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**Elite Mathematics**
Edexcel International A Level

IAL PURE 3 · WMA13

# WMA13/01 October 2023 Worked Solutions

October 2023 paper rebuilt with the worked solution after each question.

| 10 | 2020–2026 | Single-paper solutions |
|---|---|---|
| paper questions | 19 exam sessions | checked by Dr Eslam |



**Dr Eslam Ahmed**
Assistant Lecturer, Faculty of Engineering, Cairo University
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PAST PAPER

# WMA13/01 October 2023

October 2023

| 10 | 75 | 1 h 30 |
| --- | --- | --- |
| questions | marks | exam time |

Each question is followed by its worked solution and final answer.

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# Question 1

5 marks | Numerical Methods

**1.** A curve has equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = x^2 - 5x + \mathrm{e}^x \qquad x \in \mathbb{R}$$

(a) Show that the equation $\mathrm{f}(x) = 0$ has a root, $\alpha$, in the interval $[1, 2]$

**(2)**

The iterative formula

$$x_{n+1} = \sqrt{5x_n - \mathrm{e}^{x_n}}$$

with $x_1 = 1$ is used to find an approximate value for the root $\alpha$.

(b) (i) Find the value of $x_2$ to 4 decimal places.

(ii) Find, by repeated iteration, the value of $\alpha$, giving your answer to 4 decimal places.

**(3)**

## WORKED SOLUTION

Question 1 · WMA13/01 October 2023

**1** **Part (a)**

The function $f(x) = x^2 - 5x + e^x$ is continuous on $[1, 2]$.

$$f(1) = e - 4 = -1.2817\ldots < 0, \qquad f(2) = e^2 - 6 = 1.3891\ldots > 0.$$

Since the endpoint values have opposite signs, there is a root strictly between 1 and 2.

$$\boxed{1 < \alpha < 2}$$

**2** **Part (b)**

Using the given iteration with $x_1 = 1$,

$$x_2 = \sqrt{5 - e} = 1.5105357\ldots.$$

Continue with unrounded values:

$$x_3 = 1.7388278\ldots, \quad x_4 = 1.7330522\ldots, \quad x_5 = 1.7341754\ldots, \quad x_6 = 1.7339613\ldots.$$

Subsequent iterates are $1.7340023\ldots, 1.7339945\ldots, 1.7339960\ldots$, which settle to $1.7339957\ldots$. Thus (i) $x_2 = 1.5105$ and (ii) $\alpha = 1.7340$, each to 4 decimal places.

$$\boxed{x_2 = 1.5105, \qquad \alpha = 1.7340}$$

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**FINAL ANSWER**

(a) $1 < \alpha < 2$ (b) $x_2 = 1.5105, \qquad \alpha = 1.7340$

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# Question 2

8 marks | Functions, Composition & Inverses

**2.** The function f is defined by

$$\mathrm{f}(x) = \frac{x+3}{x-4} \qquad x \in \mathbb{R}, x \neq 4$$

(a) Find $\mathrm{ff}(6)$

**(2)**

(b) Find $\mathrm{f}^{-1}$

**(3)**

The function g is defined by

$$\mathrm{g}(x) = x^2 + 5 \qquad x \in \mathbb{R}, x > 0$$

(c) Find the exact value of $a$ for which

$$\mathrm{gf}(a) = 7$$

**(3)**

## WORKED SOLUTION

Question 2 · WMA13/01 October 2023

**1** **Part (a)**

Apply $f$ twice, in order:

$$f(6) = \frac{6+3}{6-4} = \frac{9}{2}, \qquad ff(6) = f\left(\frac{9}{2}\right) = \frac{9/2+3}{9/2-4} = 15.$$

$$\boxed{ff(6) = 15}$$

**2** **Part (b)**

Let $y = (x+3)/(x-4)$. Then

$$yx - 4y = x + 3 \implies x(y-1) = 4y + 3 \implies x = \frac{4y+3}{y-1}.$$

Interchanging $x$ and $y$ gives the inverse. The value 1 is never an output of $f$, since $(x+3)/(x-4) = 1$ would require $3 = -4$.

$$\boxed{f^{-1}(x) = \frac{4x+3}{x-1}, \qquad x \neq 1}$$

3

### Part (c)

Since $g$ is defined for positive inputs, $gf(a) = 7$ requires $f(a) > 0$.

$$(f(a))^2 + 5 = 7 \quad \Longrightarrow \quad f(a) = \sqrt{2}.$$

The negative square root is not in the domain of $g$. Hence

$$\frac{a+3}{a-4} = \sqrt{2} \quad \Longrightarrow \quad a = \frac{4\sqrt{2}+3}{\sqrt{2}-1} = \frac{(4\sqrt{2}+3)(\sqrt{2}+1)}{2-1} = 11 + 7\sqrt{2}.$$

This value is not 4 and gives $f(a) = \sqrt{2} > 0$, as required.

$$\boxed{a = 11 + 7\sqrt{2}}$$

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**FINAL ANSWER**

(a) $ff(6) = 15$ (b) $f^{-1}(x) = \frac{4x+3}{x-1}, \qquad x \neq 1$ (c) $a = 11 + 7\sqrt{2}$

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# Question 3

6 marks | Trig Identities & Exact Values

**3.** (a) Using the identity for $\cos(A+B)$, prove that

$$\cos 2A \equiv 2\cos^2 A - 1$$

**(2)**

(b) Hence, using algebraic integration, find the exact value of

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{8}} (5 - 4\cos^2 3x)\,\mathrm{d}x$$

**(4)**

## WORKED SOLUTION

Question 3 · WMA13/01 October 2023

### 1 Part (a)

Apply the cosine addition formula:

$$\cos 2A = \cos(A+A) = \cos A\cos A - \sin A\sin A = \cos^2 A - \sin^2 A.$$

Using $\sin^2 A = 1 - \cos^2 A$ gives the required identity.

$$\boxed{\cos 2A = 2\cos^2 A - 1}$$

### 2 Part (b)

From part (a), $\cos^2 3x = (1+\cos 6x)/2$. Therefore

$$\int_{\pi/12}^{\pi/8} (5 - 4\cos^2 3x)\,dx = \int_{\pi/12}^{\pi/8} (3 - 2\cos 6x)\,dx = \left[3x - \frac{1}{3}\sin 6x\right]_{\pi/12}^{\pi/8}.$$

Since $\sin(3\pi/4) = \sqrt{2}/2$ and $\sin(\pi/2) = 1$, the result is

$$\frac{3\pi}{8} - \frac{\sqrt{2}}{6} - \left(\frac{\pi}{4} - \frac{1}{3}\right).$$

$$\boxed{\frac{\pi}{8} + \frac{1}{3} - \frac{\sqrt{2}}{6}}$$

**FINAL ANSWER**

(a) $\cos 2A = 2\cos^2 A - 1$ (b) $\frac{\pi}{8} + \frac{1}{3} - \frac{\sqrt{2}}{6}$

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# Question 4

7 marks | Log Graphs & Exponential Models

**4.** A new mobile phone is released for sale.

The total sales $N$ of this phone, in **thousands**, is modelled by the equation

$$N = 125 - Ae^{-0.109t} \qquad t \geqslant 0$$

where $A$ is a constant and $t$ is the time in months after the phone was released for sale.

Given that when $t = 0$, $N = 32$

(a) state the value of $A$.

**(1)**

Given that when $t = T$ the total sales of the phone was 100 000

(b) find, according to the model, the value of $T$. Give your answer to 2 decimal places.

**(3)**

(c) Find, according to the model, the rate of increase in total sales when $t = 7$, giving your answer to 3 significant figures.

(*Solutions relying entirely on calculator technology are not acceptable.*)

**(2)**

The total sales of the mobile phone is expected to reach 150 000

Using this information,

(d) give a reason why the given equation is not suitable for modelling the total sales of the phone.

**(1)**

## WORKED SOLUTION

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**Part (a)**

Initially $t = 0$ and $N = 32$, because $N$ is measured in thousands.

$$32 = 125 - Ae^{0} = 125 - A.$$

$$\boxed{A = 93}$$

## 2 Part (b)

For 100000 phones, use $N = 100$:

$$100 = 125 - 93e^{-0.109T} \quad \Longrightarrow \quad e^{-0.109T} = \frac{25}{93}.$$

Taking natural logarithms,

$$T = \frac{\ln(93/25)}{0.109} = 12.0525107\ldots \text{ months}.$$

$$\boxed{T = 12.05 \text{ months}}$$

---

## 3 Part (c)

Differentiate the model:

$$\frac{dN}{dt} = 93(0.109)e^{-0.109t}.$$

At $t = 7$, this is $4.7265337\ldots$ thousand phones per month. Multiplying by 1000 gives $4726.5337\ldots$ phones per month.

$$\boxed{4.73 \times 10^3 \text{ phones per month}}$$

---

## 4 Part (d)

For $t \geq 0$, $93e^{-0.109t} > 0$, so $N < 125$ and $N \to 125$ as $t \to \infty$. The model can therefore never reach 150000 phones: its limiting count is 125000.

$$\boxed{\text{No: the limiting count is 125000 phones.}}$$

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**FINAL ANSWER**

(a) $A = 93$ (b) $T = 12.05$ months (c) $4.73 \times 10^3$ phones per month (d) No: the limiting count is 125000 phones.

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# Question 5

7 marks | Applications of Differentiation

**5.** The curve $C$ has equation

$$y = \frac{\ln(x^2 + k)}{x^2 + k} \qquad x \in \mathbb{R}$$

where $k$ is a positive constant.

(a) Show that

$$\frac{dy}{dx} = \frac{Ax(B - \ln(x^2 + k))}{(x^2 + k)^2}$$

where $A$ and $B$ are constants to be found.

**(3)**

Given that $C$ has exactly three turning points,

(b) find the $x$ coordinate of each of these points. Give your answer in terms of $k$ where appropriate.

**(3)**

(c) find the upper limit to the value for $k$.

**(1)**

## WORKED SOLUTION

Question 5 · WMA13/01 October 2023

**1** **Part (a)**

Set $u = x^2 + k > 0$. Applying the quotient and chain rules,

$$\frac{dy}{dx} = \frac{(2x/u)u - 2x\ln u}{u^2} = \frac{2x(1 - \ln(x^2 + k))}{(x^2 + k)^2}.$$

Comparing with the required form identifies both constants.

$$\boxed{A = 2, \qquad B = 1}$$

## 2 Part (b)

The denominator is positive, so stationary points satisfy

$$x = 0 \quad \text{or} \quad \ln(x^2 + k) = 1 \quad \Longrightarrow \quad x = 0 \quad \text{or} \quad x = \pm\sqrt{e - k}.$$

There are three distinct real stationary points precisely when $e - k > 0$. In that case the factor $1 - \ln(x^2 + k)$ is positive between the outer roots and negative outside them; $2x$ changes sign at zero. Thus the derivative changes sign at all three points, so all are turning points. Together with $k > 0$, the three coordinates below are real and distinct; the restriction is $0 < k < e$.

$$\boxed{x = 0, \quad x = -\sqrt{e - k}, \quad x = \sqrt{e - k}}$$

## 3 Part (c)

At $k = e$, the two outer stationary points merge with $x = 0$, so there are no longer three distinct turning points. Values of $k$ can approach $e$ from below, but cannot equal it under this condition.

$$\boxed{\text{Upper limit } e}$$

**FINAL ANSWER**

(a) $A = 2, \qquad B = 1$ (b) $x = 0, \quad x = -\sqrt{e - k}, \quad x = \sqrt{e - k}$ (c) $\text{Upper limit } e$

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# Question 6

6 marks | Log Graphs & Exponential Models

**6.** An area of sea floor is being monitored.

The area of the sea floor, $S\,\text{km}^2$, covered by coral reefs is modelled by the equation

$$S = pq^t$$

where $p$ and $q$ are constants and $t$ is the number of years after monitoring began.

Given that

$$\log_{10} S = 4.5 - 0.006t$$

(a) find, according to the model, the area of sea floor covered by coral reefs when $t = 2$

**(2)**

(b) find a complete equation for the model in the form

$$S = pq^t$$

giving the value of $p$ and the value of $q$ each to 3 significant figures.

**(3)**

(c) With reference to the model, interpret the value of the constant $q$

**(1)**

## WORKED SOLUTION

Question 6 · WMA13/01 October 2023

**1** **Part (a)**

Substitute $t = 2$ into the logarithmic model:

$$\log_{10} S = 4.5 - 0.006(2) = 4.488 \quad \Longrightarrow \quad S = 10^{4.488}.$$

Thus $S = 30760.968\ldots\ \text{km}^2$. To 3 significant figures, the area is $3.08 \times 10^4\ \text{km}^2$.

$$\boxed{S = 3.08 \times 10^4\ \text{km}^2}$$

## 2 Part (b)

Exponentiating and using index laws,

$$S = 10^{4.5-0.006t} = 10^{4.5}(10^{-0.006})^t.$$

Thus $p = 31622.7766\ldots$ and $q = 0.9862795\ldots$. To 3 significant figures, $p = 3.16 \times 10^4$ and $q = 0.986$. Hence the complete model, with both constants rounded as requested, is $S = 31600(0.986)^t$, where $S$ is in $\text{km}^2$ and $t$ is in years.

$$\boxed{S = 31600(0.986)^t}$$

## 3 Part (c)

The factor $q$ is the yearly multiplier for the sea-floor area covered by coral reefs. Each year, about $98.6\%$ of the previous year's coral-covered sea-floor area remains. This is a yearly decrease of about $1.4\%$ (or $1.372\ldots\%$ using the unrounded value of $q$).

$$\boxed{\text{About } 98.6\% \text{ of the coral-covered sea-floor area remains each year.}}$$

**FINAL ANSWER**

(a) $S = 3.08 \times 10^4\ \text{km}^2$ (b) $S = 31600(0.986)^t$ (c) About $98.6\%$ of the coral-covered sea-floor area remains each year.

# Question 7

10 marks | Functions, Composition & Inverses

7.

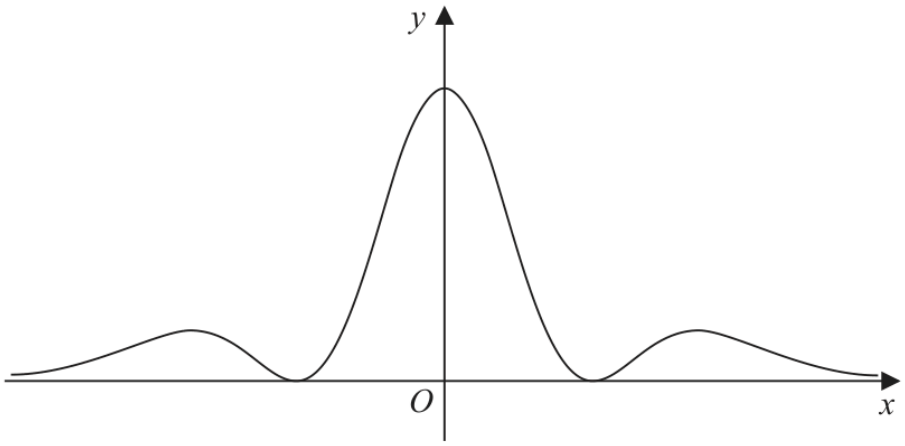



**Figure 1**

Figure 1 shows a sketch of the curve $C$ with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \mathrm{e}^{-x^2}(2x^2 - 3)^2$$

(a) Find the range of f

**(2)**

(b) Show that

$$\mathrm{f}'(x) = 2x(2x^2 - 3)\mathrm{e}^{-x^2}(A - Bx^2)$$

where $A$ and $B$ are constants to be found.

**(4)**

Given that the line $y = k$, where $k$ is a constant, $k > 0$, intersects the curve at exactly two distinct points,

(c) find the exact range of values of $k$

**(4)**

## 1 Part (a)

The exponential is positive and the squared factor is nonnegative, so $f(x) \geq 0$. The value zero is attained at $x = \pm\sqrt{3/2}$. The graph has its highest point at $x = 0$, where $f(0) = 9$; the outer maxima are lower. Thus both endpoints of the range are included.

$$\boxed{0 \leq f(x) \leq 9}$$

## 2 Part (b)

For $f(x) = e^{-x^2}(2x^2 - 3)^2$, product and chain rules give

$$f'(x) = -2xe^{-x^2}(2x^2 - 3)^2 + e^{-x^2}\,2(2x^2 - 3)\,4x.$$

Factoring yields

$$f'(x) = 2x(2x^2 - 3)e^{-x^2}\big(-(2x^2 - 3) + 4\big) = 2x(2x^2 - 3)e^{-x^2}(7 - 2x^2).$$

$$\boxed{A = 7, \qquad B = 2}$$

## 3 Part (c)

The stationary abscissae are $0$, $\pm\sqrt{3/2}$ and $\pm\sqrt{7/2}$. At the outer local maxima,

$$f\left(\pm\sqrt{7/2}\right) = e^{-7/2}(7 - 3)^2 = 16e^{-7/2}.$$

The central maximum is 9 and the two minima are zero. Also $f(x) \to 0$ as $x \to \pm\infty$. A positive horizontal line above both outer maxima but below 9 meets only the two sides of the central peak. At $k = 16e^{-7/2}$ it also touches both outer peaks, giving four distinct intersections; at $k = 9$ there is only one. Both bounds must be excluded.

$$\boxed{16e^{-7/2} < k < 9}$$

**FINAL ANSWER**

(a) $0 \leq f(x) \leq 9$ (b) $A = 7, \qquad B = 2$ (c) $16e^{-7/2} < k < 9$

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# Question 8

8 marks | Trig Identities & Exact Values

**8.** (a) Prove that

$$2\operatorname{cosec}^2 2\theta(1-\cos 2\theta) \equiv 1+\tan^2\theta$$

**(4)**

(b) Hence solve for $0 < x < 360°$, where $x \neq (90n)°$, $n \in \mathbb{N}$, the equation

$$2\operatorname{cosec}^2 2x(1-\cos 2x) = 4+3\sec x$$

giving your answers to one decimal place.

(*Solutions relying entirely on calculator technology are not acceptable.*)

**(4)**

## WORKED SOLUTION

Question 8 · WMA13/01 October 2023

### 1 Part (a)

Starting from the left side and using double-angle identities,

$$2\csc^2 2\theta(1-\cos 2\theta) = \frac{2(2\sin^2\theta)}{(2\sin\theta\cos\theta)^2} = \frac{1}{\cos^2\theta} = \sec^2\theta = 1+\tan^2\theta.$$

These steps hold wherever the original expression is defined, so $\sin 2\theta \neq 0$; no claim is made at the excluded angles.

$$\boxed{2\csc^2 2\theta(1-\cos 2\theta) = 1+\tan^2\theta}$$

### 2 Part (b)

By part (a), the equation becomes

$$\sec^2 x = 4+3\sec x \quad\Longrightarrow\quad (\sec x-4)(\sec x+1) = 0.$$

The branch $\sec x = 4$ gives $\cos x = 1/4$ and hence $x = 75.5225\ldots°$ or $284.4775\ldots°$. The other branch gives $\cos x = -1$, hence $x = 180°$ in the stated interval. Reject $180°$: it is a multiple of $90°$ and the original $\csc 2x$ is undefined there. The two remaining angles satisfy the exclusions.

$$\boxed{x = 75.5°, \qquad 284.5°}$$

**FINAL ANSWER**

(a) $2\csc^2 2\theta(1-\cos 2\theta) = 1+\tan^2\theta$ (b) $x = 75.5°, \qquad 284.5°$

# Question 9

9 marks | Exponentials & Logarithms

**9.** **In this question you must show all stages of your working.**

**Solutions relying on calculator technology are not acceptable.**

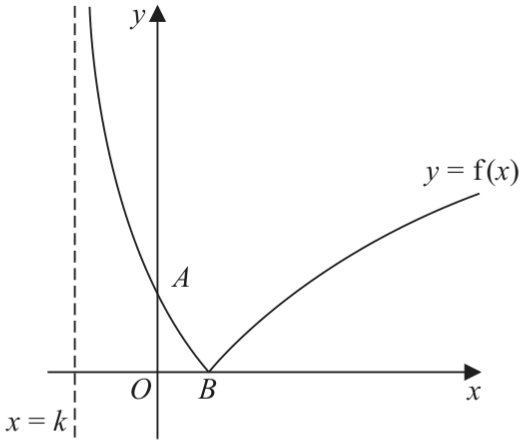



**Figure 2**

Figure 2 shows a sketch of the curve with equation

$$y = |2 - 4\ln(x + 1)| \qquad x > k$$

where $k$ is a constant.

Given that the curve

- has an asymptote at $x = k$
- cuts the $y$-axis at point $A$
- meets the $x$-axis at point $B$

as shown in Figure 2,

(a) state the value of $k$

**(1)**

(b) (i) find the $y$ coordinate of $A$

(ii) find the exact $x$ coordinate of $B$

**(3)**

(c) Using algebra and showing your working, find the set of values of $x$ such that

$$|2 - 4\ln(x + 1)| > 3$$

**(5)**

## 1 Part (a)

The logarithm requires $x+1>0$, so the domain is $x>-1$. As $x\to -1^{+}$, $\ln(x+1)\to -\infty$ and the ordinate tends to infinity.

$$\boxed{k=-1}$$

## 2 Part (b)

(i) At the $y$-axis, $x=0$, giving $y=|2-4\ln 1|=2$.
(ii) At the $x$-axis, $|2-4\ln(x+1)|=0$, so

$$\ln(x+1)=\frac{1}{2}\quad\Longrightarrow\quad x=e^{1/2}-1.$$

Thus $A=(0,2)$ and $B=(e^{1/2}-1,0)$.

$$\boxed{y_A=2,\qquad x_B=e^{1/2}-1}$$

## 3 Part (c)

On $x>-1$, split the absolute-value inequality into two alternatives:

$$2-4\ln(x+1)>3\quad\text{or}\quad 2-4\ln(x+1)<-3.$$

Dividing by $-4$ reverses the inequality in each branch:

$$\ln(x+1)<-\frac{1}{4}\quad\text{or}\quad\ln(x+1)>\frac{5}{4}.$$

Since the exponential is increasing, this becomes $x+1<e^{-1/4}$ or $x+1>e^{5/4}$. Intersect with the logarithmic domain; neither boundary is included because the source inequality is strict.

$$\boxed{-1<x<e^{-1/4}-1\quad\text{or}\quad x>e^{5/4}-1}$$

**FINAL ANSWER**

(a) $k=-1$ (b) $y_A=2,\qquad x_B=e^{1/2}-1$ (c) $-1<x<e^{-1/4}-1\quad\text{or}\quad x>e^{5/4}-1$

# Question 10

9 marks | Differentiation Techniques

**10.** **In this question you must show all stages of your working.**

**Solutions relying on calculator technology are not acceptable.**

A curve $C$ has equation

$$x = \sin^2 4y \qquad 0 \leqslant y \leqslant \frac{\pi}{8} \qquad 0 \leqslant x \leqslant 1$$

The point $P$ with $x$ coordinate $\frac{1}{4}$ lies on $C$

(a) Find the exact $y$ coordinate of $P$

**(2)**

(b) Find $\frac{dx}{dy}$

**(2)**

(c) Hence show that $\frac{dy}{dx}$ can be written in the form

$$\frac{dy}{dx} = \frac{1}{\sqrt{q + r(x + s)^2}}$$

where $q$, $r$ and $s$ are constants to be found.

**(3)**

Using the answer to part (c),

(d) (i) state the $x$ coordinate of the point where the value of $\frac{dy}{dx}$ is a minimum,

(ii) state the value of $\frac{dy}{dx}$ at this point.

**(2)**

## 1 Part (a)

At $x = 1/4$, $\sin^2 4y = 1/4$. Since $0 \le 4y \le \pi/2$,

$$\sin 4y = \frac{1}{2} \quad \Longrightarrow \quad 4y = \frac{\pi}{6}.$$

Only this branch lies in the given range of $y$.

$$\boxed{y = \frac{\pi}{24}}$$

## 2 Part (b)

Differentiate $x = (\sin 4y)^2$ with respect to $y$:

$$\frac{dx}{dy} = 2\sin 4y\,(4\cos 4y).$$

$$\boxed{\frac{dx}{dy} = 8\sin 4y\cos 4y}$$

## 3 Part (c)

On the interior of the given interval, $\sin 4y = \sqrt{x}$ and $\cos 4y = \sqrt{1-x}$ are positive. Taking the reciprocal,

$$\frac{dy}{dx} = \frac{1}{8\sqrt{x}\sqrt{1-x}} = \frac{1}{\sqrt{64x - 64x^2}}.$$

Complete the square:

$$64x - 64x^2 = 16 - 64\left(x - \frac{1}{2}\right)^2.$$

Thus $dy/dx = (q + r(x+s)^2)^{-1/2}$ has the constants below, with $0 < x < 1$ for a finite derivative; the endpoints have vertical tangents.

$$\boxed{q = 16, \qquad r = -64, \qquad s = -\frac{1}{2}}$$

## 4 Part (d)

The derivative is positive. Its minimum occurs when the square root in the denominator is largest. Since $(x - 1/2)^2 \ge 0$, the largest radicand is 16, attained at $x = 1/2$. Hence the smallest gradient is $1/\sqrt{16}$.

$$\boxed{\text{(i) } x = \frac{1}{2}, \qquad \text{(ii) } \min \frac{dy}{dx} = \frac{1}{4}}$$

**FINAL ANSWER**

(a) $y = \frac{\pi}{24}$ (b) $\frac{dx}{dy} = 8\sin 4y \cos 4y$ (c) $q = 16, \qquad r = -64, \qquad s = -\frac{1}{2}$ (d)
(i) $x = \frac{1}{2}$, (ii) $\min \frac{dy}{dx} = \frac{1}{4}$