

$$f'(x) = \frac{k(5x^2 + 9x - 2)}{(4x + 9)^{\frac{1}{2}}}$$

where k is an integer to be found.

(4)

(b) Hence, find the values of x for which $f'(x) = 0$

(1)

Figure 3 shows a sketch of the curve C with equation $y = f(x)$.

The curve has a local maximum at the point P

(c) Find the exact coordinates of P

(2)

The function g is defined by

$$g(x) = 2f(x) + 4 \qquad -\frac{9}{4} \leqslant x \leqslant 0$$

(d) Find the range of g

(3)

Question 8

12 marks | Differentiation Techniques | Main topic: Numerical Methods

8.

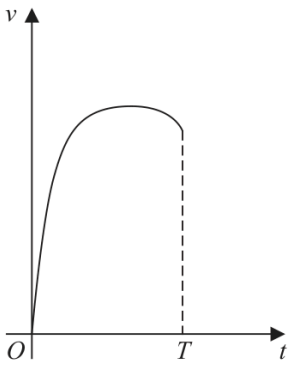


Figure 4

Figure 4 is a graph showing the velocity of a sprinter during a 100 m race.

The sprinter's velocity during the race, $v\,\text{m s}^{-1}$, is modelled by the equation

$$v = 12 - \text{e}^{t-10} - 12\text{e}^{-0.75t} \qquad t \geqslant 0$$

where t seconds is the time after the sprinter begins to run.

According to the model,

(a) find, using calculus, the sprinter's maximum velocity during the race.

(5)

Given that the sprinter runs 100 m in T seconds, such that

$$\int_0^T v\,\text{d}t = 100$$

(b) show that T is a solution of the equation

$$T = \frac{1}{12}\left(116 - 16\text{e}^{-0.75T} + \text{e}^{T-10} - \text{e}^{-10}\right)$$

(4)

The iteration formula

$$T_{n+1} = \frac{1}{12}\left(116 - 16\text{e}^{-0.75T_n} + \text{e}^{T_n-10} - \text{e}^{-10}\right)$$

is used to find an approximate value for T

Using this iteration formula with $T_1 = 10$

(c) find, to 4 decimal places,

(i) the value of T_2

(ii) the time taken by the sprinter to run the race, according to the model.

(3)

Question 9

9 marks | Differentiation Techniques | Main topic: Trig Equations & R-Form

9.

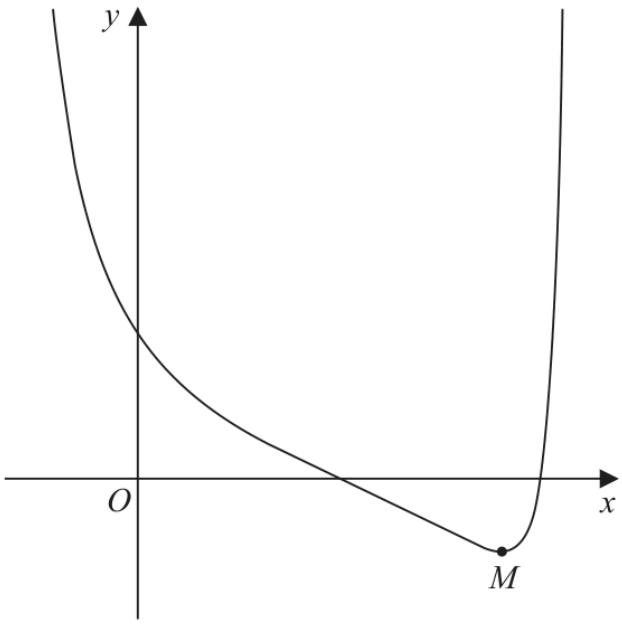


Figure 5

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 5 shows the curve with equation

$$y = \frac{1 + 2\cos x}{1 + \sin x} \qquad -\frac{\pi}{2} < x < \frac{3\pi}{2}$$

The point M, shown in Figure 5, is the minimum point on the curve.

(a) Show that the x coordinate of M is a solution of the equation

$$2\sin x + \cos x = -2$$

(4)

(b) Hence find, to 3 significant figures, the x coordinate of M.

(5)

Question 3

7 marks | Differentiation Techniques | Main topic: Applications of Differentiation

3. In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

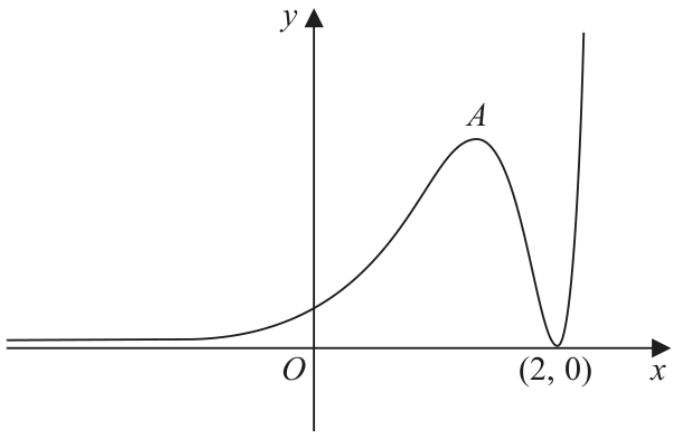


Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = (x - 2)^2 \mathrm{e}^{3x} \qquad x \in \mathbb{R}$$

The curve has a maximum turning point at A and a minimum turning point at $(2, 0)$

(a) Use calculus to find the exact coordinates of A.

(5)

Given that the equation $\mathrm{f}(x) = k$, where k is a constant, has **at least** two distinct roots,

(b) state the range of possible values for k.

(2)

Question 7

9 marks | Differentiation Techniques | Main topic: Applications of Differentiation

7. The curve C has equation

$$x = 3\tan\left(y - \frac{\pi}{6}\right) \qquad x \in \mathbb{R} \quad -\frac{\pi}{3} < y < \frac{2\pi}{3}$$

(a) Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a}{x^2 + b}$$

where a and b are integers to be found.

(4)

The point P with y coordinate $\frac{\pi}{3}$ lies on C.

Given that the tangent to C at P crosses the x-axis at the point Q.

(b) find, in simplest form, the exact x coordinate of Q.

(5)

Question 9

11 marks | Differentiation Techniques | Main topic: Numerical Methods

9.

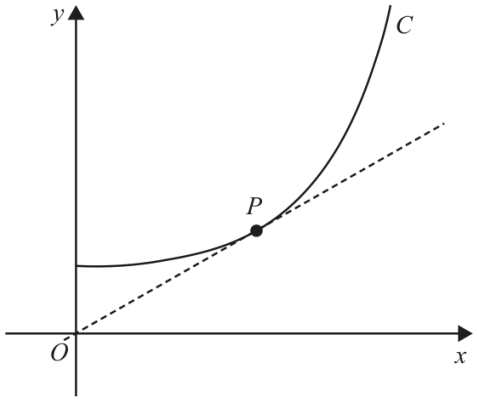


Figure 3

Figure 3 shows a sketch of part of the curve C with equation

$$y = \sqrt{3 + 4e^{x^2}} \qquad x \geqslant 0$$

(a) Find $\frac{dy}{dx}$, giving your answer in simplest form.

(2)

The point P with x coordinate α lies on C.

Given that the tangent to C at P passes through the origin, as shown in Figure 3,

(b) show that $x = \alpha$ is a solution of the equation

$$4x^2e^{x^2} - 4e^{x^2} - 3 = 0$$

(3)

(c) Hence show that α lies between 1 and 2

(2)

(d) Show that the equation in part (b) can be written in the form

$$x = \frac{1}{2}\sqrt{4 + 3e^{-x^2}}$$

(1)

The iteration formula

$$x_{n+1} = \frac{1}{2}\sqrt{4 + 3e^{-x_n^2}}$$

with $x_1 = 1$ is used to find an approximation for α.

(e) Use the iteration formula to find, to 4 decimal places, the value of

(i) x_3

(ii) α

(3)

Question 10

9 marks | Differentiation Techniques | Main topic: Log Graphs & Exponential Models

10. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

A population of fruit flies is being studied.

The number of fruit flies, F, in the population, t days after the start of the study, is modelled by the equation

$$F = \frac{350e^{kt}}{9 + e^{kt}}$$

where k is a constant.

Use the equation of the model to answer parts (a), (b) and (c).

(a) Find the number of fruit flies in the population at the start of the study.

(1)

Given that there are 200 fruit flies in the population 15 days after the start of the study,

(b) show that $k = \frac{1}{15}\ln 12$

(3)

Given also that, when $t = T$, the number of fruit flies in the population is increasing at a rate of 10 per day,

(c) find the possible values of T, giving your answers to one decimal place.

(5)

Question 7

8 marks | Differentiation Techniques | Main topic: Log Graphs & Exponential Models

7. A scientist is studying two different populations of bacteria.

The number of bacteria N in the first population is modelled by the equation

$$N = Ae^{kt} \qquad t \geqslant 0$$

where A and k are positive constants and t is the time in hours from the start of the study.

Given that

- there were 2500 bacteria in this population at the start of the study
- there were 10 000 bacteria 8 hours later

(a) find the exact value of A and the value of k to 4 significant figures.

(3)

The number of bacteria N in the second population is modelled by the equation

$$N = 60\,000e^{-0.6t} \qquad t \geqslant 0$$

where t is the time in hours from the start of the study.

(b) Find the rate of decrease of bacteria in this population exactly 5 hours from the start of the study. Give your answer to 3 significant figures.

(2)

When $t = T$, the number of bacteria in the two different populations was the same.

(c) Find the value of T, giving your answer to 3 significant figures.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(3)

Question 8

9 marks | Differentiation Techniques | Main topic: Applications of Differentiation

8.

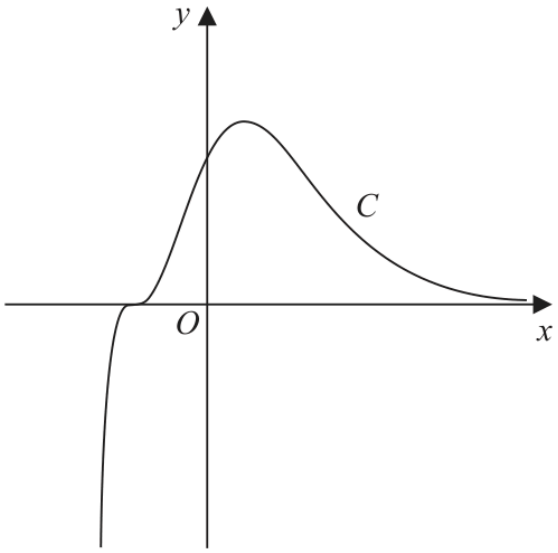


Figure 3

Figure 3 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = (2x + 1)^3 \mathrm{e}^{-4x}$$

(a) Show that

$$\mathrm{f}'(x) = A(2x + 1)^2 (1 - 4x)\, \mathrm{e}^{-4x}$$

where A is a constant to be found.

(4)

(b) Hence find the exact coordinates of the two stationary points on C.

(3)

The function g is defined by

$$\mathrm{g}(x) = 8\,\mathrm{f}(x - 2)$$

(c) Find the coordinates of the maximum stationary point on the curve with equation $y = \mathrm{g}(x)$.

(2)

Question 4

7 marks | Differentiation Techniques | Main topic: Log Graphs & Exponential Models

4. A new mobile phone is released for sale.

The total sales N of this phone, in **thousands**, is modelled by the equation

$$N = 125 - Ae^{-0.109t} \qquad t \geqslant 0$$

where A is a constant and t is the time in months after the phone was released for sale.

Given that when $t = 0$, $N = 32$

(a) state the value of A.

(1)

Given that when $t = T$ the total sales of the phone was 100 000

(b) find, according to the model, the value of T. Give your answer to 2 decimal places.

(3)

(c) Find, according to the model, the rate of increase in total sales when $t = 7$, giving your answer to 3 significant figures.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(2)

The total sales of the mobile phone is expected to reach 150 000

Using this information,

(d) give a reason why the given equation is not suitable for modelling the total sales of the phone.

(1)

Question 5

7 marks | Differentiation Techniques | Main topic: Applications of Differentiation

5. The curve C has equation

$$y = \frac{\ln(x^2 + k)}{x^2 + k} \qquad x \in \mathbb{R}$$

where k is a positive constant.

(a) Show that

$$\frac{dy}{dx} = \frac{Ax(B - \ln(x^2 + k))}{(x^2 + k)^2}$$

where A and B are constants to be found.

(3)

Given that C has exactly three turning points,

(b) find the x coordinate of each of these points. Give your answer in terms of k where appropriate.

(3)

(c) find the upper limit to the value for k.

(1)

Question 7

10 marks | Differentiation Techniques | Main topic: Functions, Composition & Inverses

7.

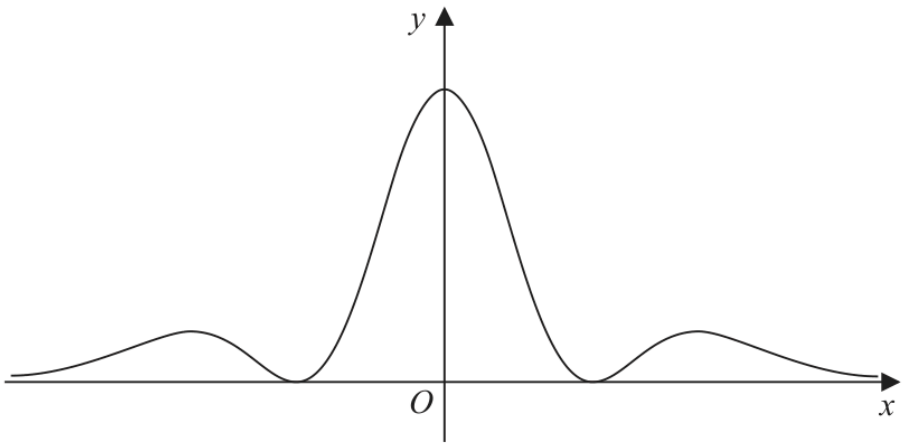


Figure 1

Figure 1 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \mathrm{e}^{-x^2}(2x^2 - 3)^2$$

(a) Find the range of f

(2)

(b) Show that

$$\mathrm{f}'(x) = 2x(2x^2 - 3)\mathrm{e}^{-x^2}(A - Bx^2)$$

where A and B are constants to be found.

(4)

Given that the line $y = k$, where k is a constant, $k > 0$, intersects the curve at exactly two distinct points,

(c) find the exact range of values of k

(4)

Question 4

13 marks | Differentiation Techniques | Main topic: Rational Algebra

4. The function f is defined by

$$\mathrm{f}(x) = \frac{2x^2 - 32}{3x^2 + 7x - 20} + \frac{8}{3x - 5} \qquad x \in \mathbb{R} \quad x > 2$$

(a) Show that $\mathrm{f}(x) = \dfrac{2x}{3x-5}$

(3)

(b) Show, using calculus, that f is a decreasing function.
You must make your reasoning clear.

(3)

The function g is defined by

$$\mathrm{g}(x) = 3 + 2\ln x \qquad x \geqslant 1$$

(c) Find g^{-1}

(3)

(d) Find the exact value of a for which

$$\mathrm{gf}(a) = 5$$

(4)

Question 5

7 marks | Differentiation Techniques

Main topic: Log Graphs & Exponential Models

5. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

The temperature, T °C, of the air in a room t minutes after a heat source is switched off, is modelled by the equation

$$T = 10 + A\mathrm{e}^{-Bt}$$

where A and B are constants.

Given that the temperature of the air in the room at the instant the heat source was switched off was 18 °C,

(a) find the value of A

(1)

Given also that, exactly 45 minutes after the heat source was switched off, the temperature of the air in the room was 16 °C,

(b) find the value of B to 3 significant figures.

(3)

Using the values for A and B,

(c) find, according to the model, the rate of change of the temperature of the air in the room exactly two minutes after the heat source was switched off.
Give your answer in °C min^{-1} to 3 significant figures.

(2)

(d) Explain why, according to the model, the temperature of the air in the room cannot fall to 5 °C

(1)

Question 6

7 marks | Differentiation Techniques | Main topic: Applications of Differentiation

6.

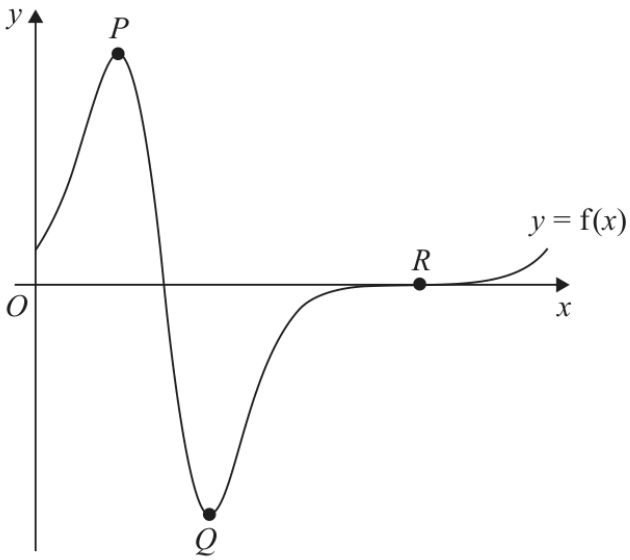


Figure 1

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 1 shows a sketch of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 2\mathrm{e}^{3\sin x}\cos x \qquad 0 \leqslant x \leqslant 2\pi$$

The curve intersects the x-axis at point R, as shown in Figure 1.

(a) State the coordinates of R

(1)

The curve has two turning points, at point P and point Q, also shown in Figure 1.

(b) Show that, at points P and Q,

$$a\sin^2 x + b\sin x + c = 0$$

where a, b and c are integers to be found.

(4)

(c) Hence find the x coordinate of point Q, giving your answer to 3 decimal places.

(2)

Question 7

12 marks | Differentiation Techniques | Main topic: Applications of Differentiation

7. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

The curve C has equation

$$y = \frac{16}{9(3x - k)} \qquad x \neq \frac{k}{3}$$

where k is a positive constant not equal to 3

(a) Find $\frac{dy}{dx}$ giving your answer in simplest form in terms of k.

(2)

The point P with x coordinate 1 lies on C.

Given that the gradient of the curve at P is -12

(b) find the two possible values of k

(3)

Given also that $k < 3$

(c) find the equation of the normal to C at P, writing your answer in the form $ax + by + c = 0$, where a, b and c are integers to be found.

(3)

(d) show, using algebraic integration that,

$$\int_1^3 \frac{16}{9(3x - k)} dx = \lambda \ln 10$$

where λ is a constant to be found.

(4)

Question 5

10 marks | Differentiation Techniques

Main topic: Functions, Composition & Inverses

5. The functions f and g are defined by

$$\mathrm{f}(x) = 2 + 5\ln x \qquad x > 0$$

$$\mathrm{g}(x) = \frac{6x - 2}{2x + 1} \qquad x > \frac{1}{3}$$

(a) Find $\mathrm{f}^{-1}(22)$

(2)

(b) Use differentiation to prove that g is an increasing function.

(3)

(c) Find g^{-1}

(3)

(d) Find the range of fg

(2)

Question 6

9 marks | Differentiation Techniques | Main topic: Applications of Differentiation

6.

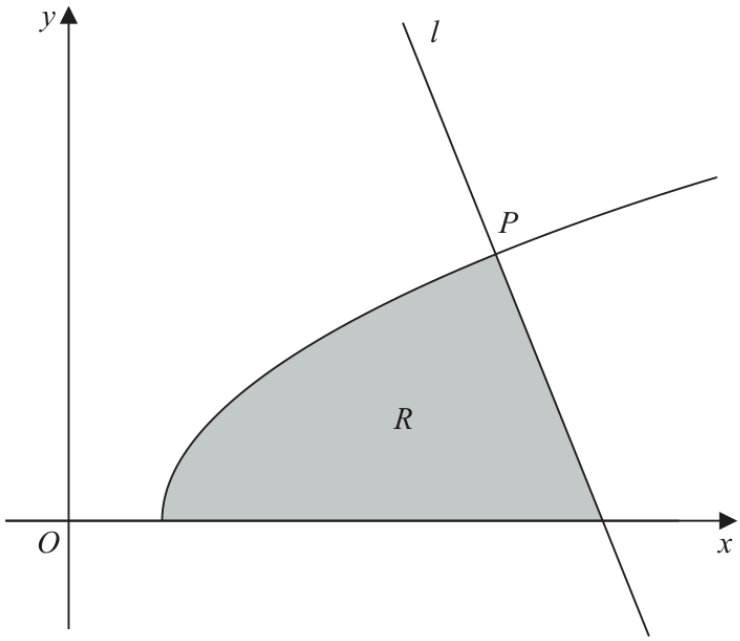


Figure 3

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 3 shows a sketch of part of the curve with equation

$$y = \sqrt{4x - 7}$$

The line l, shown in Figure 3, is the normal to the curve at the point $P(8, 5)$

(a) Use calculus to show that an equation of l is

$$5x + 2y - 50 = 0$$

(5)

The region R, shown shaded in Figure 3, is bounded by the curve, the x-axis and l.

(b) Use algebraic integration to find the exact area of R.

(4)

Question 8

10 marks | Differentiation Techniques | Main topic: Log Graphs & Exponential Models

8.

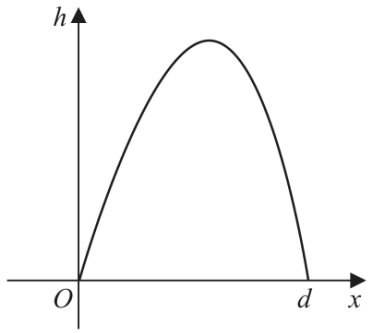


Figure 4

Figure 4 is a graph showing the path of a golf ball after the ball has been hit until it first hits the ground.

The vertical height, h metres, of the ball above the ground has been plotted against the horizontal distance travelled, x metres, measured from where the ball was hit.

The ball travels a horizontal distance of d metres before it first hits the ground.

The ball is modelled as a particle travelling in a vertical plane above horizontal ground.

The path of the ball is modelled by the equation

$$h = 1.5x - 0.5x\,e^{0.02x} \qquad 0 \leqslant x \leqslant d$$

Use the model to answer parts (a), (b) and (c).

(a) Find the value of d, giving your answer to 2 decimal places.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(3)

(b) Show that the maximum value of h occurs when

$$x = 50\ln\left(\frac{150}{x + 50}\right)$$

(4)

Using the iteration formula

$$x_{n+1} = 50\ln\left(\frac{150}{x_n + 50}\right) \qquad \text{with } x_1 = 30$$

(c) (i) find the value of x_2 to 2 decimal places,

(ii) find, by repeated iteration, the horizontal distance travelled by the golf ball before it reaches its maximum height. Give your answer to 2 decimal places.

(3)

Question 9

10 marks | Differentiation Techniques | Main topic: Applications of Differentiation

9.

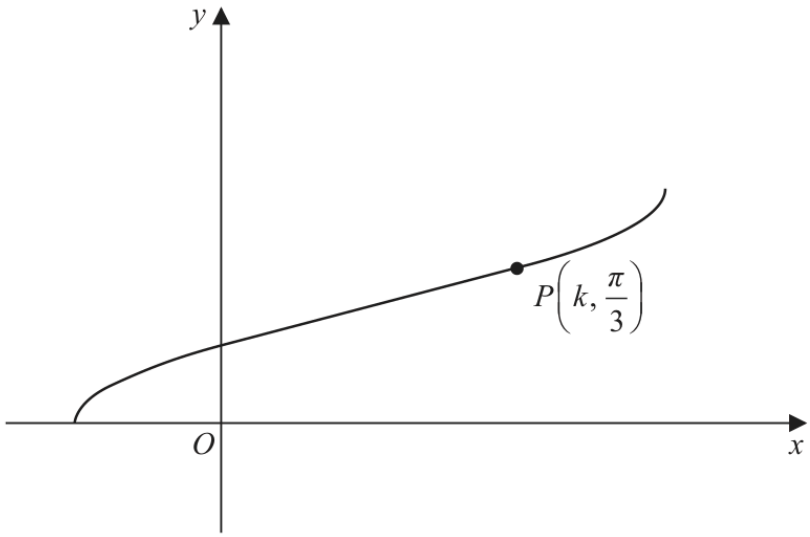


Figure 5

The curve shown in Figure 5 has equation

$$x = 4\sin^2 y - 1 \qquad 0 \leqslant y \leqslant \frac{\pi}{2}$$

The point $P\left(k, \frac{\pi}{3}\right)$ lies on the curve.

(a) Verify that $k = 2$

(1)

(b) (i) Find $\frac{\mathrm{d}x}{\mathrm{d}y}$ in terms of y

(ii) Hence show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\sqrt{x+1}\sqrt{3-x}}$

(6)

The normal to the curve at P cuts the x-axis at the point N.

(c) Find the exact area of triangle OPN, where O is the origin.

Give your answer in the form $a\pi + b\pi^2$ where a and b are constants.

(3)

Question 2

5 marks | Differentiation Techniques | Main topic: Applications of Differentiation

2.

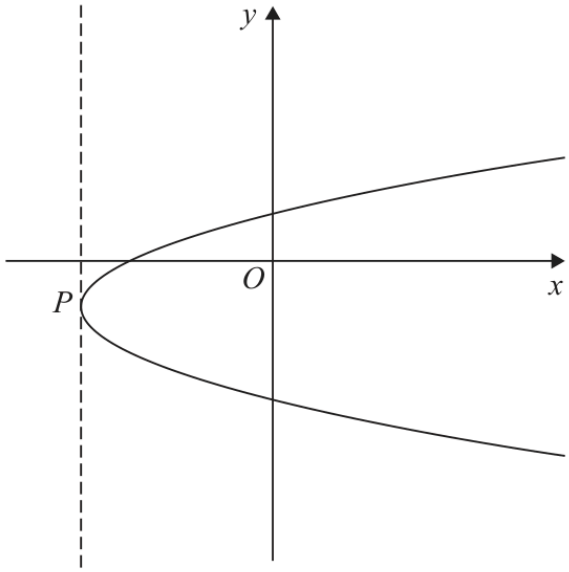


Figure 1

Figure 1 shows a sketch of the curve with equation

$$x = 2y^2 + 5y - 6$$

(a) Find $\frac{dy}{dx}$ in terms of y.

(2)

The point P lies on the curve and is shown in Figure 1.

Given that the tangent to the curve at P is parallel to the y-axis,

(b) find the coordinates of P.

(3)

Question 4

6 marks | Differentiation Techniques | Main topic: Log Graphs & Exponential Models

4. The number of bacteria on a surface is being monitored.

The number of bacteria, N, on the surface, t hours after monitoring began is modelled by the equation

$$\log_{10} N = 0.35t + 2$$

Use the equation of the model to answer parts (a) to (c).

(a) Find the initial number of bacteria on the surface.

(1)

(b) Show that the equation of the model can be written in the form

$$N = ab^t$$

where a and b are constants to be found. Give the value of b to 2 decimal places.

(3)

(c) Hence find the rate of growth of bacteria on the surface exactly 5 hours after monitoring began.

(2)

Question 7

10 marks | Differentiation Techniques | Main topic: Graph Transformations

7.

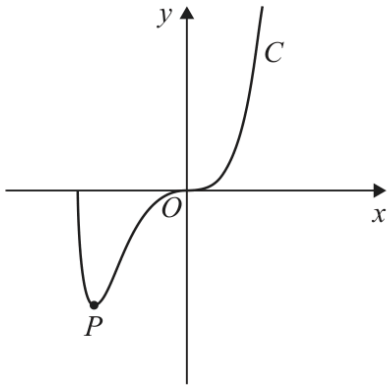


Figure 3

The curve C has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x^3\sqrt{4x+7} \qquad x \geqslant -\frac{7}{4}$$

(a) Show that

$$\mathrm{f}'(x) = \frac{kx^2(2x+3)}{\sqrt{4x+7}}$$

where k is a constant to be found.

(4)

The point P, shown in Figure 3, is the minimum turning point on C.

(b) Find the coordinates of P.

(2)

(c) Hence find the range of the function g defined by

$$\mathrm{g}(x) = -4\,\mathrm{f}(x) \qquad x \geqslant -\frac{7}{4}$$

(2)

The point Q with coordinates $\left(\frac{1}{2}, \frac{3}{8}\right)$ lies on C.

(d) Find the coordinates of the point to which Q is mapped when C is transformed to the curve with equation

$$y = 40\,\mathrm{f}\left(x - \frac{3}{2}\right) - 8$$

(2)

Question 8

12 marks | Differentiation Techniques | Main topic: Log Graphs & Exponential Models

8.

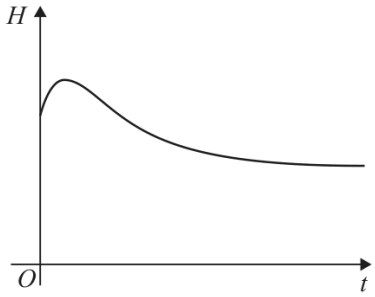


Figure 4

The heart rate of a horse is being monitored.
The heart rate H, measured in beats per minute (bpm), is modelled by the equation

$$H = 32 + 40e^{-0.2t} - 20e^{-0.9t}$$

where t minutes is the time after monitoring began.

Figure 4 is a sketch of H against t.

Use the equation of the model to answer parts (a) to (e).

(a) State the initial heart rate of the horse.

(1)

In the long term, the heart rate of the horse approaches L bpm.

(b) State the value of L.

(1)

The heart rate of the horse reaches its maximum value after T minutes.

(c) Find the value of T, giving your answer to 3 decimal places.

(*Solutions based entirely on calculator technology are not acceptable.*)

(5)

The heart rate of the horse is 37 bpm after M minutes.

(d) Show that M is a solution of the equation

$$t = 5\ln\left(\frac{8}{1 + 4e^{-0.9t}}\right)$$

(2)

Using the iteration formula

$$t_{n+1} = 5\ln\left(\frac{8}{1 + 4e^{-0.9t_n}}\right) \quad \text{with} \quad t_1 = 10$$

(e) (i) find, to 4 decimal places, the value of t_2

(ii) find, to 4 decimal places, the value of M

(3)

Question 9

14 marks | Differentiation Techniques | Main topic: Integration

9.

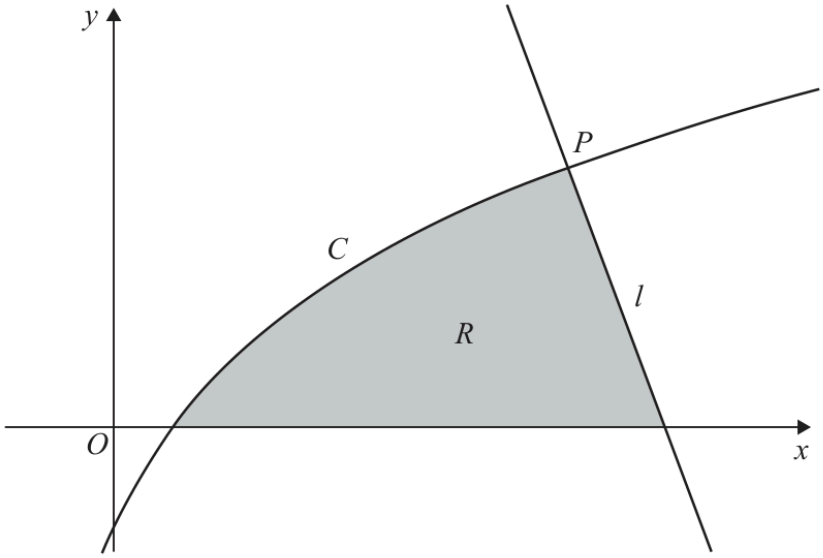


Figure 5

Figure 5 shows a sketch of part of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{6x^2 + 4x - 2}{2x + 1} \qquad x > -\frac{1}{2}$$

(a) Find $\mathrm{f}'(x)$, giving the answer in simplest form.

(3)

The line l is the normal to C at the point $P(2, 6)$

(b) Show that an equation for l is

$$16y + 5x = 106$$

(3)

(c) Write $\mathrm{f}(x)$ in the form $Ax + B + \dfrac{D}{2x + 1}$ where A, B and D are constants.

(3)

The region R, shown shaded in Figure 5, is bounded by C, l and the x-axis.

(d) Use algebraic integration to find the exact area of R, giving your answer in the form $P + Q\ln 3$, where P and Q are rational constants.

(*Solutions based entirely on calculator technology are not acceptable.*)

(5)

Question 3

6 marks | Differentiation Techniques | Main topic: Applications of Differentiation

3. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

A curve has equation

$$y = \frac{4x + 1}{(x + 3)^2} \qquad x \neq -3 \qquad x \in \mathbb{R}$$

Use calculus to find the range of values of x for which y is increasing.

(6)

Question 5

9 marks | Differentiation Techniques | Main topic: Log Graphs & Exponential Models

5. A hot piece of metal is cooled by dropping it into water. The temperature, H°C, of the metal, t minutes after it is dropped into the water, is modelled by the equation

$$H = 280e^{-0.05t} + 24 \qquad t \geqslant 0$$

Use the equation of the model to answer parts (a) to (d).

(a) Find the initial temperature of the piece of metal.

(1)

(b) On Diagram 1, sketch the graph of H against t. On your sketch, state the equation of the asymptote to the curve.

(2)

(c) Find the value of t for which $H = 144$, giving your answer to 2 decimal places.

(*Solutions based entirely on calculator technology are not acceptable.*)

(3)

(d) Show by differentiation that

$$\frac{dH}{dt} = a + bH$$

where a and b are constants to be found.

(3)

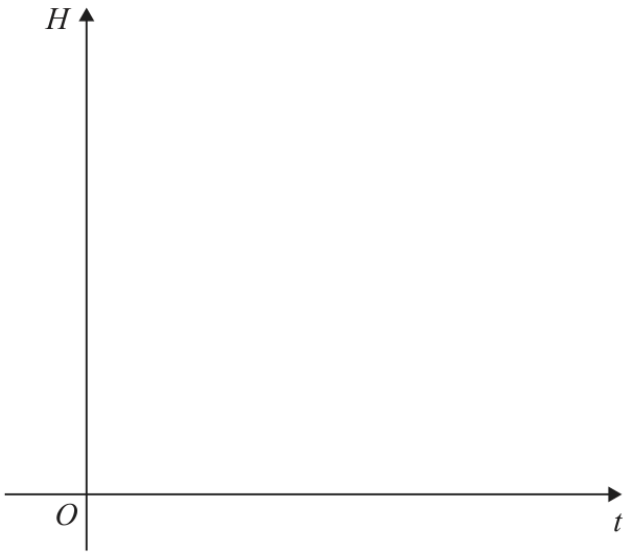


Diagram 1

Question 9

8 marks | Differentiation Techniques | Main topic: Applications of Differentiation

9.

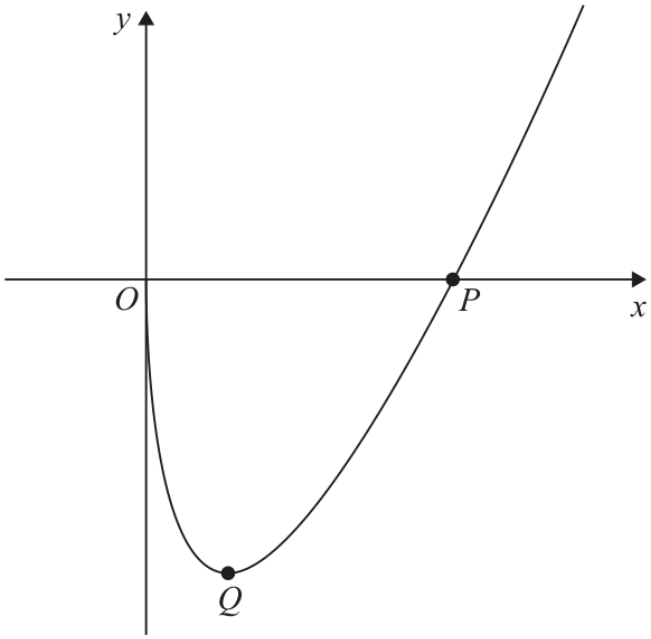


Figure 1

Figure 1 shows a sketch of part of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 6\sqrt{x}\ \ln(4x) \qquad x > 0$$

The curve cuts the x-axis at point P

(a) State the x coordinate of P

(1)

The point Q, shown in Figure 1, is the stationary point on C

(b) Use calculus to find the exact coordinates of Q

(5)

(c) Hence find the range of the function $\mathrm{g}(x)$ where

$$\mathrm{g}(x) = -2\,\mathrm{f}(x)$$

(2)

WMA13/01 MAY/JUNE 2025

Question 4

5 marks | Differentiation Techniques | Main topic: Rational Algebra

4: The function f is defined by

$$f(x) = \frac{49x}{x^2 + x - 12} + \frac{7x}{x + 4} \qquad x > 3$$

(a) Show that

$$f(x) = \frac{7x}{x - 3} \qquad x > 3$$

(3)

(b) Hence find $f'(x)$ giving your answer in simplest form.

(2)

WMA13/01 MAY/JUNE 2025

Question 5

4 marks | Differentiation Techniques | Main topic: Integration

5: Find

(i) $\int \sin^2 3x \, dx$

(2)

(ii) $\int x(x^2 + 4)^{\frac{3}{2}} dx$

(2)

Question 6

8 marks | Differentiation Techniques | Main topic: Log Graphs & Exponential Models

6: **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The temperature, θ°C, of a computer processor, t minutes after the computer is switched off, is modelled by the equation

$$\theta = 21 + Ae^{-kt}$$

where A and k are positive constants.

Given that the temperature of the processor was 75°C when the computer was switched off,

(a) find the value of A.

(2)

Given also that it takes 5 minutes for the temperature of the processor to decrease from 75°C to 25°C,

(b) find the value of k, giving your answer to 3 significant figures.

(3)

At time T minutes, the temperature of the processor is decreasing at a rate of 9°C per minute.

(c) Find the value of T according to the model, giving your answer to 2 decimal places.

(3)

Question 7

9 marks | Differentiation Techniques | Main topic: Numerical Methods

7: A continuous curve has equation

$$y = e^{-x^2} \sin 3x \qquad 0 \leqslant x \leqslant \frac{\pi}{3}$$

The curve has a stationary point at the point P.

(a) Show, using calculus, that the x coordinate of P is a solution of the equation

$$x = \frac{1}{3} \arctan\left(\frac{3}{2x}\right)$$

(4)

Using the iteration formula

$$x_{n+1} = \frac{1}{3} \arctan\left(\frac{3}{2x_n}\right) \qquad x_1 = 0.4$$

(b) find the value of

(i) x_2

(ii) x_4

giving your answers to 4 decimal places.

(3)

(c) Using a suitable interval and a suitable function which should be stated, show that the x coordinate of P is 0.430 correct to 3 decimal places.

(2)

Question 3

9 marks | Differentiation Techniques | Main topic: Log Graphs & Exponential Models

3. The share price, £V, of a company is being monitored.

A graph is drawn of $\log_{10} V$ against t, where t is the number of years after monitoring began.

The graph is a straight line passing through the points $(0, 2)$ and $(5, 2.25)$

Using this information,

(a) find an equation for the line in the form

$$\log_{10} V = mt + c$$

where m and c are constants.

(2)

(b) Write the answer to part (a) in the form

$$V = ab^t$$

where a and b are constants to be found.

Give the exact value of a and the value of b to 3 significant figures.

(3)

When $t = T$, the rate of increase in the share price of the company was £50 per year.

(c) Find the value of T, giving your answer to the nearest integer.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(4)

Question 5

8 marks | Differentiation Techniques | Main topic: Log Graphs & Exponential Models

5. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The number of squirrels in a forest is being studied.

The number of squirrels, N, in the forest, t years after the start of the study, is modelled by the equation

$$N = \frac{4000e^{0.1t}}{19 + e^{0.2t}} \qquad t \geqslant 0$$

Use the equation of the model to answer parts (a), (b), (c) and (d).

(a) Find the number of squirrels in the forest at the start of the study.

(1)

(b) Find $\frac{dN}{dt}$

(2)

The number of squirrels in the forest is at a maximum when $t = T$

Using the answer to part (b),

(c) show that $e^{0.2T} = A$, where A is a constant to be found.

(2)

(d) Hence find the maximum number of squirrels in the forest.

Show your working and give your answer to the nearest whole number.

(3)

Question 6

7 marks | Differentiation Techniques | Main topic: Integration

6. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(i) Given $y = \ln(2x^2 + 5)$ find $\dfrac{dy}{dx}$

(2)

(ii) A curve C has equation $y = f(x)$ where

$$f(x) = \frac{21x}{3x^2 + k} \qquad x \in \mathbb{R}$$

where k is a positive integer, $k > 1$

Given that

$$\int_1^k f(x)\,dx < 7\ln 8$$

find the greatest possible value of k

(5)

WMA13/01 OCTOBER 2025

Question 7

7 marks | Differentiation Techniques | Main topic: Applications of Differentiation

7. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

A curve C has equation

$$y = 2xe^{x^2+(3k-2)x}$$

where k is a constant.

Given that C has two distinct turning points, find the range of possible values of k.

(7)

Question 10

9 marks | Differentiation Techniques | Main topic: Log Graphs & Exponential Models

10. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

Two populations of insects are being studied over the same period of time.

The number of insects N_A in population A, t years after the start of the study, is modelled by the equation

$$N_A = 8000 + 900e^{0.2t}$$

According to the model,

(a) find the rate of growth in the number of insects in population A exactly 5 years after the start of the study.

(2)

The number of insects N_B in population B, t years after the start of the study, is modelled by the equation

$$N_B = 8000 + Pe^{kt}$$

where P and k are positive constants.

Given that

- there are 10 000 insects in population B at the start of the study
- there are 11 570 insects in population B exactly 4 years after the start of the study
- there are the same number of insects in population A and population B after T years

find, according to the models,

(b) the exact value of P and the value of k to 3 decimal places,

(4)

(c) the value of T, giving your answer to one decimal place.

(3)

Question 1

8 marks | Differentiation Techniques | Main topic: Numerical Methods

1. A curve has equation $y = f(x)$ where

$$f(x) = \frac{2}{3}x^3 - 3x^2 + 5x + e^{2x}(1 - 2x) \qquad x \in \mathbb{R}$$

(a) Find $f'(x)$ giving the answer in simplest form.

(3)

The equation $f'(x) = 0$ has a single root, α

(b) Show that α lies in the interval $[0, 1]$

(2)

Given that the equation $f'(x) = 0$ can be written as

$$x = \frac{2x^2 + 5}{4e^{2x} + 6}$$

(c) use the iteration formula

$$x_{n+1} = \frac{2x_n^{\ 2} + 5}{4e^{2x_n} + 6} \qquad \text{with } x_1 = 0$$

to find, to 4 decimal places,

(i) the value of x_3

(ii) the value of α

(3)

Question 5

7 marks | Differentiation Techniques | Main topic: Log Graphs & Exponential Models

5.

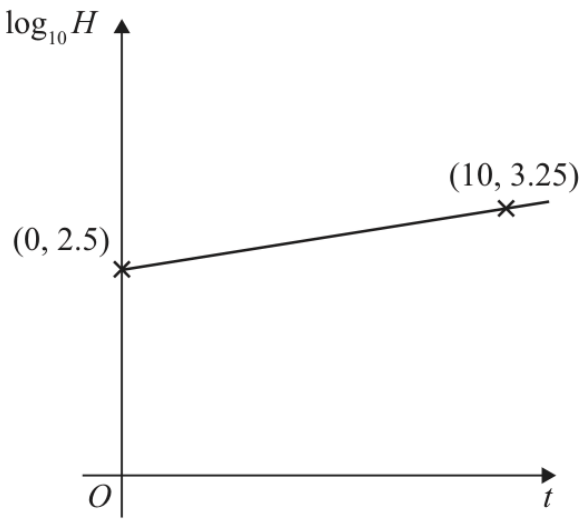


Figure 1

The recycling of household waste in a particular town is being monitored.

The number of households in the town, H, recycling their household waste, t years after monitoring began, is modelled by the equation

$$H = ab^t$$

where a and b are constants.

The line in Figure 1 shows the linear relationship between t and $\log_{10} H$

The line passes through the points (0, 2.5) and (10, 3.25)

Using this information,

(a) find, to 3 significant figures, the value of a and the value of b

(4)

(b) With reference to the model, interpret the value of the constant b

(1)

(c) Find the value of $\frac{dH}{dt}$ when $t = 5$, giving the answer to 2 significant figures.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(2)

Question 7

12 marks | Differentiation Techniques | Main topic: Log Graphs & Exponential Models

7. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The selling price of a car is monitored over a 15-year period.

The selling price P_C of the car, in thousands of pounds, t years after monitoring began, is modelled by the equation

$$P_C = 7 + 16e^{-0.1t} \qquad 0 \leqslant t \leqslant 15$$

Given that when $t = T$ the selling price of the car is £15 000

(a) find, according to the model, the value of T. Give the answer to 2 decimal places.

(3)

(b) Find the value of $\frac{dP_C}{dt}$ when $t = 5$. Give the answer to 2 decimal places.

(2)

The selling price of a van is monitored over the same 15-year period.

The selling price P_V of the van, in thousands of pounds, t years after monitoring began, is modelled by the equation

$$P_V = 4 + me^{-0.2t} \qquad 0 \leqslant t \leqslant 15$$

where m is a constant.

Use the equations of these models to answer parts (c) and (d).

Given that $P_V = 36$ when $t = 0$

(c) (i) state the value of m

(ii) find the difference between the initial selling price of the car and the initial selling price of the van.

(3)

When $t = \alpha$, the selling price of the car is equal to the selling price of the van.

(d) Find, using algebra and showing your working, the value of α, giving the answer to 3 significant figures.

(4)

TOPIC

Applications of Differentiation

Every past-paper question on this topic, January 2020 to June 2026.

Question 7

11 marks | Applications of Differentiation

7.

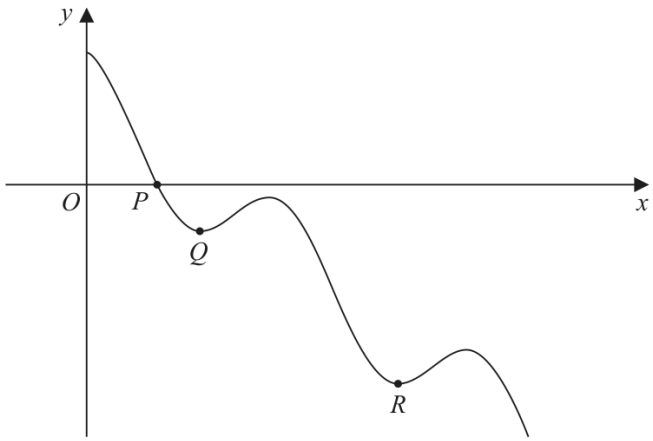


Figure 3

Figure 3 shows a sketch of part of the curve with equation

$$y = 2\cos 3x - 3x + 4 \qquad x > 0$$

where x is measured in radians.

The curve crosses the x-axis at the point P, as shown in Figure 3.

Given that the x coordinate of P is α,

(a) show that α lies between 0.8 and 0.9

(2)

The iteration formula

$$x_{n+1} = \frac{1}{3}\arccos(1.5x_n - 2)$$

can be used to find an approximate value for α.

(b) Using this iteration formula with $x_1 = 0.8$ find, to 4 decimal places, the value of

(i) x_2

(ii) x_5

(3)

The point Q and the point R are local minimum points on the curve, as shown in Figure 3.

Given that the x coordinates of Q and R are β and λ respectively, and that they are the two smallest values of x at which local minima occur,

(c) find, using calculus, the exact value of β and the exact value of λ.

(6)

Question 9

14 marks | Applications of Differentiation

9. (a) Given that

$$\frac{x^4 - x^3 - 10x^2 + 3x - 9}{x^2 - x - 12} \equiv x^2 + P + \frac{Q}{x - 4} \qquad x > -3$$

find the value of the constant P and show that $Q = 5$

(4)

The curve C has equation $y = \mathrm{g}(x)$, where

$$\mathrm{g}(x) = \frac{x^4 - x^3 - 10x^2 + 3x - 9}{x^2 - x - 12} \qquad -3 < x < 3.5 \qquad x \in \mathbb{R}$$

(b) Find the equation of the tangent to C at the point where $x = 2$
Give your answer in the form $y = mx + c$, where m and c are constants to be found.

(5)

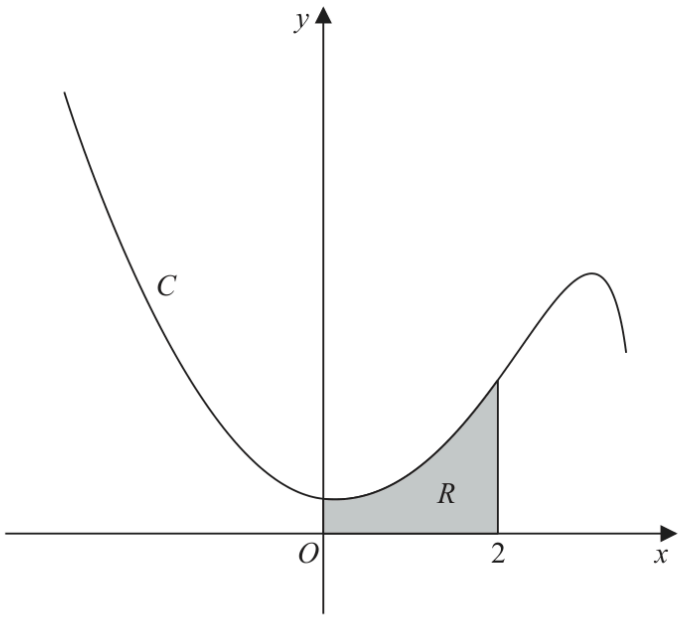


Figure 4

Figure 4 shows a sketch of the curve C.
The region R, shown shaded in Figure 4, is bounded by C, the y-axis, the x-axis and the line with equation $x = 2$

(c) Find the exact area of R, writing your answer in the form $a + b\ln 2$, where a and b are constants to be found.

(5)

Question 1

4 marks | Applications of Differentiation

1. Find, using calculus, the x coordinate of the stationary point on the curve with equation

$$y = (2x + 5)e^{3x}$$

(4)

Question 5

9 marks | Applications of Differentiation

5.

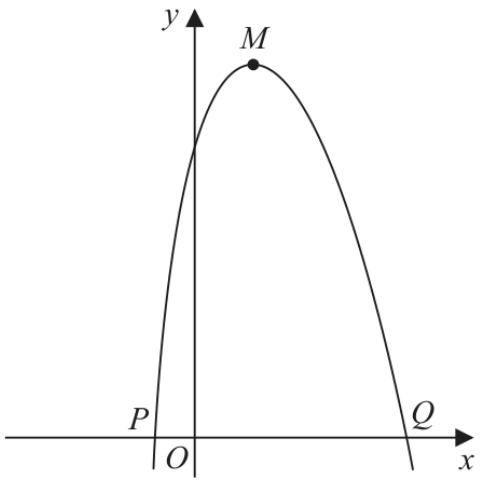


Figure 1

Figure 1 shows a sketch of part of the curve with equation

$$y = 6\ln(2x + 3) - \frac{1}{2}x^2 + 4 \qquad x > -\frac{3}{2}$$

The curve cuts the negative x-axis at the point P, as shown in Figure 1.

(a) Show that the x coordinate of P lies in the interval $[-1.25, -1.2]$

(2)

The curve cuts the positive x-axis at the point Q, also shown in Figure 1.

Using the iterative formula

$$x_{n+1} = \sqrt{12\ln(2x_n + 3) + 8} \quad \text{with} \quad x_1 = 6$$

(b) (i) find, to 4 decimal places, the value of x_2

(ii) find, by continued iteration, the x coordinate of Q. Give your answer to 4 decimal places.

(3)

The curve has a maximum turning point at M, as shown in Figure 1.

(c) Using calculus and showing each stage of your working, find the x coordinate of M.

(4)

Question 1

6 marks | Applications of Differentiation

1. The curve C has equation

$$y = (3x - 2)^6$$

(a) Find $\dfrac{dy}{dx}$

(2)

Given that the point $P\left(\frac{1}{3}, 1\right)$ lies on C,

(b) find the equation of the normal to C at P. Write your answer in the form $ax + by + c = 0$ where a, b and c are integers to be found.

(4)

Question 6

10 marks | Applications of Differentiation

6.

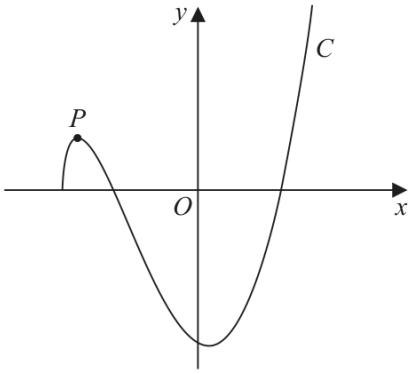


Figure 3

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

The function f is defined by

$$f(x) = 5(x^2 - 2)(4x + 9)^{\frac{1}{2}} \qquad x \geqslant -\frac{9}{4}$$

(a) Show that

$$f'(x) = \frac{k(5x^2 + 9x - 2)}{(4x + 9)^{\frac{1}{2}}}$$

where k is an integer to be found.

(4)

(b) Hence, find the values of x for which $f'(x) = 0$

(1)

Figure 3 shows a sketch of the curve C with equation $y = f(x)$.

The curve has a local maximum at the point P

(c) Find the exact coordinates of P

(2)

The function g is defined by

$$g(x) = 2f(x) + 4 \qquad -\frac{9}{4} \leqslant x \leqslant 0$$

(d) Find the range of g

(3)

Question 3

7 marks | Applications of Differentiation

3. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

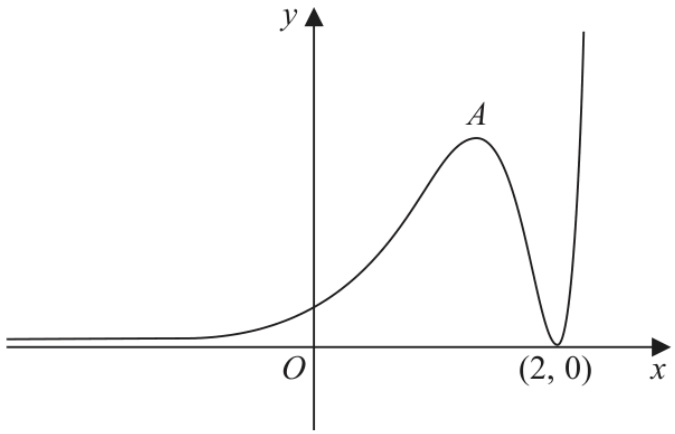


Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = (x-2)^2 \mathrm{e}^{3x} \qquad x \in \mathbb{R}$$

The curve has a maximum turning point at A and a minimum turning point at $(2, 0)$

(a) Use calculus to find the exact coordinates of A.

(5)

Given that the equation $\mathrm{f}(x) = k$, where k is a constant, has **at least** two distinct roots,

(b) state the range of possible values for k.

(2)

Question 7

9 marks | Applications of Differentiation

7. The curve C has equation

$$x = 3\tan\left(y - \frac{\pi}{6}\right) \qquad x \in \mathbb{R} \quad -\frac{\pi}{3} < y < \frac{2\pi}{3}$$

(a) Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a}{x^2 + b}$$

where a and b are integers to be found.

(4)

The point P with y coordinate $\frac{\pi}{3}$ lies on C.

Given that the tangent to C at P crosses the x-axis at the point Q.

(b) find, in simplest form, the exact x coordinate of Q.

(5)

Question 8

9 marks | Applications of Differentiation

8.

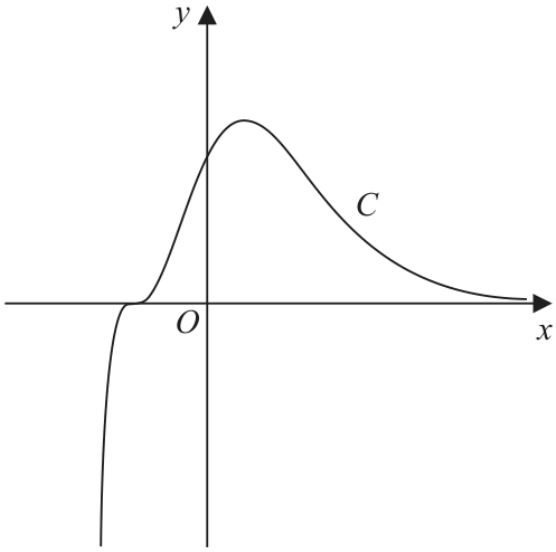


Figure 3

Figure 3 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = (2x + 1)^3 \mathrm{e}^{-4x}$$

(a) Show that

$$\mathrm{f}'(x) = A(2x + 1)^2 (1 - 4x)\, \mathrm{e}^{-4x}$$

where A is a constant to be found.

(4)

(b) Hence find the exact coordinates of the two stationary points on C.

(3)

The function g is defined by

$$\mathrm{g}(x) = 8\,\mathrm{f}(x - 2)$$

(c) Find the coordinates of the maximum stationary point on the curve with equation $y = \mathrm{g}(x)$.

(2)

Question 5

7 marks | Applications of Differentiation

5. The curve C has equation

$$y = \frac{\ln(x^2 + k)}{x^2 + k} \qquad x \in \mathbb{R}$$

where k is a positive constant.

(a) Show that

$$\frac{dy}{dx} = \frac{Ax(B - \ln(x^2 + k))}{(x^2 + k)^2}$$

where A and B are constants to be found.

(3)

Given that C has exactly three turning points,

(b) find the x coordinate of each of these points. Give your answer in terms of k where appropriate.

(3)

(c) find the upper limit to the value for k.

(1)

Question 6

7 marks | Applications of Differentiation

6.

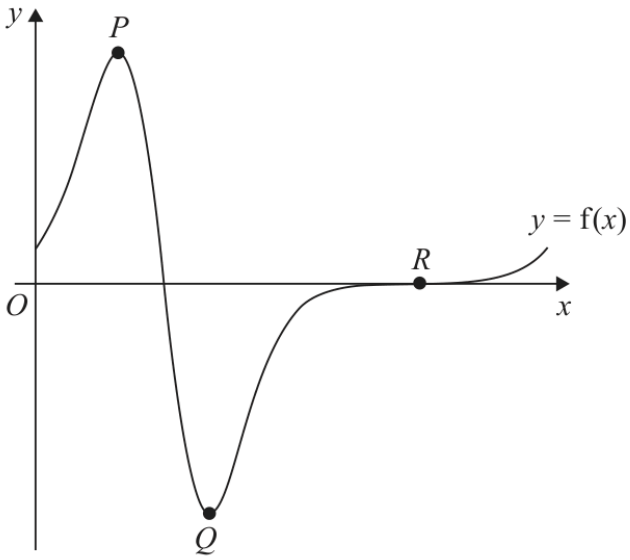


Figure 1

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 1 shows a sketch of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 2\mathrm{e}^{3\sin x}\cos x \qquad 0 \leqslant x \leqslant 2\pi$$

The curve intersects the x-axis at point R, as shown in Figure 1.

(a) State the coordinates of R

(1)

The curve has two turning points, at point P and point Q, also shown in Figure 1.

(b) Show that, at points P and Q,

$$a\sin^2 x + b\sin x + c = 0$$

where a, b and c are integers to be found.

(4)

(c) Hence find the x coordinate of point Q, giving your answer to 3 decimal places.

(2)

Question 7

12 marks | Applications of Differentiation

7. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

The curve C has equation

$$y = \frac{16}{9(3x - k)} \qquad x \neq \frac{k}{3}$$

where k is a positive constant not equal to 3

(a) Find $\frac{dy}{dx}$ giving your answer in simplest form in terms of k.

(2)

The point P with x coordinate 1 lies on C.

Given that the gradient of the curve at P is -12

(b) find the two possible values of k

(3)

Given also that $k < 3$

(c) find the equation of the normal to C at P, writing your answer in the form $ax + by + c = 0$, where a, b and c are integers to be found.

(3)

(d) show, using algebraic integration that,

$$\int_1^3 \frac{16}{9(3x - k)} dx = \lambda \ln 10$$

where λ is a constant to be found.

(4)

Question 6

9 marks | Applications of Differentiation

6.

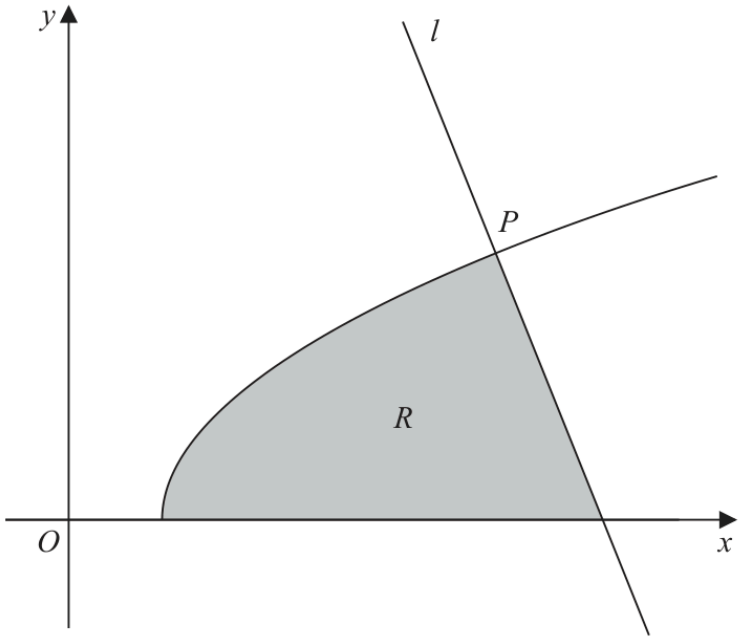


Figure 3

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 3 shows a sketch of part of the curve with equation

$$y = \sqrt{4x - 7}$$

The line l, shown in Figure 3, is the normal to the curve at the point $P(8, 5)$

(a) Use calculus to show that an equation of l is

$$5x + 2y - 50 = 0$$

(5)

The region R, shown shaded in Figure 3, is bounded by the curve, the x-axis and l.

(b) Use algebraic integration to find the exact area of R.

(4)

Question 9

10 marks | Applications of Differentiation

9.

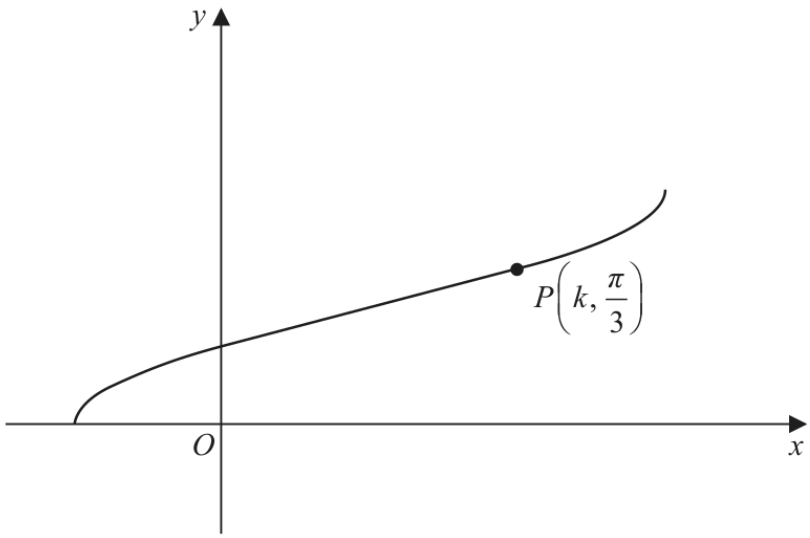


Figure 5

The curve shown in Figure 5 has equation

$$x = 4\sin^2 y - 1 \qquad 0 \leqslant y \leqslant \frac{\pi}{2}$$

The point $P\left(k, \frac{\pi}{3}\right)$ lies on the curve.

(a) Verify that $k = 2$

(1)

(b) (i) Find $\frac{\mathrm{d}x}{\mathrm{d}y}$ in terms of y

(ii) Hence show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\sqrt{x+1}\sqrt{3-x}}$

(6)

The normal to the curve at P cuts the x-axis at the point N.

(c) Find the exact area of triangle OPN, where O is the origin.

Give your answer in the form $a\pi + b\pi^2$ where a and b are constants.

(3)

Question 2

5 marks | Applications of Differentiation

2.

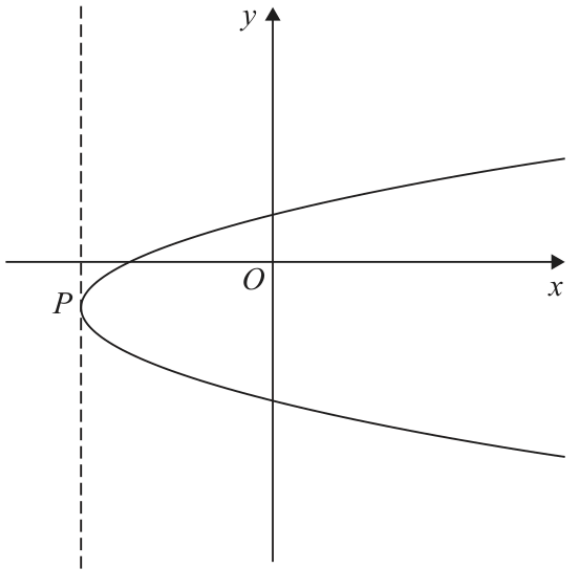


Figure 1

Figure 1 shows a sketch of the curve with equation

$$x = 2y^2 + 5y - 6$$

(a) Find $\frac{dy}{dx}$ in terms of y.

(2)

The point P lies on the curve and is shown in Figure 1.

Given that the tangent to the curve at P is parallel to the y-axis,

(b) find the coordinates of P.

(3)

Question 3

6 marks | Applications of Differentiation

3. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

A curve has equation

$$y = \frac{4x + 1}{(x + 3)^2} \qquad x \neq -3 \qquad x \in \mathbb{R}$$

Use calculus to find the range of values of x for which y is increasing.

(6)

Question 9

8 marks | Applications of Differentiation

9.

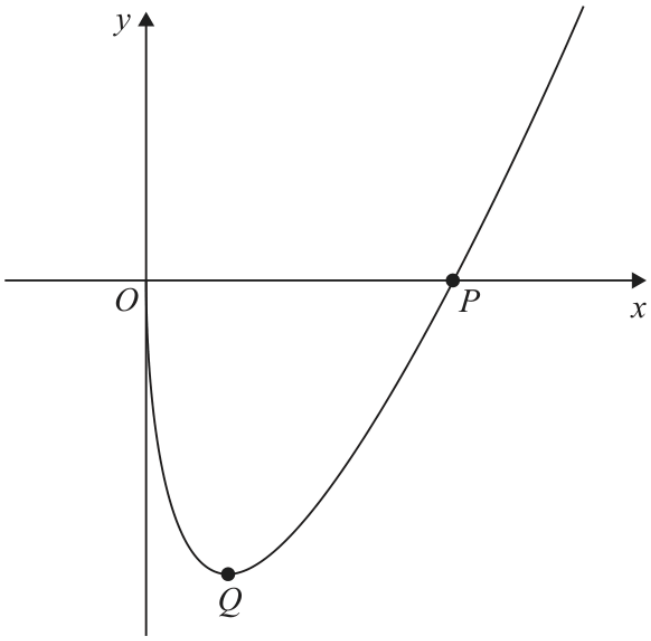


Figure 1

Figure 1 shows a sketch of part of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 6\sqrt{x}\ln(4x) \qquad x > 0$$

The curve cuts the x-axis at point P

(a) State the x coordinate of P

(1)

The point Q, shown in Figure 1, is the stationary point on C

(b) Use calculus to find the exact coordinates of Q

(5)

(c) Hence find the range of the function $\mathrm{g}(x)$ where

$$\mathrm{g}(x) = -2\,\mathrm{f}(x)$$

(2)

Question 7

7 marks | Applications of Differentiation

7. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

A curve C has equation

$$y = 2xe^{x^2+(3k-2)x}$$

where k is a constant.

Given that C has two distinct turning points, find the range of possible values of k.

(7)

Question 4

11 marks | Applications of Differentiation | Main topic: Differentiation Techniques

4. (i) $$f(x) = \frac{(2x+5)^2}{x-3} \qquad x \neq 3$$

(a) Find $f'(x)$ in the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are fully factorised quadratic expressions.

(b) Hence find the range of values of x for which $f(x)$ is increasing.

(6)

(ii) $$g(x) = x\sqrt{\sin 4x} \qquad 0 \leqslant x < \frac{\pi}{4}$$

The curve with equation $y = g(x)$ has a maximum at the point M.

Show that the x coordinate of M satisfies the equation

$$\tan 4x + kx = 0$$

where k is a constant to be found.

(5)

Question 3

7 marks | Applications of Differentiation | Main topic: Differentiation Techniques

3.

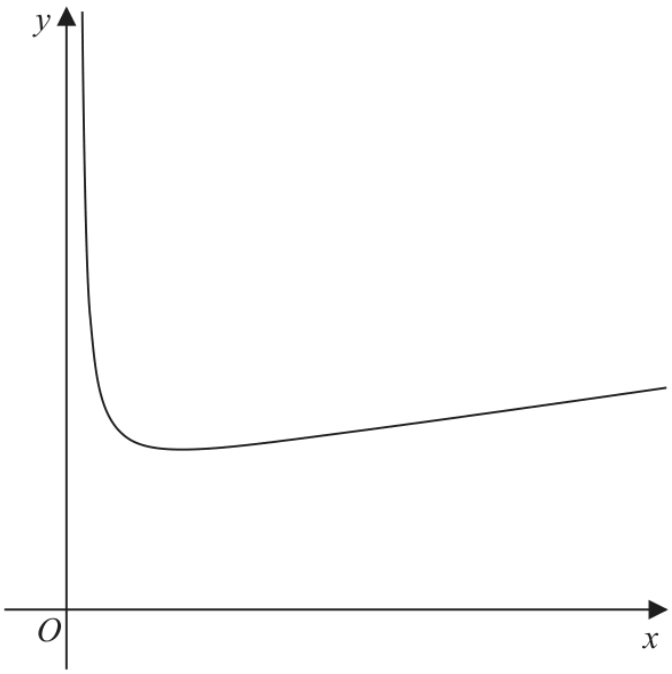


Figure 1

Figure 1 shows a sketch of a curve with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{2x + 3}{\sqrt{4x - 1}} \qquad x > \frac{1}{4}$$

(a) Find, in simplest form, $\mathrm{f}'(x)$.

(4)

(b) Hence find the range of f.

(3)

Question 8

9 marks | Applications of Differentiation | Main topic: Differentiation Techniques

8. (i) The curve C has equation $y = g(x)$ where

$$g(x) = e^{3x} \sec 2x \qquad -\frac{\pi}{4} < x < \frac{\pi}{4}$$

(a) Find $g'(x)$

(2)

(b) Hence find the x coordinate of the stationary point of C.

(3)

(ii) A different curve has equation

$$x = \ln(\sin y) \qquad 0 < y < \frac{\pi}{2}$$

Show that

$$\frac{dy}{dx} = \frac{e^x}{f(x)}$$

where $f(x)$ is a function of e^x that should be found.

(4)

Question 5

11 marks | Applications of Differentiation | Main topic: Log Graphs & Exponential Models

5. The temperature, θ°C, inside an oven, t minutes after the oven is switched on, is given by

$$\theta = A - 180e^{-kt}$$

where A and k are positive constants.

Given that the temperature inside the oven is initially 18 °C,

(a) find the value of A.

(2)

The temperature inside the oven, 5 minutes after the oven is switched on, is 90 °C.

(b) Show that $k = p \ln q$ where p and q are rational numbers to be found.

(4)

Hence find

(c) the temperature inside the oven 9 minutes after the oven is switched on, giving your answer to 3 significant figures,

(2)

(d) the rate of increase of the temperature inside the oven 9 minutes after the oven is switched on. Give your answer in °C min^{-1} to 3 significant figures.

(3)

Question 6

8 marks | Applications of Differentiation | Main topic: Numerical Methods

6.

$$f(x) = x\cos\left(\frac{x}{3}\right) \qquad x > 0$$

(a) Find $f'(x)$

(2)

(b) Show that the equation $f'(x) = 0$ can be written as

$$x = k\arctan\left(\frac{k}{x}\right)$$

where k is an integer to be found.

(2)

(c) Starting with $x_1 = 2.5$ use the iteration formula

$$x_{n+1} = k\arctan\left(\frac{k}{x_n}\right)$$

with the value of k found in part (b), to calculate the values of x_2 and x_6 giving your answers to 3 decimal places.

(2)

(d) Using a suitable interval and a suitable function that should be stated, show that a root of $f'(x) = 0$ is 2.581 correct to 3 decimal places.

(2)

Question 10

10 marks | Applications of Differentiation | Main topic: Differentiation Techniques

10. The curve C has equation

$$x = 3\sec^2 2y \qquad x > 3 \qquad 0 < y < \frac{\pi}{4}$$

(a) Find $\frac{dx}{dy}$ in terms of y.

(2)

(b) Hence show that

$$\frac{dy}{dx} = \frac{p}{qx\sqrt{x-3}}$$

where p is irrational and q is an integer, stating the values of p and q.

(3)

(c) Find the equation of the normal to C at the point where $y = \frac{\pi}{12}$, giving your answer in the form $y = mx + c$, giving m and c as exact irrational numbers.

(5)

Question 1

7 marks | Applications of Differentiation | Main topic: Numerical Methods

1. The curve C has equation

$$y = x^2 \cos\left(\frac{1}{2}x\right) \qquad 0 < x \leqslant \pi$$

The curve has a stationary point at the point P.

(a) Show, using calculus, that the x coordinate of P is a solution of the equation

$$x = 2\arctan\left(\frac{4}{x}\right)$$

(4)

Using the iteration formula

$$x_{n+1} = 2\arctan\left(\frac{4}{x_n}\right) \qquad x_1 = 2$$

(b) find the value of x_2 and the value of x_6, giving your answers to 3 decimal places.

(3)

Question 5

7 marks | Applications of Differentiation | Main topic: Log Graphs & Exponential Models

5.

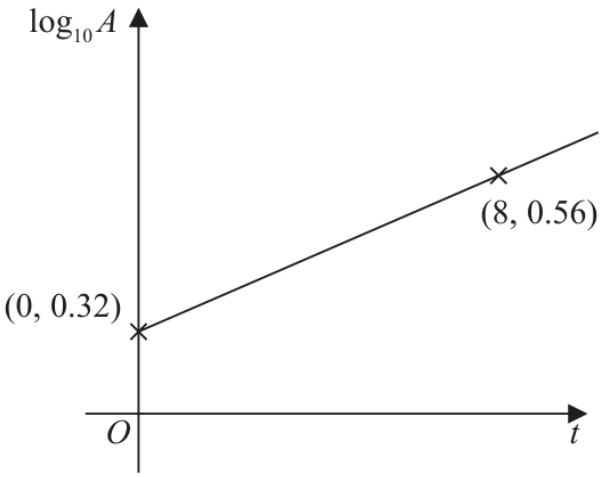


Figure 1

The growth of duckweed on a pond is being studied.

The surface area of the pond covered by duckweed, $A\text{m}^2$, at a time t days after the start of the study is modelled by the equation

$$A = pq^t \qquad \text{where } p \text{ and } q \text{ are positive constants}$$

Figure 1 shows the linear relationship between $\log_{10} A$ and t.

The points (0, 0.32) and (8, 0.56) lie on the line as shown.

(a) Find, to 3 decimal places, the value of p and the value of q.

(4)

Using the model with the values of p and q found in part (a),

(b) find the rate of increase of the surface area of the pond covered by duckweed, in m^2/day, exactly 6 days after the start of the study.
Give your answer to 2 decimal places.

(3)

Question 8

13 marks | Applications of Differentiation | Main topic: Log Graphs & Exponential Models

8. A scientist is studying a population of fish in a lake. The number of fish, N, in the population, t years after the start of the study, is modelled by the equation

$$N = \frac{600e^{0.3t}}{2 + e^{0.3t}} \qquad t \geqslant 0$$

Use the equation of the model to answer parts (a), (b), (c), (d) and (e).

(a) Find the number of fish in the lake at the start of the study.

(1)

(b) Find the upper limit to the number of fish in the lake.

(1)

(c) Find the time, after the start of the study, when there are predicted to be 500 fish in the lake. Give your answer in years and months to the nearest month.

(4)

(d) Show that

$$\frac{dN}{dt} = \frac{Ae^{0.3t}}{(2 + e^{0.3t})^2}$$

where A is a constant to be found.

(3)

Given that when $t = T$, $\frac{dN}{dt} = 8$

(e) find the value of T to one decimal place.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(4)

WMA13/01 OCTOBER 2021

Question 1

9 marks | Applications of Differentiation | Main topic: Rational Algebra

1. The function f is defined by

$$\mathrm{f}(x) = \frac{5x}{x^2 + 7x + 12} + \frac{5x}{x + 4} \qquad x > 0$$

(a) Show that $\mathrm{f}(x) = \dfrac{5x}{x + 3}$

(3)

(b) Find f^{-1}

(3)

(c) (i) Find, in simplest form, $\mathrm{f}'(x)$.

(ii) Hence, state whether f is an increasing or a decreasing function, giving a reason for your answer.

(3)

Question 6

8 marks | Applications of Differentiation | Main topic: Differentiation Techniques

6. (i) The curve C_1 has equation

$$y = 3\ln(x^2 - 5) - 4x^2 + 15 \qquad x > \sqrt{5}$$

Show that C_1 has a stationary point at $x = \frac{\sqrt{p}}{2}$ where p is a constant to be found.

(4)

(ii) A different curve C_2 has equation

$$y = 4x - 12\sin^2 x$$

(a) Show that, for this curve,

$$\frac{dy}{dx} = A + B\sin 2x$$

where A and B are constants to be found.

(b) Hence, state the maximum gradient of this curve.

(4)

Question 8

7 marks | Applications of Differentiation | Main topic: Differentiation Techniques

8. A curve C has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \arcsin\left(\frac{1}{2}x\right) \qquad -2 \leqslant x \leqslant 2 \qquad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

(a) Sketch C.

(1)

(b) Given $x = 2\sin y$, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{A - x^2}}$$

where A is a constant to be found.

(3)

The point P lies on C and has y coordinate $\frac{\pi}{4}$

(c) Find the equation of the tangent to C at P. Write your answer in the form $y = mx + c$, where m and c are constants to be found.

(3)

Question 9

9 marks | Applications of Differentiation | Main topic: Numerical Methods

9.

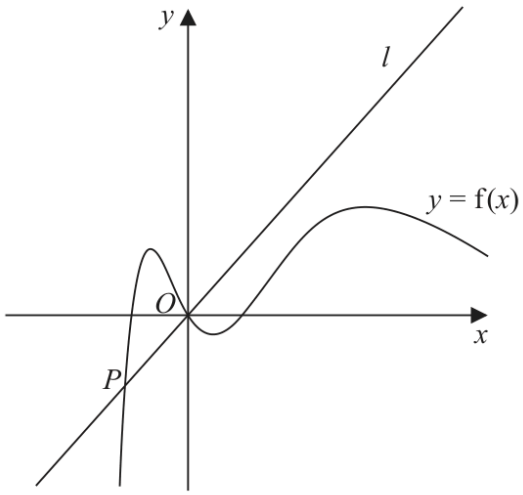


Figure 3

Figure 3 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x(x^2 - 4)\mathrm{e}^{-\frac{1}{2}x}$$

(a) Find $\mathrm{f}'(x)$.

(2)

The line l is the normal to the curve at O and meets the curve again at the point P.

The point P lies in the 3rd quadrant, as shown in Figure 3.

(b) Show that the x coordinate of P is a solution of the equation

$$x = -\frac{1}{2}\sqrt{16 + \mathrm{e}^{\frac{1}{2}x}}$$

(4)

(c) Using the iterative formula

$$x_{n+1} = -\frac{1}{2}\sqrt{16 + \mathrm{e}^{\frac{1}{2}x_n}} \quad \text{with } x_1 = -2$$

find, to 4 decimal places,

(i) the value of x_2

(ii) the x coordinate of P.

(3)

Question 6

11 marks | Applications of Differentiation | Main topic: Functions, Composition & Inverses

6. The function f is defined by

$$\mathrm{f}(x) = \frac{5x - 3}{x - 4} \qquad x > 4$$

(a) Show, by using calculus, that f is a decreasing function.

(3)

(b) Find f^{-1}

(3)

(c) (i) Show that $\mathrm{ff}(x) = \dfrac{ax + b}{x + c}$ where a, b and c are constants to be found.

(ii) Deduce the range of ff.

(5)

Question 8

8 marks | Applications of Differentiation | Main topic: Log Graphs & Exponential Models

8. A dose of antibiotics is given to a patient.

The amount of the antibiotic, x milligrams, in the patient's bloodstream t hours after the dose was given, is found to satisfy the equation

$$\log_{10} x = 2.74 - 0.079t$$

(a) Show that this equation can be written in the form

$$x = pq^{-t}$$

where p and q are constants to be found. Give the value of p to the nearest whole number and the value of q to 2 significant figures.

(4)

(b) With reference to the equation in part (a), interpret the value of the constant p.

(1)

When a different dose of the antibiotic is given to another patient, the values of x and t satisfy the equation

$$x = 400 \times 1.4^{-t}$$

(c) Use calculus to find, to 2 significant figures, the value of $\dfrac{dx}{dt}$ when $t = 5$

(3)

Question 10

7 marks | Applications of Differentiation | Main topic: Differentiation Techniques

10.

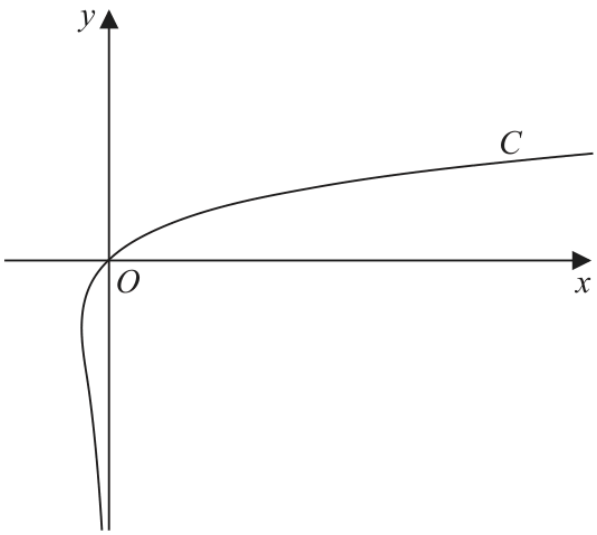


Figure 3

Figure 3 shows a sketch of the curve C with equation

$$x = ye^{2y} \qquad y \in \mathbb{R}$$

(a) Show that

$$\frac{dy}{dx} = \frac{y}{x(1+2y)}$$

(4)

Given that the straight line with equation $x = k$, where k is a constant, cuts C at exactly two points,

(b) find the range of possible values for k.

(3)

Question 8

12 marks | Applications of Differentiation | Main topic: Numerical Methods

8.

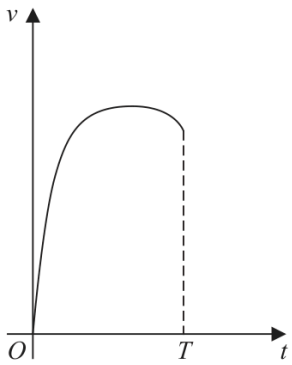


Figure 4

Figure 4 is a graph showing the velocity of a sprinter during a 100 m race.

The sprinter's velocity during the race, $v\,\text{m s}^{-1}$, is modelled by the equation

$$v = 12 - \text{e}^{t-10} - 12\text{e}^{-0.75t} \qquad t \geqslant 0$$

where t seconds is the time after the sprinter begins to run.

According to the model,

(a) find, using calculus, the sprinter's maximum velocity during the race.

(5)

Given that the sprinter runs 100 m in T seconds, such that

$$\int_0^T v\,\text{d}t = 100$$

(b) show that T is a solution of the equation

$$T = \frac{1}{12}\left(116 - 16\text{e}^{-0.75T} + \text{e}^{T-10} - \text{e}^{-10}\right)$$

(4)

The iteration formula

$$T_{n+1} = \frac{1}{12}\left(116 - 16\text{e}^{-0.75T_n} + \text{e}^{T_n-10} - \text{e}^{-10}\right)$$

is used to find an approximate value for T

Using this iteration formula with $T_1 = 10$

(c) find, to 4 decimal places,

(i) the value of T_2

(ii) the time taken by the sprinter to run the race, according to the model.

(3)

Question 9

9 marks | Applications of Differentiation | Main topic: Trig Equations & R-Form

9.

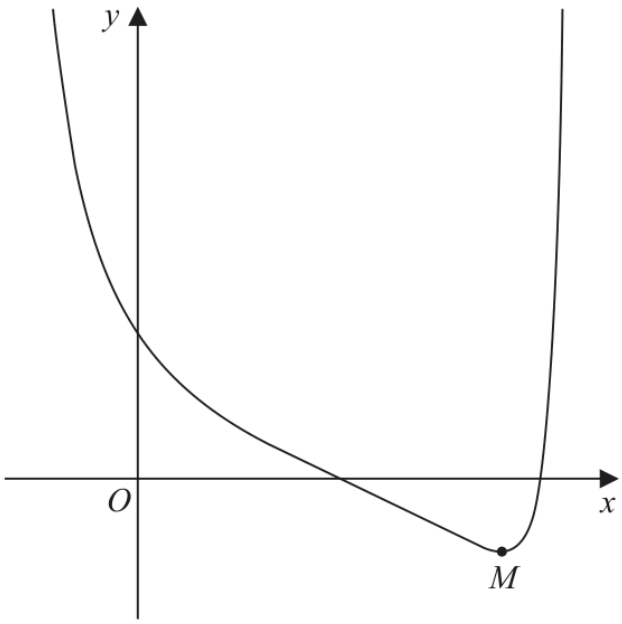


Figure 5

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 5 shows the curve with equation

$$y = \frac{1 + 2\cos x}{1 + \sin x} \qquad -\frac{\pi}{2} < x < \frac{3\pi}{2}$$

The point M, shown in Figure 5, is the minimum point on the curve.

(a) Show that the x coordinate of M is a solution of the equation

$$2\sin x + \cos x = -2$$

(4)

(b) Hence find, to 3 significant figures, the x coordinate of M.

(5)

Question 9

11 marks | Applications of Differentiation | Main topic: Numerical Methods

9.

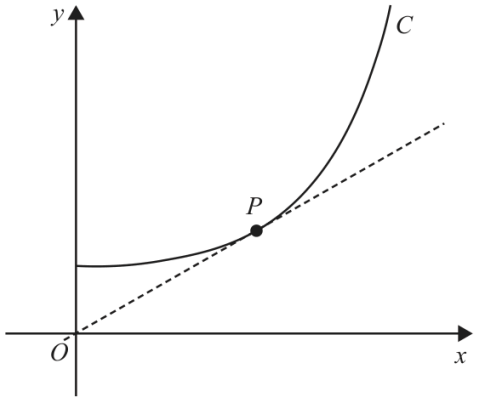


Figure 3

Figure 3 shows a sketch of part of the curve C with equation

$$y = \sqrt{3 + 4e^{x^2}} \qquad x \geqslant 0$$

(a) Find $\frac{dy}{dx}$, giving your answer in simplest form.

(2)

The point P with x coordinate α lies on C.

Given that the tangent to C at P passes through the origin, as shown in Figure 3,

(b) show that $x = \alpha$ is a solution of the equation

$$4x^2e^{x^2} - 4e^{x^2} - 3 = 0$$

(3)

(c) Hence show that α lies between 1 and 2

(2)

(d) Show that the equation in part (b) can be written in the form

$$x = \frac{1}{2}\sqrt{4 + 3e^{-x^2}}$$

(1)

The iteration formula

$$x_{n+1} = \frac{1}{2}\sqrt{4 + 3e^{-x_n^2}}$$

with $x_1 = 1$ is used to find an approximation for α.

(e) Use the iteration formula to find, to 4 decimal places, the value of

(i) x_3

(ii) α

(3)

Question 10

9 marks | Applications of Differentiation | Main topic: Log Graphs & Exponential Models

10. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

A population of fruit flies is being studied.

The number of fruit flies, F, in the population, t days after the start of the study, is modelled by the equation

$$F = \frac{350e^{kt}}{9 + e^{kt}}$$

where k is a constant.

Use the equation of the model to answer parts (a), (b) and (c).

(a) Find the number of fruit flies in the population at the start of the study.

(1)

Given that there are 200 fruit flies in the population 15 days after the start of the study,

(b) show that $k = \frac{1}{15}\ln 12$

(3)

Given also that, when $t = T$, the number of fruit flies in the population is increasing at a rate of 10 per day,

(c) find the possible values of T, giving your answers to one decimal place.

(5)

Question 7

8 marks | Applications of Differentiation | Main topic: Log Graphs & Exponential Models

7. A scientist is studying two different populations of bacteria.

The number of bacteria N in the first population is modelled by the equation

$$N = A\mathrm{e}^{kt} \qquad t \geqslant 0$$

where A and k are positive constants and t is the time in hours from the start of the study.

Given that

- there were 2500 bacteria in this population at the start of the study
- there were 10 000 bacteria 8 hours later

(a) find the exact value of A and the value of k to 4 significant figures.

(3)

The number of bacteria N in the second population is modelled by the equation

$$N = 60\,000\mathrm{e}^{-0.6t} \qquad t \geqslant 0$$

where t is the time in hours from the start of the study.

(b) Find the rate of decrease of bacteria in this population exactly 5 hours from the start of the study. Give your answer to 3 significant figures.

(2)

When $t = T$, the number of bacteria in the two different populations was the same.

(c) Find the value of T, giving your answer to 3 significant figures.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(3)

Question 10

8 marks | Applications of Differentiation | Main topic: Differentiation Techniques

10.

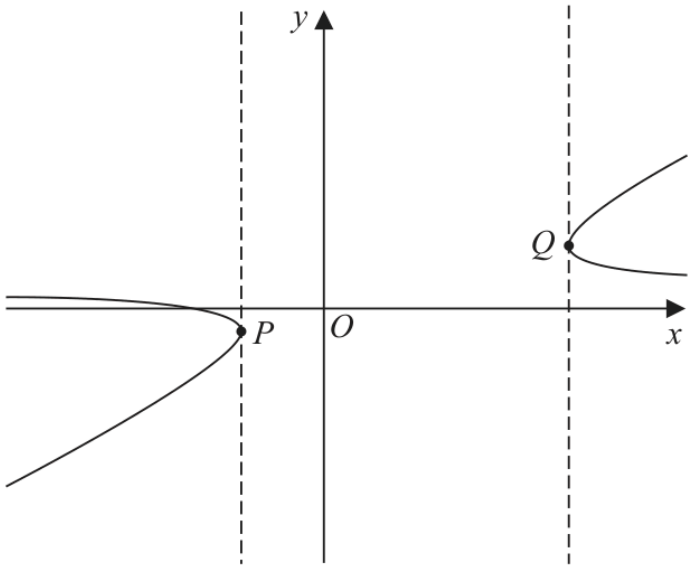


Figure 4

Figure 4 shows a sketch of the curve with equation

$$x = \frac{2y^2 + 6}{3y - 3}$$

(a) Find $\frac{dx}{dy}$ giving your answer as a fully simplified fraction.

(4)

The tangents at points P and Q on the curve are parallel to the y-axis, as shown in Figure 4.

(b) Use the answer to part (a) to find the equations of these two tangents.

(4)

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Question 4

7 marks | Applications of Differentiation | Main topic: Log Graphs & Exponential Models

4. A new mobile phone is released for sale.

The total sales N of this phone, in **thousands**, is modelled by the equation

$$N = 125 - Ae^{-0.109t} \qquad t \geqslant 0$$

where A is a constant and t is the time in months after the phone was released for sale.

Given that when $t = 0$, $N = 32$

(a) state the value of A.

(1)

Given that when $t = T$ the total sales of the phone was 100 000

(b) find, according to the model, the value of T. Give your answer to 2 decimal places.

(3)

(c) Find, according to the model, the rate of increase in total sales when $t = 7$, giving your answer to 3 significant figures.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(2)

The total sales of the mobile phone is expected to reach 150 000

Using this information,

(d) give a reason why the given equation is not suitable for modelling the total sales of the phone.

(1)

Question 7

10 marks | Applications of Differentiation | Main topic: Functions, Composition & Inverses

7.

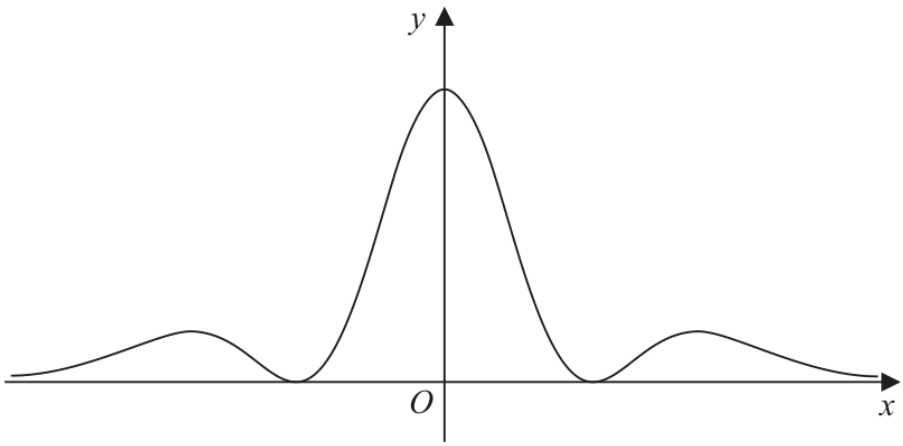


Figure 1

Figure 1 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \mathrm{e}^{-x^2}(2x^2 - 3)^2$$

(a) Find the range of f

(2)

(b) Show that

$$\mathrm{f}'(x) = 2x(2x^2 - 3)\mathrm{e}^{-x^2}(A - Bx^2)$$

where A and B are constants to be found.

(4)

Given that the line $y = k$, where k is a constant, $k > 0$, intersects the curve at exactly two distinct points,

(c) find the exact range of values of k

(4)

Question 10

9 marks | Applications of Differentiation | Main topic: Differentiation Techniques

10. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

A curve C has equation

$$x = \sin^2 4y \qquad 0 \leqslant y \leqslant \frac{\pi}{8} \qquad 0 \leqslant x \leqslant 1$$

The point P with x coordinate $\frac{1}{4}$ lies on C

(a) Find the exact y coordinate of P

(2)

(b) Find $\frac{\mathrm{d}x}{\mathrm{d}y}$

(2)

(c) Hence show that $\frac{\mathrm{d}y}{\mathrm{d}x}$ can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{q + r(x + s)^2}}$$

where q, r and s are constants to be found.

(3)

Using the answer to part (c),

(d) (i) state the x coordinate of the point where the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ is a minimum,

(ii) state the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at this point.

(2)

Question 4

13 marks | Applications of Differentiation | Main topic: Rational Algebra

4. The function f is defined by

$$f(x) = \frac{2x^2 - 32}{3x^2 + 7x - 20} + \frac{8}{3x - 5} \qquad x \in \mathbb{R} \quad x > 2$$

(a) Show that $f(x) = \frac{2x}{3x-5}$

(3)

(b) Show, using calculus, that f is a decreasing function.
You must make your reasoning clear.

(3)

The function g is defined by

$$g(x) = 3 + 2\ln x \qquad x \geqslant 1$$

(c) Find g^{-1}

(3)

(d) Find the exact value of a for which

$$gf(a) = 5$$

(4)

Question 5

7 marks | Applications of Differentiation | Main topic: Log Graphs & Exponential Models

5. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

The temperature, T °C, of the air in a room t minutes after a heat source is switched off, is modelled by the equation

$$T = 10 + A\mathrm{e}^{-Bt}$$

where A and B are constants.

Given that the temperature of the air in the room at the instant the heat source was switched off was 18 °C,

(a) find the value of A

(1)

Given also that, exactly 45 minutes after the heat source was switched off, the temperature of the air in the room was 16 °C,

(b) find the value of B to 3 significant figures.

(3)

Using the values for A and B,

(c) find, according to the model, the rate of change of the temperature of the air in the room exactly two minutes after the heat source was switched off.
Give your answer in °C min^{-1} to 3 significant figures.

(2)

(d) Explain why, according to the model, the temperature of the air in the room cannot fall to 5 °C

(1)

Question 5

10 marks | Applications of Differentiation | Main topic: Functions, Composition & Inverses

5. The functions f and g are defined by

$$\mathrm{f}(x) = 2 + 5\ln x \qquad x > 0$$

$$\mathrm{g}(x) = \frac{6x - 2}{2x + 1} \qquad x > \frac{1}{3}$$

(a) Find $\mathrm{f}^{-1}(22)$

(2)

(b) Use differentiation to prove that g is an increasing function.

(3)

(c) Find g^{-1}

(3)

(d) Find the range of fg

(2)

Question 8

10 marks | Applications of Differentiation | Main topic: Log Graphs & Exponential Models

8.

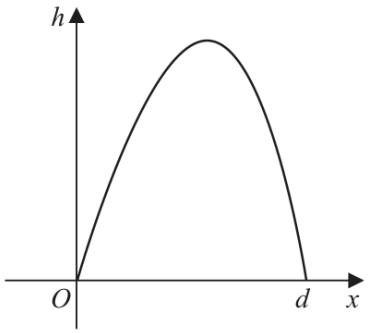


Figure 4

Figure 4 is a graph showing the path of a golf ball after the ball has been hit until it first hits the ground.

The vertical height, h metres, of the ball above the ground has been plotted against the horizontal distance travelled, x metres, measured from where the ball was hit.

The ball travels a horizontal distance of d metres before it first hits the ground.

The ball is modelled as a particle travelling in a vertical plane above horizontal ground.

The path of the ball is modelled by the equation

$$h = 1.5x - 0.5x\,e^{0.02x} \qquad 0 \leqslant x \leqslant d$$

Use the model to answer parts (a), (b) and (c).

(a) Find the value of d, giving your answer to 2 decimal places.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(3)

(b) Show that the maximum value of h occurs when

$$x = 50\ln\left(\frac{150}{x + 50}\right)$$

(4)

Using the iteration formula

$$x_{n+1} = 50\ln\left(\frac{150}{x_n + 50}\right) \qquad \text{with } x_1 = 30$$

(c) (i) find the value of x_2 to 2 decimal places,

(ii) find, by repeated iteration, the horizontal distance travelled by the golf ball before it reaches its maximum height. Give your answer to 2 decimal places.

(3)

Question 4

6 marks | Applications of Differentiation | Main topic: Log Graphs & Exponential Models

4. The number of bacteria on a surface is being monitored.

The number of bacteria, N, on the surface, t hours after monitoring began is modelled by the equation

$$\log_{10} N = 0.35t + 2$$

Use the equation of the model to answer parts (a) to (c).

(a) Find the initial number of bacteria on the surface.

(1)

(b) Show that the equation of the model can be written in the form

$$N = ab^t$$

where a and b are constants to be found. Give the value of b to 2 decimal places.

(3)

(c) Hence find the rate of growth of bacteria on the surface exactly 5 hours after monitoring began.

(2)

Question 7

10 marks | Applications of Differentiation | Main topic: Graph Transformations

7.

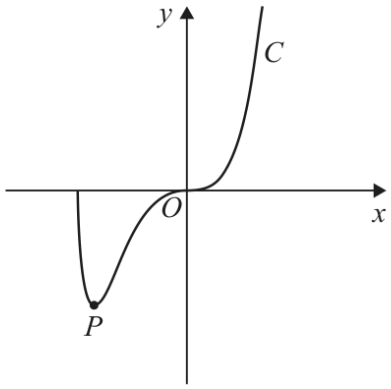


Figure 3

The curve C has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x^3\sqrt{4x+7} \qquad x \geqslant -\frac{7}{4}$$

(a) Show that

$$\mathrm{f}'(x) = \frac{kx^2(2x+3)}{\sqrt{4x+7}}$$

where k is a constant to be found.

(4)

The point P, shown in Figure 3, is the minimum turning point on C.

(b) Find the coordinates of P.

(2)

(c) Hence find the range of the function g defined by

$$\mathrm{g}(x) = -4\,\mathrm{f}(x) \qquad x \geqslant -\frac{7}{4}$$

(2)

The point Q with coordinates $\left(\frac{1}{2}, \frac{3}{8}\right)$ lies on C.

(d) Find the coordinates of the point to which Q is mapped when C is transformed to the curve with equation

$$y = 40\,\mathrm{f}\left(x - \frac{3}{2}\right) - 8$$

(2)

Question 8

12 marks | Applications of Differentiation | Main topic: Log Graphs & Exponential Models

8.

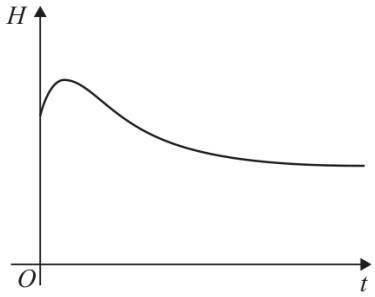


Figure 4

The heart rate of a horse is being monitored.
The heart rate H, measured in beats per minute (bpm), is modelled by the equation

$$H = 32 + 40\mathrm{e}^{-0.2t} - 20\,\mathrm{e}^{-0.9t}$$

where t minutes is the time after monitoring began.

Figure 4 is a sketch of H against t.

Use the equation of the model to answer parts (a) to (e).

(a) State the initial heart rate of the horse.

(1)

In the long term, the heart rate of the horse approaches L bpm.

(b) State the value of L.

(1)

The heart rate of the horse reaches its maximum value after T minutes.

(c) Find the value of T, giving your answer to 3 decimal places.

(*Solutions based entirely on calculator technology are not acceptable.*)

(5)

The heart rate of the horse is 37 bpm after M minutes.

(d) Show that M is a solution of the equation

$$t = 5\ln\left(\frac{8}{1 + 4\mathrm{e}^{-0.9t}}\right)$$

(2)

Using the iteration formula

$$t_{n+1} = 5\ln\left(\frac{8}{1 + 4\mathrm{e}^{-0.9t_n}}\right) \quad \text{with} \quad t_1 = 10$$

(e) (i) find, to 4 decimal places, the value of t_2

(ii) find, to 4 decimal places, the value of M

(3)

Question 9

14 marks | Applications of Differentiation | Main topic: Integration

9.

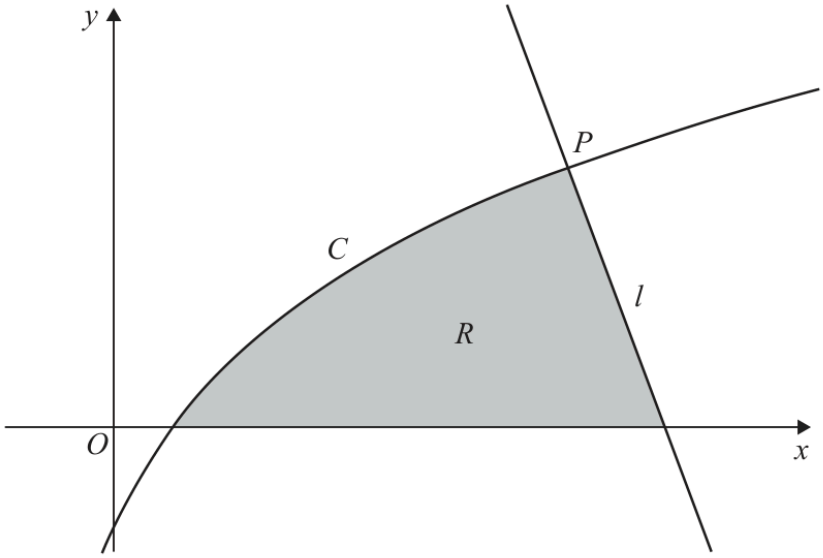


Figure 5

Figure 5 shows a sketch of part of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{6x^2 + 4x - 2}{2x + 1} \qquad x > -\frac{1}{2}$$

(a) Find $\mathrm{f}'(x)$, giving the answer in simplest form.

(3)

The line l is the normal to C at the point $P(2, 6)$

(b) Show that an equation for l is

$$16y + 5x = 106$$

(3)

(c) Write $\mathrm{f}(x)$ in the form $Ax + B + \dfrac{D}{2x + 1}$ where A, B and D are constants.

(3)

The region R, shown shaded in Figure 5, is bounded by C, l and the x-axis.

(d) Use algebraic integration to find the exact area of R, giving your answer in the form $P + Q\ln 3$, where P and Q are rational constants.

(*Solutions based entirely on calculator technology are not acceptable.*)

(5)

Question 10

9 marks | Applications of Differentiation | Main topic: Differentiation Techniques

10.

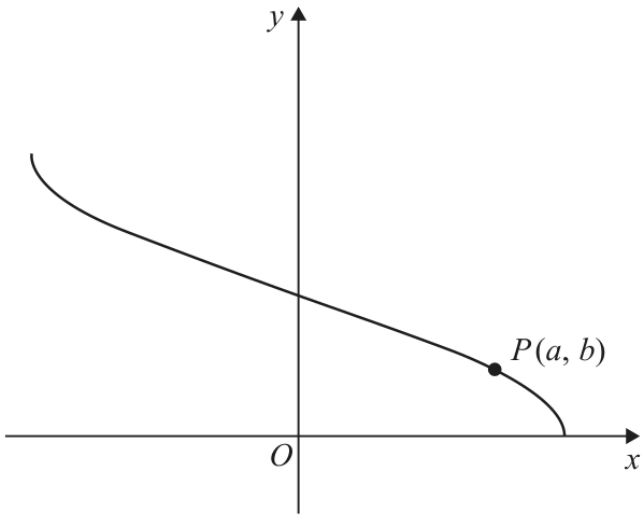


Figure 2

Figure 2 shows a sketch of the curve with equation

$$x = 3\cos 2y \qquad -3 \leqslant x \leqslant 3 \qquad 0 \leqslant y \leqslant \frac{\pi}{2}$$

(a) Find $\frac{\mathrm{d}x}{\mathrm{d}y}$ in terms of y.

(2)

(b) Hence show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{k}{\sqrt{9 - x^2}}$$

where k is a constant to be found.

(3)

The point $P(a, b)$ lies on the curve and is shown in Figure 2.

Given that

- the gradient of the curve at P is $-\frac{1}{4}$
- both a and b are positive

(c) find the exact values of a and b.

(4)

Question 1

8 marks | Applications of Differentiation | Main topic: Functions, Composition & Inverses

1: **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The functions f and g are defined by

$$f(x) = \frac{2x}{3x+1} \qquad x \in \mathbb{R} \quad x \geqslant 0$$

$$g(x) = 4 - x^2 \qquad x \in \mathbb{R} \quad x \geqslant 0$$

(a) Find the value of $gf(1)$

(2)

(b) Find the range of f

(2)

(c) Find $f^{-1}(x)$

(2)

(d) Solve $f^{-1}(x) = f(x)$

(2)

Question 6

8 marks | Applications of Differentiation | Main topic: Log Graphs & Exponential Models

6: **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The temperature, θ°C, of a computer processor, t minutes after the computer is switched off, is modelled by the equation

$$\theta = 21 + Ae^{-kt}$$

where A and k are positive constants.

Given that the temperature of the processor was 75°C when the computer was switched off,

(a) find the value of A.

(2)

Given also that it takes 5 minutes for the temperature of the processor to decrease from 75°C to 25°C,

(b) find the value of k, giving your answer to 3 significant figures.

(3)

At time T minutes, the temperature of the processor is decreasing at a rate of 9°C per minute.

(c) Find the value of T according to the model, giving your answer to 2 decimal places.

(3)

Question 7

9 marks | Applications of Differentiation | Main topic: Numerical Methods

7: A continuous curve has equation

$$y = e^{-x^2} \sin 3x \qquad 0 \leqslant x \leqslant \frac{\pi}{3}$$

The curve has a stationary point at the point P.

(a) Show, using calculus, that the x coordinate of P is a solution of the equation

$$x = \frac{1}{3}\arctan\left(\frac{3}{2x}\right)$$

(4)

Using the iteration formula

$$x_{n+1} = \frac{1}{3}\arctan\left(\frac{3}{2x_n}\right) \qquad x_1 = 0.4$$

(b) find the value of

(i) x_2

(ii) x_4

giving your answers to 4 decimal places.

(3)

(c) Using a suitable interval and a suitable function which should be stated, show that the x coordinate of P is 0.430 correct to 3 decimal places.

(2)

Question 9

11 marks | Applications of Differentiation | Main topic: Differentiation Techniques

9:

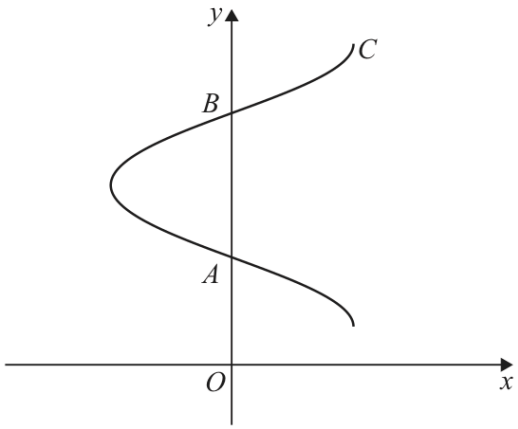


Figure 2

Figure 2 shows a sketch of the curve C with equation

$$x = \frac{2}{3}\sin\left(3y + \frac{\pi}{4}\right) \qquad \frac{\pi}{12} < y < \frac{3\pi}{4}$$

The curve intersects the y-axis at the points A and B as shown.

(a) Find the exact value of the y coordinate of

- point A
- point B

(3)

(b) Show that

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{p - qx^2}$$

where p and q are integers to be found.

(4)

The **normal** to C at A and the **tangent** to C at B intersect at the point D.

Using

- the answer to part (b)
- the sketch of curve C in Figure 2

(c) find, in simplest form, the exact x coordinate of D.

(4)

Question 3

9 marks | Applications of Differentiation | Main topic: Log Graphs & Exponential Models

3. The share price, £V, of a company is being monitored.

A graph is drawn of $\log_{10} V$ against t, where t is the number of years after monitoring began.

The graph is a straight line passing through the points $(0, 2)$ and $(5, 2.25)$

Using this information,

(a) find an equation for the line in the form

$$\log_{10} V = mt + c$$

where m and c are constants.

(2)

(b) Write the answer to part (a) in the form

$$V = ab^t$$

where a and b are constants to be found.

Give the exact value of a and the value of b to 3 significant figures.

(3)

When $t = T$, the rate of increase in the share price of the company was £50 per year.

(c) Find the value of T, giving your answer to the nearest integer.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(4)

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Question 5

8 marks | Applications of Differentiation | Main topic: Log Graphs & Exponential Models

5. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The number of squirrels in a forest is being studied.

The number of squirrels, N, in the forest, t years after the start of the study, is modelled by the equation

$$N = \frac{4000e^{0.1t}}{19 + e^{0.2t}} \qquad t \geqslant 0$$

Use the equation of the model to answer parts (a), (b), (c) and (d).

(a) Find the number of squirrels in the forest at the start of the study.

(1)

(b) Find $\frac{dN}{dt}$

(2)

The number of squirrels in the forest is at a maximum when $t = T$

Using the answer to part (b),

(c) show that $e^{0.2T} = A$, where A is a constant to be found.

(2)

(d) Hence find the maximum number of squirrels in the forest.

Show your working and give your answer to the nearest whole number.

(3)

Question 10

9 marks | Applications of Differentiation | Main topic: Log Graphs & Exponential Models

10. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

Two populations of insects are being studied over the same period of time.

The number of insects N_A in population A, t years after the start of the study, is modelled by the equation

$$N_A = 8000 + 900e^{0.2t}$$

According to the model,

(a) find the rate of growth in the number of insects in population A exactly 5 years after the start of the study.

(2)

The number of insects N_B in population B, t years after the start of the study, is modelled by the equation

$$N_B = 8000 + Pe^{kt}$$

where P and k are positive constants.

Given that

- there are 10 000 insects in population B at the start of the study
- there are 11 570 insects in population B exactly 4 years after the start of the study
- there are the same number of insects in population A and population B after T years

find, according to the models,

(b) the exact value of P and the value of k to 3 decimal places,

(4)

(c) the value of T, giving your answer to one decimal place.

(3)

Question 3

8 marks | Applications of Differentiation | Main topic: Differentiation Techniques

3. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The curve C has equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{4x^2 - 14x + 15}{x - 3} \qquad x < 3$$

(a) Find $\mathrm{f}'(x)$, giving the answer in the form

$$\frac{ax^2 + bx + c}{(x-3)^2}$$

where a, b and c are constants.

(4)

(b) Hence find the coordinates of the turning point of C.

(2)

The point $P\left(\frac{5}{2}, -10\right)$ lies on C.

(c) Find the coordinates of the point to which P is mapped, when the curve with equation $y = \mathrm{f}(x)$ is transformed to the curve with equation $y = \mathrm{f}(2x) + 3$

(2)

Question 5

7 marks | Applications of Differentiation | Main topic: Log Graphs & Exponential Models

5.

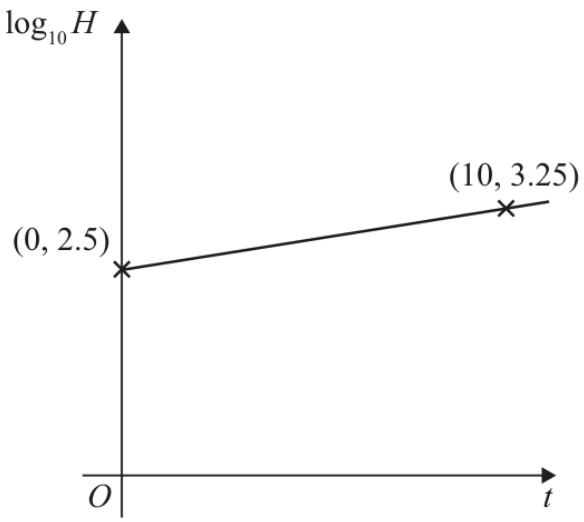


Figure 1

The recycling of household waste in a particular town is being monitored.

The number of households in the town, H, recycling their household waste, t years after monitoring began, is modelled by the equation

$$H = ab^t$$

where a and b are constants.

The line in Figure 1 shows the linear relationship between t and $\log_{10} H$

The line passes through the points (0, 2.5) and (10, 3.25)

Using this information,

(a) find, to 3 significant figures, the value of a and the value of b

(4)

(b) With reference to the model, interpret the value of the constant b

(1)

(c) Find the value of $\frac{dH}{dt}$ when $t = 5$, giving the answer to 2 significant figures.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(2)

Question 6

6 marks | Applications of Differentiation | Main topic: Differentiation Techniques

6. The curve C has equation

$$x = \left(3\sin 2y - \frac{6y}{\pi}\right)^2$$

Given that

- the point P with y coordinate $\frac{\pi}{4}$ lies on C
- the tangent to C at P cuts the x-axis at the point Q

use calculus to find the x coordinate of Q

(6)

Question 7

12 marks | Applications of Differentiation | Main topic: Log Graphs & Exponential Models

7. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The selling price of a car is monitored over a 15-year period.

The selling price P_C of the car, in thousands of pounds, t years after monitoring began, is modelled by the equation

$$P_C = 7 + 16e^{-0.1t} \qquad 0 \leqslant t \leqslant 15$$

Given that when $t = T$ the selling price of the car is £15 000

(a) find, according to the model, the value of T. Give the answer to 2 decimal places.

(3)

(b) Find the value of $\frac{dP_C}{dt}$ when $t = 5$. Give the answer to 2 decimal places.

(2)

The selling price of a van is monitored over the same 15-year period.

The selling price P_V of the van, in thousands of pounds, t years after monitoring began, is modelled by the equation

$$P_V = 4 + me^{-0.2t} \qquad 0 \leqslant t \leqslant 15$$

where m is a constant.

Use the equations of these models to answer parts (c) and (d).

Given that $P_V = 36$ when $t = 0$

(c) (i) state the value of m

(ii) find the difference between the initial selling price of the car and the initial selling price of the van.

(3)

When $t = \alpha$, the selling price of the car is equal to the selling price of the van.

(d) Find, using algebra and showing your working, the value of α, giving the answer to 3 significant figures.

(4)

TOPIC

Integration

Every past-paper question on this topic, January 2020 to June 2026.

Question 8

10 marks | Integration

8. (i) Find, using algebraic integration, the exact value of

$$\int_3^{42} \frac{2}{3x-1}\,\mathrm{d}x$$

giving your answer in simplest form.

(4)

(ii) $$\mathrm{h}(x) = \frac{2x^3 - 7x^2 + 8x + 1}{(x-1)^2} \qquad x > 1$$

Given $\mathrm{h}(x) = Ax + B + \frac{C}{(x-1)^2}$ where A, B and C are constants to be found, find

$$\int \mathrm{h}(x)\,\mathrm{d}x$$

(6)

Question 1

3 marks | Integration

1. Find

$$\int \frac{x^2 - 5}{2x^3} \mathrm{d}x \qquad x > 0$$

giving your answer in simplest form.

(3)

Question 9

4 marks | Integration

9. Find

(i) $\displaystyle\int \frac{3x-2}{3x^2-4x+5}\,\mathrm{d}x$

(2)

(ii) $\displaystyle\int \frac{\mathrm{e}^{2x}}{(\mathrm{e}^{2x}-1)^3}\,\mathrm{d}x \qquad x \neq 0$

(2)

Question 3

8 marks | Integration

3. (i) Find

$$\int \frac{12}{(2x-1)^2}\,dx$$

giving your answer in simplest form.

(2)

(ii) (a) Write $\frac{4x+3}{x+2}$ in the form

$$A + \frac{B}{x+2}$$ where A and B are constants to be found

(b) Hence find, using algebraic integration, the exact value of

$$\int_{-8}^{-5} \frac{4x+3}{x+2}\,dx$$

giving your answer in simplest form.

(6)

Question 5

6 marks | Integration

5. (i) Find, by algebraic integration, the exact value of

$$\int_2^4 \frac{8}{(2x-3)^3}\,\mathrm{d}x$$

(4)

(ii) Find, in simplest form,

$$\int x\left(x^2+3\right)^7\,\mathrm{d}x$$

(2)

Question 10

7 marks | Integration

10.

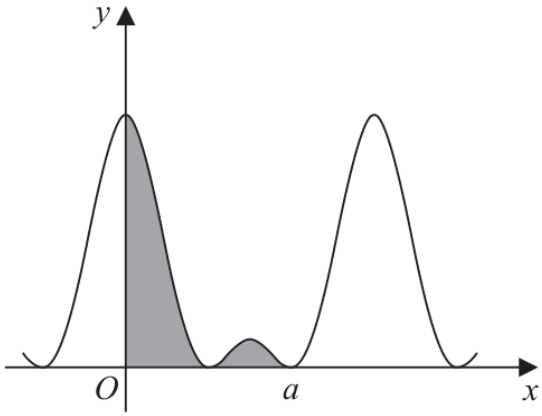


Figure 4

Figure 4 shows a sketch of part of the curve with equation

$$y = (1 + 2\cos 2x)^2$$

(a) Show that

$$(1 + 2\cos 2x)^2 \equiv p + q\cos 2x + r\cos 4x$$

where p, q and r are constants to be found.

(2)

The curve touches the positive x-axis for the second time when $x = a$, as shown in Figure 4.

The regions bounded by the curve, the y-axis and the x-axis up to $x = a$ are shown shaded in Figure 4.

(b) Find, using algebraic integration and making your method clear, the exact total area of the shaded regions. Write your answer in simplest form.

(5)

Question 3

6 marks | Integration

3. (i) Find, in simplest form,

$$\int (2x - 5)^7 \, dx$$

(2)

(ii) Show, by algebraic integration, that

$$\int_0^{\frac{\pi}{3}} \frac{4 \sin x}{1 + 2 \cos x} dx = \ln a$$

where a is a rational constant to be found.

(4)

Question 3

6 marks | Integration

3. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

Given that k is a positive constant,

(a) find

$$\int \frac{9x}{3x^2 + k} \, dx$$

(2)

Given also that

$$\int_2^5 \frac{9x}{3x^2 + k} \, dx = \ln 8$$

(b) find the value of k

(4)

Question 1

9 marks | Integration

1. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

$$\mathrm{f}(x) = \frac{2x^3 - 4x - 15}{x^2 + 3x + 4}$$

(a) Show that

$$\mathrm{f}(x) \equiv Ax + B + \frac{C(2x + 3)}{x^2 + 3x + 4}$$

where A, B and C are integers to be found.

(4)

(b) Hence, find

$$\int_3^5 \mathrm{f}(x)\,\mathrm{d}x$$

giving your answer in the form $p + \ln q$, where p and q are integers.

(5)

Question 8

5 marks | Integration

8. Find, in simplest form,

$$\int (2\cos x - \sin x)^2 \, dx$$

(5)

Question 2

7 marks | Integration

2. $$g(x) = \frac{2x^2 - 5x + 8}{x - 2}$$

(a) Write $g(x)$ in the form

$$Ax + B + \frac{C}{x - 2}$$

where A, B and C are integers to be found.

(3)

(b) Hence use algebraic integration to show that

$$\int_4^8 g(x)\,dx = \alpha + \beta \ln 3$$

where α and β are integers to be found.

(4)

Question 9

14 marks | Integration

9.

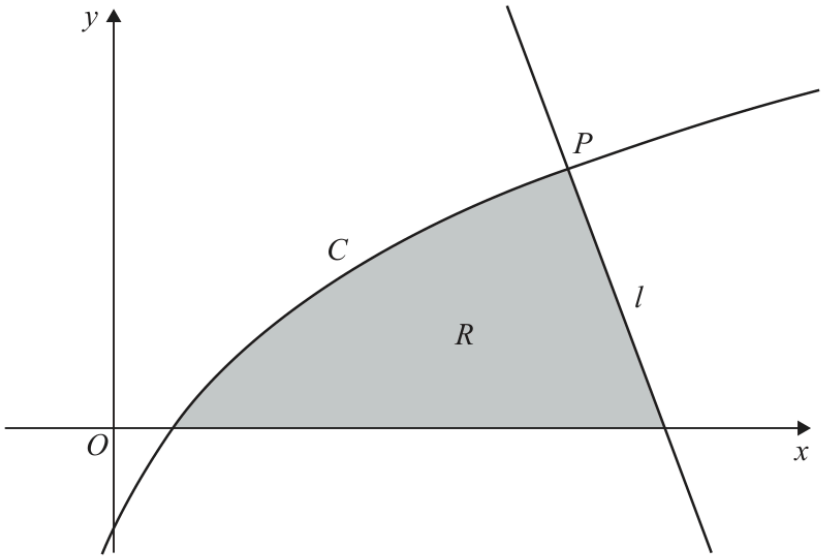


Figure 5

Figure 5 shows a sketch of part of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{6x^2 + 4x - 2}{2x + 1} \qquad x > -\frac{1}{2}$$

(a) Find $\mathrm{f}'(x)$, giving the answer in simplest form.

(3)

The line l is the normal to C at the point $P(2, 6)$

(b) Show that an equation for l is

$$16y + 5x = 106$$

(3)

(c) Write $\mathrm{f}(x)$ in the form $Ax + B + \frac{D}{2x + 1}$ where A, B and D are constants.

(3)

The region R, shown shaded in Figure 5, is bounded by C, l and the x-axis.

(d) Use algebraic integration to find the exact area of R, giving your answer in the form $P + Q\ln 3$, where P and Q are rational constants.

(*Solutions based entirely on calculator technology are not acceptable.*)

(5)

Question 4

9 marks | Integration

4. Given that

$$\frac{4x^3 + 2x^2 + 3x + 8}{x^2 + 4} \equiv Ax + B + \frac{Cx + D}{x^2 + 4}$$

(a) (i) find the values of the constants A, B and C

(ii) show that $D = 0$

(4)

(b) Hence, using algebraic integration, find

$$\int_1^4 \frac{4x^3 + 2x^2 + 3x + 8}{x^2 + 4} \, \mathrm{d}x$$

giving your answer in the form $p + q \ln 2$, where p and q are integers.

(5)

WMA13/01 MAY/JUNE 2025

Question 5

4 marks | Integration

5: Find

(i) $\int \sin^2 3x \, dx$

(2)

(ii) $\int x(x^2 + 4)^{\frac{3}{2}} dx$

(2)

Question 6

7 marks | Integration

6. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(i) Given $y = \ln(2x^2 + 5)$ find $\dfrac{dy}{dx}$

(2)

(ii) A curve C has equation $y = f(x)$ where

$$f(x) = \frac{21x}{3x^2 + k} \qquad x \in \mathbb{R}$$

where k is a positive integer, $k > 1$

Given that

$$\int_1^k f(x)\,dx < 7\ln 8$$

find the greatest possible value of k

(5)

Question 3

10 marks | Integration

3. (i) Find, in simplest form,

$$\int 2(3x+5)^7 \, dx$$

(2)

(ii) $$f(x) = \frac{4x^3 + x^2 - 7x + 14}{(x-1)(x+2)} \qquad x > 1$$

(a) Write $f(x)$ in the form

$$ax + b + \frac{c}{x-1}$$

where a, b and c are constants to be found.

(4)

(b) Hence, using algebraic integration, find

$$\int_3^9 f(x) \, dx$$

writing the answer in simplest form.

(4)

Question 4

5 marks | Integration

4. **In this question you must show all stages of your working.**
Solutions relying on calculator technology are not acceptable.

Using algebraic integration, find the exact value of

$$\int_{\frac{\pi}{18}}^{\frac{\pi}{12}} (\sin 3x + \cos 3x)^2 \, \mathrm{d}x$$

(5)

WMA13/01 OCTOBER 2020

Question 5

8 marks | Integration

Main topic: Trig Identities & Exact Values

5. (a) Show that

$$\sin 3x \equiv 3\sin x - 4\sin^3 x$$

(4)

(b) Hence find, using algebraic integration,

$$\int_0^{\frac{\pi}{3}} \sin^3 x \, dx$$

(4)

Question 9

14 marks | Integration

Main topic: Applications of Differentiation

9. (a) Given that

$$\frac{x^4 - x^3 - 10x^2 + 3x - 9}{x^2 - x - 12} \equiv x^2 + P + \frac{Q}{x - 4} \qquad x > -3$$

find the value of the constant P and show that $Q = 5$

(4)

The curve C has equation $y = \mathrm{g}(x)$, where

$$\mathrm{g}(x) = \frac{x^4 - x^3 - 10x^2 + 3x - 9}{x^2 - x - 12} \qquad -3 < x < 3.5 \qquad x \in \mathbb{R}$$

(b) Find the equation of the tangent to C at the point where $x = 2$
Give your answer in the form $y = mx + c$, where m and c are constants to be found.

(5)

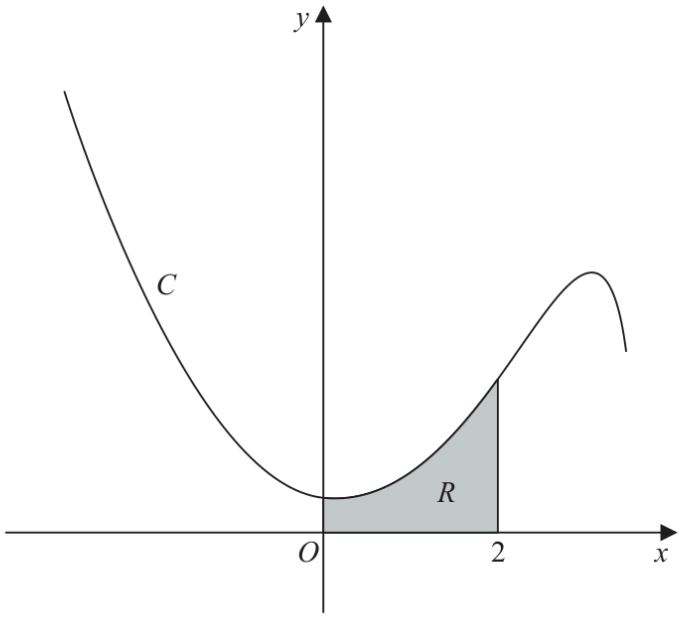


Figure 4

Figure 4 shows a sketch of the curve C.
The region R, shown shaded in Figure 4, is bounded by C, the y-axis, the x-axis and the line with equation $x = 2$

(c) Find the exact area of R, writing your answer in the form $a + b\ln 2$, where a and b are constants to be found.

(5)

Question 8

12 marks | Integration | Main topic: Numerical Methods

8.

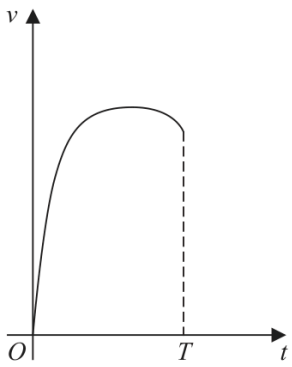


Figure 4

Figure 4 is a graph showing the velocity of a sprinter during a 100 m race.

The sprinter's velocity during the race, $v\,\text{m s}^{-1}$, is modelled by the equation

$$v = 12 - \text{e}^{t-10} - 12\text{e}^{-0.75t} \qquad t \geqslant 0$$

where t seconds is the time after the sprinter begins to run.

According to the model,

(a) find, using calculus, the sprinter's maximum velocity during the race.

(5)

Given that the sprinter runs 100 m in T seconds, such that

$$\int_0^T v\,\text{d}t = 100$$

(b) show that T is a solution of the equation

$$T = \frac{1}{12}\left(116 - 16\text{e}^{-0.75T} + \text{e}^{T-10} - \text{e}^{-10}\right)$$

(4)

The iteration formula

$$T_{n+1} = \frac{1}{12}\left(116 - 16\text{e}^{-0.75T_n} + \text{e}^{T_n-10} - \text{e}^{-10}\right)$$

is used to find an approximate value for T

Using this iteration formula with $T_1 = 10$

(c) find, to 4 decimal places,

(i) the value of T_2

(ii) the time taken by the sprinter to run the race, according to the model.

(3)

Question 4

7 marks | Integration | Main topic: Rational Algebra

4. $$f(x) = \frac{2x^4 + 15x^3 + 35x^2 + 21x - 4}{(x+3)^2} \qquad x \in \mathbb{R} \quad x > -3$$

(a) Find the values of the constants A, B, C and D such that

$$f(x) = Ax^2 + Bx + C + \frac{D}{(x+3)^2}$$

(4)

(b) Hence find,

$$\int f(x)\,dx$$

(3)

Question 3

7 marks | Integration | Main topic: Differentiation Techniques

3. (i) Find $\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\sin^2 3x\right)$ writing your answer in simplest form.

(2)

(ii)(a) Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(3x^2-4\right)^6$

(2)

(b) Hence show that

$$\int_0^{\sqrt{2}} x\left(3x^2-4\right)^5 \mathrm{d}x = R$$

where R is an integer to be found.

(*Solutions relying on calculator technology are not acceptable.*)

(3)

Question 9

10 marks | Integration

Main topic: Trig Identities & Exact Values

9. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that

$$\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} \equiv \operatorname{cosec} x \qquad x \neq \frac{n\pi}{2} \qquad n \in \mathbb{Z}$$

(3)

(b) Hence solve, for $0 < \theta < \frac{\pi}{2}$

$$\left(\frac{\cos 2\theta}{\sin \theta} + \frac{\sin 2\theta}{\cos \theta}\right)^2 = 6\cot\theta - 4$$

giving your answers to 3 significant figures as appropriate.

(5)

(c) Using the result from part (a), or otherwise, find the exact value of

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x}\right) \cot x \, dx$$

(2)

WMA13/01 OCTOBER 2023

Question 3

6 marks | Integration

Main topic: Trig Identities & Exact Values

3. (a) Using the identity for $\cos(A + B)$, prove that

$$\cos 2A \equiv 2\cos^2 A - 1$$

(2)

(b) Hence, using algebraic integration, find the exact value of

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{8}} (5 - 4\cos^2 3x)\,\mathrm{d}x$$

(4)

Question 7

12 marks | Integration

Main topic: Applications of Differentiation

7. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

The curve C has equation

$$y = \frac{16}{9(3x - k)} \qquad x \neq \frac{k}{3}$$

where k is a positive constant not equal to 3

(a) Find $\frac{dy}{dx}$ giving your answer in simplest form in terms of k.

(2)

The point P with x coordinate 1 lies on C.

Given that the gradient of the curve at P is -12

(b) find the two possible values of k

(3)

Given also that $k < 3$

(c) find the equation of the normal to C at P, writing your answer in the form $ax + by + c = 0$, where a, b and c are integers to be found.

(3)

(d) show, using algebraic integration that,

$$\int_1^3 \frac{16}{9(3x - k)} dx = \lambda \ln 10$$

where λ is a constant to be found.

(4)

Question 6

9 marks | Integration

Main topic: Applications of Differentiation

6.

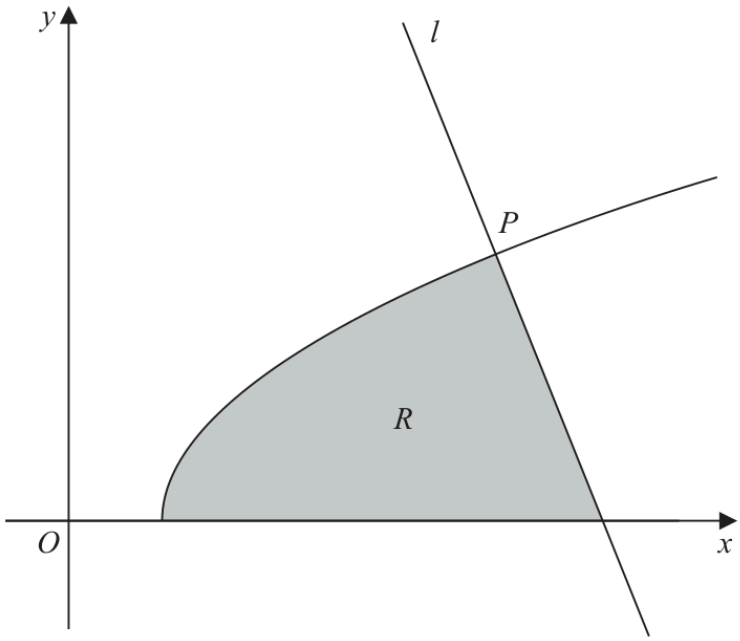


Figure 3

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 3 shows a sketch of part of the curve with equation

$$y = \sqrt{4x - 7}$$

The line l, shown in Figure 3, is the normal to the curve at the point $P(8, 5)$

(a) Use calculus to show that an equation of l is

$$5x + 2y - 50 = 0$$

(5)

The region R, shown shaded in Figure 3, is bounded by the curve, the x-axis and l.

(b) Use algebraic integration to find the exact area of R.

(4)

Question 9

11 marks | Integration | Main topic: Modulus

9.

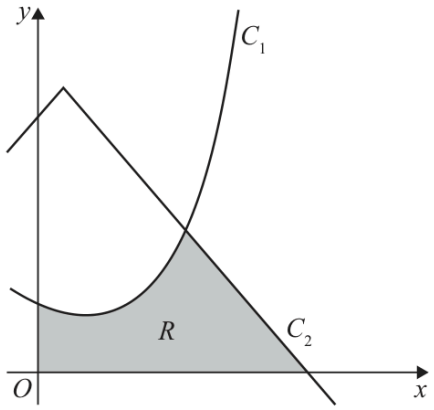


Figure 2

In this question you must show all stages of your working.
Solutions relying entirely on calculator technology are not acceptable.

(a) Find $\int \left(\frac{11}{2x+5} + 4e^{x-3} \right) dx$

(2)

Figure 2 shows a sketch of part of the curve C_1 and part of the graph C_2
The curve C_1 has equation

$$y = \frac{11}{2x+5} + 4e^{x-3}$$

The graph C_2 has equation

$$y = a - |bx - 1|$$

where a and b are positive integers.

(b) State, in terms of a and/or b, the coordinates of the maximum point of C_2

(2)

Given that

- C_1 intersects C_2 at $x = 3$
- C_2 crosses the y-axis at $(0, 9)$

(c) find the value of a and the value of b

(3)

The region R, shown shaded in Figure 2, is bounded by C_1, C_2, the x-axis and the y-axis.

(d) Find the area of R, giving the answer to 3 significant figures.

(4)

TOPIC

Numerical Methods

Every past-paper question on this topic, January 2020 to June 2026.

Question 6

9 marks | Numerical Methods

6.

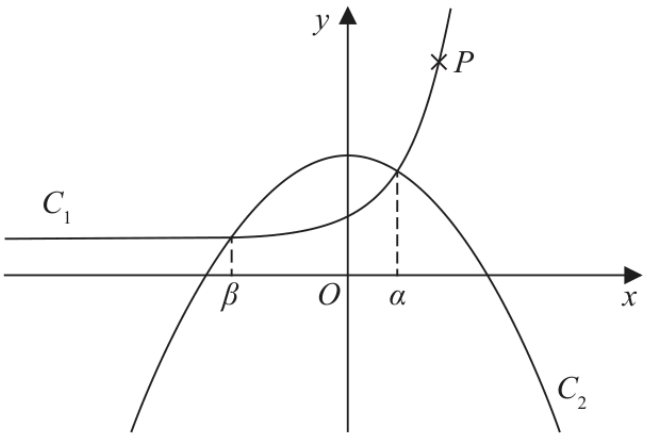


Figure 3

Figure 3 shows a sketch of curve C_1 with equation $y = 5e^{x-1} + 3$

and curve C_2 with equation $y = 10 - x^2$

The point P lies on C_1 and has y coordinate 18

(a) Find the x coordinate of P, writing your answer in the form $\ln k$, where k is a constant to be found.

(3)

The curve C_1 meets the curve C_2 at $x = \alpha$ and at $x = \beta$, as shown in Figure 3.

(b) Using a suitable interval and a suitable function that should be stated, show that to 3 decimal places $\alpha = 1.134$

(3)

The iterative equation

$$x_{n+1} = -\sqrt{7 - 5e^{x_n - 1}}$$

is used to find an approximation to β.

Using this iterative formula with $x_1 = -3$

(c) find the value of x_2 and the value of β, giving each answer to 6 decimal places.

(3)

Question 6

8 marks | Numerical Methods

6.

$$f(x) = x\cos\left(\frac{x}{3}\right) \qquad x > 0$$

(a) Find $f'(x)$

(2)

(b) Show that the equation $f'(x) = 0$ can be written as

$$x = k\arctan\left(\frac{k}{x}\right)$$

where k is an integer to be found.

(2)

(c) Starting with $x_1 = 2.5$ use the iteration formula

$$x_{n+1} = k\arctan\left(\frac{k}{x_n}\right)$$

with the value of k found in part (b), to calculate the values of x_2 and x_6 giving your answers to 3 decimal places.

(2)

(d) Using a suitable interval and a suitable function that should be stated, show that a root of $f'(x) = 0$ is 2.581 correct to 3 decimal places.

(2)

Question 1

7 marks | Numerical Methods

1. The curve C has equation

$$y = x^2 \cos\left(\frac{1}{2}x\right) \qquad 0 < x \leqslant \pi$$

The curve has a stationary point at the point P.

(a) Show, using calculus, that the x coordinate of P is a solution of the equation

$$x = 2\arctan\left(\frac{4}{x}\right)$$

(4)

Using the iteration formula

$$x_{n+1} = 2\arctan\left(\frac{4}{x_n}\right) \qquad x_1 = 2$$

(b) find the value of x_2 and the value of x_6, giving your answers to 3 decimal places.

(3)

Question 9

9 marks | Numerical Methods

9.

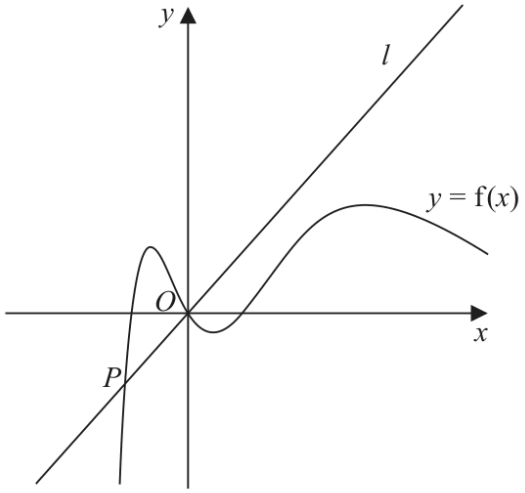


Figure 3

Figure 3 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x(x^2 - 4)\mathrm{e}^{-\frac{1}{2}x}$$

(a) Find $\mathrm{f}'(x)$.

(2)

The line l is the normal to the curve at O and meets the curve again at the point P.

The point P lies in the 3rd quadrant, as shown in Figure 3.

(b) Show that the x coordinate of P is a solution of the equation

$$x = -\frac{1}{2}\sqrt{16 + \mathrm{e}^{\frac{1}{2}x}}$$

(4)

(c) Using the iterative formula

$$x_{n+1} = -\frac{1}{2}\sqrt{16 + \mathrm{e}^{\frac{1}{2}x_n}} \quad \text{with } x_1 = -2$$

find, to 4 decimal places,

(i) the value of x_2

(ii) the x coordinate of P.

(3)

Question 8

12 marks | Numerical Methods

8.

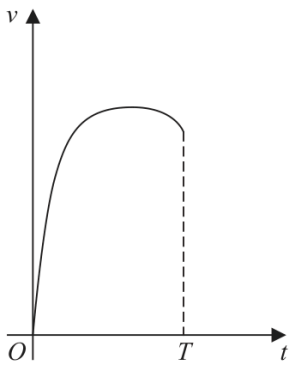


Figure 4

Figure 4 is a graph showing the velocity of a sprinter during a 100 m race.

The sprinter's velocity during the race, $v\,\text{m s}^{-1}$, is modelled by the equation

$$v = 12 - \text{e}^{t-10} - 12\text{e}^{-0.75t} \qquad t \geqslant 0$$

where t seconds is the time after the sprinter begins to run.

According to the model,

(a) find, using calculus, the sprinter's maximum velocity during the race.

(5)

Given that the sprinter runs 100 m in T seconds, such that

$$\int_0^T v\,\text{d}t = 100$$

(b) show that T is a solution of the equation

$$T = \frac{1}{12}\left(116 - 16\text{e}^{-0.75T} + \text{e}^{T-10} - \text{e}^{-10}\right)$$

(4)

The iteration formula

$$T_{n+1} = \frac{1}{12}\left(116 - 16\text{e}^{-0.75T_n} + \text{e}^{T_n-10} - \text{e}^{-10}\right)$$

is used to find an approximate value for T

Using this iteration formula with $T_1 = 10$

(c) find, to 4 decimal places,

(i) the value of T_2

(ii) the time taken by the sprinter to run the race, according to the model.

(3)

Question 9

11 marks | Numerical Methods

9.

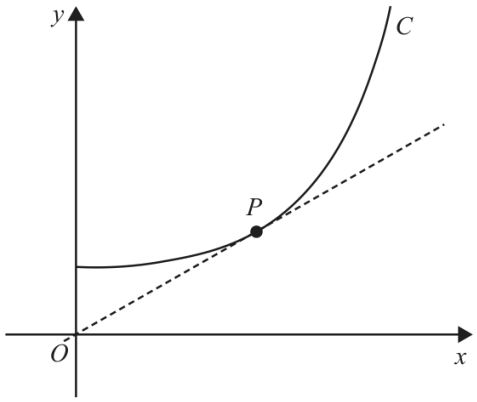


Figure 3

Figure 3 shows a sketch of part of the curve C with equation

$$y = \sqrt{3 + 4e^{x^2}} \qquad x \geqslant 0$$

(a) Find $\dfrac{dy}{dx}$, giving your answer in simplest form.

(2)

The point P with x coordinate α lies on C.

Given that the tangent to C at P passes through the origin, as shown in Figure 3,

(b) show that $x = \alpha$ is a solution of the equation

$$4x^2e^{x^2} - 4e^{x^2} - 3 = 0$$

(3)

(c) Hence show that α lies between 1 and 2

(2)

(d) Show that the equation in part (b) can be written in the form

$$x = \frac{1}{2}\sqrt{4 + 3e^{-x^2}}$$

(1)

The iteration formula

$$x_{n+1} = \frac{1}{2}\sqrt{4 + 3e^{-x_n^2}}$$

with $x_1 = 1$ is used to find an approximation for α.

(e) Use the iteration formula to find, to 4 decimal places, the value of

(i) x_3

(ii) α

(3)

Question 1

5 marks | Numerical Methods

1. $g(x) = x^6 + 2x - 1000$

(a) Show that $g(x) = 0$ has a root α in the interval $[3, 4]$

(2)

Using the iteration formula

$$x_{n+1} = \sqrt[6]{1000 - 2x_n} \qquad \text{with } x_1 = 3$$

(b) (i) find, to 4 decimal places, the value of x_2

(ii) find, by repeated iteration, the value of α.
Give your answer to 4 decimal places.

(3)

Question 1

5 marks | Numerical Methods

1. A curve has equation $y = f(x)$ where

$$f(x) = x^2 - 5x + e^x \qquad x \in \mathbb{R}$$

(a) Show that the equation $f(x) = 0$ has a root, α, in the interval $[1, 2]$

(2)

The iterative formula

$$x_{n+1} = \sqrt{5x_n - e^{x_n}}$$

with $x_1 = 1$ is used to find an approximate value for the root α.

(b) (i) Find the value of x_2 to 4 decimal places.

(ii) Find, by repeated iteration, the value of α, giving your answer to 4 decimal places.

(3)

Question 2

6 marks | Numerical Methods

2. A curve has equation $y = f(x)$ where

$$f(x) = x^4 - 5x^2 + 4x - 7 \qquad x \in \mathbb{R}$$

(a) Show that the equation $f(x) = 0$ has a root, α, in the interval $[2, 3]$

(2)

(b) Show that the equation $f(x) = 0$ can be written as

$$x = \sqrt[3]{\frac{5x^2 - 4x + 7}{x}}$$

(1)

The iterative formula

$$x_{n+1} = \sqrt[3]{\frac{5x_n^{\,2} - 4x_n + 7}{x_n}}$$

is used to find α

(c) Starting with $x_1 = 2$ and using the iterative formula,

(i) find, to 4 decimal places, the value of x_2

(ii) find, to 4 decimal places, the value of α

(3)

Question 1

6 marks | Numerical Methods

1. $f(x) = 2\sec x + 6x - 3 \qquad 0 < x < \frac{\pi}{2}$

The equation $f(x) = 0$ has a single root α

(a) Show that $0.1 < \alpha < 0.2$

(2)

(b) Show that α is a solution of

$$x = \frac{1}{2} - \frac{1}{3\cos x}$$

(1)

The iterative formula

$$x_{n+1} = \frac{1}{2} - \frac{1}{3\cos x_n}$$

is used to find α

(c) Starting with $x_1 = 0.15$ and using the iterative formula,

(i) find, to 4 decimal places, the value of x_2

(ii) find, to 4 decimal places, the value of α

(3)

Question 7

9 marks | Numerical Methods

7: A continuous curve has equation

$$y = e^{-x^2} \sin 3x \qquad 0 \leqslant x \leqslant \frac{\pi}{3}$$

The curve has a stationary point at the point P.

(a) Show, using calculus, that the x coordinate of P is a solution of the equation

$$x = \frac{1}{3} \arctan\left(\frac{3}{2x}\right)$$

(4)

Using the iteration formula

$$x_{n+1} = \frac{1}{3} \arctan\left(\frac{3}{2x_n}\right) \qquad x_1 = 0.4$$

(b) find the value of

(i) x_2

(ii) x_4

giving your answers to 4 decimal places.

(3)

(c) Using a suitable interval and a suitable function which should be stated, show that the x coordinate of P is 0.430 correct to 3 decimal places.

(2)

Question 2

7 marks | Numerical Methods

2.

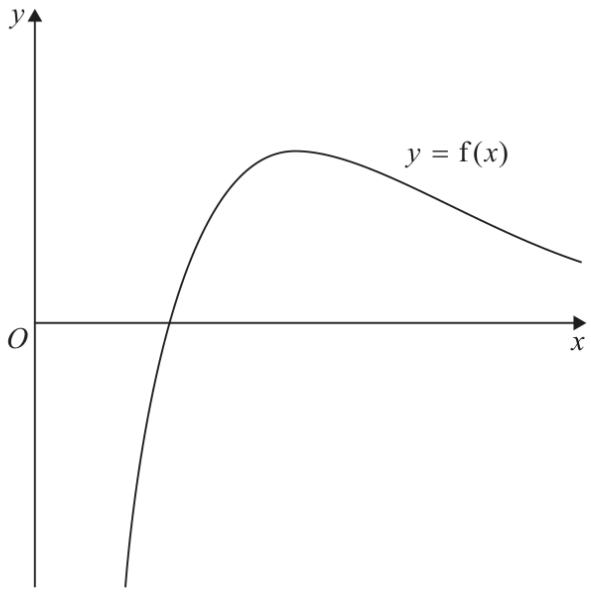


Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{2x^2 + 3x - 4}{\mathrm{e}^x} - \frac{1}{x^2} \qquad x \in \mathbb{R} \quad x \neq 0$$

(a) Show that $\mathrm{f}(x) = 0$ has a root α in the interval $[1, 2]$

(2)

(b) Show that the equation $\mathrm{f}(x) = 0$ can be written in the form

$$x = \sqrt[3]{\frac{\mathrm{e}^x + 4x^2}{2x + 3}}$$

(2)

Using the iteration formula

$$x_{n+1} = \sqrt[3]{\frac{\mathrm{e}^{x_n} + 4x_n^{\,2}}{2x_n + 3}} \qquad \text{with } x_1 = 1$$

find, to 4 decimal places,

(c) (i) the value of x_3

(ii) the value of α

(3)

Question 1

6 marks | Numerical Methods

1. $$g(x) = 3x - 5 + 5\sin\left(x^2\right)$$

where x is measured in radians.

(a) Show that $g(x) = 0$ has a root α in the interval $[0.7, 0.8]$

(2)

(b) Show that the equation $g(x) = 0$ can be written in the form

$$x = \left[\arcsin\left(1 - px\right)\right]^{\frac{1}{2}}$$

where p is a constant to be found.

(1)

Using the iteration formula

$$x_{n+1} = \left[\arcsin\left(1 - px_n\right)\right]^{\frac{1}{2}}$$

with the value of p found in part (b) and starting with $x_1 = 0.7$

(c) (i) find, to 4 decimal places, the value of x_2

(ii) find, by repeated iteration, the value of α, giving your answer to 4 decimal places.

(3)

Question 1

8 marks | Numerical Methods

1. A curve has equation $y = f(x)$ where

$$f(x) = \frac{2}{3}x^3 - 3x^2 + 5x + e^{2x}(1 - 2x) \qquad x \in \mathbb{R}$$

(a) Find $f'(x)$ giving the answer in simplest form.

(3)

The equation $f'(x) = 0$ has a single root, α

(b) Show that α lies in the interval $[0, 1]$

(2)

Given that the equation $f'(x) = 0$ can be written as

$$x = \frac{2x^2 + 5}{4e^{2x} + 6}$$

(c) use the iteration formula

$$x_{n+1} = \frac{2x_n^{\ 2} + 5}{4e^{2x_n} + 6} \qquad \text{with } x_1 = 0$$

to find, to 4 decimal places,

(i) the value of x_3

(ii) the value of α

(3)

Question 7

11 marks | Numerical Methods

Main topic: Applications of Differentiation

7.

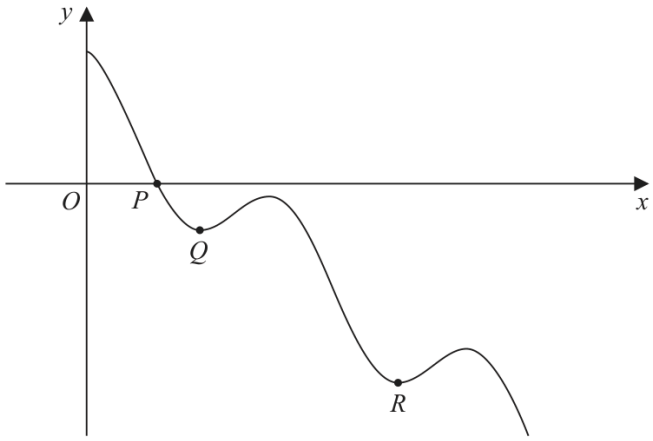


Figure 3

Figure 3 shows a sketch of part of the curve with equation

$$y = 2\cos 3x - 3x + 4 \qquad x > 0$$

where x is measured in radians.

The curve crosses the x-axis at the point P, as shown in Figure 3.

Given that the x coordinate of P is α,

(a) show that α lies between 0.8 and 0.9

(2)

The iteration formula

$$x_{n+1} = \frac{1}{3}\arccos(1.5x_n - 2)$$

can be used to find an approximate value for α.

(b) Using this iteration formula with $x_1 = 0.8$ find, to 4 decimal places, the value of

(i) x_2

(ii) x_5

(3)

The point Q and the point R are local minimum points on the curve, as shown in Figure 3.

Given that the x coordinates of Q and R are β and λ respectively, and that they are the two smallest values of x at which local minima occur,

(c) find, using calculus, the exact value of β and the exact value of λ.

(6)

Question 5

9 marks | Numerical Methods

Main topic: Applications of Differentiation

5.

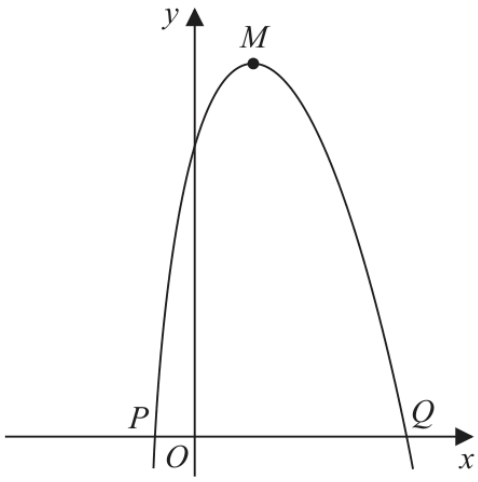


Figure 1

Figure 1 shows a sketch of part of the curve with equation

$$y = 6\ln(2x + 3) - \frac{1}{2}x^2 + 4 \qquad x > -\frac{3}{2}$$

The curve cuts the negative x-axis at the point P, as shown in Figure 1.

(a) Show that the x coordinate of P lies in the interval $[-1.25, -1.2]$

(2)

The curve cuts the positive x-axis at the point Q, also shown in Figure 1.

Using the iterative formula

$$x_{n+1} = \sqrt{12\ln(2x_n + 3) + 8} \quad \text{with} \quad x_1 = 6$$

(b) (i) find, to 4 decimal places, the value of x_2

(ii) find, by continued iteration, the x coordinate of Q. Give your answer to 4 decimal places.

(3)

The curve has a maximum turning point at M, as shown in Figure 1.

(c) Using calculus and showing each stage of your working, find the x coordinate of M.

(4)

Question 5

11 marks | Numerical Methods

Main topic: Log Graphs & Exponential Models

5.

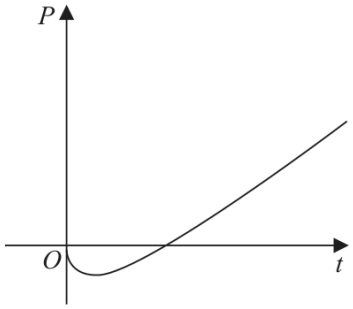


Figure 2

The profit made by a company, £P million, t years after the company started trading, is modelled by the equation

$$P = \frac{4t-1}{10} + \frac{3}{4}\ln\left[\frac{t+1}{(2t+1)^2}\right]$$

The graph of P against t is shown in Figure 2.

According to the model,

(a) show that exactly one year after it started trading, the company had made a loss of approximately £830 000

(2)

A manager of the company wants to know the value of t for which $P = 0$

(b) Show that this value of t occurs in the interval $[6, 7]$

(2)

(c) Show that the equation $P = 0$ can be expressed in the form

$$t = \frac{1}{4} + \frac{15}{8}\ln\left[\frac{(2t+1)^2}{t+1}\right]$$

(2)

(d) Using the iteration formula

$$t_{n+1} = \frac{1}{4} + \frac{15}{8}\ln\left[\frac{(2t_n+1)^2}{t_n+1}\right] \quad \text{with} \quad t_1 = 6$$

find the value of t_2 and the value of t_6, giving your answers to 3 decimal places.

(3)

(e) Hence find, according to the model, how many months it takes in total, from when the company started trading, for it to make a profit.

(2)

Question 8

10 marks | Numerical Methods

Main topic: Log Graphs & Exponential Models

8.

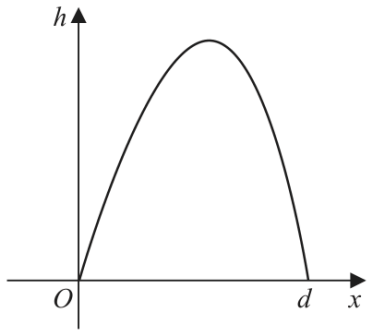


Figure 4

Figure 4 is a graph showing the path of a golf ball after the ball has been hit until it first hits the ground.

The vertical height, h metres, of the ball above the ground has been plotted against the horizontal distance travelled, x metres, measured from where the ball was hit.

The ball travels a horizontal distance of d metres before it first hits the ground.

The ball is modelled as a particle travelling in a vertical plane above horizontal ground.

The path of the ball is modelled by the equation

$$h = 1.5x - 0.5x\,e^{0.02x} \qquad 0 \leqslant x \leqslant d$$

Use the model to answer parts (a), (b) and (c).

(a) Find the value of d, giving your answer to 2 decimal places.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(3)

(b) Show that the maximum value of h occurs when

$$x = 50\ln\left(\frac{150}{x + 50}\right)$$

(4)

Using the iteration formula

$$x_{n+1} = 50\ln\left(\frac{150}{x_n + 50}\right) \qquad \text{with } x_1 = 30$$

(c) (i) find the value of x_2 to 2 decimal places,

(ii) find, by repeated iteration, the horizontal distance travelled by the golf ball before it reaches its maximum height. Give your answer to 2 decimal places.

(3)

Question 8

12 marks | Numerical Methods

Main topic: Log Graphs & Exponential Models

8.

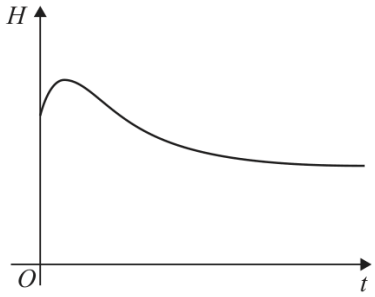


Figure 4

The heart rate of a horse is being monitored.
The heart rate H, measured in beats per minute (bpm), is modelled by the equation

$$H = 32 + 40e^{-0.2t} - 20e^{-0.9t}$$

where t minutes is the time after monitoring began.

Figure 4 is a sketch of H against t.

Use the equation of the model to answer parts (a) to (e).

(a) State the initial heart rate of the horse.

(1)

In the long term, the heart rate of the horse approaches L bpm.

(b) State the value of L.

(1)

The heart rate of the horse reaches its maximum value after T minutes.

(c) Find the value of T, giving your answer to 3 decimal places.

(*Solutions based entirely on calculator technology are not acceptable.*)

(5)

The heart rate of the horse is 37 bpm after M minutes.

(d) Show that M is a solution of the equation

$$t = 5\ln\left(\frac{8}{1 + 4e^{-0.9t}}\right)$$

(2)

Using the iteration formula

$$t_{n+1} = 5\ln\left(\frac{8}{1 + 4e^{-0.9t_n}}\right) \quad \text{with} \quad t_1 = 10$$

(e) (i) find, to 4 decimal places, the value of t_2

(ii) find, to 4 decimal places, the value of M

(3)