Elite Mathematics
Edexcel International A Level

IAL PURE 3 · WMA13

WMA13/01 May/June 2022 Worked Solutions

May/June 2022 paper rebuilt with the worked solution after each question.

9	2020–2026	Single-paper solutions
paper questions	19 exam sessions	checked by Dr Eslam



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PAST PAPER

WMA13/01 May/June 2022

May/June 2022

9	75	1 h 30
questions	marks	exam time

Each question is followed by its worked solution and final answer.

WMA13/01 MAY/JUNE 2022

Question 1

6 marks | Applications of Differentiation

1. The curve C has equation

$$y = (3x - 2)^6$$

(a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$

(2)

Given that the point $P\left(\dfrac{1}{3}, 1\right)$ lies on C,

(b) find the equation of the normal to C at P. Write your answer in the form $ax + by + c = 0$ where a, b and c are integers to be found.

(4)

WORKED SOLUTION

Question 1 · WMA13/01 May/June 2022

1 **Part (a)**

Differentiate the sixth power and multiply by the derivative of $3x - 2$: $dy/dx = 6(3x-2)^5 \cdot 3$.

$$\boxed{\frac{dy}{dx} = 18(3x-2)^5}$$

2 **Part (b)**

At $P = (1/3, 1)$, the tangent gradient is $18(-1)^5 = -18$. The normal gradient is $-1/(-18) = 1/18$. Thus

$$y - 1 = \frac{1}{18}(x - 1/3) \implies 54y - 54 = 3x - 1.$$

Rearranging with integer coefficients gives the required equation.

$$\boxed{3x - 54y + 53 = 0}$$

✓ **FINAL ANSWER**

(a) $\frac{dy}{dx} = 18(3x-2)^5$ (b) $3x - 54y + 53 = 0$

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Question 2

9 marks | Functions, Composition & Inverses

2. The functions f and g are defined by

$$\mathrm{f}(x) = \frac{5-x}{3x+2} \qquad x \in \mathbb{R}, x \neq -\frac{2}{3}$$

$$\mathrm{g}(x) = 2x - 7 \qquad x \in \mathbb{R}$$

(a) Find the value of $\mathrm{fg}(5)$

(2)

(b) Find f^{-1}

(3)

(c) Solve the equation

$$\mathrm{f}\left(\frac{1}{a}\right) = \mathrm{g}(a+3)$$

(4)

WORKED SOLUTION

Question 2 · WMA13/01 May/June 2022

1 **Part (a)**

$g(5) = 2(5) - 7 = 3$, then $fg(5) = f(3) = (5-3)/(9+2) = 2/11$.

$$\boxed{fg(5) = \frac{2}{11}}$$

2 **Part (b)**

Let $y = (5-x)/(3x+2)$. Then $3xy + 2y = 5 - x$, so $x(3y+1) = 5 - 2y$ and $x = (5-2y)/(3y+1)$. Exchange x and y. The inverse cannot have input $-1/3$, since this is not an output of f: $(5-x)/(3x+2) = -1/3$ would imply $15 = -2$.

$$\boxed{f^{-1}(x) = \frac{5-2x}{3x+1}, \qquad x \neq -\frac{1}{3}}$$

3 **Part (c)**

Here $a \neq 0$ and $a \neq -3/2$ are required by the input and denominator. Substitute into both functions:

$$f(1/a) = \frac{5a-1}{3+2a}, \qquad g(a+3) = 2a - 1.$$

Thus $5a - 1 = (2a-1)(2a+3) = 4a^2 + 4a - 3$, giving $4a^2 - a - 2 = 0$. By the quadratic formula, $a = (1 \pm \sqrt{33})/8$. Neither value is excluded.

$$\boxed{a = \frac{1+\sqrt{33}}{8} \quad \text{or} \quad a = \frac{1-\sqrt{33}}{8}}$$

✓ **FINAL ANSWER**

(a) $fg(5) = \frac{2}{11}$ (b) $f^{-1}(x) = \frac{5-2x}{3x+1}, \quad x \neq -\frac{1}{3}$ (c) $a = \frac{1+\sqrt{33}}{8}$ or $a = \frac{1-\sqrt{33}}{8}$

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Question 3

6 marks | Integration

3. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

Given that k is a positive constant,

(a) find

$$\int \frac{9x}{3x^2 + k}\ \mathrm{d}x$$

(2)

Given also that

$$\int_2^5 \frac{9x}{3x^2 + k}\ \mathrm{d}x = \ln 8$$

(b) find the value of k

(4)

WORKED SOLUTION

Question 3 · WMA13/01 May/June 2022

1 **Part (a)**

Since $d(3x^2 + k)/dx = 6x$, the numerator is $3/2$ times this derivative. The denominator is positive because $k > 0$, so

$$\int \frac{9x}{3x^2 + k}\,dx = \frac{3}{2}\ln(3x^2 + k) + C.$$

$$\boxed{\frac{3}{2}\ln(3x^2 + k) + C}$$

2 **Part (b)**

Apply the limits to (a) and combine the logarithms:

$$\frac{3}{2}\ln\frac{75+k}{12+k}=\ln 8.$$

Hence $\ln((75+k)/(12+k))=(2/3)\ln 8=\ln 4$, so $75+k=4(12+k)$. This gives $27=3k$ and $k=9$, consistent with $k>0$.

$$\boxed{k=9}$$

✓ **FINAL ANSWER**

(a) $\frac{3}{2}\ln(3x^2+k)+C$ (b) $k=9$

Question 4

7 marks | Log Graphs & Exponential Models

4.

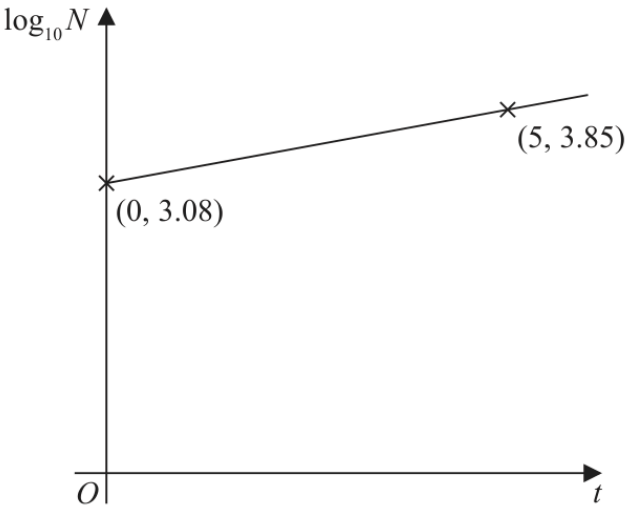


Figure 1

The number of subscribers to an online video streaming service, N, is modelled by the equation

$$N = ab^t$$

where a and b are constants and t is the number of years since monitoring began.

The line in Figure 1 shows the linear relationship between t and $\log_{10} N$

The line passes through the points (0, 3.08) and (5, 3.85)

Using this information,

(a) find an equation for this line.

(2)

(b) Find the value of a and the value of b, giving your answers to 3 significant figures.

(3)

When $t = T$ the number of subscribers is 500 000

According to the model,

(c) find the value of T

(2)

1 Part (a)

The slope is $(3.85 - 3.08)/(5 - 0) = 0.154$ and the vertical intercept is 3.08. The plotted vertical coordinate is $\log_{10} N$, not N.

$$\boxed{\log_{10} N = 0.154t + 3.08}$$

2 Part (b)

Taking logs of $N = ab^t$ gives $\log_{10} N = \log_{10} a + t \log_{10} b$. Compare with (a): $\log_{10} a = 3.08$ and $\log_{10} b = 0.154$. Thus $a = 10^{3.08} = 1202.2644\ldots$ and $b = 10^{0.154} = 1.4256076\ldots$. To three significant figures:

$$\boxed{a = 1.20 \times 10^3, \qquad b = 1.43}$$

3 Part (c)

Use the unrounded logarithmic model to avoid rounding drift:

$$\log_{10}(500000) = 0.154T + 3.08 \quad \Longrightarrow \quad T = \frac{\log_{10}(500000) - 3.08}{0.154} = 17.0062987\ldots.$$

$$\boxed{T = 17.0 \text{ years}}$$

FINAL ANSWER

(a) $\log_{10} N = 0.154t + 3.08$ (b) $a = 1.20 \times 10^3, \qquad b = 1.43$ (c) $T = 17.0$ years

Question 5

8 marks | Modulus

5.

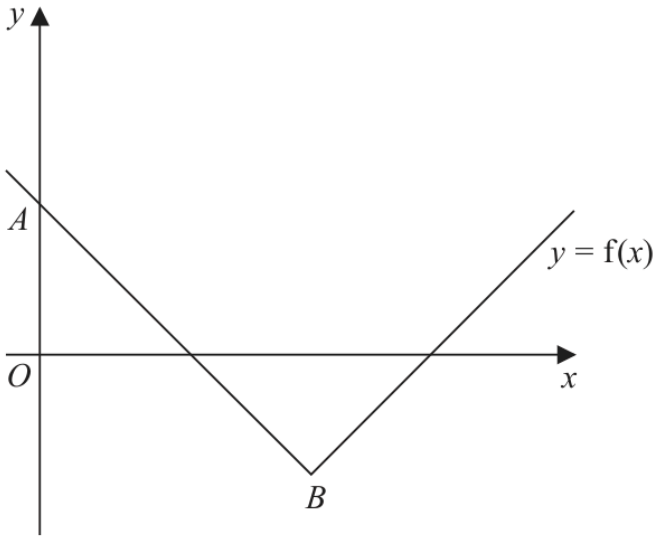


Figure 2

Figure 2 shows part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = |kx - 9| - 2 \qquad x \in \mathbb{R}$$

and k is a positive constant.

The graph intersects the y-axis at the point A and has a minimum point at B as shown.

(a) (i) Find the y coordinate of A

(ii) Find, in terms of k, the x coordinate of B

(2)

(b) Find, in terms of k, the range of values of x that satisfy the inequality

$$|kx - 9| - 2 < 0$$

(3)

Given that the line $y = 3 - 2x$ intersects the graph $y = \mathrm{f}(x)$ at two distinct points,

(c) find the range of possible values of k

(3)

1 Part (a)

(i) At $x = 0$, $f(0) = |-9| - 2 = 7$, so the y coordinate of A is 7. (ii) The minimum occurs when the modulus vanishes: $kx - 9 = 0$, hence $x_B = 9/k$.

$$\boxed{y_A = 7, \qquad x_B = \frac{9}{k}}$$

2 Part (b)

$|kx - 9| - 2 < 0$ means $|kx - 9| < 2$, or $-2 < kx - 9 < 2$. Add 9 and divide by the positive k, preserving both strict inequality signs.

$$\boxed{\frac{7}{k} < x < \frac{11}{k}}$$

3 Part (c)

For $x \le 9/k$, the graph is $y = 7 - kx$; for $x \ge 9/k$, it is $y = kx - 11$. The intersection of the right branch with $3 - 2x$ has $x = 14/(k+2)$. For it to lie strictly to the right of the vertex,

$$\frac{14}{k+2} > \frac{9}{k} \iff 14k > 9k + 18 \iff k > 18/5.$$

For these k, the left-branch intersection has $x = 4/(k-2) < 9/k$, so both intersections exist and differ. At $k = 18/5$ the two points coincide at the vertex. Each straight branch can supply at most one intersection; when the right branch is inadmissible, two distinct points cannot occur (k=2 gives parallel, unequal left lines).

$$\boxed{k > \frac{18}{5}}$$

FINAL ANSWER

(a) $y_A = 7, \qquad x_B = \frac{9}{k}$ (b) $\frac{7}{k} < x < \frac{11}{k}$ (c) $k > \frac{18}{5}$

Question 6

10 marks | Applications of Differentiation

6.

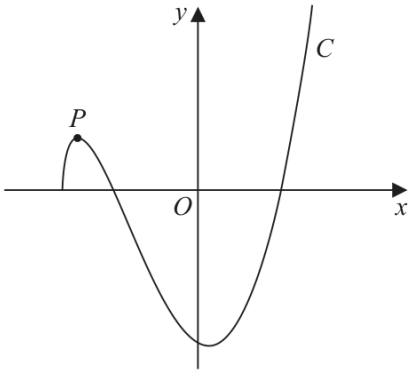


Figure 3

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

The function f is defined by

$$f(x) = 5(x^2 - 2)(4x + 9)^{\frac{1}{2}} \qquad x \geqslant -\frac{9}{4}$$

(a) Show that

$$f'(x) = \frac{k(5x^2 + 9x - 2)}{(4x + 9)^{\frac{1}{2}}}$$

where k is an integer to be found.

(4)

(b) Hence, find the values of x for which $f'(x) = 0$

(1)

Figure 3 shows a sketch of the curve C with equation $y = f(x)$.

The curve has a local maximum at the point P

(c) Find the exact coordinates of P

(2)

The function g is defined by

$$g(x) = 2f(x) + 4 \qquad -\frac{9}{4} \leqslant x \leqslant 0$$

(d) Find the range of g

(3)

1 Part (a)

Use product and chain rules:

$$f'(x) = 10x(4x+9)^{1/2} + 10(x^2-2)(4x+9)^{-1/2}.$$

With a common denominator,

$$f'(x) = \frac{10x(4x+9) + 10(x^2-2)}{\sqrt{4x+9}} = \frac{10(5x^2+9x-2)}{\sqrt{4x+9}}.$$

This derivative formula applies for $x > -9/4$; the endpoint derivative is not finite.

$$\boxed{k = 10}$$

2 Part (b)

The denominator is positive in the derivative domain. Thus $5x^2+9x-2 = (5x-1)(x+2) = 0$.

$$\boxed{x = -2, \quad \frac{1}{5}}$$

3 Part (c)

The derivative changes from positive to negative at $x = -2$, so this is the local maximum P. Substitute into f: $f(-2) = 5(4-2)\sqrt{-8+9} = 10$.

$$\boxed{P = (-2, 10)}$$

4 Part (d)

On $[-9/4, 0]$, f rises from $f(-9/4) = 0$ to $f(-2) = 10$, then decreases to $f(0) = -30$. The other stationary point $1/5$ is outside this restricted interval. Thus the minimum and maximum of f on it are -30 and 10. Applying the positive vertical transformation $g = 2f + 4$ gives endpoints $2(-30)+4 = -56$ and $2(10)+4 = 24$, both attained.

$$\boxed{-56 \le g(x) \le 24}$$

FINAL ANSWER

(a) $k = 10$ (b) $x = -2, \quad \frac{1}{5}$ (c) $P = (-2, 10)$ (d) $-56 \le g(x) \le 24$

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Question 7

8 marks | Trig Identities & Exact Values

7. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that the equation

$$2\sin\theta\,(3\cot^2 2\theta - 7) = 13\sec\theta$$

can be written as

$$3\,\mathrm{cosec}^2 2\theta - 13\,\mathrm{cosec}\,2\theta - 10 = 0$$

(4)

(b) Hence solve, for $0 < \theta < \frac{\pi}{2}$, the equation

$$2\sin\theta\,(3\cot^2 2\theta - 7) = 13\sec\theta$$

giving your answers to 3 significant figures.

(4)

WORKED SOLUTION

Question 7 · WMA13/01 May/June 2022

1 Part (a)

On the domain of the original equation, multiply by $\cos\theta$:

$$2\sin\theta\cos\theta(3\cot^2 2\theta - 7) = 13 \implies \sin 2\theta(3\cot^2 2\theta - 7) = 13.$$

Divide by the nonzero $\sin 2\theta$ and use $\cot^2 u = \csc^2 u - 1$:

$$3(\csc^2 2\theta - 1) - 7 = 13\csc 2\theta.$$

Collect the terms to obtain the displayed equation.

$$\boxed{3\csc^2 2\theta - 13\csc 2\theta - 10 = 0}$$

2

Part (b)

Put $u = \csc 2\theta$. Then $(3u + 2)(u - 5) = 0$, so $u = 5$ or $-2/3$. The second would give $\sin 2\theta = -3/2$, which is impossible. For $0 < \theta < \pi/2$, we have $0 < 2\theta < \pi$. Thus

$$2\theta = \arcsin(1/5) \quad \text{or} \quad \pi - \arcsin(1/5).$$

Hence $\theta = 0.1006789604\ldots$ or $1.4701173664\ldots$ radians. Both satisfy the original denominator restrictions.

$$\boxed{\theta = 0.101, \quad 1.47 \text{ radians}}$$

FINAL ANSWER

(a) $3\csc^2 2\theta - 13\csc 2\theta - 10 = 0$ (b) $\theta = 0.101, \quad 1.47$ radians

Question 8

12 marks | Numerical Methods

8.

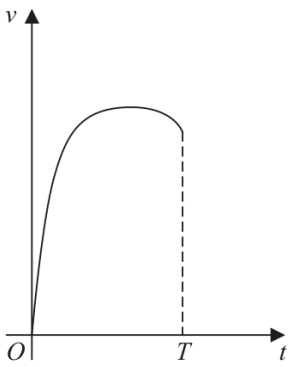


Figure 4

Figure 4 is a graph showing the velocity of a sprinter during a 100 m race.

The sprinter's velocity during the race, $v\,\text{m s}^{-1}$, is modelled by the equation

$$v = 12 - \text{e}^{t-10} - 12\text{e}^{-0.75t} \qquad t \geqslant 0$$

where t seconds is the time after the sprinter begins to run.

According to the model,

(a) find, using calculus, the sprinter's maximum velocity during the race.

(5)

Given that the sprinter runs 100 m in T seconds, such that

$$\int_0^T v\,\text{d}t = 100$$

(b) show that T is a solution of the equation

$$T = \frac{1}{12}\left(116 - 16\text{e}^{-0.75T} + \text{e}^{T-10} - \text{e}^{-10}\right)$$

(4)

The iteration formula

$$T_{n+1} = \frac{1}{12}\left(116 - 16\text{e}^{-0.75T_n} + \text{e}^{T_n-10} - \text{e}^{-10}\right)$$

is used to find an approximate value for T

Using this iteration formula with $T_1 = 10$

(c) find, to 4 decimal places,

(i) the value of T_2

(ii) the time taken by the sprinter to run the race, according to the model.

(3)

1 Part (a)

Differentiate the velocity:

$$v'(t) = -e^{t-10} + 9e^{-0.75t}.$$

At a stationary velocity, $e^{t-10} = 9e^{-0.75t}$, so $e^{1.75t} = 9e^{10}$ and $t = (10 + \ln 9)/1.75 = 6.9698426\ldots$ seconds. The second derivative $-e^{t-10} - 6.75e^{-0.75t}$ is negative, so this is a maximum. Substitute this unrounded time into v to get $11.88728125\ldots$ metres per second.

$$\boxed{v_{\max} = 11.9\ \mathrm{m\,s^{-1}}}$$

2 Part (b)

Integrating the velocity gives $12t - e^{t-10} + 16e^{-0.75t}$. Evaluate between 0 and T:

$$100 = 12T - e^{T-10} + 16e^{-0.75T} - (-e^{-10} + 16).$$

Rearrange to $12T = 116 - 16e^{-0.75T} + e^{T-10} - e^{-10}$ and divide by 12.

$$\boxed{T = \frac{1}{12}(116 - 16e^{-0.75T} + e^{T-10} - e^{-10})}$$

3 Part (c)

With $T_1 = 10$, substitution gives $T_2 = 9.7492587708\ldots$. Continuing with unrounded values gives $T_3 = 9.7306248365\ldots$, $T_4 = 9.7294150526\ldots$, $T_5 = 9.7293372715\ldots$, $T_6 = 9.7293322738\ldots$.

Further iteration settles at $9.7293319307\ldots$ seconds. Thus (i) $T_2 = 9.7493$ and (ii) the race time is 9.7293 seconds, to four decimal places.

$$\boxed{T_2 = 9.7493, \qquad T = 9.7293\ \mathrm{s}}$$

FINAL ANSWER

(a) $v_{\max} = 11.9\ \mathrm{m\,s^{-1}}$ (b) $T = \frac{1}{12}(116 - 16e^{-0.75T} + e^{T-10} - e^{-10})$ (c) $T_2 = 9.7493, \qquad T = 9.7293\ \mathrm{s}$

Question 9

9 marks | Trig Equations & R-Form

9.

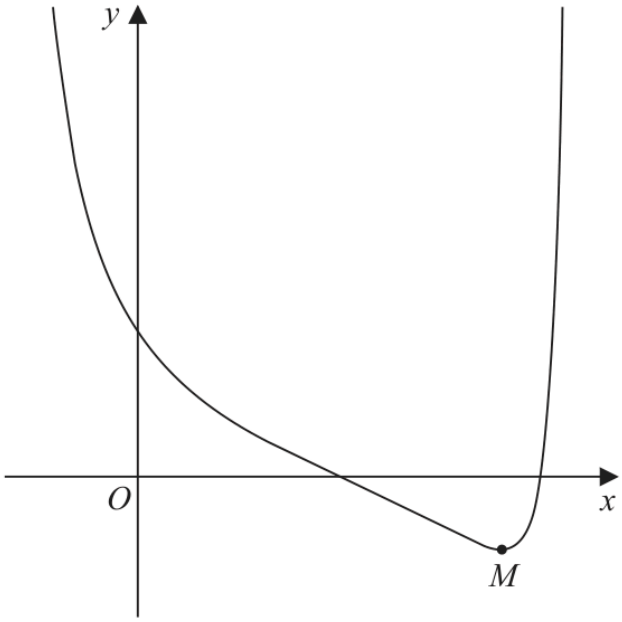


Figure 5

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 5 shows the curve with equation

$$y = \frac{1 + 2\cos x}{1 + \sin x} \qquad -\frac{\pi}{2} < x < \frac{3\pi}{2}$$

The point M, shown in Figure 5, is the minimum point on the curve.

(a) Show that the x coordinate of M is a solution of the equation

$$2\sin x + \cos x = -2$$

(4)

(b) Hence find, to 3 significant figures, the x coordinate of M.

(5)

1 Part (a)

Differentiate by the quotient rule:

$$y' = \frac{-2\sin x(1+\sin x) - (1+2\cos x)\cos x}{(1+\sin x)^2} = \frac{-2\sin x - \cos x - 2(\sin^2 x + \cos^2 x)}{(1+\sin x)^2} = \frac{-2\sin x - \cos x - 2}{(1+\sin x)^2}.$$

Inside the stated interval, $1+\sin x > 0$. At the interior minimum M, $y' = 0$, so its numerator is zero, giving the required equation.

$$\boxed{2\sin x + \cos x = -2}$$

2 Part (b)

Write $2\sin x + \cos x = \sqrt{5}\sin(x+\beta)$, where $\cos\beta = 2/\sqrt{5}$, $\sin\beta = 1/\sqrt{5}$ and $\beta = \arctan(1/2)$. The equation becomes $\sin(x+\beta) = -2/\sqrt{5}$. Let $\gamma = \arcsin(2/\sqrt{5})$, so $\gamma + \beta = \pi/2$. The solutions in the closed boundary range include $x = -\pi/2$, $3\pi/2$, and

$$x = \pi + \gamma - \beta = 3.7850937624\ldots.$$

Both boundary values are outside the original open domain (and make its denominator zero). The sole interior stationary point is M; the derivative changes from negative to positive there.

$$\boxed{x_M = 3.79}$$

FINAL ANSWER

(a) $2\sin x + \cos x = -2$ (b) $x_M = 3.79$