Elite Mathematics
Edexcel International A Level

IAL PURE 3 · WMA13

WMA13/01 January 2023 Worked Solutions

January 2023 paper rebuilt with the worked solution after each question.

10	2020–2026	Single-paper solutions
paper questions	19 exam sessions	checked by Dr Eslam



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PAST PAPER

WMA13/01 January 2023

January 2023

10	75	1 h 30
questions	marks	exam time

Each question is followed by its worked solution and final answer.

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Question 1

6 marks | Functions, Composition & Inverses

1. The functions f and g are defined by

$$\mathrm{f}(x) = 9 - x^2 \qquad x \in \mathbb{R} \quad x \geqslant 0$$

$$\mathrm{g}(x) = \frac{3}{2x+1} \qquad x \in \mathbb{R} \quad x \geqslant 0$$

(a) Write down the range of f

(1)

(b) Find the value of $\mathrm{fg}(1.5)$

(2)

(c) Find g^{-1}

(3)

WORKED SOLUTION

Question 1 · WMA13/01 January 2023

1 **Part (a)**

For $x \geq 0$, $f(x) = 9 - x^2$ has maximum $f(0) = 9$ and decreases without a lower bound.

$$\boxed{\text{range}(f) = (-\infty, 9]}$$

2 **Part (b)**

Apply g before f:

$$g(1.5) = \frac{3}{2(1.5)+1} = \frac{3}{4}, \qquad fg(1.5) = f(3/4) = 9 - \left(\frac{3}{4}\right)^2 = \frac{135}{16}.$$

The intermediate value $3/4$ is in the nonnegative domain of f.

$$\boxed{fg(1.5) = \frac{135}{16}}$$

3 **Part (c)**

Let $y = 3/(2x+1)$. Then $y(2x+1) = 3$, so $2xy = 3 - y$ and $x = (3-y)/(2y)$. Exchange the variables. Since g decreases from 3 towards 0 on $[0, \infty)$, its range is $(0, 3]$, which is the inverse domain.

$$\boxed{g^{-1}(x) = \frac{3-x}{2x}, \qquad 0 < x \leq 3}$$

FINAL ANSWER

(a) $\textbf{range}(f) = (-\infty, 9]$ (b) $fg(1.5) = \frac{135}{16}$ (c) $g^{-1}(x) = \frac{3-x}{2x}, \qquad 0 < x \leq 3$

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Question 2

6 marks | Trig Equations & R-Form

2. $$\mathrm{f}(x) = \cos x + 2\sin x$$

(a) Express $\mathrm{f}(x)$ in the form $R\cos(x - \alpha)$, where R and α are constants,

$R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

$$\mathrm{g}(x) = 3 - 7\mathrm{f}(2x)$$

(b) Using the answer to part (a),

(i) write down the exact maximum value of $\mathrm{g}(x)$,

(ii) find the smallest positive value of x for which this maximum value occurs, giving your answer to 2 decimal places.

(3)

WORKED SOLUTION

Question 2 · WMA13/01 January 2023

1 Part (a)

Expand $R\cos(x-\alpha) = R\cos\alpha\cos x + R\sin\alpha\sin x$. Comparing coefficients gives $R\cos\alpha = 1$ and $R\sin\alpha = 2$. Thus $R^2 = 1^2 + 2^2 = 5$ and $\tan\alpha = 2$. The specified first-quadrant phase is $\alpha = \arctan 2 = 1.1071487\ldots$ radians.

$$\boxed{R = \sqrt{5}, \qquad \alpha = 1.107 \text{ radians}}$$

2 Part (b)

Using the exact phase $\alpha = \arctan 2$, write $g(x) = 3 - 7\sqrt{5}\cos(2x - \alpha)$. (i) The negative cosine coefficient means the maximum occurs when the cosine is -1; the maximum is $3 + 7\sqrt{5}$. (ii) $2x - \alpha = \pi + 2n\pi$, so $x = (\pi + \alpha)/2 + n\pi$. The $n = -1$ value is negative; $n = 0$ is the smallest positive one, $x = 2.1243706857\ldots$, or 2.12 to two decimal places.

$$\boxed{\max g = 3 + 7\sqrt{5}, \qquad x = 2.12}$$

FINAL ANSWER

(a) $R = \sqrt{5}, \qquad \alpha = 1.107$ radians (b) $\max g = 3 + 7\sqrt{5}, \qquad x = 2.12$

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Question 3

5 marks | Log Graphs & Exponential Models

3.

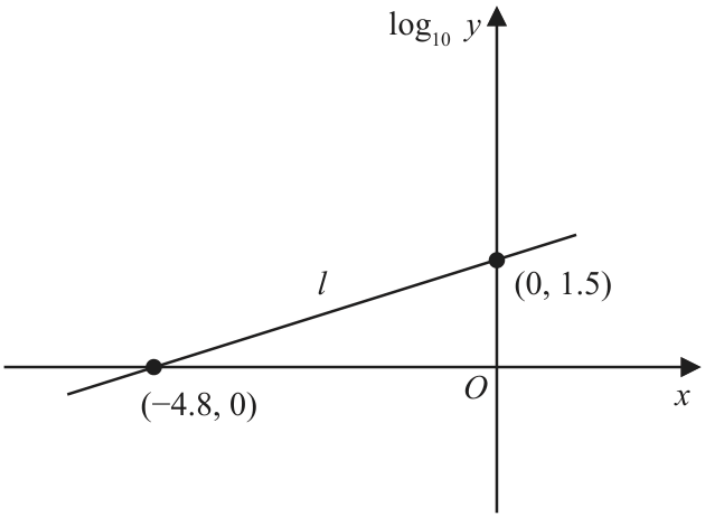


Figure 1

The line l in Figure 1 shows a linear relationship between $\log_{10} y$ and x.

The line passes through the points (0, 1.5) and (−4.8, 0) as shown.

(a) Write down an equation for l.

(2)

(b) Hence, or otherwise, express y in the form kb^x, giving the values of the constants k and b to 3 significant figures.

(3)

WORKED SOLUTION

Question 3 · WMA13/01 January 2023

1 Part (a)

The gradient in the $(x, \log_{10} y)$ coordinates is $(1.5 - 0)/(0 - (-4.8)) = 1.5/4.8 = 5/16$. The vertical intercept is $1.5 = 3/2$, giving the equation below.

$$\boxed{\log_{10} y = \frac{5}{16}x + \frac{3}{2}}$$

2 **Part (b)**

Exponentiating base 10 gives

$$y = 10^{(5/16)x + 3/2} = 10^{3/2}(10^{5/16})^x.$$

Therefore $k = 10^{3/2} = 31.62277660\ldots$ and $b = 10^{5/16} = 2.053525026\ldots$. To three significant figures the model constants are 31.6 and 2.05.

$$\boxed{k = 31.6, \qquad b = 2.05 \quad \left(y \approx 31.6(2.05)^x\right)}$$

✓ **FINAL ANSWER**

(a) $\log_{10} y = \frac{5}{16}x + \frac{3}{2}$ (b) $k = 31.6, \qquad b = 2.05 \quad \left(y \approx 31.6(2.05)^x\right)$

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Question 4

7 marks | Rational Algebra

4. $$\mathrm{f}(x) = \frac{2x^4 + 15x^3 + 35x^2 + 21x - 4}{(x+3)^2} \qquad x \in \mathbb{R} \quad x > -3$$

(a) Find the values of the constants A, B, C and D such that

$$\mathrm{f}(x) = Ax^2 + Bx + C + \frac{D}{(x+3)^2}$$

(4)

(b) Hence find,

$$\int \mathrm{f}(x)\,\mathrm{d}x$$

(3)

WORKED SOLUTION

Question 4 · WMA13/01 January 2023

1 Part (a)

Multiply the proposed identity by $(x+3)^2 = x^2 + 6x + 9$ and compare coefficients:

$$A = 2, \quad 6A + B = 15 \Rightarrow B = 3, \quad 9A + 6B + C = 35 \Rightarrow C = -1.$$

The coefficient of x checks: $9B + 6C = 27 - 6 = 21$. The constant equation is $9C + D = -4$, so $D = 5$. Hence $f(x) = 2x^2 + 3x - 1 + 5(x+3)^{-2}$.

$$\boxed{A = 2, \quad B = 3, \quad C = -1, \quad D = 5}$$

2 Part (b)

Integrate each term in the decomposition:

$$\int f(x)\,dx = \int \left[2x^2 + 3x - 1 + 5(x+3)^{-2}\right] dx = \frac{2}{3}x^3 + \frac{3}{2}x^2 - x - \frac{5}{x+3} + K.$$

Here K is an arbitrary constant; the domain $x > -3$ avoids the denominator zero.

$$\boxed{\frac{2}{3}x^3 + \frac{3}{2}x^2 - x - \frac{5}{x+3} + K}$$

FINAL ANSWER

(a) $A = 2, \quad B = 3, \quad C = -1, \quad D = 5$ (b) $\frac{2}{3}x^3 + \frac{3}{2}x^2 - x - \frac{5}{x+3} + K$

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Question 5

9 marks | Trig Equations & R-Form

5. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Prove that

$$\cot^2 x - \tan^2 x \equiv 4\cot 2x \operatorname{cosec} 2x \qquad x \neq \frac{n\pi}{2} \quad n \in \mathbb{Z}$$

(4)

(b) Hence solve, for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$4\cot 2\theta \operatorname{cosec} 2\theta = 2\tan^2 \theta$$

giving your answers to 2 decimal places.

(5)

WORKED SOLUTION

Question 5 · WMA13/01 January 2023

1 Part (a)

For $x \neq n\pi/2$, both sine and cosine are nonzero. Then

$$\cot^2 x - \tan^2 x = \frac{\cos^4 x - \sin^4 x}{\sin^2 x \cos^2 x} = \frac{(\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)}{\sin^2 x \cos^2 x} = \frac{\cos 2x}{\frac{1}{4}\sin^2 2x} = 4\frac{\cos 2x}{\sin 2x}\frac{1}{\sin 2x} = 4\cot 2x \csc 2x.$$

$$\boxed{\cot^2 x - \tan^2 x = 4\cot 2x \csc 2x}$$

2 Part (b)

By part (a), the equation is $\cot^2\theta - \tan^2\theta = 2\tan^2\theta$, so $\cot^2\theta = 3\tan^2\theta$. The original expression excludes $\theta = 0$ in the stated interval, so $\tan\theta \neq 0$ and multiplication by $\tan^2\theta$ gives

$$1 = 3\tan^4\theta \quad \Longrightarrow \quad \tan\theta = \pm 3^{-1/4}.$$

Tangent is one-to-one on $(-\pi/2, \pi/2)$; the two angles are $\pm\arctan(3^{-1/4}) = \pm 0.6497662865\ldots$ radians. Both are valid and nonzero. To two decimal places:

$$\boxed{\theta = -0.65, \quad 0.65 \text{ radians}}$$

FINAL ANSWER

(a) $\cot^2 x - \tan^2 x = 4 \cot 2x \csc 2x$ (b) $\theta = -0.65, \quad 0.65 \text{ radians}$

Question 6

8 marks | Modulus

6.

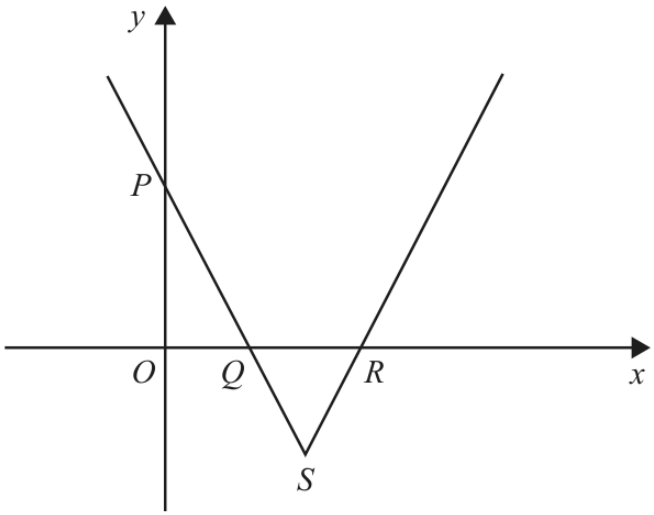


Figure 2

Figure 2 shows a sketch of the graph with equation

$$y = |3x - 5a| - 2a$$

where a is a positive constant.

The graph

- cuts the y-axis at the point P
- cuts the x-axis at the points Q and R
- has a minimum point at S

(a) Find, in simplest form in terms of a, the coordinates of

(i) point P

(ii) points Q and R

(iii) point S

(4)

(b) Find, in simplest form in terms of a, the values of x for which

$$|3x - 5a| - 2a = |x - 2a|$$

(4)

1 Part (a)

For $y = |3x - 5a| - 2a$ with $a > 0$: (i) At $x = 0$, $y = 5a - 2a = 3a$, so $P = (0, 3a)$. (ii) At $y = 0$, $|3x - 5a| = 2a$ gives $3x - 5a = \pm 2a$. Thus $x = a$ or $7a/3$; since $a < 7a/3$, $Q = (a, 0)$ and $R = (7a/3, 0)$. (iii) The modulus vanishes at $x = 5a/3$, giving the minimum $S = (5a/3, -2a)$.

$$\boxed{P = (0, 3a), \quad Q = (a, 0), \quad R = (7a/3, 0), \quad S = (5a/3, -2a)}$$

2 Part (b)

Split $|3x - 5a| - 2a = |x - 2a|$ at $x = 5a/3$ and $x = 2a$; their order uses $a > 0$. For $x < 5a/3$: $3a - 3x = 2a - x$, so $x = a/2$, which is in this region. For $5a/3 \le x < 2a$: $3x - 7a = 2a - x$, so $x = 9a/4$; this is outside the assumed region and is rejected. For $x \ge 2a$: $3x - 7a = x - 2a$, so $x = 5a/2$, which is valid. These three regions cover the whole real line, so there are exactly two solutions.

$$\boxed{x = \frac{a}{2}, \qquad x = \frac{5a}{2}}$$

FINAL ANSWER

(a) $P = (0, 3a), \quad Q = (a, 0), \quad R = (7a/3, 0), \quad S = (5a/3, -2a)$ (b) $x = \frac{a}{2}, \qquad x = \frac{5a}{2}$

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Question 7

9 marks | Applications of Differentiation

7. The curve C has equation

$$x = 3\tan\left(y - \frac{\pi}{6}\right) \qquad x \in \mathbb{R} \quad -\frac{\pi}{3} < y < \frac{2\pi}{3}$$

(a) Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a}{x^2 + b}$$

where a and b are integers to be found.

(4)

The point P with y coordinate $\frac{\pi}{3}$ lies on C.

Given that the tangent to C at P crosses the x-axis at the point Q.

(b) find, in simplest form, the exact x coordinate of Q.

(5)

WORKED SOLUTION

Question 7 · WMA13/01 January 2023

1 **Part (a)**

Put $u = y - \pi/6$. Since $x = 3\tan u$,

$$\frac{dx}{dy} = 3\sec^2 u = 3(1 + \tan^2 u) = 3\left(1 + \frac{x^2}{9}\right) = \frac{x^2 + 9}{3}.$$

This is positive for all real x, so we may take the reciprocal: $dy/dx = 3/(x^2 + 9)$. Thus the required integers are $a = 3$, $b = 9$.

$$\boxed{\frac{dy}{dx} = \frac{3}{x^2 + 9}, \qquad a = 3, \quad b = 9}$$

2 **Part (b)**

At $y = \pi/3$, $x = 3\tan(\pi/3 - \pi/6) = 3\tan(\pi/6) = \sqrt{3}$. The tangent gradient is $3/((\sqrt{3})^2 + 9) = 1/4$. Its point-gradient equation is

$$y - \frac{\pi}{3} = \frac{1}{4}(x - \sqrt{3}).$$

At its x-axis crossing $y = 0$, $-4\pi/3 = x - \sqrt{3}$.

$$\boxed{x_Q = \sqrt{3} - \frac{4\pi}{3}}$$

✓ **FINAL ANSWER**

(a) $\frac{dy}{dx} = \frac{3}{x^2+9}$, $\quad a = 3, \quad b = 9$ (b) $x_Q = \sqrt{3} - \frac{4\pi}{3}$

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Question 8

5 marks | Integration

8. Find, in simplest form,

$$\int (2\cos x - \sin x)^2 \, \mathrm{d}x$$

(5)

WORKED SOLUTION

Question 8 · WMA13/01 January 2023

1 Part (main)

Expand the square first:

$$(2\cos x - \sin x)^2 = 4\cos^2 x - 4\sin x \cos x + \sin^2 x.$$

Use $\cos^2 x = (1 + \cos 2x)/2$, $\sin^2 x = (1 - \cos 2x)/2$, and $2\sin x \cos x = \sin 2x$. The integrand becomes

$$\frac{5}{2} + \frac{3}{2}\cos 2x - 2\sin 2x.$$

Integrating with the angle-2 chain factors gives $5x/2 + 3\sin 2x/4 + \cos 2x + K$.

$$\boxed{\int (2\cos x - \sin x)^2 \, dx = \frac{5}{2}x + \frac{3}{4}\sin 2x + \cos 2x + K}$$

FINAL ANSWER

(main) $\int (2\cos x - \sin x)^2 \, dx = \frac{5}{2}x + \frac{3}{4}\sin 2x + \cos 2x + K$

Question 9

11 marks | Numerical Methods

9.

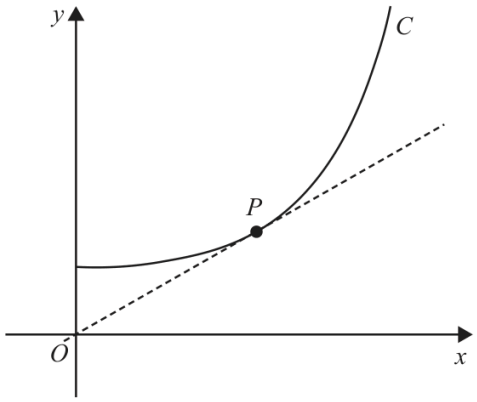


Figure 3

Figure 3 shows a sketch of part of the curve C with equation

$$y = \sqrt{3 + 4e^{x^2}} \qquad x \geqslant 0$$

(a) Find $\dfrac{dy}{dx}$, giving your answer in simplest form.

(2)

The point P with x coordinate α lies on C.

Given that the tangent to C at P passes through the origin, as shown in Figure 3,

(b) show that $x = \alpha$ is a solution of the equation

$$4x^2e^{x^2} - 4e^{x^2} - 3 = 0$$

(3)

(c) Hence show that α lies between 1 and 2

(2)

(d) Show that the equation in part (b) can be written in the form

$$x = \frac{1}{2}\sqrt{4 + 3e^{-x^2}}$$

(1)

The iteration formula

$$x_{n+1} = \frac{1}{2}\sqrt{4 + 3e^{-x_n^2}}$$

with $x_1 = 1$ is used to find an approximation for α.

(e) Use the iteration formula to find, to 4 decimal places, the value of

(i) x_3

(ii) α

(3)

1 Part (a)

Differentiate $y = (3 + 4e^{x^2})^{1/2}$ using the chain rule:

$$\frac{dy}{dx} = \frac{1}{2}(3 + 4e^{x^2})^{-1/2} \cdot 4e^{x^2} \cdot 2x = \frac{4xe^{x^2}}{\sqrt{3 + 4e^{x^2}}}.$$

$$\boxed{\frac{dy}{dx} = \frac{4xe^{x^2}}{\sqrt{3 + 4e^{x^2}}}}$$

2 Part (b)

Since the tangent at $P = (\alpha, \sqrt{3 + 4e^{\alpha^2}})$ passes through the origin, its gradient is y_P/α, with $\alpha > 0$. Equate this to the derivative:

$$\frac{4\alpha e^{\alpha^2}}{\sqrt{3 + 4e^{\alpha^2}}} = \frac{\sqrt{3 + 4e^{\alpha^2}}}{\alpha} \quad \Longrightarrow \quad 4\alpha^2 e^{\alpha^2} = 3 + 4e^{\alpha^2}.$$

Consequently $x = \alpha$ solves the displayed equation.

$$\boxed{4x^2e^{x^2} - 4e^{x^2} - 3 = 0}$$

3 Part (c)

Let $h(x) = 4x^2e^{x^2} - 4e^{x^2} - 3$, a continuous function. Then $h(1) = -3 < 0$ and $h(2) = 12e^4 - 3 = 652.1778\ldots > 0$, so a root lies between 1 and 2. Moreover $h'(x) = 8x^3e^{x^2} > 0$ for $x > 0$, so this is the unique positive root corresponding to the tangent abscissa α.

$$\boxed{1 < \alpha < 2}$$

4 Part (d)

From $4x^2e^{x^2} = 4e^{x^2} + 3$, divide by the positive e^{x^2} to get $4x^2 = 4 + 3e^{-x^2}$. The source domain has $x \geq 0$, so take the positive square root.

$$\boxed{x = \frac{1}{2}\sqrt{4 + 3e^{-x^2}}}$$

5 **Part (e)**

With $x_1 = 1$, the unrounded iterates start

$$x_2 = 1.1295616764\ldots, \quad x_3 = 1.0997193683\ldots, \quad x_4 = 1.1062486441\ldots, \quad x_5 = 1.1048012899\ldots.$$

Continue: $x_6 = 1.1051212304\ldots$, $x_7 = 1.1050504628\ldots$, $x_8 = 1.1050661137\ldots$, $x_9 = 1.1050626523\ldots$; further iteration settles at $1.1050632792\ldots$. Therefore (i) $x_3 = 1.0997$ and (ii) $\alpha = 1.1051$, both to four decimal places.

$$\boxed{x_3 = 1.0997, \qquad \alpha = 1.1051}$$

✓ **FINAL ANSWER**

(a) $\frac{dy}{dx} = \frac{4xe^{x^2}}{\sqrt{3+4e^{x^2}}}$ (b) $4x^2e^{x^2} - 4e^{x^2} - 3 = 0$ (c) $1 < \alpha < 2$ (d) $x = \frac{1}{2}\sqrt{4 + 3e^{-x^2}}$ (e) $x_3 = 1.0997, \qquad \alpha = 1.1051$

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Question 10

9 marks | Log Graphs & Exponential Models

10. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

A population of fruit flies is being studied.

The number of fruit flies, F, in the population, t days after the start of the study, is modelled by the equation

$$F = \frac{350e^{kt}}{9 + e^{kt}}$$

where k is a constant.

Use the equation of the model to answer parts (a), (b) and (c).

(a) Find the number of fruit flies in the population at the start of the study.

(1)

Given that there are 200 fruit flies in the population 15 days after the start of the study,

(b) show that $k = \frac{1}{15}\ln 12$

(3)

Given also that, when $t = T$, the number of fruit flies in the population is increasing at a rate of 10 per day,

(c) find the possible values of T, giving your answers to one decimal place.

(5)

WORKED SOLUTION

Question 10 · WMA13/01 January 2023

1 **Part (a)**

At the start $t = 0$, so $e^{kt} = 1$ and the model gives $F(0) = 350/(9 + 1) = 35$ fruit flies.

$$\boxed{F(0) = 35}$$

2 Part (b)

Using $F(15) = 200$,

$$200 = \frac{350e^{15k}}{9 + e^{15k}} \quad \Longrightarrow \quad 1800 + 200e^{15k} = 350e^{15k} \quad \Longrightarrow \quad e^{15k} = 12.$$

Take natural logs to obtain $15k = \ln 12$, as required.

$$\boxed{k = \frac{\ln 12}{15}}$$

3 Part (c)

By the quotient rule,

$$F'(t) = \frac{350ke^{kt}(9 + e^{kt}) - 350ke^{2kt}}{(9 + e^{kt})^2} = \frac{3150ke^{kt}}{(9 + e^{kt})^2}.$$

Set $F'(T) = 10$ and let $u = e^{kT} > 0$. Then

$$315ku = (9 + u)^2 \quad \Longrightarrow \quad u^2 + (18 - 315k)u + 81 = 0.$$

With the exact $k = \ln 12/15$, the quadratic formula gives

$$u = \frac{315k - 18 \pm \sqrt{(315k - 18)^2 - 324}}{2} = 2.5615490777\ldots \text{ or } 31.6214905678\ldots.$$

Both values are positive and at least 1, hence both correspond to $T \geq 0$. Finally $T = \ln u/k$ gives $5.6779528361\ldots$ or $20.8488936858\ldots$ days. To one decimal place the two times are as follows.

$$\boxed{T = 5.7 \text{ days}, \qquad T = 20.8 \text{ days}}$$

FINAL ANSWER

(a) $F(0) = 35$ (b) $k = \frac{\ln 12}{15}$ (c) $T = 5.7$ days, $\qquad T = 20.8$ days