Elite Mathematics
Edexcel International A Level

IAL PURE 3 · WMA13

WMA13/01 May/June 2024 Worked Solutions

May/June 2024 paper rebuilt with the worked solution after each question.

9	2020–2026	Single-paper solutions
paper questions	19 exam sessions	checked by Dr Eslam



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PAST PAPER

WMA13/01 May/June 2024

May/June 2024

9	75	1 h 30
questions	marks	exam time

Each question is followed by its worked solution and final answer.

WMA13/01 MAY/JUNE 2024

Question 1

6 marks | Modulus

1.

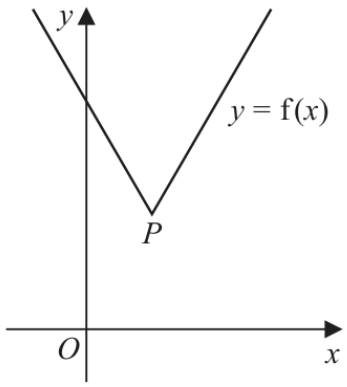


Figure 1

Figure 1 shows a sketch of the graph with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 2|x - 5| + 10$$

The point P, shown in Figure 1, is the vertex of the graph.

(a) State the coordinates of P

(2)

(b) Use algebra to solve

$$2|x - 5| + 10 > 6x$$

(*Solutions relying on calculator technology are not acceptable.*)

(2)

(c) Find the point to which P is mapped, when the graph with equation $y = \mathrm{f}(x)$ is transformed to the graph with equation $y = 3\mathrm{f}(x - 2)$

(2)

WORKED SOLUTION

Question 1 · WMA13/01 May/June 2024

1 Part (a)

The modulus is smallest when $x - 5 = 0$. At $x = 5$, $f(5) = 10$.

$$\boxed{P = (5, 10)}$$

2 Part (b)

For $x < 5$, $|x - 5| = 5 - x$, so

$$20 - 2x > 6x \quad \Longrightarrow \quad x < \frac{5}{2}.$$

This is consistent with $x < 5$. For $x \geq 5$, $|x - 5| = x - 5$, so $2x > 6x$, which requires $x < 0$ and is inconsistent with this branch. The equality point $x = 5/2$ is excluded because the inequality is strict.

$$\boxed{x < \frac{5}{2}}$$

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3 Part (c)

The transformation $y = 3f(x - 2)$ moves a point (x, y) to $(x + 2, 3y)$. Thus $P = (5, 10)$ moves to $(5 + 2, 3 \cdot 10)$.

$$\boxed{(7, 30)}$$

✓ FINAL ANSWER

(a) $P = (5, 10)$ (b) $x < \frac{5}{2}$ (c) $(7, 30)$

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Question 2

7 marks | Integration

2. $$\mathrm{g}(x) = \frac{2x^2 - 5x + 8}{x - 2}$$

(a) Write $\mathrm{g}(x)$ in the form

$$Ax + B + \frac{C}{x - 2}$$

where A, B and C are integers to be found.

(3)

(b) Hence use algebraic integration to show that

$$\int_4^8 \mathrm{g}(x)\,\mathrm{d}x = \alpha + \beta \ln 3$$

where α and β are integers to be found.

(4)

WORKED SOLUTION

Question 2 · WMA13/01 May/June 2024

Part (a)

Multiplying the proposed identity by $x - 2$ gives

$$2x^2 - 5x + 8 = (Ax + B)(x - 2) + C = Ax^2 + (B - 2A)x + C - 2B.$$

Thus $A = 2$, $B - 4 = -5$ and $C - 2(-1) = 8$, giving $B = -1$, $C = 6$. Hence $g(x) = 2x - 1 + 6/(x - 2)$, for $x \neq 2$.

$$\boxed{A = 2, \quad B = -1, \quad C = 6}$$

Part (b)

Since $x - 2 > 0$ throughout $[4, 8]$,

$$\int_4^8 g(x)\,dx = \left[x^2 - x + 6\ln(x - 2)\right]_4^8 = (64 - 8 + 6\ln 6) - (16 - 4 + 6\ln 2) = 44 + 6\ln(6/2).$$

$$\boxed{44 + 6\ln 3, \qquad \alpha = 44, \quad \beta = 6}$$

FINAL ANSWER

(a) $A = 2, \quad B = -1, \quad C = 6$ (b) $44 + 6\ln 3, \qquad \alpha = 44, \quad \beta = 6$

Question 3

6 marks | Log Graphs & Exponential Models

3. (i) The variables x and y are connected by the equation

$$y = \frac{10^6}{x^3} \qquad x > 0$$

Sketch the graph of $\log_{10} y$ against $\log_{10} x$

Show on your sketch the coordinates of the points of intersection of the graph with the axes.

(3)

(ii)

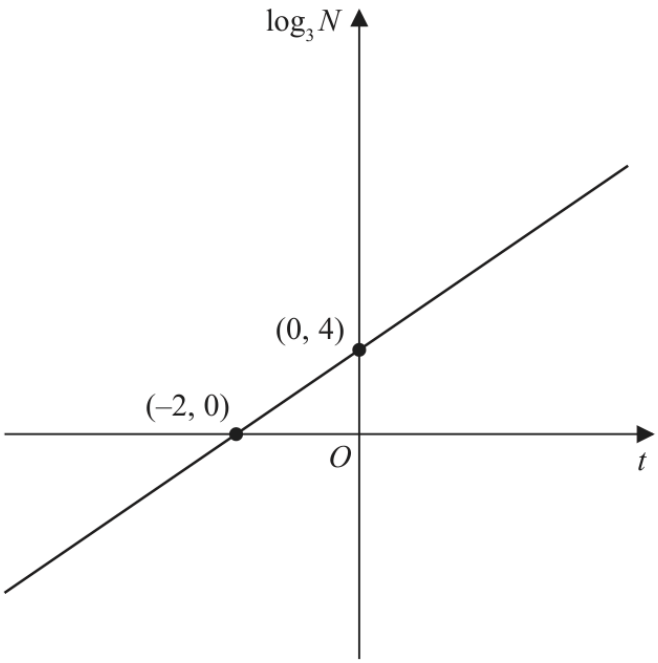


Figure 2

Figure 2 shows the linear relationship between $\log_3 N$ and t.

Show that $N = ab^t$ where a and b are constants to be found.

(3)

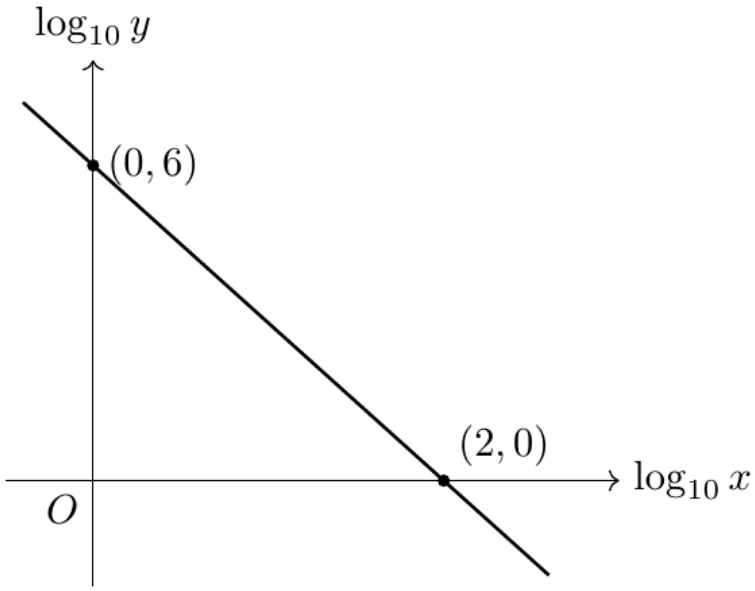
1 Part (i)

Put $X = \log_{10} x$ and $Y = \log_{10} y$. Taking logarithms of $y = 10^6/x^3$ gives

$$Y = 6 - 3X.$$

At $X = 0$, $Y = 6$; at $Y = 0$, $X = 2$. The graph is a straight line of gradient -3.

$$\boxed{Y = 6 - 3X; \quad (0, 6),\ (2, 0)}$$



2 Part (ii)

The gradient in the given plot is $(4 - 0)/(0 - (-2)) = 2$ and its vertical intercept is 4. Therefore $\log_3 N = 2t + 4$. Exponentiating in base 3,

$$N = 3^{2t+4} = 3^4 (3^2)^t = 81 \cdot 9^t.$$

$$\boxed{a = 81, \quad b = 9}$$

FINAL ANSWER

(i) $Y = 6 - 3X; \quad (0, 6),\ (2, 0)$ (ii) $a = 81, \quad b = 9$

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Question 4

9 marks | Trig Identities & Exact Values

4. $$\mathrm{f}(x) = 8\sin x\cos x + 4\cos^2 x - 3$$

(a) Write $\mathrm{f}(x)$ in the form

$$a\sin 2x + b\cos 2x + c$$

where a, b and c are integers to be found.

(3)

(b) Use the answer to part (a) to write $\mathrm{f}(x)$ in the form

$$R\sin(2x + \alpha) + c$$

where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α in radians to 3 significant figures.

(3)

(c) Hence, or otherwise,

(i) state the maximum value of $\mathrm{f}(x)$

(ii) find the **second** smallest positive value of x at which a maximum value of $\mathrm{f}(x)$ occurs. Give your answer to 3 significant figures.

(3)

WORKED SOLUTION

Question 4 · WMA13/01 May/June 2024

1 **Part (a)**

Use $\sin 2x = 2\sin x\cos x$ and $2\cos^2 x = 1 + \cos 2x$:

$$f(x) = 4\sin 2x + 2(1 + \cos 2x) - 3 = 4\sin 2x + 2\cos 2x - 1.$$

$$\boxed{a = 4, \quad b = 2, \quad c = -1}$$

2 Part (b)

Expanding $R\sin(2x+\alpha)$ and matching coefficients gives

$$R\cos\alpha = 4, \qquad R\sin\alpha = 2.$$

Squaring and adding, $R^2 = 20$, so $R = 2\sqrt{5}$ since $R > 0$. Also $\tan\alpha = 1/2$, and $0 < \alpha < \pi/2$ selects $\alpha = \arctan(1/2) = 0.463647609\ldots$ radians. Thus $f(x) = 2\sqrt{5}\sin(2x + \arctan(1/2)) - 1$. With the phase displayed to 3 significant figures, $f(x) \approx 2\sqrt{5}\sin(2x + 0.464) - 1$.

$$\boxed{R = 2\sqrt{5}, \quad \alpha = 0.464 \text{ radians}, \quad c = -1}$$

3 Part (c)

(i) The sine term has maximum 1, hence the maximum of f is $2\sqrt{5} - 1$. (ii) Maxima occur when

$$2x + \alpha = \frac{\pi}{2} + 2k\pi, \qquad x = \frac{\pi}{4} - \frac{\alpha}{2} + k\pi.$$

The value for $k = 0$ is $0.553574\ldots > 0$, while $k = -1$ is negative. The second positive value therefore has $k = 1$:

$$x = \frac{5\pi}{4} - \frac{1}{2}\arctan(1/2) = 3.695167\ldots.$$

$$\boxed{\max f = 2\sqrt{5} - 1, \quad x = 3.70 \text{ radians}}$$

FINAL ANSWER

(a) $a = 4, \quad b = 2, \quad c = -1$ (b) $R = 2\sqrt{5}, \quad \alpha = 0.464 \text{ radians}, \quad c = -1$ (c) $\max f = 2\sqrt{5} - 1, \quad x = 3.70 \text{ radians}$

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Question 5

10 marks | Functions, Composition & Inverses

5. The functions f and g are defined by

$$\mathrm{f}(x) = 2 + 5\ln x \qquad x > 0$$

$$\mathrm{g}(x) = \frac{6x - 2}{2x + 1} \qquad x > \frac{1}{3}$$

(a) Find $\mathrm{f}^{-1}(22)$

(2)

(b) Use differentiation to prove that g is an increasing function.

(3)

(c) Find g^{-1}

(3)

(d) Find the range of fg

(2)

WORKED SOLUTION

Question 5 · WMA13/01 May/June 2024

1 **Part (a)**

To find $f^{-1}(22)$, solve $f(u) = 22$: $2 + 5\ln u = 22$, so $\ln u = 4$ and $u = e^4 > 0$.

$$\boxed{f^{-1}(22) = e^4}$$

2 **Part (b)**

Using the quotient rule,

$$g'(x) = \frac{6(2x+1) - 2(6x-2)}{(2x+1)^2} = \frac{10}{(2x+1)^2}.$$

For every $x > 1/3$, the denominator is positive, so $g'(x) > 0$. Thus g is strictly increasing on its entire domain.

$$\boxed{g \text{ is strictly increasing for } x > \frac{1}{3}}$$

3

Part (c)

Set $y = (6x-2)/(2x+1)$. Then

$$2xy + y = 6x - 2, \qquad (6-2y)x = y + 2, \qquad x = \frac{y+2}{6-2y}.$$

As $x \downarrow 1/3$, $g(x) \to 0$; as $x \to \infty$, $g(x) \to 3$. Since g is strictly increasing, its range is $(0, 3)$; neither limiting value is attained. Interchange the input and output symbols.

$$\boxed{g^{-1}(x) = \frac{x+2}{6-2x}, \quad 0 < x < 3}$$

4

Part (d)

The inner function g takes every value in $(0, 3)$. Since $f(u) = 2 + 5\ln u$ is increasing for $u > 0$, its values on this interval run from $-\infty$ to, but not including, $2 + 5\ln 3$.

$$\boxed{\text{Range}(fg) = (-\infty,\ 2 + 5\ln 3)}$$

✓ **FINAL ANSWER**

(a) $f^{-1}(22) = e^4$ (b) g **is strictly increasing for** $x > \frac{1}{3}$ (c) $g^{-1}(x) = \frac{x+2}{6-2x}, \quad 0 < x < 3$ (d) $\text{Range}(fg) = (-\infty,\ 2 + 5\ln 3)$

Question 6

9 marks | Applications of Differentiation

6.

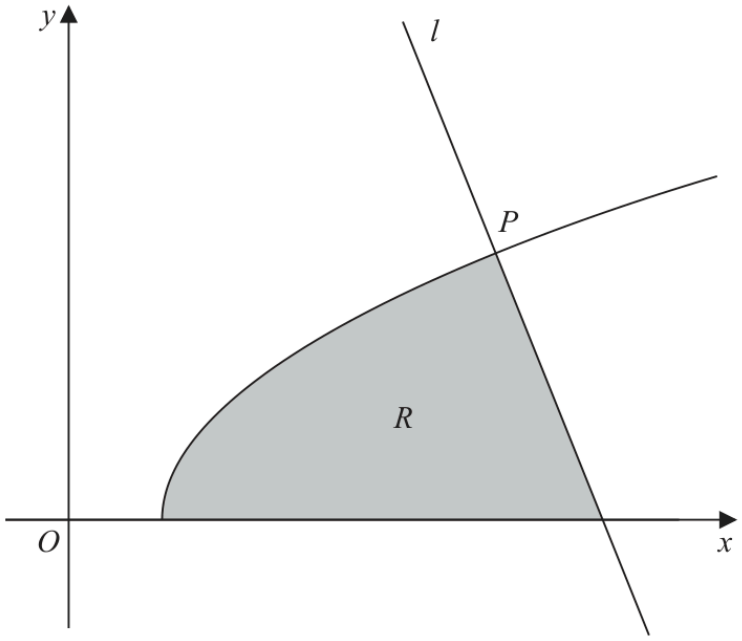


Figure 3

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 3 shows a sketch of part of the curve with equation

$$y = \sqrt{4x - 7}$$

The line l, shown in Figure 3, is the normal to the curve at the point $P(8, 5)$

(a) Use calculus to show that an equation of l is

$$5x + 2y - 50 = 0$$

(5)

The region R, shown shaded in Figure 3, is bounded by the curve, the x-axis and l.

(b) Use algebraic integration to find the exact area of R.

(4)

1 Part (a)

For $y = (4x-7)^{1/2}$,

$$\frac{dy}{dx} = \frac{1}{2}(4x-7)^{-1/2}\cdot 4 = \frac{2}{\sqrt{4x-7}}.$$

At $P=(8,5)$, the tangent gradient is $2/5$, so the normal gradient is $-5/2$. The point-gradient equation is

$$y-5 = -\frac{5}{2}(x-8), \qquad 2y-10 = -5x+40.$$

$$\boxed{5x+2y-50=0}$$

2 Part (b)

The curve meets the x-axis at $x=7/4$. The normal meets it at $x=10$. Split the shaded region at P, into a curve area from $7/4$ to 8 and a triangle from 8 to 10:

$$\int_{7/4}^{8}\sqrt{4x-7}\,dx = \left[\frac{(4x-7)^{3/2}}{6}\right]_{7/4}^{8} = \frac{125}{6}.$$

The triangle has area $\frac{1}{2}(10-8)\cdot 5 = 5$. Hence the total area is $125/6+5$.

$$\boxed{\text{Area}(R) = \frac{155}{6}}$$

FINAL ANSWER

(a) $5x+2y-50=0$ (b) $\text{Area}(R)=\frac{155}{6}$

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Question 7

8 marks | Trig Identities & Exact Values

7. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Given that

$$\sqrt{2}\sin(x+45°) = \cos(x-60°)$$

show that

$$\tan x = -2-\sqrt{3}$$

(4)

(b) Hence or otherwise, solve, for $0 \leqslant \theta < 180°$

$$\sqrt{2}\sin(2\theta) = \cos(2\theta-105°)$$

(4)

WORKED SOLUTION

Question 7 · WMA13/01 May/June 2024

1 **Part (a)**

Expand both compound angles:

$$\sin x + \cos x = \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x.$$

Thus $(2-\sqrt{3})\sin x = -\cos x$. Here $\cos x \neq 0$: otherwise this equation would require $\sin x = 0$ as well, which is impossible. Dividing by $(2-\sqrt{3})\cos x$ gives

$$\tan x = -\frac{1}{2-\sqrt{3}} = -\frac{2+\sqrt{3}}{(2-\sqrt{3})(2+\sqrt{3})}.$$

$$\boxed{\tan x = -2-\sqrt{3}}$$

2 **Part (b)**

Set $x = 2\theta - 45^\circ$. Then $x + 45^\circ = 2\theta$ and $x - 60^\circ = 2\theta - 105^\circ$, so part (a) gives $\tan(2\theta - 45^\circ) = -2 - \sqrt{3}$. Since $\tan 75^\circ = 2 + \sqrt{3}$,

$$2\theta - 45^\circ = -75^\circ + 180^\circ k, \qquad \theta = -15^\circ + 90^\circ k.$$

In $0 \le \theta < 180^\circ$, only $k = 1, 2$ are allowed. Substitution in the original equation confirms both values.

$$\boxed{\theta = 75^\circ,\ 165^\circ}$$

FINAL ANSWER

(a) $\tan x = -2 - \sqrt{3}$ (b) $\theta = 75^\circ,\ 165^\circ$

Question 8

10 marks | Log Graphs & Exponential Models

8.

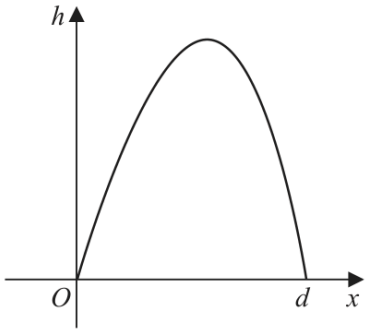


Figure 4

Figure 4 is a graph showing the path of a golf ball after the ball has been hit until it first hits the ground.

The vertical height, h metres, of the ball above the ground has been plotted against the horizontal distance travelled, x metres, measured from where the ball was hit.

The ball travels a horizontal distance of d metres before it first hits the ground.

The ball is modelled as a particle travelling in a vertical plane above horizontal ground.

The path of the ball is modelled by the equation

$$h = 1.5x - 0.5x\,e^{0.02x} \qquad 0 \leqslant x \leqslant d$$

Use the model to answer parts (a), (b) and (c).

(a) Find the value of d, giving your answer to 2 decimal places.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(3)

(b) Show that the maximum value of h occurs when

$$x = 50\ln\left(\frac{150}{x + 50}\right)$$

(4)

Using the iteration formula

$$x_{n+1} = 50\ln\left(\frac{150}{x_n + 50}\right) \qquad \text{with } x_1 = 30$$

(c) (i) find the value of x_2 to 2 decimal places,

(ii) find, by repeated iteration, the horizontal distance travelled by the golf ball before it reaches its maximum height. Give your answer to 2 decimal places.

(3)

1 Part (a)

At the first landing, $h = 0$ and $x = d > 0$:

$$0 = d(1.5 - 0.5e^{0.02d}), \qquad e^{0.02d} = 3.$$

Taking logarithms, $0.02d = \ln 3$, hence $d = 50\ln 3 = 54.930614\ldots$. The root $x = 0$ is the launch point, not the landing point.

$$\boxed{d = 54.93 \text{ m}}$$

2 Part (b)

By the product and chain rules,

$$h'(x) = 1.5 - 0.5e^{0.02x} - 0.01xe^{0.02x}.$$

At a stationary point,

$$1.5 = 0.01(x + 50)e^{0.02x}, \qquad e^{0.02x} = \frac{150}{x + 50}.$$

Since $x + 50 > 0$, logarithms give $0.02x = \ln(150/(x + 50))$. Also $h''(x) = -(0.02 + 0.0002x)e^{0.02x} < 0$ on $[0, d]$. Thus the stationary point is the unique maximum; the endpoints both have zero height.

$$\boxed{x = 50\ln\left(\frac{150}{x + 50}\right)}$$

3 Part (c)

(i) With $x_1 = 30$,

$$x_2 = 50\ln\left(\frac{150}{30 + 50}\right) = 31.43043297\ldots.$$

(ii) Continuing with full intermediate precision gives $x_3 = 30.54431108\ldots$, $x_4 = 31.09139065\ldots$, $x_5 = 30.75292479\ldots$ and $x_6 = 30.96205562\ldots$. Continued iteration converges to $30.88212\ldots$, which lies in $(0, d)$. This is the horizontal distance at maximum height, not the maximum height itself.

$$\boxed{x_2 = 31.43, \quad x_{\text{maximum}} = 30.88 \text{ m}}$$

FINAL ANSWER

(a) $d = 54.93$ m (b) $x = 50\ln\left(\frac{150}{x+50}\right)$ (c) $x_2 = 31.43, \quad x_{\text{maximum}} = 30.88$ m

Question 9

10 marks | Applications of Differentiation

9.

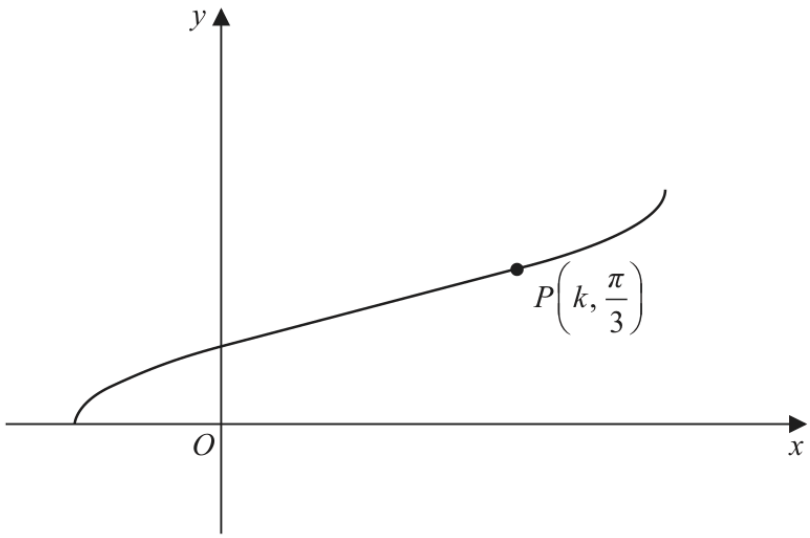


Figure 5

The curve shown in Figure 5 has equation

$$x = 4\sin^2 y - 1 \qquad 0 \leqslant y \leqslant \frac{\pi}{2}$$

The point $P\left(k, \frac{\pi}{3}\right)$ lies on the curve.

(a) Verify that $k = 2$

(1)

(b) (i) Find $\frac{dx}{dy}$ in terms of y

(ii) Hence show that $\frac{dy}{dx} = \frac{1}{2\sqrt{x+1}\sqrt{3-x}}$

(6)

The normal to the curve at P cuts the x-axis at the point N.

(c) Find the exact area of triangle OPN, where O is the origin.

Give your answer in the form $a\pi + b\pi^2$ where a and b are constants.

(3)

1 Part (a)

At $y = \pi/3$, $x = 4(\sqrt{3}/2)^2 - 1 = 3 - 1 = 2$.

$$\boxed{k = 2}$$

2 Part (b)

(i) Differentiating with respect to y gives

$$\frac{dx}{dy} = 8\sin y \cos y.$$

(ii) For $0 < y < \pi/2$, both sine and cosine are positive, and $-1 < x < 3$. From $x + 1 = 4\sin^2 y$,

$$\sin y = \frac{\sqrt{x+1}}{2}, \qquad \cos y = \sqrt{1 - \frac{x+1}{4}} = \frac{\sqrt{3-x}}{2}.$$

Consequently $dx/dy = 2\sqrt{x+1}\sqrt{3-x}$. Taking the reciprocal proves the required formula on $-1 < x < 3$. At $x = -1$ and $x = 3$ the denominator is zero: these are endpoint vertical tangents, not finite derivatives.

$$\boxed{\frac{dx}{dy} = 8\sin y\cos y, \qquad \frac{dy}{dx} = \frac{1}{2\sqrt{x+1}\sqrt{3-x}}}$$

3 Part (c)

At $P = (2, \pi/3)$ the tangent gradient is $1/(2\sqrt{3})$, so the normal gradient is $-2\sqrt{3}$. Thus the normal is $y - \pi/3 = -2\sqrt{3}(x - 2)$. Setting $y = 0$ gives $N = (2 + \pi/(6\sqrt{3}), 0)$. Triangle OPN has horizontal base $ON = 2 + \pi/(6\sqrt{3})$ and perpendicular height $\pi/3$:

$$\text{Area}(OPN) = \frac{1}{2}\left(2 + \frac{\pi}{6\sqrt{3}}\right)\frac{\pi}{3} = \frac{\pi}{3} + \frac{\pi^2}{36\sqrt{3}}.$$

$$\boxed{\text{Area}(OPN) = \frac{1}{3}\pi + \frac{\sqrt{3}}{108}\pi^2}$$

FINAL ANSWER

(a) $k = 2$ (b) $\frac{dx}{dy} = 8\sin y\cos y$, $\frac{dy}{dx} = \frac{1}{2\sqrt{x+1}\sqrt{3-x}}$ (c) $\text{Area}(OPN) = \frac{1}{3}\pi + \frac{\sqrt{3}}{108}\pi^2$