Elite Mathematics
Edexcel International A Level

IAL PURE 3 · WMA13

WMA13/01 October 2022 Worked Solutions

October 2022 paper rebuilt with the worked solution after each question.

9	2020–2026	Single-paper solutions
paper questions	19 exam sessions	checked by Dr Eslam



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PAST PAPER

WMA13/01 October 2022

October 2022

9	75	1 h 30
questions	marks	exam time

Each question is followed by its worked solution and final answer.

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Question 1

9 marks | Integration

1. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

$$\mathrm{f}(x) = \frac{2x^3 - 4x - 15}{x^2 + 3x + 4}$$

(a) Show that

$$\mathrm{f}(x) \equiv Ax + B + \frac{C(2x+3)}{x^2 + 3x + 4}$$

where A, B and C are integers to be found.

(4)

(b) Hence, find

$$\int_3^5 \mathrm{f}(x)\,\mathrm{d}x$$

giving your answer in the form $p + \ln q$, where p and q are integers.

(5)

WORKED SOLUTION

Question 1 · WMA13/01 October 2022

Part (a)

Clear the denominator in the proposed identity:

$$2x^3 - 4x - 15 = (Ax + B)(x^2 + 3x + 4) + C(2x + 3).$$

Matching coefficients gives $A = 2$, $3A + B = 0$, hence $B = -6$. Then $4A + 3B + 2C = -4$ gives $C = 3$; the constant checks as $4B + 3C = -24 + 9 = -15$. Therefore $f(x) = 2x - 6 + 3(2x + 3)/(x^2 + 3x + 4)$.

$$\boxed{A = 2, \quad B = -6, \quad C = 3}$$

2 **Part (b)**

The numerator $2x + 3$ is the derivative of $x^2 + 3x + 4$. This quadratic is $(x + 3/2)^2 + 7/4 > 0$, so

$$\int f(x)\,dx = x^2 - 6x + 3\ln(x^2 + 3x + 4) + K.$$

Using both limits,

$$\int_3^5 f(x)\,dx = (-5 + 3\ln 44) - (-9 + 3\ln 22) = 4 + 3\ln 2 = 4 + \ln 8.$$

Thus $p = 4$ and $q = 8$ are integers, as required.

$$\boxed{4 + \ln 8}$$

FINAL ANSWER

(a) $A = 2, \quad B = -6, \quad C = 3$ (b) $4 + \ln 8$

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Question 2

7 marks | Functions, Composition & Inverses

2. The functions f and g are defined by

$$\mathrm{f}(x) = 5 - \frac{4}{3x+2} \qquad x \geqslant 0$$

$$\mathrm{g}(x) = \left| 4\sin\left(\frac{x}{3} + \frac{\pi}{6}\right) \right| \qquad x \in \mathbb{R}$$

(a) Find the range of f

(2)

(b) (i) Find $\mathrm{f}^{-1}(x)$

(ii) Write down the domain of f^{-1}

(3)

(c) Find $\mathrm{fg}(-\pi)$

(2)

WORKED SOLUTION

Question 2 · WMA13/01 October 2022

1 Part (a)

For $x \geq 0$, $0 < 4/(3x+2) \leq 2$. Thus $f(0) = 3$ is attained, while $f(x)$ increases towards 5 without reaching it.

$$\boxed{3 \leq f(x) < 5}$$

2 Part (b)

(i) Put $y = 5 - 4/(3x+2)$. Rearranging gives $(5-y)(3x+2) = 4$, hence

$$x = \frac{4/(5-y) - 2}{3} = \frac{2y-6}{15-3y}.$$

Exchange variables. (ii) The inverse domain is the range of f found above.

$$\boxed{f^{-1}(x) = \frac{2x-6}{15-3x}, \qquad 3 \leq x < 5}$$

3 Part (c)

The original function has an absolute value and angle $x/3 + \pi/6$:

$$g(-\pi) = |4\sin(-\pi/3 + \pi/6)| = |4\sin(-\pi/6)| = 2.$$

Then $fg(-\pi) = f(2) = 5 - 4/(3\cdot 2 + 2) = 5 - 1/2 = 9/2$.

$$\boxed{fg(-\pi) = \frac{9}{2}}$$

FINAL ANSWER

(a) $3 \le f(x) < 5$ (b) $f^{-1}(x) = \frac{2x-6}{15-3x}, \qquad 3 \le x < 5$ (c) $fg(-\pi) = \frac{9}{2}$

Question 3

7 marks | Applications of Differentiation

3. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

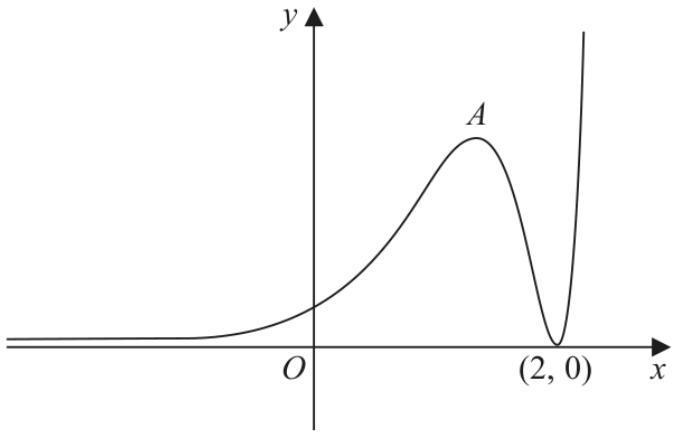


Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = (x-2)^2 \mathrm{e}^{3x} \qquad x \in \mathbb{R}$$

The curve has a maximum turning point at A and a minimum turning point at $(2, 0)$

(a) Use calculus to find the exact coordinates of A.

(5)

Given that the equation $\mathrm{f}(x) = k$, where k is a constant, has **at least** two distinct roots,

(b) state the range of possible values for k.

(2)

1 Part (a)

By the product rule,

$$f'(x) = 2(x-2)e^{3x} + 3(x-2)^2 e^{3x} = e^{3x}(x-2)(3x-4).$$

Since the exponential is positive, stationary points have $x = 2$ or $x = 4/3$. The sign of f' changes from positive to negative at $4/3$, so this is the maximum A.

$$f(4/3) = (-2/3)^2 e^4 = \frac{4}{9}e^4.$$

$$\boxed{A = \left(\frac{4}{3}, \frac{4e^4}{9}\right)}$$

2 Part (b)

The curve is nonnegative and has just one zero, $x = 2$; hence $k = 0$ is excluded. It tends to 0 from above as $x \to -\infty$, rises to $4e^4/9$, falls to 0, then rises without bound. For $0 < k < 4e^4/9$ there are three intersections. At the maximum level there are two distinct intersections, including A. For a larger positive k there is only one.

$$\boxed{0 < k \leq \frac{4e^4}{9}}$$

FINAL ANSWER

(a) $A = \left(\frac{4}{3}, \frac{4e^4}{9}\right)$ (b) $0 < k \leq \frac{4e^4}{9}$

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Question 4

5 marks | Differentiation Techniques

4. $y = \log_{10}(2x + 1)$

(a) Express x in terms of y.

(2)

(b) Hence, giving your answer in terms of x, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$

(3)

WORKED SOLUTION

Question 4 · WMA13/01 October 2022

1 Part (a)

The logarithm is base 10. Thus $10^y = 2x + 1$ and $2x = 10^y - 1$.

$$\boxed{x = \frac{10^y - 1}{2}}$$

2 Part (b)

Differentiate the result of (a) with respect to y:

$$\frac{dx}{dy} = \frac{1}{2}\,10^y \ln 10.$$

Take the reciprocal and use $10^y = 2x + 1$:

$$\frac{dy}{dx} = \frac{2}{10^y \ln 10} = \frac{2}{(2x+1)\ln 10}.$$

The original logarithm requires $x > -1/2$.

$$\boxed{\frac{dy}{dx} = \frac{2}{(2x+1)\ln 10}}$$

FINAL ANSWER

(a) $x = \frac{10^y - 1}{2}$ (b) $\frac{dy}{dx} = \frac{2}{(2x+1)\ln 10}$

Question 5

11 marks | Log Graphs & Exponential Models

5.

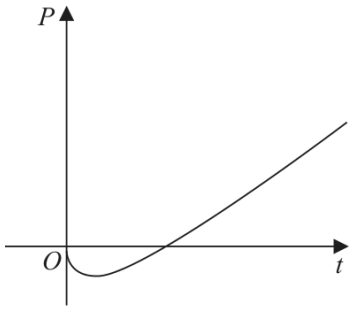


Figure 2

The profit made by a company, £P million, t years after the company started trading, is modelled by the equation

$$P = \frac{4t-1}{10} + \frac{3}{4}\ln\left[\frac{t+1}{(2t+1)^2}\right]$$

The graph of P against t is shown in Figure 2.

According to the model,

(a) show that exactly one year after it started trading, the company had made a loss of approximately £830 000

(2)

A manager of the company wants to know the value of t for which $P = 0$

(b) Show that this value of t occurs in the interval $[6, 7]$

(2)

(c) Show that the equation $P = 0$ can be expressed in the form

$$t = \frac{1}{4} + \frac{15}{8}\ln\left[\frac{(2t+1)^2}{t+1}\right]$$

(2)

(d) Using the iteration formula

$$t_{n+1} = \frac{1}{4} + \frac{15}{8}\ln\left[\frac{(2t_n+1)^2}{t_n+1}\right] \quad \text{with} \quad t_1 = 6$$

find the value of t_2 and the value of t_6, giving your answers to 3 decimal places.

(3)

(e) Hence find, according to the model, how many months it takes in total, from when the company started trading, for it to make a profit.

(2)

1 Part (a)

At one year,

$$P(1) = \frac{3}{10} + \frac{3}{4}\ln\left(\frac{2}{9}\right) = -0.8280580476\ldots \quad \text{million pounds.}$$

The negative sign indicates a loss. Its magnitude is about 828058 pounds, which is approximately 830000 pounds.

$$\boxed{\text{Loss} \approx \pounds 830000}$$

2 Part (b)

The model is continuous for $t \geq 0$. Evaluate

$$P(6) = \frac{23}{10} + \frac{3}{4}\ln\frac{7}{169} = -0.0879914\ldots < 0, \qquad P(7) = \frac{27}{10} + \frac{3}{4}\ln\frac{8}{225} = 0.1975059\ldots > 0.$$

Continuity and the sign change give a zero between 6 and 7.

$$\boxed{t \in (6,7) \subset [6,7]}$$

3 Part (c)

Set $P = 0$ and move the logarithm to the other side:

$$\frac{4t-1}{10} = -\frac{3}{4}\ln\frac{t+1}{(2t+1)^2} = \frac{3}{4}\ln\frac{(2t+1)^2}{t+1}.$$

Multiply by 10 and divide by 4 to obtain the required form.

$$\boxed{t = \frac{1}{4} + \frac{15}{8}\ln\frac{(2t+1)^2}{t+1}}$$

4 Part (d)

Starting with $t_1 = 6$, retain full calculator precision at each substitution:

$$t_2 = 6.2199785610\ldots, \quad t_3 = 6.2867730738\ldots, \quad t_4 = 6.3065963643\ldots, \quad t_5 = 6.3124397510\ldots, \quad t_6 = 6.3141587873\ldots.$$

Therefore the requested values to three decimal places are as follows.

$$\boxed{t_2 = 6.220, \qquad t_6 = 6.314}$$

5 Part (e)

The break-even time is approximately 6.315 years, or $6.315 \times 12 = 75.78$ months. Profit starts just after that time; the first complete number of months is therefore 76, not 75. Indeed $P(75/12) < 0$ and $P(76/12) > 0$.

$$\boxed{76 \text{ months}}$$

FINAL ANSWER

(a) $\text{Loss} \approx \pounds 830000$ (b) $t \in (6,7) \subset [6,7]$ (c) $t = \frac{1}{4} + \frac{15}{8} \ln \frac{(2t+1)^2}{t+1}$ (d) $t_2 = 6.220, \qquad t_6 = 6.314$ (e) 76 months

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Question 6

6 marks | Differentiation Techniques

6.

$$y = \frac{2 + 3\sin x}{\cos x + \sin x}$$

Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a\tan x + b\sec x + c}{\sec x + 2\sin x}$$

where a, b and c are integers to be found.

(6)

WORKED SOLUTION

Question 6 · WMA13/01 October 2022

1 **Part (main)**

With $y = (2 + 3\sin x)/(\cos x + \sin x)$, the quotient rule gives

$$\frac{dy}{dx} = \frac{3\cos x(\cos x + \sin x) - (2 + 3\sin x)(\cos x - \sin x)}{(\cos x + \sin x)^2}.$$

Expand the numerator and cancel the opposite mixed terms:

$$3\cos^2 x + 3\sin^2 x - 2\cos x + 2\sin x = 3 - 2\cos x + 2\sin x.$$

The denominator is $1 + 2\sin x\cos x$. Divide numerator and denominator by $\cos x$ to obtain

$$\frac{dy}{dx} = \frac{2\tan x + 3\sec x - 2}{\sec x + 2\sin x}.$$

This displayed form is used where its trigonometric terms are defined; the preceding quotient form also covers points with $\cos x = 0$ when the original denominator is nonzero.

$$\boxed{a = 2, \quad b = 3, \quad c = -2}$$

FINAL ANSWER

(main) $a = 2, \quad b = 3, \quad c = -2$

Question 7

12 marks | Modulus

7.

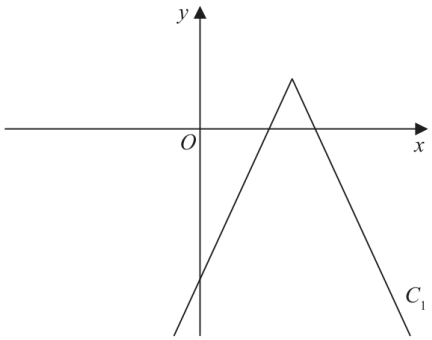


Figure 3

Figure 3 shows a sketch of the graph of C_1 with equation

$$y = 5 - |3x - 22|$$

(a) Write down the coordinates of

(i) the vertex of C_1

(ii) the intersection of C_1 with the y-axis.

(2)

(b) Find the x coordinates of the intersections of C_1 with the x-axis.

(2)

Diagram 1, shown on page 21, is a copy of Figure 3.

(c) On Diagram 1, sketch the curve C_2 with equation

$$y = \frac{1}{9}x^2 - 9$$

Identify clearly the coordinates of any points of intersection of C_2 with the coordinate axes.

(3)

(d) Find the coordinates of the points of intersection of C_1 and C_2
(*Solutions relying entirely on calculator technology are not acceptable.*)

(5)

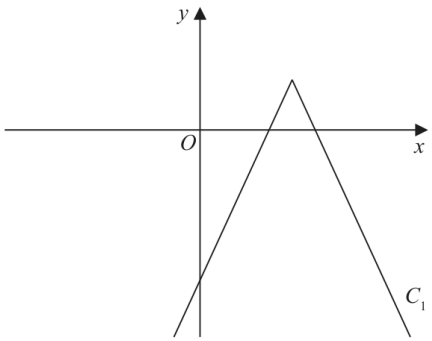


Diagram 1

1 Part (a)

(i) The modulus is zero at $3x - 22 = 0$, giving the vertex $(22/3, 5)$. (ii) At $x = 0$, $y = 5 - 22 = -17$, so the y-axis intersection is $(0, -17)$.

$$\boxed{(i)\ (22/3, 5), \qquad (ii)\ (0, -17)}$$

2 Part (b)

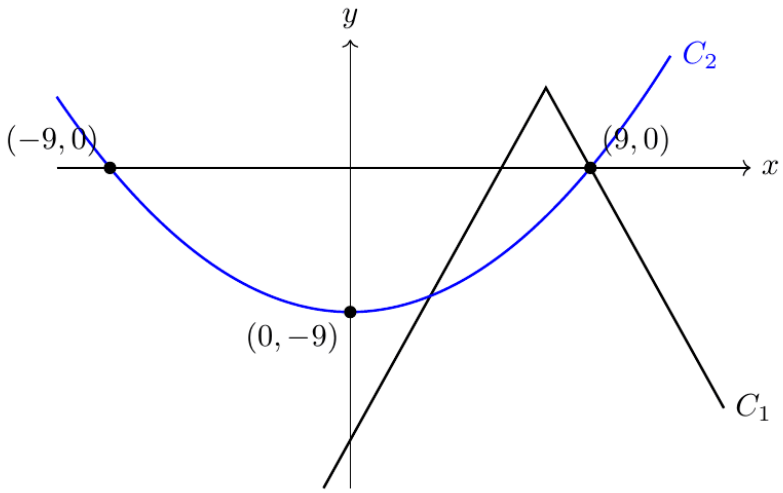
Set $y = 0$: $|3x - 22| = 5$. Thus $3x - 22 = 5$ or -5, giving $x = 9$ or $17/3$.

$$\boxed{x = \frac{17}{3}, \quad 9}$$

3 Part (c)

The curve $C_2 : y = x^2/9 - 9$ is an upward-opening parabola, symmetric about the y-axis. Its vertex and y-intercept are $(0, -9)$; setting y=0 gives $x^2 = 81$, hence x-intercepts $(-9, 0)$ and $(9, 0)$. The sketch below includes C_1 for context and labels all three axis intersections of C_2.

$$\boxed{(-9, 0), \quad (9, 0), \quad (0, -9)}$$



4 Part (d)

For $x \le 22/3$, C_1 is $y = 3x - 17$. Equating to C_2 gives

$$x^2/9 - 9 = 3x - 17 \implies x^2 - 27x + 72 = (x - 3)(x - 24) = 0.$$

Only $x = 3$ belongs to this branch; $y = -8$. For $x \ge 22/3$, C_1 is $y = 27 - 3x$. Then

$$x^2/9 - 9 = 27 - 3x \implies x^2 + 27x - 324 = (x - 9)(x + 36) = 0.$$

Only $x = 9$ belongs to this branch; $y = 0$. The rejected candidates 24 and -36 lie outside their assumed branch regions.

$$\boxed{(3, -8), \qquad (9, 0)}$$

FINAL ANSWER

(a) $(i)\ (22/3, 5), \quad (ii)\ (0, -17)$ (b) $x = \frac{17}{3}, \quad 9$ (c) $(-9, 0), \quad (9, 0), \quad (0, -9)$ (d) $(3, -8), \quad (9, 0)$

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Question 8

9 marks | Trig Equations & R-Form

8. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Express $8\sin x - 15\cos x$ in the form $R\sin(x-\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R, and give the value of α, in radians, to 4 significant figures.

(3)

$$f(x) = \frac{15}{41 + 16\sin x - 30\cos x} \qquad x > 0$$

(b) Find

(i) the minimum value of $f(x)$

(ii) the smallest value of x at which this minimum value occurs.

(4)

(c) State the y coordinate of the minimum points on the curve with equation

$$y = 2f(x) - 5 \qquad x > 0$$

(1)

(d) State the smallest value of x at which a maximum point occurs for the curve with equation

$$y = -f(2x) \qquad x > 0$$

(1)

WORKED SOLUTION

Question 8 · WMA13/01 October 2022

Part (a)

Expand $R\sin(x-\alpha) = R\cos\alpha\sin x - R\sin\alpha\cos x$. Thus $R\cos\alpha = 8$, $R\sin\alpha = 15$, so $R = \sqrt{64+225} = 17$ and $\alpha = \arctan(15/8) = 1.0808390005\ldots$ radians, in the required quadrant.

$$\boxed{R = 17, \qquad \alpha = 1.081 \text{ radians}}$$

2 Part (b)

Using the exact phase from (a),

$$f(x) = \frac{15}{41 + 34\sin(x - \alpha)}.$$

The denominator lies between 7 and 75, so it is always positive. (i) The smallest fraction occurs at its largest denominator: $f_{\min} = 15/75 = 1/5$. (ii) This occurs when $x - \alpha = \pi/2 + 2n\pi$. The smallest positive value is $x = \alpha + \pi/2 = 2.6516353273\ldots$, or 2.65 to three significant figures.

$$\boxed{f_{\min} = \frac{1}{5}, \qquad x = 2.65}$$

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3 Part (c)

The transformation $2f(x) - 5$ preserves the locations of minima and maps their y coordinate to $2(1/5) - 5 = -23/5$.

$$\boxed{y = -\frac{23}{5}}$$

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4 Part (d)

Reflection in the x-axis changes minima of f into maxima of $-f$. The argument $2x$ halves their x coordinates, so the first positive maximum is at

$$x = \frac{\alpha + \pi/2}{2} = 1.3258176637\ldots \approx 1.33.$$

$$\boxed{x = 1.33}$$

✓ FINAL ANSWER

(a) $R = 17, \qquad \alpha = 1.081$ radians (b) $f_{\min} = \frac{1}{5}, \qquad x = 2.65$ (c) $y = -\frac{23}{5}$ (d) $x = 1.33$

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Question 9

9 marks | Trig Equations & R-Form

9. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

Given that $\cos 2\theta - \sin 3\theta \neq 0$

(a) prove that

$$\frac{\cos^2\theta}{\cos 2\theta - \sin 3\theta} \equiv \frac{1+\sin\theta}{1-2\sin\theta-4\sin^2\theta}$$

(4)

(b) Hence solve, for $0 < \theta \leqslant 360°$

$$\frac{\cos^2\theta}{\cos 2\theta - \sin 3\theta} = 2\operatorname{cosec}\theta$$

Give your answers to one decimal place.

(5)

WORKED SOLUTION

Question 9 · WMA13/01 October 2022

1 Part (a)

Write $s = \sin\theta$. The compound-angle formula gives

$$\sin 3\theta = \sin(2\theta+\theta) = 2\sin\theta\cos^2\theta + (1-2\sin^2\theta)\sin\theta = 3s - 4s^3.$$

Hence

$$\cos 2\theta - \sin 3\theta = 1 - 2s^2 - 3s + 4s^3 = (1-s)(1-2s-4s^2), \quad \cos^2\theta = (1-s)(1+s).$$

The given nonzero denominator ensures that $1 - s \neq 0$ and the other factor is nonzero. Cancel the common factor to obtain the required identity.

$$\boxed{\frac{\cos^2\theta}{\cos 2\theta - \sin 3\theta} = \frac{1+\sin\theta}{1-2\sin\theta-4\sin^2\theta}}$$

2 Part (b)

Use (a) and $\csc\theta = 1/\sin\theta$. With $s = \sin\theta \neq 0$,

$$\frac{1+s}{1-2s-4s^2} = \frac{2}{s} \implies s + s^2 = 2 - 4s - 8s^2 \implies 9s^2 + 5s - 2 = 0.$$

Thus $s = (-5 \pm \sqrt{97})/18$, approximately 0.269380989 and -0.824936545. Both lie in $[-1, 1]$, are nonzero, and satisfy the original nonzero-denominator condition. The positive value gives $15.6274\ldots$° and $164.3726\ldots$°; the negative value gives $235.5821\ldots$° and $304.4179\ldots$°. All four angles lie in $0 < \theta \leq 360$°.

$$\boxed{\theta = 15.6^\circ,\ 164.4^\circ,\ 235.6^\circ,\ 304.4^\circ}$$

FINAL ANSWER

(a) $\frac{\cos^2\theta}{\cos 2\theta - \sin 3\theta} = \frac{1+\sin\theta}{1-2\sin\theta-4\sin^2\theta}$ (b) $\theta = 15.6^\circ,\ 164.4^\circ,\ 235.6^\circ,\ 304.4^\circ$