Elite Mathematics
Edexcel International A Level

IAL PURE 3 · WMA13

WMA13/01 May/June 2023 Worked Solutions

May/June 2023 paper rebuilt with the worked solution after each question.

10	2020–2026	Single-paper solutions
paper questions	19 exam sessions	checked by Dr Eslam



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PAST PAPER

WMA13/01 May/June 2023

May/June 2023

10	75	1 h 30
questions	marks	exam time

Each question is followed by its worked solution and final answer.

WMA13/01 MAY/JUNE 2023

Question 1

5 marks | Numerical Methods

1. $g(x) = x^6 + 2x - 1000$

(a) Show that $g(x) = 0$ has a root α in the interval $[3, 4]$

(2)

Using the iteration formula

$$x_{n+1} = \sqrt[6]{1000 - 2x_n} \qquad \text{with } x_1 = 3$$

(b) (i) find, to 4 decimal places, the value of x_2

(ii) find, by repeated iteration, the value of α.
Give your answer to 4 decimal places.

(3)

WORKED SOLUTION

Question 1 · WMA13/01 May/June 2023

1 Part (a)

The polynomial g is continuous on $[3, 4]$.

$$g(3) = 729 + 6 - 1000 = -265 < 0, \qquad g(4) = 4096 + 8 - 1000 = 3104 > 0.$$

There is a sign change, so the intermediate value theorem gives a root strictly between 3 and 4.

$$\boxed{3 < \alpha < 4}$$

2 Part (b)

(i) With $x_1 = 3$, $x_2 = \sqrt[6]{994} = 3.1591074477\ldots$. (ii) Retaining unrounded values gives

$$x_3 = 3.1589388680\ldots, \quad x_4 = 3.1589390467\ldots, \quad x_5 = 3.1589390465\ldots.$$

Further iteration leaves the displayed four decimal places unchanged.

$$\boxed{x_2 = 3.1591, \qquad \alpha = 3.1589}$$

FINAL ANSWER

(a) $3 < \alpha < 4$ (b) $x_2 = 3.1591, \qquad \alpha = 3.1589$

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Question 2

6 marks | Log Graphs & Exponential Models

2.

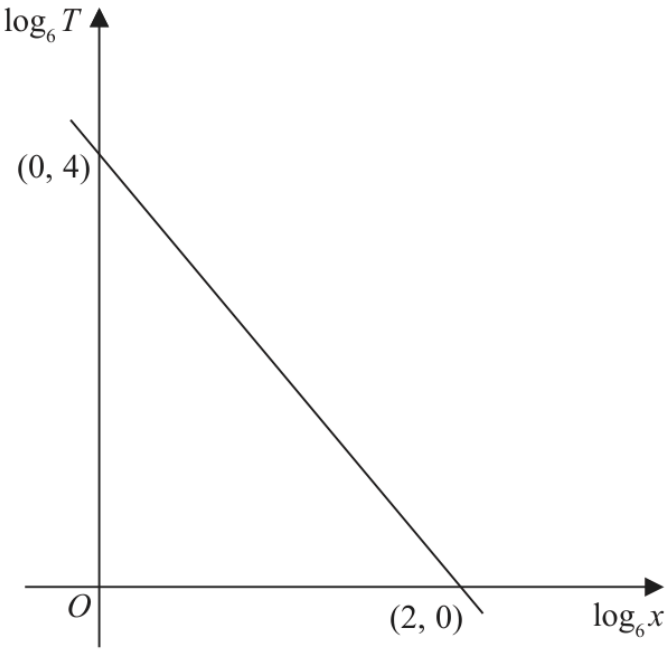


Figure 1

Figure 1 shows the linear relationship between $\log_6 T$ and $\log_6 x$

The line passes through the points (0, 4) and (2, 0) as shown.

(a) (i) Find an equation linking $\log_6 T$ and $\log_6 x$

(ii) Hence find the exact value of T when $x = 216$

(3)

(b) Find an equation, not involving logs, linking T with x

(3)

WORKED SOLUTION

Question 2 · WMA13/01 May/June 2023

1 **Part (a)**

(i) Write $Y = \log_6 T$, $X = \log_6 x$. The gradient is $(0-4)/(2-0) = -2$ and the intercept is 4, so $Y = 4 - 2X$. (ii) Since $216 = 6^3$, $\log_6 T = 4 - 2(3) = -2$, hence $T = 6^{-2} = 1/36$.

$$\boxed{\log_6 T = 4 - 2\log_6 x, \qquad T = \frac{1}{36}}$$

2 **Part (b)**

The log equation gives $\log_6 T + 2\log_6 x = 4$. For $x, T > 0$, combine logs to obtain $\log_6(Tx^2) = 4$. Thus $Tx^2 = 6^4 = 1296$, or $T = 1296/x^2$.

$$\boxed{T = \frac{1296}{x^2}, \qquad x > 0}$$

✓ **FINAL ANSWER**

(a) $\log_6 T = 4 - 2\log_6 x, \qquad T = \frac{1}{36}$ (b) $T = \frac{1296}{x^2}, \qquad x > 0$

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Question 3

7 marks | Differentiation Techniques

3. (i) Find $\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\sin^2 3x\right)$ writing your answer in simplest form.

(2)

(ii)(a) Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(3x^2-4\right)^6$

(2)

(b) Hence show that

$$\int_0^{\sqrt{2}} x\left(3x^2-4\right)^5 \mathrm{d}x = R$$

where R is an integer to be found.

(Solutions relying on calculator technology are not acceptable.)

(3)

WORKED SOLUTION

Question 3 · WMA13/01 May/June 2023

1 Part (i)

Differentiate the logarithm directly on its domain $\sin 3x \neq 0$:

$$\frac{d}{dx}\ln(\sin^2 3x) = \frac{2\sin 3x\cos 3x\cdot 3}{\sin^2 3x} = 6\cot 3x.$$

This does not assume $\sin 3x > 0$; the logarithm's argument is its square.

$$\boxed{6\cot 3x}$$

2 Part (ii)(a)

Apply the power and chain rules:

$$\frac{d}{dx}(3x^2-4)^6 = 6(3x^2-4)^5(6x) = 36x(3x^2-4)^5.$$

$$\boxed{36x(3x^2-4)^5}$$

3 Part (ii)(b)

From part (ii)(a), an antiderivative of $x(3x^2-4)^5$ is $(3x^2-4)^6/36$.

$$R = \left[\frac{(3x^2-4)^6}{36}\right]_0^{\sqrt{2}} = \frac{2^6-(-4)^6}{36} = \frac{64-4096}{36} = -112.$$

$$\boxed{R = -112}$$

FINAL ANSWER

(i) $6\cot 3x$ (ii)(a) $36x(3x^2-4)^5$ (ii)(b) $R=-112$

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Question 4

6 marks | Functions, Composition & Inverses

4. The function f is defined by

$$f(x) = 2x^2 - 5 \qquad x \geqslant 0 \quad x \in \mathbb{R}$$

(a) State the range of f

(1)

On the following page there is a diagram, labelled Diagram 1, which shows a sketch of the curve with equation $y = f(x)$.

(b) On Diagram 1, sketch the curve with equation $y = f^{-1}(x)$.

(2)

The curve with equation $y = f(x)$ meets the curve with equation $y = f^{-1}(x)$ at the point P

Using algebra and showing your working,

(c) find the exact x coordinate of P

(3)

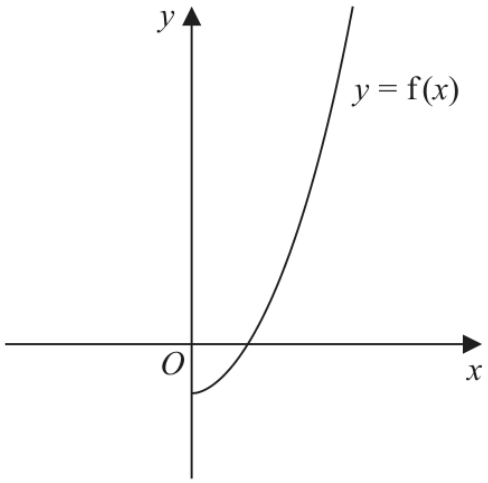


Diagram 1

WORKED SOLUTION

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1 **Part (a)**

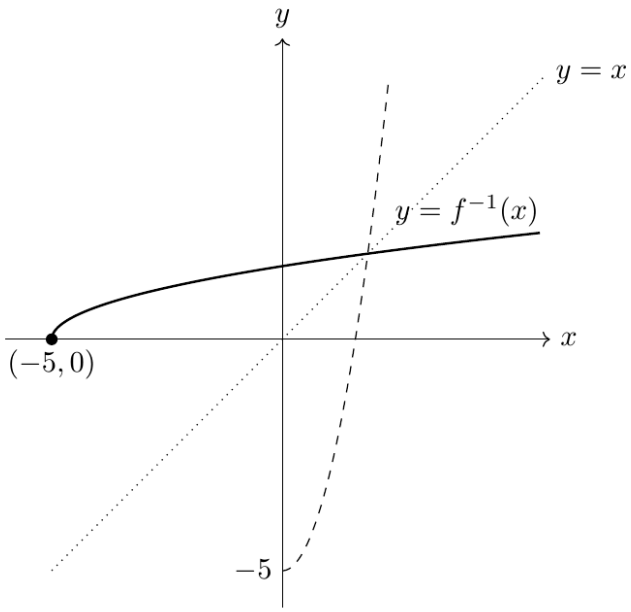
Since $x \geq 0$, $2x^2 \geq 0$. The minimum is $f(0) = -5$ and there is no upper bound.

$$\boxed{\text{range}(f) = [-5, \infty)}$$

2 Part (b)

Reflect the given curve in $y = x$. Solving $x = 2y^2 - 5$ with $y \geq 0$ gives $y = \sqrt{(x+5)/2}$, $x \geq -5$. The inverse starts at $(-5, 0)$, passes through $(0, \sqrt{5/2})$, and is increasing and concave down. The solid inverse curve below is paired with the dashed original curve.

$$\boxed{f^{-1}(x) = \sqrt{\frac{x+5}{2}}, \quad x \geq -5}$$



3 Part (c)

Here f is strictly increasing on $[0, \infty)$, so an intersection of f and its inverse must be on $y = x$: if $y = f(x) > x$, then $f(y) > f(x) = y$, contradicting $f(y) = x$; the case $y < x$ is similar. Hence

$$2x^2 - 5 = x \quad \Longrightarrow \quad 2x^2 - x - 5 = 0 \quad \Longrightarrow \quad x = \frac{1 \pm \sqrt{41}}{4}.$$

Reject the negative root because the original curve has $x \geq 0$.

$$\boxed{x_P = \frac{1 + \sqrt{41}}{4}}$$

✓ FINAL ANSWER

(a) $\mathbf{range}(f) = [-5, \infty)$ (b) $f^{-1}(x) = \sqrt{\frac{x+5}{2}}, \quad x \geq -5$ (c) $x_P = \frac{1+\sqrt{41}}{4}$

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Question 5

7 marks | Trig Equations & R-Form

5. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(i) Solve, for $0 < x < \pi$

$$(x - 2)\left(\sqrt{3}\sec x + 2\right) = 0$$

(3)

(ii) Solve, for $0 < \theta < 360°$

$$10\sin\theta = 3\cos 2\theta$$

(4)

WORKED SOLUTION

Question 5 · WMA13/01 May/June 2023

1 Part (i)

The zero-product rule gives $x - 2 = 0$ or $\sqrt{3}\sec x + 2 = 0$. The first gives $x = 2$, which lies in $(0, \pi)$ and has $\cos x \neq 0$. The second gives $\cos x = -\sqrt{3}/2$; the only angle in the interval is $5\pi/6$.

$$\boxed{x = 2, \quad x = \frac{5\pi}{6}}$$

2 Part (ii)

Use $\cos 2\theta = 1 - 2\sin^2\theta$ and set $s = \sin\theta$:

$$10s = 3 - 6s^2 \quad \Longrightarrow \quad 6s^2 + 10s - 3 = 0 \quad \Longrightarrow \quad s = \frac{-5 \pm \sqrt{43}}{6}.$$

The negative root is less than -1, so reject it. The positive root is $0.2595730874\ldots$. Its principal angle is $15.04473218\ldots°$; the other solution is $180° - 15.04473218\ldots°$. Both are in the given interval. To three significant figures:

$$\boxed{\theta = 15.0°, \quad 165°}$$

FINAL ANSWER

(i) $x = 2, \quad x = \frac{5\pi}{6}$ (ii) $\theta = 15.0°, \quad 165°$

Question 6

9 marks | Modulus

6.

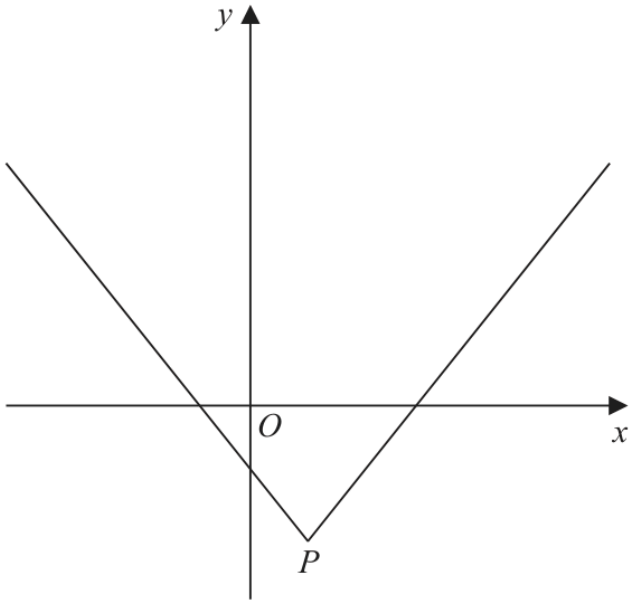


Figure 2

Figure 2 shows a sketch of the graph $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 3|x - 2| - 10$$

The vertex of the graph is at point P, shown in Figure 2.

(a) Find the coordinates of P

(2)

(b) Find $\mathrm{ff}(0)$

(2)

(c) Solve the inequality

$$3|x - 2| - 10 < 5x + 10$$

(2)

(d) Solve the equation

$$\mathrm{f}(|x|) = 0$$

(3)

1 Part (a)

The modulus is smallest when $x - 2 = 0$, giving $x = 2$ and $f(2) = -10$. The two straight-line branches meet there, so this is the vertex of the V-shaped graph.

$$\boxed{P = (2, -10)}$$

2 Part (b)

First $f(0) = 3|0 - 2| - 10 = -4$. Then $ff(0) = f(-4) = 3|-4 - 2| - 10 = 18 - 10 = 8$. In particular $ff(0)$ means $f(f(0))$, not the square of $f(0)$.

$$\boxed{ff(0) = 8}$$

3 Part (c)

For $x < 2$, the inequality becomes $3(2 - x) - 10 < 5x + 10$, so $-4 - 3x < 5x + 10$, giving $x > -7/4$. This yields $-7/4 < x < 2$. For $x \geq 2$, it becomes $3x - 16 < 5x + 10$, giving $x > -13$; every $x \geq 2$ satisfies this. The union is $x > -7/4$.

$$\boxed{x > -\frac{7}{4}}$$

4 Part (d)

Let $u = |x| \geq 0$. Then $f(|x|) = 0$ gives

$$3|u - 2| - 10 = 0 \quad \Longrightarrow \quad u - 2 = \pm\frac{10}{3}.$$

Thus $u = 16/3$ or $u = -4/3$. Reject $u = -4/3$ because $u \geq 0$; $|x| = 16/3$ has the two solutions below.

$$\boxed{x = -\frac{16}{3}, \quad \frac{16}{3}}$$

FINAL ANSWER

(a) $P = (2, -10)$ (b) $ff(0) = 8$ (c) $x > -\frac{7}{4}$ (d) $x = -\frac{16}{3}, \quad \frac{16}{3}$

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Question 7

8 marks | Log Graphs & Exponential Models

7. A scientist is studying two different populations of bacteria.

The number of bacteria N in the first population is modelled by the equation

$$N = Ae^{kt} \qquad t \geqslant 0$$

where A and k are positive constants and t is the time in hours from the start of the study.

Given that

- there were 2500 bacteria in this population at the start of the study
- there were 10 000 bacteria 8 hours later

(a) find the exact value of A and the value of k to 4 significant figures.

(3)

The number of bacteria N in the second population is modelled by the equation

$$N = 60\,000e^{-0.6t} \qquad t \geqslant 0$$

where t is the time in hours from the start of the study.

(b) Find the rate of decrease of bacteria in this population exactly 5 hours from the start of the study. Give your answer to 3 significant figures.

(2)

When $t = T$, the number of bacteria in the two different populations was the same.

(c) Find the value of T, giving your answer to 3 significant figures.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(3)

WORKED SOLUTION

Question 7 · WMA13/01 May/June 2023

1 **Part (a)**

At $t = 0$, $2500 = Ae^0$, so $A = 2500$ exactly. At $t = 8$, $10000 = 2500e^{8k}$; hence $e^{8k} = 4$ and $k = \ln 4/8 = 0.1732867951\ldots$. To four significant figures:

$$\boxed{A = 2500, \qquad k = 0.1733}$$

2 Part (b)

For the second population,

$$\frac{dN}{dt} = -36000e^{-0.6t}, \qquad \left.\frac{dN}{dt}\right|_{t=5} = -36000e^{-3} = -1792.334461\ldots.$$

The requested rate of decrease is the positive magnitude, to three significant figures.

$$\boxed{1.79 \times 10^3 \text{ bacteria per hour}}$$

3 Part (c)

Retain the exact value $k = \ln 4/8$. Equating populations gives

$$2500e^{kT} = 60000e^{-0.6T} \quad \Longrightarrow \quad e^{(k+0.6)T} = 24.$$

Take natural logs and solve:

$$T = \frac{\ln 24}{0.6 + \ln 4/8} = 4.109799689\ldots \text{ hours}.$$

$$\boxed{T = 4.11 \text{ hours}}$$

FINAL ANSWER

(a) $A = 2500, \qquad k = 0.1733$ (b) 1.79×10^3 bacteria per hour (c) $T = 4.11$ hours

Question 8

9 marks | Applications of Differentiation

8.

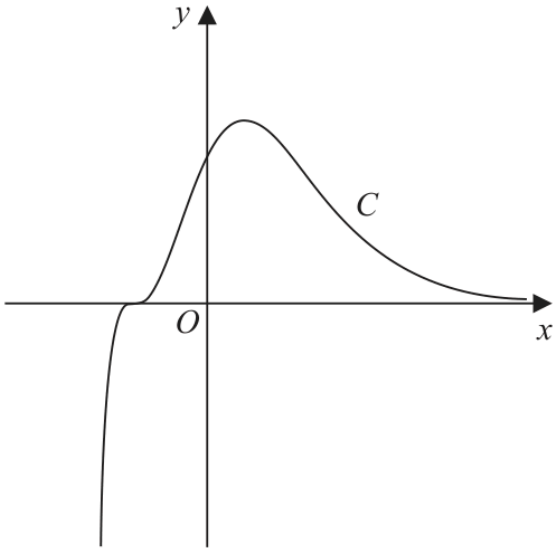


Figure 3

Figure 3 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = (2x + 1)^3 \mathrm{e}^{-4x}$$

(a) Show that

$$\mathrm{f}'(x) = A(2x + 1)^2 (1 - 4x)\, \mathrm{e}^{-4x}$$

where A is a constant to be found.

(4)

(b) Hence find the exact coordinates of the two stationary points on C.

(3)

The function g is defined by

$$\mathrm{g}(x) = 8\,\mathrm{f}(x - 2)$$

(c) Find the coordinates of the maximum stationary point on the curve with equation $y = \mathrm{g}(x)$.

(2)

1 Part (a)

The product rule gives

$$f'(x) = 6(2x+1)^2e^{-4x} - 4(2x+1)^3e^{-4x} = (2x+1)^2e^{-4x}[6-4(2x+1)] = 2(2x+1)^2(1-4x)e^{-4x}.$$

Comparison with the stated form gives $A = 2$.

$$\boxed{A = 2}$$

2 Part (b)

Since $e^{-4x} > 0$, $f'(x) = 0$ means $2x+1=0$ or $1-4x=0$. Thus $x = -1/2$ or $x = 1/4$.
Substitution gives $f(-1/2) = 0$ and $f(1/4) = (3/2)^3e^{-1} = 27/(8e)$.

$$\boxed{\left(-\frac{1}{2}, 0\right), \quad \left(\frac{1}{4}, \frac{27}{8e}\right)}$$

3 Part (c)

The derivative changes from positive to negative at $x = 1/4$, so this is the maximum; at $x = -1/2$ it does not change sign. The transformation $g(x) = 8f(x-2)$ sends (x, y) to $(x+2, 8y)$. Hence the maximum stationary point is $(1/4 + 2, 8 \cdot 27/(8e)) = (9/4, 27/e)$.

$$\boxed{\left(\frac{9}{4}, \frac{27}{e}\right)}$$

FINAL ANSWER

(a) $A = 2$ (b) $\left(-\frac{1}{2}, 0\right), \quad \left(\frac{1}{4}, \frac{27}{8e}\right)$ (c) $\left(\frac{9}{4}, \frac{27}{e}\right)$

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Question 9

10 marks | Trig Identities & Exact Values

9. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that

$$\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} \equiv \operatorname{cosec} x \qquad x \neq \frac{n\pi}{2} \qquad n \in \mathbb{Z}$$

(3)

(b) Hence solve, for $0 < \theta < \frac{\pi}{2}$

$$\left(\frac{\cos 2\theta}{\sin \theta} + \frac{\sin 2\theta}{\cos \theta}\right)^2 = 6\cot\theta - 4$$

giving your answers to 3 significant figures as appropriate.

(5)

(c) Using the result from part (a), or otherwise, find the exact value of

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x}\right) \cot x \, dx$$

(2)

WORKED SOLUTION

Question 9 · WMA13/01 May/June 2023

1 Part (a)

Using a common denominator,

$$\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} = \frac{\cos 2x \cos x + \sin 2x \sin x}{\sin x \cos x} = \frac{\cos(2x - x)}{\sin x \cos x} = \frac{1}{\sin x} = \csc x.$$

The stated exclusion $x \neq n\pi/2$ ensures both denominators are nonzero.

$$\boxed{\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} = \csc x}$$

2 Part (b)

By part (a), the squared bracket is $\csc^2\theta$. Using $\csc^2\theta = 1 + \cot^2\theta$ and setting $u = \cot\theta$ gives

$$1 + u^2 = 6u - 4 \quad \Longrightarrow \quad u^2 - 6u + 5 = (u-1)(u-5) = 0.$$

On $0 < \theta < \pi/2$, both positive cotangents are permitted: $u = 1$ gives $\theta = \pi/4$, while $u = 5$ gives $\tan\theta = 1/5$, $\theta = 0.1973955598\ldots$ radians.

$$\boxed{\theta = \frac{\pi}{4}, \quad 0.197 \text{ radians}}$$

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3 Part (c)

The bracket in part (a) equals $\csc x$. Therefore

$$\int_{\pi/6}^{\pi/4} \csc x \cot x \, dx = \left[-\csc x\right]_{\pi/6}^{\pi/4} = -\sqrt{2} - (-2) = 2 - \sqrt{2}.$$

$$\boxed{2 - \sqrt{2}}$$

FINAL ANSWER

(a) $\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} = \csc x$ (b) $\theta = \frac{\pi}{4}, \quad 0.197$ radians (c) $2 - \sqrt{2}$

Question 10

8 marks | Differentiation Techniques

10.

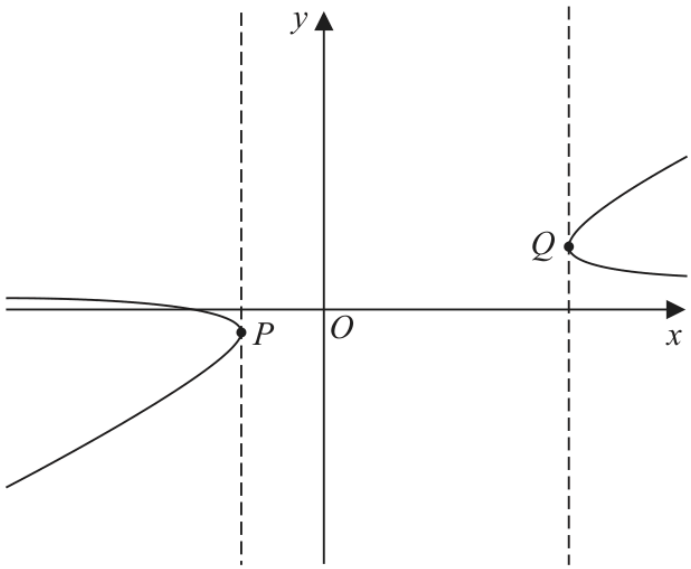


Figure 4

Figure 4 shows a sketch of the curve with equation

$$x = \frac{2y^2 + 6}{3y - 3}$$

(a) Find $\frac{dx}{dy}$ giving your answer as a fully simplified fraction.

(4)

The tangents at points P and Q on the curve are parallel to the y-axis, as shown in Figure 4.

(b) Use the answer to part (a) to find the equations of these two tangents.

(4)

1 Part (a)

The quotient rule gives, for $y \neq 1$,

$$\frac{dx}{dy} = \frac{4y(3y-3) - 3(2y^2+6)}{(3y-3)^2} = \frac{6y^2 - 12y - 18}{9(y-1)^2} = \frac{2(y-3)(y+1)}{3(y-1)^2}.$$

$$\boxed{\frac{dx}{dy} = \frac{2(y-3)(y+1)}{3(y-1)^2}, \quad y \neq 1}$$

2 Part (b)

A tangent parallel to the y-axis is vertical, so $dx/dy = 0$. The numerator gives $(y-3)(y+1) = 0$, hence $y = 3$ or $y = -1$; neither makes the denominator zero. Then

$$x(3) = \frac{18+6}{9-3} = 4, \qquad x(-1) = \frac{2+6}{-3-3} = -\frac{4}{3}.$$

Thus $Q = (4, 3)$ and $P = (-4/3, -1)$, with the vertical tangent equations below.

$$\boxed{x = -\frac{4}{3}, \qquad x = 4}$$

FINAL ANSWER

(a) $\frac{dx}{dy} = \frac{2(y-3)(y+1)}{3(y-1)^2}, \quad y \neq 1$ (b) $x = -\frac{4}{3}, \qquad x = 4$