Elite Mathematics
Edexcel International A Level

IAL PURE 3 · WMA13

WMA13/01 January 2024 Worked Solutions

January 2024 paper rebuilt with the worked solution after each question.

9	2020–2026	Single-paper solutions
paper questions	19 exam sessions	checked by Dr Eslam



Dr Eslam Ahmed
Assistant Lecturer, Faculty of Engineering, Cairo University
eliteigcse.com · free for every student

PAST PAPER

WMA13/01 January 2024

January 2024

9	75	1 h 30
questions	marks	exam time

Each question is followed by its worked solution and final answer.

WMA13/01 JANUARY 2024

Question 1

4 marks | Graph Transformations

1. The point $P(-4, -3)$ lies on the curve with equation $y = \mathrm{f}(x)$, $x \in \mathbb{R}$

Find the point to which P is mapped when the curve with equation $y = \mathrm{f}(x)$ is transformed to the curve with equation

(a) $y = \mathrm{f}(2x)$

(1)

(b) $y = 3\mathrm{f}(x - 1)$

(2)

(c) $y = |\mathrm{f}(x)|$

(1)

WORKED SOLUTION

Question 1 · WMA13/01 January 2024

1 Part (a)

For $y = f(2x)$ the horizontal coordinate is divided by 2 while the vertical coordinate is unchanged: $(-4, -3) \mapsto (-4/2, -3)$.

$$\boxed{(-2, -3)}$$

2 Part (b)

The inside shift moves the point right by 1; the outside factor multiplies its ordinate by 3. Thus $(-4, -3) \mapsto (-4 + 1, 3(-3))$.

$$\boxed{(-3, -9)}$$

3 Part (c)

Output modulus retains the input and replaces the output by its absolute value: $(-4, -3) \mapsto (-4, |-3|)$.

$$\boxed{(-4, 3)}$$

FINAL ANSWER

(a) $(-2, -3)$ (b) $(-3, -9)$ (c) $(-4, 3)$

WMA13/01 JANUARY 2024

Question 2

6 marks | Numerical Methods

2. A curve has equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = x^4 - 5x^2 + 4x - 7 \qquad x \in \mathbb{R}$$

(a) Show that the equation $\mathrm{f}(x) = 0$ has a root, α, in the interval $[2, 3]$

(2)

(b) Show that the equation $\mathrm{f}(x) = 0$ can be written as

$$x = \sqrt[3]{\frac{5x^2 - 4x + 7}{x}}$$

(1)

The iterative formula

$$x_{n+1} = \sqrt[3]{\frac{5x_n^{\,2} - 4x_n + 7}{x_n}}$$

is used to find α

(c) Starting with $x_1 = 2$ and using the iterative formula,

(i) find, to 4 decimal places, the value of x_2

(ii) find, to 4 decimal places, the value of α

(3)

WORKED SOLUTION

Question 2 · WMA13/01 January 2024

1 **Part (a)**

For $f(x) = x^4 - 5x^2 + 4x - 7$,

$$f(2) = 16 - 20 + 8 - 7 = -3, \qquad f(3) = 81 - 45 + 12 - 7 = 41.$$

The polynomial is continuous on $[2, 3]$ and the endpoint values have opposite signs. Therefore a root lies strictly between 2 and 3.

$$\boxed{2 < \alpha < 3}$$

2 Part (b)

Rearrange $x^4 - 5x^2 + 4x - 7 = 0$ to $x^4 = 5x^2 - 4x + 7$. Since $f(0) = -7$, zero is not a root, so division by x is valid at a root:

$$x^3 = \frac{5x^2 - 4x + 7}{x}.$$

Taking the real cube root gives the requested form.

$$\boxed{x = \sqrt[3]{\frac{5x^2 - 4x + 7}{x}}}$$

3 Part (c)

With $x_1 = 2$,

$$x_2 = \sqrt[3]{\frac{5(2)^2 - 4(2) + 7}{2}} = \sqrt[3]{\frac{19}{2}} = 2.117911792\ldots.$$

Continue using unrounded intermediate values: $x_3 = 2.146846021\ldots$, $x_4 = 2.154063147\ldots$, $x_5 = 2.155869187\ldots$, $x_6 = 2.156321484\ldots$. Further iteration gives $x_{10} = 2.156472046\ldots$ and $x_{13} = 2.156472632\ldots$.

$$\boxed{x_2 = 2.1179, \quad \alpha = 2.1565}$$

FINAL ANSWER

(a) $2 < \alpha < 3$ (b) $x = \sqrt[3]{\frac{5x^2-4x+7}{x}}$ (c) $x_2 = 2.1179, \quad \alpha = 2.1565$

WMA13/01 JANUARY 2024

Question 3

7 marks | Log Graphs & Exponential Models

3. The amount of money raised for a charity is being monitored.

The total amount raised in the t months after monitoring began, £D, is modelled by the equation

$$\log_{10} D = 1.04 + 0.38t$$

(a) Write this equation in the form

$$D = ab^t$$

where a and b are constants to be found. Give each value to 4 significant figures.

(3)

When $t = T$, the total amount of money raised is £45 000

According to the model,

(b) find the value of T, giving your answer to 3 significant figures.

(2)

The charity aims to raise a total of £350 000 within the first 12 months of monitoring.

According to the model,

(c) determine whether or not the charity will achieve its aim.

(2)

WORKED SOLUTION

Question 3 · WMA13/01 January 2024

Part (a)

Exponentiating in base 10,

$$D = 10^{1.04+0.38t} = 10^{1.04}(10^{0.38})^t.$$

Thus $a = 10^{1.04} = 10.96478196\ldots$ and $b = 10^{0.38} = 2.398832919\ldots$. To four significant figures the displayed model is $D \approx 10.96(2.399)^t$.

$$\boxed{a = 10.96, \quad b = 2.399}$$

2 Part (b)

Use the original unrounded logarithmic model:

$$\log_{10} 45000 = 1.04 + 0.38T, \qquad T = \frac{\log_{10} 45000 - 1.04}{0.38} = 9.508453984\ldots.$$

$$\boxed{T = 9.51 \text{ months}}$$

3 Part (c)

At 12 months,

$$D(12) = 10^{1.04+0.38(12)} = 10^{5.6} = 398107.17\ldots.$$

This exceeds the target of 350000 pounds. The model is increasing because $10^{0.38} > 1$, so the target is reached within the first 12 months.

$$\boxed{\text{Yes: } D(12) \approx 398107 > 350000}$$

FINAL ANSWER

(a) $a = 10.96, \quad b = 2.399$ (b) $T = 9.51$ months (c) Yes: $D(12) \approx 398107 > 350000$

WMA13/01 JANUARY 2024

Question 4

13 marks | Rational Algebra

4. The function f is defined by

$$\mathrm{f}(x) = \frac{2x^2 - 32}{3x^2 + 7x - 20} + \frac{8}{3x - 5} \qquad x \in \mathbb{R} \quad x > 2$$

(a) Show that $\mathrm{f}(x) = \dfrac{2x}{3x-5}$

(3)

(b) Show, using calculus, that f is a decreasing function.
You must make your reasoning clear.

(3)

The function g is defined by

$$\mathrm{g}(x) = 3 + 2\ln x \qquad x \geqslant 1$$

(c) Find g^{-1}

(3)

(d) Find the exact value of a for which

$$\mathrm{gf}(a) = 5$$

(4)

WORKED SOLUTION

Question 4 · WMA13/01 January 2024

1 Part (a)

Factor the first denominator:

$$3x^2 + 7x - 20 = (3x - 5)(x + 4), \qquad 2x^2 - 32 = 2(x - 4)(x + 4).$$

On the stated domain $x > 2$, both denominators and $x + 4$ are nonzero. Hence

$$f(x) = \frac{2(x - 4)(x + 4)}{(3x - 5)(x + 4)} + \frac{8}{3x - 5} = \frac{2x - 8 + 8}{3x - 5}.$$

$$\boxed{f(x) = \frac{2x}{3x - 5}, \quad x > 2}$$

2 Part (b)

The quotient rule gives

$$f'(x) = \frac{2(3x-5) - 3(2x)}{(3x-5)^2} = -\frac{10}{(3x-5)^2}.$$

For every $x > 2$, the squared denominator is positive; hence $f'(x) < 0$. Therefore f is strictly decreasing on its whole domain.

$$\boxed{f \text{ is strictly decreasing on } (2, \infty)}$$

3 Part (c)

Set $y = 3 + 2\ln x$. Then $\ln x = (y-3)/2$, so $x = e^{(y-3)/2}$. For $x \geq 1$, the original range is $y \geq 3$. Interchanging the input and output symbols gives the inverse with its domain.

$$\boxed{g^{-1}(x) = e^{(x-3)/2}, \quad x \geq 3}$$

4 Part (d)

The equation $g(f(a)) = 5$ gives $3 + 2\ln(f(a)) = 5$, so $\ln(f(a)) = 1$ and $f(a) = e$. Using the simplified f,

$$\frac{2a}{3a-5} = e \quad \Longrightarrow \quad 2a = 3ea - 5e \quad \Longrightarrow \quad (3e-2)a = 5e.$$

The resulting value is approximately $2.2083 > 2$, and the inner output $e \geq 1$ is admissible for g.

$$\boxed{a = \frac{5e}{3e-2}}$$

FINAL ANSWER

(a) $f(x) = \frac{2x}{3x-5}, \quad x > 2$ (b) f is strictly decreasing on $(2, \infty)$ (c) $g^{-1}(x) = e^{(x-3)/2}, \quad x \geq 3$ (d) $a = \frac{5e}{3e-2}$

WMA13/01 JANUARY 2024

Question 5

7 marks | Log Graphs & Exponential Models

5. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

The temperature, T °C, of the air in a room t minutes after a heat source is switched off, is modelled by the equation

$$T = 10 + Ae^{-Bt}$$

where A and B are constants.

Given that the temperature of the air in the room at the instant the heat source was switched off was 18 °C,

(a) find the value of A

(1)

Given also that, exactly 45 minutes after the heat source was switched off, the temperature of the air in the room was 16 °C,

(b) find the value of B to 3 significant figures.

(3)

Using the values for A and B,

(c) find, according to the model, the rate of change of the temperature of the air in the room exactly two minutes after the heat source was switched off.
Give your answer in °C min^{-1} to 3 significant figures.

(2)

(d) Explain why, according to the model, the temperature of the air in the room cannot fall to 5 °C

(1)

WORKED SOLUTION

Question 5 · WMA13/01 January 2024

1 **Part (a)**

At $t = 0$, $18 = 10 + Ae^0 = 10 + A$.

$$\boxed{A = 8}$$

2 Part (b)

At $t = 45$, $16 = 10 + 8e^{-45B}$, so $e^{-45B} = 6/8 = 3/4$. Taking logarithms,

$$-45B = \ln(3/4), \qquad B = \frac{\ln(4/3)}{45} = 0.006392934943\ldots.$$

$$\boxed{B = 0.00639}$$

3 Part (c)

Differentiate $T = 10 + 8e^{-Bt}$:

$$\frac{dT}{dt} = -8Be^{-Bt}.$$

At $t = 2$, using the unrounded value $B = \ln(4/3)/45$, $dT/dt = -0.05049372834\ldots$. The negative sign means the temperature is falling.

$$\boxed{\frac{dT}{dt} = -0.0505\ ^\circ\mathrm{C\,min^{-1}}}$$

4 Part (d)

For every finite $t \geq 0$, $8e^{-Bt} > 0$ because $B > 0$. Thus $T = 10 + 8e^{-Bt} > 10$; as $t \to \infty$, T approaches 10 from above. It therefore cannot reach 5 degrees Celsius.

$$\boxed{T > 10 > 5}$$

FINAL ANSWER

(a) $A = 8$ (b) $B = 0.00639$ (c) $\frac{dT}{dt} = -0.0505\ ^\circ\mathrm{C\,min^{-1}}$ (d) $T > 10 > 5$

Question 6

7 marks | Applications of Differentiation

6.

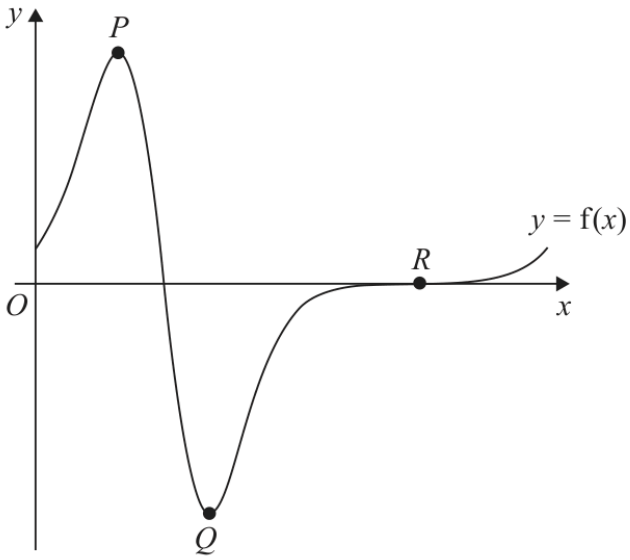


Figure 1

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 1 shows a sketch of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 2\mathrm{e}^{3\sin x}\cos x \qquad 0 \leqslant x \leqslant 2\pi$$

The curve intersects the x-axis at point R, as shown in Figure 1.

(a) State the coordinates of R

(1)

The curve has two turning points, at point P and point Q, also shown in Figure 1.

(b) Show that, at points P and Q,

$$a\sin^2 x + b\sin x + c = 0$$

where a, b and c are integers to be found.

(4)

(c) Hence find the x coordinate of point Q, giving your answer to 3 decimal places.

(2)

1 Part (a)

Since $2e^{3\sin x} > 0$, zeros occur when $\cos x = 0$. The labelled point R is the right-hand crossing, at $x = 3\pi/2$.

$$\boxed{R = \left(\frac{3\pi}{2}, 0\right)}$$

2 Part (b)

For $f(x) = 2e^{3\sin x}\cos x$, the product and chain rules give

$$f'(x) = 2\left(3e^{3\sin x}\cos^2 x - e^{3\sin x}\sin x\right) = 2e^{3\sin x}(3\cos^2 x - \sin x).$$

At a turning point the derivative is zero. The exponential factor is nonzero, so $3\cos^2 x - \sin x = 0$. Substitute $\cos^2 x = 1 - \sin^2 x$ and rearrange: $3 - 3\sin^2 x - \sin x = 0$.

$$\boxed{3\sin^2 x + \sin x - 3 = 0, \quad a = 3,\ b = 1,\ c = -3}$$

3 Part (c)

Solving $3s^2 + s - 3 = 0$ for $s = \sin x$ gives

$$s = \frac{-1 \pm \sqrt{37}}{6}.$$

The negative root is less than -1, so it is impossible for a real sine. Let $\beta = \arcsin\left((\sqrt{37} - 1)/6\right)$ in the first quadrant. The labelled minimum Q lies in the second quadrant, hence $x_Q = \pi - \beta = 2.131037293\ldots$.

$$\boxed{x_Q = 2.131}$$

FINAL ANSWER

(a) $R = \left(\frac{3\pi}{2}, 0\right)$ (b) $3\sin^2 x + \sin x - 3 = 0, \quad a = 3,\ b = 1,\ c = -3$ (c) $x_Q = 2.131$

Question 7

12 marks | Applications of Differentiation

7. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

The curve C has equation

$$y = \frac{16}{9(3x - k)} \qquad x \neq \frac{k}{3}$$

where k is a positive constant not equal to 3

(a) Find $\frac{dy}{dx}$ giving your answer in simplest form in terms of k.

(2)

The point P with x coordinate 1 lies on C.

Given that the gradient of the curve at P is -12

(b) find the two possible values of k

(3)

Given also that $k < 3$

(c) find the equation of the normal to C at P, writing your answer in the form $ax + by + c = 0$, where a, b and c are integers to be found.

(3)

(d) show, using algebraic integration that,

$$\int_1^3 \frac{16}{9(3x - k)} dx = \lambda \ln 10$$

where λ is a constant to be found.

(4)

1 Part (a)

Write $y = (16/9)(3x-k)^{-1}$. The chain rule gives

$$\frac{dy}{dx} = \frac{16}{9}(-1)(3x-k)^{-2}\cdot 3.$$

$$\boxed{\frac{dy}{dx} = -\frac{16}{3(3x-k)^2}}$$

2 Part (b)

At $x = 1$, the given gradient yields

$$-\frac{16}{3(3-k)^2} = -12 \quad\Longrightarrow\quad (3-k)^2 = \frac{4}{9}.$$

Hence $3-k = \pm 2/3$. Both resulting k-values are positive and different from 3.

$$\boxed{k = \frac{7}{3} \text{ or } \frac{11}{3}}$$

3 Part (c)

Since $k < 3$, use $k = 7/3$. At $x = 1$, $y = 16/[9(3-7/3)] = 8/3$. The tangent gradient is -12, so the normal gradient is $1/12$:

$$y - \frac{8}{3} = \frac{1}{12}(x-1), \qquad 12y - 32 = x - 1.$$

$$\boxed{x - 12y + 31 = 0}$$

4 Part (d)

With $k = 7/3$, the denominator does not vanish on $[1,3]$. Using the reciprocal-linear antiderivative,

$$\int_1^3 \frac{16}{9(3x-7/3)}\,dx = \left[\frac{16}{27}\ln|3x-7/3|\right]_1^3 = \frac{16}{27}\left(\ln\frac{20}{3} - \ln\frac{2}{3}\right) = \frac{16}{27}\ln\left(\frac{20/3}{2/3}\right).$$

$$\boxed{\frac{16}{27}\ln 10, \quad \lambda = \frac{16}{27}}$$

FINAL ANSWER

(a) $\frac{dy}{dx} = -\frac{16}{3(3x-k)^2}$ (b) $k = \frac{7}{3}$ or $\frac{11}{3}$ (c) $x - 12y + 31 = 0$ (d) $\frac{16}{27}\ln 10, \quad \lambda = \frac{16}{27}$

Question 8

11 marks | Modulus

8.

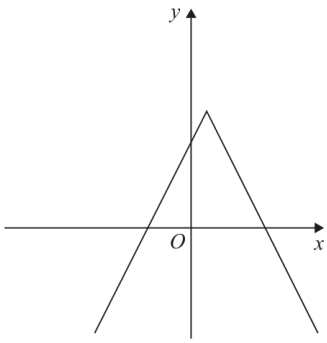


Figure 2

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

The graph shown in Figure 2 has equation

$$y = a - |2x - b|$$

where a and b are positive constants, $a > b$

(a) Find, giving your answer in terms of a and b,

(i) the coordinates of the maximum point of the graph,

(ii) the coordinates of the point of intersection of the graph with the y-axis,

(iii) the coordinates of the points of intersection of the graph with the x-axis.

(5)

On page 24 there is a copy of Figure 2 called Diagram 1.

(b) On Diagram 1, sketch the graph with equation

$$y = |x| - 1$$

(2)

Given that the graphs $y = |x| - 1$ and $y = a - |2x - b|$ intersect at $x = -3$ and $x = 5$

(c) find the value of a and the value of b

(4)

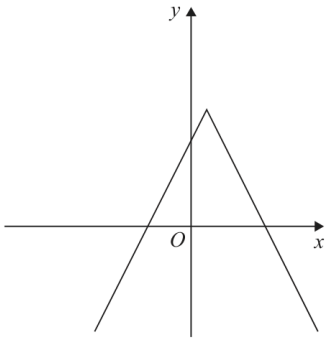


Diagram 1

1 Part (a)

(i) Since $|2x-b| \geq 0$, the maximum occurs at $2x-b=0$: $x=b/2$ and $y=a$. (ii) At $x=0$, $y=a-|-b|=a-b$ because $b>0$. (iii) At an x-axis crossing, $|2x-b|=a$, hence $2x-b=\pm a$. Since $a>b>0$, $(b-a)/2<0$ and $(b+a)/2>0$, matching the sketch.

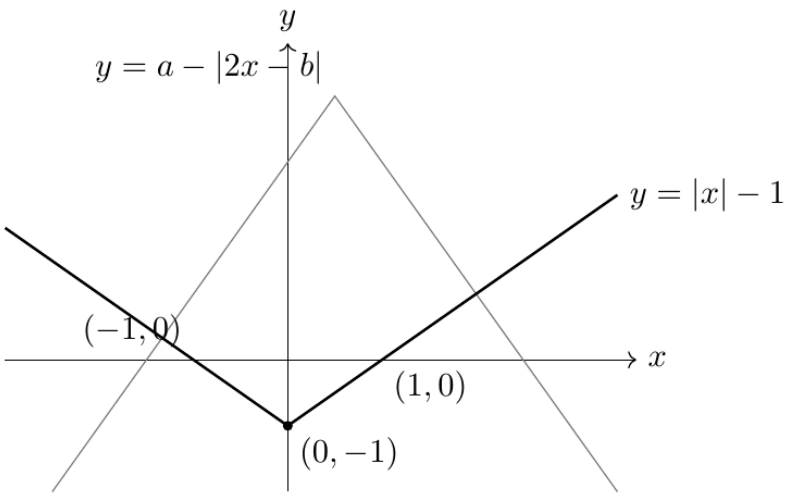
$$\boxed{\left(\frac{b}{2}, a\right); \quad (0, a-b); \quad \left(\frac{b-a}{2}, 0\right), \ \left(\frac{b+a}{2}, 0\right)}$$

2 Part (b)

The graph $y=|x|-1$ is a V with vertex $(0,-1)$ and intercepts $(-1,0)$ and $(1,0)$. Its left branch is $y=-x-1$ for $x<0$ and its right branch is $y=x-1$ for $x \geq 0$. Add this V to the supplied axes; the original inverted V remains unchanged.

The grey original curve is schematic; the three labelled points belong to the added graph.

$$\boxed{\text{Vertex } (0,-1); \quad \text{zeros } x=\pm 1}$$



3 Part (c)

At $x=-3$, the common ordinate is $|-3|-1=2$. Since $b>0$, $|-6-b|=6+b$, so $a-(6+b)=2$, giving $a=b+8$. At $x=5$, the common ordinate is $|5|-1=4$, so $a-|10-b|=4$. Substitute $a=b+8$ to obtain $|10-b|=b+4$. If $b \leq 10$, then $10-b=b+4$, giving $b=3$, $a=11$. If $b>10$, then $b-10=b+4$, impossible. Both accepted values satisfy $a>b>0$.

$$\boxed{a=11, \quad b=3}$$

FINAL ANSWER

(a) $\left(\frac{b}{2}, a\right)$; $(0, a-b)$; $\left(\frac{b-a}{2}, 0\right)$, $\left(\frac{b+a}{2}, 0\right)$ (b) $\text{Vertex } (0,-1)$; $\text{zeros } x=\pm 1$ (c) $a=11, \quad b=3$

WMA13/01 JANUARY 2024

Question 9

8 marks | Trig Equations & R-Form

9. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that the equation

$$\frac{3\sin\theta\cos\theta}{\cos\theta+\sin\theta} = (2+\sec 2\theta)(\cos\theta-\sin\theta)$$

can be written in the form

$$3\sin 2\theta - 4\cos 2\theta = 2$$

(3)

(b) Hence solve for $\pi < x < \frac{3\pi}{2}$

$$\frac{3\sin x\cos x}{\cos x+\sin x} = (2+\sec 2x)(\cos x-\sin x)$$

giving the answer to 3 significant figures.

(5)

WORKED SOLUTION

Question 9 · WMA13/01 January 2024

1 Part (a)

The original equation is defined only when $\cos\theta+\sin\theta\neq 0$ and $\cos 2\theta\neq 0$. Multiplying by $\cos\theta+\sin\theta$ gives

$$3\sin\theta\cos\theta = (2+\sec 2\theta)(\cos^2\theta-\sin^2\theta) = (2+\sec 2\theta)\cos 2\theta = 2\cos 2\theta+1.$$

Using $2\sin\theta\cos\theta=\sin 2\theta$, multiply through by 2: $3\sin 2\theta = 4\cos 2\theta+2$.

$$\boxed{3\sin 2\theta - 4\cos 2\theta = 2}$$

2 **Part (b)**

Write $3\sin 2x - 4\cos 2x = 5\sin(2x - \phi)$, where $\cos\phi = 3/5$, $\sin\phi = 4/5$ and $\phi = \arctan(4/3)$. Then $\sin(2x - \phi) = 2/5$. Because $\pi < x < 3\pi/2$, $2\pi - \phi < 2x - \phi < 3\pi - \phi$, approximately $(5.356, 8.497)$. With $\gamma = \arcsin(2/5)$, the only angle in this interval is $2\pi + \gamma$; $3\pi - \gamma \approx 9.013$ is outside it. Thus

$$x = \frac{2\pi + \arcsin(2/5) + \arctan(4/3)}{2} = 3.810998686\ldots.$$

At this value $\cos x + \sin x \neq 0$ and $\cos 2x \neq 0$; direct substitution satisfies the original equation.

$$\boxed{x = 3.81}$$

FINAL ANSWER

(a) $3\sin 2\theta - 4\cos 2\theta = 2$ (b) $x = 3.81$