Elite Mathematics
Edexcel International A Level

IAL PURE 3 · WMA13

WMA13/01 October 2024 Worked Solutions

October 2024 paper rebuilt with the worked solution after each question.

9	2020–2026	Single-paper solutions
paper questions	19 exam sessions	checked by Dr Eslam



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PAST PAPER

WMA13/01 October 2024

October 2024

9	75	1 h 30
questions	marks	exam time

Each question is followed by its worked solution and final answer.

WMA13/01 OCTOBER 2024

Question 1

5 marks | Trig Equations & R-Form

1. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

Solve, for $0 < \theta \leqslant 360°$, the equation

$$3\tan^2\theta + 7\sec\theta - 3 = 0$$

giving your answers to one decimal place.

(5)

WORKED SOLUTION

Question 1 · WMA13/01 October 2024

1 **Part (main)**

Using $\tan^2\theta = \sec^2\theta - 1$ gives

$$3\sec^2\theta + 7\sec\theta - 6 = 0 \quad \Longrightarrow \quad (3\sec\theta - 2)(\sec\theta + 3) = 0.$$

The value $\sec\theta = 2/3$ would give $\cos\theta = 3/2$, impossible for a real angle. Thus $\cos\theta = -1/3$. In $0 < \theta \leq 360°$, the two angles are $\arccos(-1/3) = 109.4712\ldots°$ and $360° - 109.4712\ldots°$.

$$\boxed{\theta = 109.5°,\ 250.5°}$$

FINAL ANSWER

(main) $\theta = 109.5°,\ 250.5°$

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Question 2

5 marks | Applications of Differentiation

2.

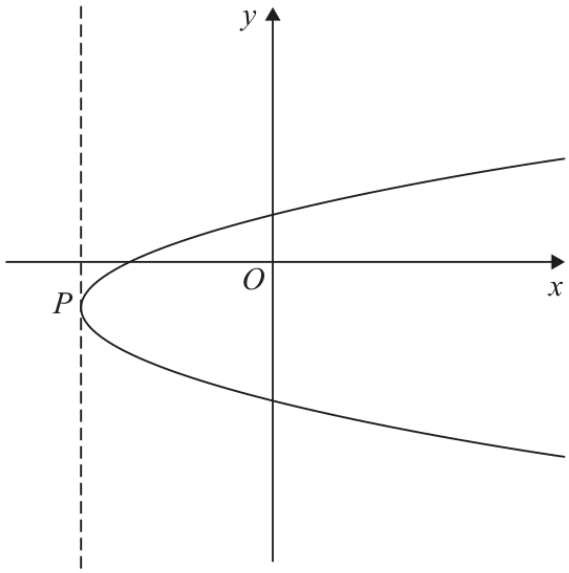


Figure 1

Figure 1 shows a sketch of the curve with equation

$$x = 2y^2 + 5y - 6$$

(a) Find $\frac{dy}{dx}$ in terms of y.

(2)

The point P lies on the curve and is shown in Figure 1.

Given that the tangent to the curve at P is parallel to the y-axis,

(b) find the coordinates of P.

(3)

WORKED SOLUTION

Question 2 · WMA13/01 October 2024

1 **Part (a)**

Differentiate $x = 2y^2 + 5y - 6$ with respect to y:

$$\frac{dx}{dy} = 4y + 5, \qquad \frac{dy}{dx} = \frac{1}{4y+5}, \quad y \neq -\frac{5}{4}.$$

$$\boxed{\frac{dy}{dx} = \frac{1}{4y+5}}$$

2 Part (b)

A tangent parallel to the y-axis is vertical, so use $dx/dy = 0$:

$$4y + 5 = 0 \quad \Longrightarrow \quad y = -\frac{5}{4}.$$

Substituting into the curve gives

$$x = 2\left(-\frac{5}{4}\right)^2 + 5\left(-\frac{5}{4}\right) - 6 = \frac{25}{8} - \frac{25}{4} - 6 = -\frac{73}{8}.$$

$$\boxed{P = \left(-\frac{73}{8}, -\frac{5}{4}\right)}$$

FINAL ANSWER

(a) $\frac{dy}{dx} = \frac{1}{4y+5}$ (b) $P = \left(-\frac{73}{8}, -\frac{5}{4}\right)$

Question 3

7 marks | Modulus

3.

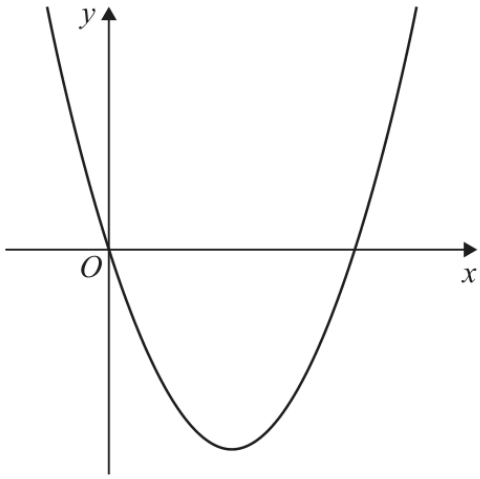


Figure 2

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 2 shows a sketch of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 2x^2 - 10x \qquad x \in \mathbb{R}$$

(a) Solve the equation

$$\mathrm{f}(|x|) = 48$$

(3)

(b) Find the set of values of x for which

$$|\mathrm{f}(x)| \geqslant \frac{5}{2}x$$

(4)

1 Part (a)

Set $u = |x| \geq 0$. Then

$$2u^2 - 10u = 48 \quad \Longrightarrow \quad u^2 - 5u - 24 = (u-8)(u+3) = 0.$$

Reject $u = -3$ since $u \geq 0$. Therefore $|x| = 8$.

$$\boxed{x = -8 \text{ or } x = 8}$$

2 Part (b)

For $x \leq 0$, the right side $\frac{5}{2}x \leq 0$ and the left side is nonnegative, so every such x satisfies the inequality. For $0 < x < 5$, $f(x) < 0$ and

$$10x - 2x^2 \geq \frac{5}{2}x \quad \Longleftrightarrow \quad x\left(\frac{15}{2} - 2x\right) \geq 0 \quad \Longrightarrow \quad 0 < x \leq \frac{15}{4}.$$

For $x \geq 5$, $f(x) \geq 0$ and

$$2x^2 - 10x \geq \frac{5}{2}x \quad \Longleftrightarrow \quad x\left(2x - \frac{25}{2}\right) \geq 0 \quad \Longrightarrow \quad x \geq \frac{25}{4}.$$

Combining the cases retains equality at both boundary points.

$$\boxed{x \leq \frac{15}{4} \text{ or } x \geq \frac{25}{4}}$$

FINAL ANSWER

(a) $x = -8$ or $x = 8$ (b) $x \leq \frac{15}{4}$ or $x \geq \frac{25}{4}$

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Question 4

6 marks | Log Graphs & Exponential Models

4. The number of bacteria on a surface is being monitored.

The number of bacteria, N, on the surface, t hours after monitoring began is modelled by the equation

$$\log_{10} N = 0.35t + 2$$

Use the equation of the model to answer parts (a) to (c).

(a) Find the initial number of bacteria on the surface.

(1)

(b) Show that the equation of the model can be written in the form

$$N = ab^t$$

where a and b are constants to be found. Give the value of b to 2 decimal places.

(3)

(c) Hence find the rate of growth of bacteria on the surface exactly 5 hours after monitoring began.

(2)

WORKED SOLUTION

Question 4 · WMA13/01 October 2024

1 **Part (a)**

At $t = 0$, $\log_{10} N = 2$, so $N = 10^2$.

$$\boxed{N(0) = 100}$$

2 **Part (b)**

Exponentiating base 10 gives

$$N = 10^{0.35t+2} = 10^2(10^{0.35})^t = 100(10^{0.35})^t.$$

Thus $a = 100$ and $b = 10^{0.35} = 2.238721\ldots$, which is 2.24 to 2 decimal places.

$$\boxed{a = 100, \qquad b = 2.24}$$

3 Part (c)

Retain the unrounded base $10^{0.35}$ when differentiating:

$$\frac{dN}{dt} = 100(10^{0.35})^t \ln(10^{0.35}) = 0.35\ln(10)\,10^{0.35t+2}.$$

At $t = 5$ this is $0.35\ln(10)\,10^{3.75} = 4531.9356\ldots$ bacteria per hour.

$$\boxed{\left.\frac{dN}{dt}\right|_{t=5} = 4.53 \times 10^3 \text{ bacteria per hour}}$$

FINAL ANSWER

(a) $N(0) = 100$ (b) $a = 100, \qquad b = 2.24$ (c) $\left.\frac{dN}{dt}\right|_{t=5} = 4.53 \times 10^3$ bacteria per hour

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Question 5

8 marks | Trig Identities & Exact Values

5. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that $\sin 3x$ can be written in the form

$$P\sin x + Q\sin^3 x$$

where P and Q are constants to be found.

(4)

(b) Hence or otherwise, solve, for $0 < \theta \leqslant 360°$, the equation

$$2\sin 3\theta = 5\sin 2\theta$$

giving your answers, in degrees, to one decimal place as appropriate.

(4)

WORKED SOLUTION

Question 5 · WMA13/01 October 2024

1 **Part (a)**

Expand $\sin(2x + x)$ using the addition formula:

$$\sin 3x = \sin 2x \cos x + \cos 2x \sin x = 2\sin x \cos^2 x + (1 - 2\sin^2 x)\sin x.$$

Now use $\cos^2 x = 1 - \sin^2 x$:

$$\sin 3x = 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x = 3\sin x - 4\sin^3 x.$$

$$\boxed{P = 3, \quad Q = -4}$$

2

Part (b)

Substitute the identity and $\sin 2\theta = 2\sin\theta\cos\theta$:

$$6\sin\theta - 8\sin^3\theta = 10\sin\theta\cos\theta.$$

Factor without dividing by $\sin\theta$:

$$2\sin\theta(4\cos^2\theta - 5\cos\theta - 1) = 0.$$

If $\sin\theta = 0$, the permitted solutions are $180°$ and $360°$ (not $0°$). Otherwise $\cos\theta = (5 \pm \sqrt{41})/8$. The plus value exceeds 1 and is rejected. The minus value gives $\theta = 100.1013\ldots°$ or $259.8986\ldots°$.

$$\boxed{\theta = 100.1°,\ 180°,\ 259.9°,\ 360°}$$

✓ **FINAL ANSWER**

(a) $P = 3, \quad Q = -4$ (b) $\theta = 100.1°,\ 180°,\ 259.9°,\ 360°$

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Question 6

8 marks | Functions, Composition & Inverses

6. The functions f and g are defined by

$$\mathrm{f}(x) = 6 - \frac{21}{2x+3} \qquad x \geqslant 0$$

$$\mathrm{g}(x) = x^2 + 5 \qquad x \in \mathbb{R}$$

(a) Find $\mathrm{gf}(2)$

(2)

(b) Find f^{-1}

(3)

(c) Solve the equation

$$\mathrm{gg}(x) = 126$$

(3)

WORKED SOLUTION

Question 6 · WMA13/01 October 2024

1 **Part (a)**

First apply f: $f(2) = 6 - 21/(2\cdot 2+3) = 3$. Then apply g: $gf(2) = g(3) = 3^2 + 5$.

$$\boxed{gf(2) = 14}$$

2 **Part (b)**

Let $y = 6 - 21/(2x+3)$. Then

$$(6-y)(2x+3) = 21, \qquad x = \frac{1}{2}\left(\frac{21}{6-y} - 3\right) = \frac{3(y+1)}{2(6-y)}.$$

Interchange x, y. On the original domain $x \geq 0$, f increases from $f(0) = -1$ towards 6, never attaining 6. Thus the inverse domain is $-1 \leq x < 6$.

$$\boxed{f^{-1}(x) = \frac{3(x+1)}{2(6-x)}, \quad -1 \leq x < 6}$$

3 Part (c)

$$gg(x) = (x^2+5)^2 + 5 = 126 \quad \Longrightarrow \quad (x^2+5)^2 = 121.$$

Since $x^2 + 5 > 0$, take $x^2 + 5 = 11$, not -11. Therefore $x^2 = 6$.

$$\boxed{x = \pm\sqrt{6}}$$

FINAL ANSWER

(a) $gf(2) = 14$ (b) $f^{-1}(x) = \frac{3(x+1)}{2(6-x)}, \quad -1 \le x < 6$ (c) $x = \pm\sqrt{6}$

Question 7

10 marks | Graph Transformations

7.

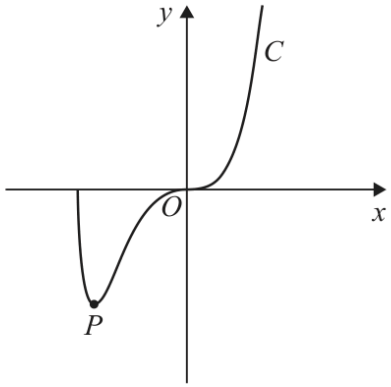


Figure 3

The curve C has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x^3\sqrt{4x+7} \qquad x \geqslant -\frac{7}{4}$$

(a) Show that

$$\mathrm{f}'(x) = \frac{kx^2(2x+3)}{\sqrt{4x+7}}$$

where k is a constant to be found.

(4)

The point P, shown in Figure 3, is the minimum turning point on C.

(b) Find the coordinates of P.

(2)

(c) Hence find the range of the function g defined by

$$\mathrm{g}(x) = -4\,\mathrm{f}(x) \qquad x \geqslant -\frac{7}{4}$$

(2)

The point Q with coordinates $\left(\frac{1}{2}, \frac{3}{8}\right)$ lies on C.

(d) Find the coordinates of the point to which Q is mapped when C is transformed to the curve with equation

$$y = 40\,\mathrm{f}\left(x - \frac{3}{2}\right) - 8$$

(2)

1 Part (a)

For $x > -7/4$, apply the product and chain rules:

$$f'(x) = 3x^2\sqrt{4x+7} + x^3\frac{2}{\sqrt{4x+7}} = \frac{3x^2(4x+7)+2x^3}{\sqrt{4x+7}} = \frac{7x^2(2x+3)}{\sqrt{4x+7}}.$$

$$\boxed{k = 7}$$

2 Part (b)

The interior stationary values satisfy $x^2(2x+3)=0$, so $x=0$ or $x=-3/2$. Near $x=-3/2$, the derivative changes from negative to positive; near $x=0$ it stays positive on both sides. Thus the minimum turning point is at $x=-3/2$.

$$f(-3/2) = (-3/2)^3\sqrt{-6+7} = -27/8.$$

$$\boxed{P = \left(-\frac{3}{2}, -\frac{27}{8}\right)}$$

3 Part (c)

The endpoint has $f(-7/4)=0$, the global minimum is $-27/8$ and $f(x)$ tends to infinity as x increases. Hence the range of f is $[-27/8, \infty)$. Multiplication by -4 reverses the inequality and makes the upper endpoint $27/2$.

$$\boxed{g(x) \le \frac{27}{2}}$$

4 Part (d)

The transformation is a shift right by $3/2$, vertical stretch by 40, then shift down by 8:

$$(x, y) \mapsto (x + 3/2,\, 40y - 8).$$

Therefore $Q = (1/2, 3/8)$ maps to $(1/2 + 3/2,\ 40\cdot 3/8 - 8)$.

$$\boxed{Q \mapsto (2, 7)}$$

FINAL ANSWER

(a) $k = 7$ (b) $P = \left(-\frac{3}{2}, -\frac{27}{8}\right)$ (c) $g(x) \le \frac{27}{2}$ (d) $Q \mapsto (2, 7)$

Question 8

12 marks | Log Graphs & Exponential Models

8.

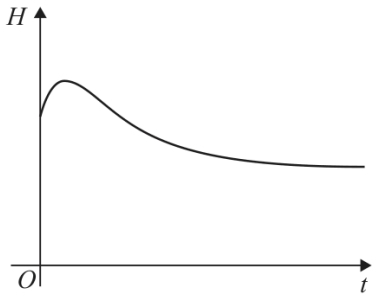


Figure 4

The heart rate of a horse is being monitored.
The heart rate H, measured in beats per minute (bpm), is modelled by the equation

$$H = 32 + 40e^{-0.2t} - 20\,e^{-0.9t}$$

where t minutes is the time after monitoring began.

Figure 4 is a sketch of H against t.

Use the equation of the model to answer parts (a) to (e).

(a) State the initial heart rate of the horse.

(1)

In the long term, the heart rate of the horse approaches L bpm.

(b) State the value of L.

(1)

The heart rate of the horse reaches its maximum value after T minutes.

(c) Find the value of T, giving your answer to 3 decimal places.

(*Solutions based entirely on calculator technology are not acceptable.*)

(5)

The heart rate of the horse is 37 bpm after M minutes.

(d) Show that M is a solution of the equation

$$t = 5\ln\left(\frac{8}{1 + 4e^{-0.9t}}\right)$$

(2)

Using the iteration formula

$$t_{n+1} = 5\ln\left(\frac{8}{1 + 4e^{-0.9t_n}}\right) \quad \text{with} \quad t_1 = 10$$

(e) (i) find, to 4 decimal places, the value of t_2

(ii) find, to 4 decimal places, the value of M

(3)

1 Part (a)

At $t = 0$, $H = 32 + 40 - 20 = 52$.

$$\boxed{H(0) = 52 \text{ bpm}}$$

2 Part (b)

As $t \to \infty$, both decaying exponential terms tend to zero.

$$\boxed{L = 32}$$

3 Part (c)

$$\frac{dH}{dt} = -8e^{-0.2t} + 18e^{-0.9t}.$$

At the maximum,

$$18e^{-0.9T} = 8e^{-0.2T} \quad \Longrightarrow \quad e^{0.7T} = \frac{9}{4} \quad \Longrightarrow \quad T = \frac{10}{7}\ln\frac{9}{4} = 1.1584717\ldots.$$

Also $H''(t) = 1.6e^{-0.2t} - 16.2e^{-0.9t}$. At the stationary point, $e^{-0.9T} = (4/9)e^{-0.2T}$, so $H''(T) = -5.6e^{-0.2T} < 0$, confirming a maximum.

$$\boxed{T = 1.158 \text{ minutes}}$$

4 Part (d)

For $H = 37$,

$$5 = 40e^{-0.2t} - 20e^{-0.9t} \quad \Longrightarrow \quad 8e^{-0.2t} = 1 + 4e^{-0.9t}.$$

Taking logarithms gives

$$-0.2t = \ln\left(\frac{1 + 4e^{-0.9t}}{8}\right) \quad \Longrightarrow \quad t = -5\ln\left(\frac{1 + 4e^{-0.9t}}{8}\right).$$

Using $-\ln u = \ln(1/u)$ for $u > 0$ then gives

$$t = 5\ln\left(\frac{8}{1 + 4e^{-0.9t}}\right).$$

$$\boxed{t = 5\ln\left(\frac{8}{1 + 4e^{-0.9t}}\right)}$$

5 **Part (e)**

Starting with $t_1 = 10$,

$$t_2 = 5\ln\left(\frac{8}{1+4e^{-9}}\right) = 10.3947401213\ldots.$$

Continuing without rounding intermediate values gives $t_3 = 10.3954778346\ldots$, $t_4 = 10.3954789826\ldots$ and $t_5 = 10.3954789844\ldots$. Thus (i) $t_2 = 10.3947$, and (ii) $M = 10.3955$ minutes, both to 4 decimal places.

$$\boxed{t_2 = 10.3947, \quad M = 10.3955 \text{ minutes}}$$

✓ **FINAL ANSWER**

(a) $H(0) = 52$ bpm (b) $L = 32$ (c) $T = 1.158$ minutes (d) $t = 5\ln\left(\frac{8}{1+4e^{-0.9t}}\right)$ (e) $t_2 = 10.3947, \quad M = 10.3955$ minutes

Question 9

14 marks | Integration

9.

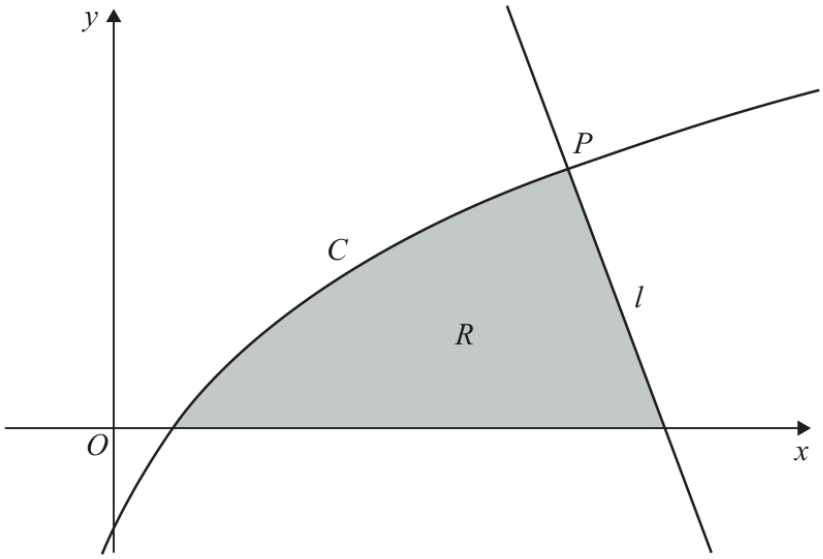


Figure 5

Figure 5 shows a sketch of part of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{6x^2 + 4x - 2}{2x + 1} \qquad x > -\frac{1}{2}$$

(a) Find $\mathrm{f}'(x)$, giving the answer in simplest form.

(3)

The line l is the normal to C at the point $P(2, 6)$

(b) Show that an equation for l is

$$16y + 5x = 106$$

(3)

(c) Write $\mathrm{f}(x)$ in the form $Ax + B + \dfrac{D}{2x + 1}$ where A, B and D are constants.

(3)

The region R, shown shaded in Figure 5, is bounded by C, l and the x-axis.

(d) Use algebraic integration to find the exact area of R, giving your answer in the form $P + Q\ln 3$, where P and Q are rational constants.

(*Solutions based entirely on calculator technology are not acceptable.*)

(5)

1 Part (a)

With $u = 6x^2 + 4x - 2$ and $v = 2x + 1$, the quotient rule gives

$$f'(x) = \frac{(12x + 4)(2x + 1) - 2(6x^2 + 4x - 2)}{(2x + 1)^2} = \frac{12x^2 + 12x + 8}{(2x + 1)^2}.$$

$$\boxed{f'(x) = \frac{4(3x^2 + 3x + 2)}{(2x + 1)^2}}$$

2 Part (b)

At $x = 2$, $f'(2) = 80/25 = 16/5$, so the normal gradient is $-5/16$. Through $P = (2, 6)$,

$$y - 6 = -\frac{5}{16}(x - 2) \quad \Longrightarrow \quad 16y - 96 = -5x + 10.$$

$$\boxed{16y + 5x = 106}$$

3 Part (c)

Multiplying the proposed identity by $2x + 1$ gives

$$6x^2 + 4x - 2 = (Ax + B)(2x + 1) + D = 2Ax^2 + (A + 2B)x + B + D.$$

Compare coefficients: $2A = 6$, $A + 2B = 4$, $B + D = -2$. Thus $f(x) = 3x + \frac{1}{2} - \frac{5}{2(2x+1)}$.

$$\boxed{A = 3, \quad B = \frac{1}{2}, \quad D = -\frac{5}{2}}$$

4 Part (d)

The curve meets the x-axis when $6x^2 + 4x - 2 = 2(3x - 1)(x + 1) = 0$. The root $x = -1$ is outside $x > -1/2$, leaving $x = 1/3$. The normal meets the axis at $x = 106/5$. Split the shaded region at P: a curve area from $1/3$ to 2, then a triangle.

$$\int_{1/3}^{2} f(x)\,dx = \left[\frac{3}{2}x^2 + \frac{1}{2}x - \frac{5}{4}\ln(2x + 1)\right]_{1/3}^{2} = 7 - \frac{1}{3} - \frac{5}{4}\ln\left(\frac{5}{5/3}\right) = \frac{20}{3} - \frac{5}{4}\ln 3.$$

The triangle has area $\frac{1}{2}(106/5 - 2) \cdot 6 = 288/5$. Adding gives the exact area of R.

$$\boxed{\text{Area}(R) = \frac{964}{15} - \frac{5}{4}\ln 3}$$

FINAL ANSWER

(a) $f'(x) = \frac{4(3x^2+3x+2)}{(2x+1)^2}$ (b) $16y + 5x = 106$ (c) $A = 3, \quad B = \frac{1}{2}, \quad D = -\frac{5}{2}$ (d) $\text{Area}(R) = \frac{964}{15} - \frac{5}{4}\ln 3$