Elite Mathematics
Edexcel International A Level

IAL PURE 3 · WMA13

WMA13/01 January 2022 Worked Solutions

January 2022 paper rebuilt with the worked solution after each question.

10	2020–2026	Single-paper solutions
paper questions	19 exam sessions	checked by Dr Eslam



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PAST PAPER

WMA13/01 January 2022

January 2022

10	75	1 h 30
questions	marks	exam time

Each question is followed by its worked solution and final answer.

WMA13/01 JANUARY 2022

Question 1

4 marks | Applications of Differentiation

1. Find, using calculus, the x coordinate of the stationary point on the curve with equation

$$y = (2x + 5)e^{3x}$$

(4)

WORKED SOLUTION

Question 1 · WMA13/01 January 2022

1 **Part (main)**

Apply the product rule:

$$y' = 2e^{3x} + 3(2x + 5)e^{3x} = e^{3x}(6x + 17).$$

At a stationary point, $y' = 0$. Since $e^{3x} > 0$ for every real x, $6x + 17 = 0$, which determines the unique stationary x coordinate.

$$\boxed{x = -\frac{17}{6}}$$

FINAL ANSWER

(main) $x = -\frac{17}{6}$

WMA13/01 JANUARY 2022

Question 2

5 marks | Trig Identities & Exact Values

2. (a) Show that the equation

$$8\cos\theta = 3\operatorname{cosec}\theta$$

can be written in the form

$$\sin 2\theta = k$$

where k is a constant to be found.

(3)

(b) Hence find the smallest positive solution of the equation

$$8\cos\theta = 3\operatorname{cosec}\theta$$

giving your answer, in degrees, to one decimal place.

(2)

WORKED SOLUTION

Question 2 · WMA13/01 January 2022

1 **Part (a)**

The original equation requires $\sin\theta \neq 0$. Using $\csc\theta = 1/\sin\theta$, multiply by $\sin\theta$ to obtain $8\sin\theta\cos\theta = 3$. Since $2\sin\theta\cos\theta = \sin 2\theta$, this becomes $4\sin 2\theta = 3$.

$$\boxed{\sin 2\theta = \frac{3}{4}, \qquad k = \frac{3}{4}}$$

2 **Part (b)**

For the smallest positive solution take the first positive sine branch: $2\theta = \arcsin(3/4) = 48.59037789\ldots^\circ$. Divide by two to obtain $\theta = 24.29518895\ldots^\circ$. The supplementary branch gives a larger positive angle.

$$\boxed{\theta = 24.3^\circ}$$

✓ **FINAL ANSWER**

(a) $\sin 2\theta = \frac{3}{4}, \qquad k = \frac{3}{4}$ (b) $\theta = 24.3^\circ$

WMA13/01 JANUARY 2022

Question 3

6 marks | Integration

3. (i) Find, in simplest form,

$$\int (2x-5)^7 \, \mathrm{d}x$$

(2)

(ii) Show, by algebraic integration, that

$$\int_0^{\frac{\pi}{3}} \frac{4\sin x}{1+2\cos x} \mathrm{d}x = \ln a$$

where a is a rational constant to be found.

(4)

WORKED SOLUTION

Question 3 · WMA13/01 January 2022

1 Part (i)

Raise the power by one, divide by the new power and by the inner derivative 2:

$$\int (2x-5)^7 \, dx = \frac{(2x-5)^8}{2 \cdot 8} + C.$$

$$\boxed{\frac{(2x-5)^8}{16} + C}$$

2 Part (ii)

The derivative of $1 + 2\cos x$ is $-2\sin x$, so a primitive is $-2\ln(1+2\cos x)$. The log argument is positive throughout $0 \le x \le \pi/3$. Hence

$$\int_0^{\pi/3} \frac{4\sin x}{1+2\cos x} \, dx = \left[-2\ln(1+2\cos x)\right]_0^{\pi/3} = -2\ln 2 + 2\ln 3 = \ln \frac{9}{4}.$$

$$\boxed{a = \frac{9}{4}}$$

FINAL ANSWER

(i) $\frac{(2x-5)^8}{16} + C$ (ii) $a = \frac{9}{4}$

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Question 4

7 marks | Log Graphs & Exponential Models

4. The growth of a weed on the surface of a pond is being studied.

The surface area of the pond covered by the weed, $A\,\text{m}^2$, is modelled by the equation

$$A = \frac{80pe^{0.15t}}{pe^{0.15t} + 4}$$

where p is a positive constant and t is the number of days after the start of the study.

Given that

- $30\,\text{m}^2$ of the surface of the pond was covered by the weed at the start of the study
- $50\,\text{m}^2$ of the surface of the pond was covered by the weed T days after the start of the study

(a) show that $p = 2.4$

(2)

(b) find the value of T, giving your answer to one decimal place.

(*Solutions relying entirely on graphical or numerical methods are not acceptable.*)

(4)

The weed grows until it covers the surface of the pond.

(c) Find, according to the model, the maximum possible surface area of the pond.

(1)

WORKED SOLUTION

Question 4 · WMA13/01 January 2022

1 **Part (a)**

At $t = 0$, the model gives $30 = 80p/(p+4)$. Thus $30p + 120 = 80p$, so $50p = 120$ and $p = 2.4 > 0$.

$$\boxed{p = 2.4}$$

2 Part (b)

Put $E = e^{0.15T}$ and use $p = 2.4$. At area $50\,\text{m}^2$,

$$50 = \frac{192E}{2.4E + 4} \quad \Longrightarrow \quad 120E + 200 = 192E \quad \Longrightarrow \quad E = \frac{25}{9}.$$

Taking natural logarithms gives $0.15T = \ln(25/9)$, so $T = \ln(25/9)/0.15 = 6.8110083\ldots$ days.

$$\boxed{T = 6.8 \text{ days}}$$

3 Part (c)

Divide numerator and denominator by $pe^{0.15t}$: $A = 80/(1 + (4/p)e^{-0.15t}) \longrightarrow 80$ as $t \longrightarrow \infty$. Thus the model's limiting pond surface area is $80\,\text{m}^2$; the finite-time covered area approaches this value from below.

$$\boxed{80\ \text{m}^2}$$

FINAL ANSWER

(a) $p = 2.4$ (b) $T = 6.8$ days (c) $80\ \text{m}^2$

Question 5

9 marks | Applications of Differentiation

5.

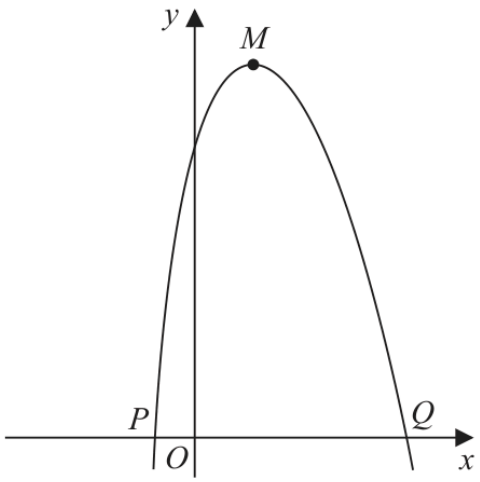


Figure 1

Figure 1 shows a sketch of part of the curve with equation

$$y = 6\ln(2x + 3) - \frac{1}{2}x^2 + 4 \qquad x > -\frac{3}{2}$$

The curve cuts the negative x-axis at the point P, as shown in Figure 1.

(a) Show that the x coordinate of P lies in the interval $[-1.25, -1.2]$

(2)

The curve cuts the positive x-axis at the point Q, also shown in Figure 1.

Using the iterative formula

$$x_{n+1} = \sqrt{12\ln(2x_n + 3) + 8} \quad \text{with} \quad x_1 = 6$$

(b) (i) find, to 4 decimal places, the value of x_2

(ii) find, by continued iteration, the x coordinate of Q. Give your answer to 4 decimal places.

(3)

The curve has a maximum turning point at M, as shown in Figure 1.

(c) Using calculus and showing each stage of your working, find the x coordinate of M.

(4)

1 Part (a)

Let $F(x) = 6\ln(2x+3) - x^2/2 + 4$. Then $F(-1.25) = -0.94013308\ldots$ and $F(-1.2) = 0.21504626\ldots$. The log argument is positive on this interval, so F is continuous there. The change of sign proves a zero between these endpoints, giving the stated bracket for P.

$$\boxed{-1.25 < x_P < -1.2}$$

2 Part (b)

Apply the given recurrence with $x_1 = 6$:

$$x_2 = \sqrt{12\ln 15 + 8} = 6.363694085\ldots.$$

Continuing without rounding the internal values,

$$x_3 = 6.408185698\ldots, \quad x_4 = 6.413466053\ldots, \quad x_5 = 6.414090476\ldots, \quad x_6 = 6.414164285\ldots, \quad x_7 = 6.414173009\ldots.$$

Further iteration settles at $6.41417418\ldots$. The positive square-root branch corresponds to Q.

$$\boxed{\text{(i) } x_2 = 6.3637, \qquad \text{(ii) } x_Q = 6.4142}$$

3 Part (c)

Differentiate: $F'(x) = 12/(2x+3) - x$. For a stationary point, $12 = x(2x+3)$, or $2x^2 + 3x - 12 = 0$. The quadratic formula gives $x = (-3 \pm \sqrt{105})/4$. The negative choice is less than $-3/2$ and outside the original domain. Also $F''(x) = -24/(2x+3)^2 - 1 < 0$, confirming the admissible point is a maximum.

$$\boxed{x_M = \frac{-3+\sqrt{105}}{4}}$$

FINAL ANSWER

(a) $-1.25 < x_P < -1.2$ (b) $\text{(i) } x_2 = 6.3637, \qquad \text{(ii) } x_Q = 6.4142$ (c) $x_M = \frac{-3+\sqrt{105}}{4}$

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Question 6

11 marks | Functions, Composition & Inverses

6. The function f is defined by

$$\mathrm{f}(x) = \frac{5x - 3}{x - 4} \qquad x > 4$$

(a) Show, by using calculus, that f is a decreasing function.

(3)

(b) Find f^{-1}

(3)

(c) (i) Show that $\mathrm{ff}(x) = \dfrac{ax + b}{x + c}$ where a, b and c are constants to be found.

(ii) Deduce the range of ff.

(5)

WORKED SOLUTION

Question 6 · WMA13/01 January 2022

1 **Part (a)**

By the quotient rule,

$$f'(x) = \frac{5(x-4) - (5x-3)}{(x-4)^2} = \frac{-17}{(x-4)^2}.$$

For every $x > 4$, the denominator is positive and the numerator is negative. Therefore $f'(x) < 0$ throughout its domain, so f is decreasing.

$$\boxed{f \text{ is decreasing for } x > 4}$$

2 **Part (b)**

Set $y = (5x - 3)/(x - 4)$. Then $yx - 4y = 5x - 3$, giving $x(y - 5) = 4y - 3$ and $x = (4y - 3)/(y - 5)$. Exchange the variable names. Since $f(x) = 5 + 17/(x - 4)$ on $x > 4$, its range is $(5, \infty)$, which becomes the inverse's domain.

$$\boxed{f^{-1}(x) = \frac{4x - 3}{x - 5}, \qquad x > 5}$$

3 **Part (c)**

(i) For $x > 4$, $f(x) > 5 > 4$, so the second application is defined.

$$ff(x) = \frac{5\left(\frac{5x-3}{x-4}\right) - 3}{\left(\frac{5x-3}{x-4}\right) - 4} = \frac{25x - 15 - 3x + 12}{5x - 3 - 4x + 16} = \frac{22x - 3}{x + 13}.$$

Thus $a = 22$, $b = -3$, $c = 13$. (ii) Rewrite $ff(x) = 22 - 289/(x + 13)$ with the original domain $x > 4$. It is strictly increasing, tends to 5 as $x \to 4^+$ and to 22 as $x \to \infty$. Neither limiting value is attained.

$$\boxed{ff(x) = \frac{22x - 3}{x + 13}, \qquad 5 < ff(x) < 22}$$

FINAL ANSWER

(a) f **is decreasing for** $x > 4$ (b) $f^{-1}(x) = \frac{4x-3}{x-5}, \qquad x > 5$ (c) $ff(x) = \frac{22x-3}{x+13}, \qquad 5 < ff(x) < 22$

Question 7

10 marks | Modulus

7.

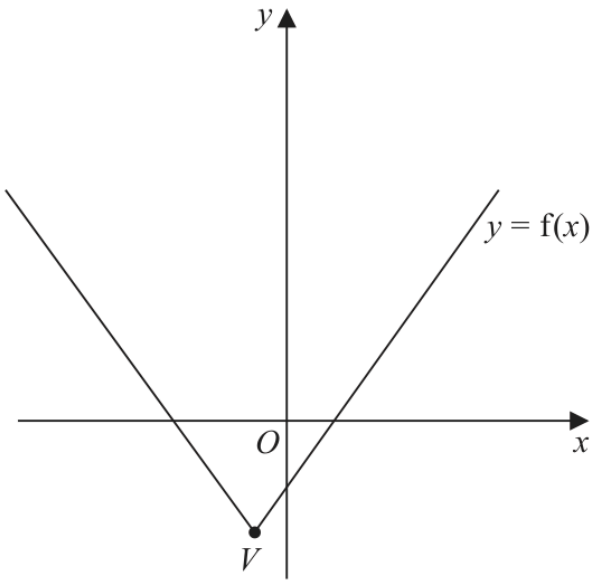


Figure 2

Figure 2 shows a sketch of part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \frac{1}{2}|2x + 7| - 10$$

(a) State the coordinates of the vertex, V, of the graph.

(2)

(b) Solve, using algebra,

$$\frac{1}{2}|2x + 7| - 10 \geqslant \frac{1}{3}x + 1$$

(4)

(c) Sketch the graph with equation

$$y = |\mathrm{f}(x)|$$

stating the coordinates of the local maximum point and each local minimum point.

(4)

1 Part (a)

At the vertex $2x + 7 = 0$, hence $x = -7/2$ and $f(x) = -10$.

$$\boxed{V = (-7/2, -10)}$$

2 Part (b)

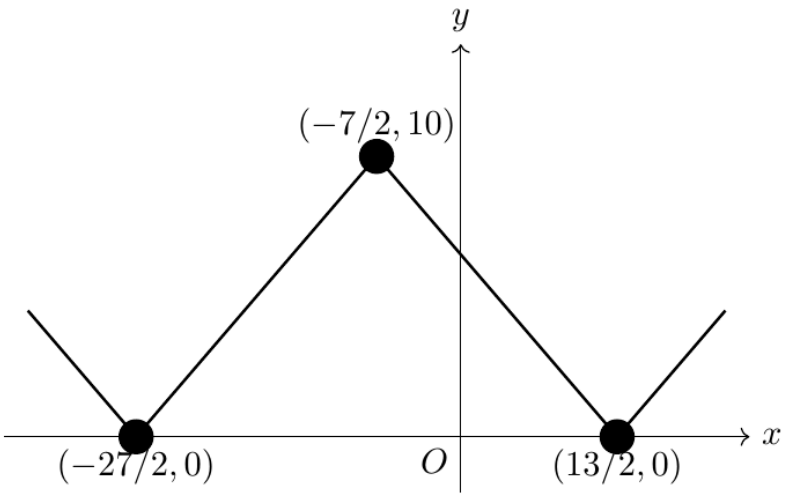
For $x \geq -7/2$, $f(x) = x - 13/2$. The inequality becomes $x - 13/2 \geq x/3 + 1$, giving $x \geq 45/4$, which satisfies this branch. For $x < -7/2$, $f(x) = -x - 27/2$. Then $-x - 27/2 \geq x/3 + 1$, giving $x \leq -87/8$, which satisfies that branch. Join the two permitted outer intervals, including the equality endpoints.

$$\boxed{x \leq -\frac{87}{8} \quad \text{or} \quad x \geq \frac{45}{4}}$$

3 Part (c)

The zeros of f satisfy $|2x + 7| = 20$, giving $x = -27/2$ and $x = 13/2$. Reflect the part below the x-axis upward, leaving the positive branches unchanged. This creates minima at $(-27/2, 0)$ and $(13/2, 0)$ and a local maximum at $(-7/2, 10)$.

$$\boxed{\text{Maximum } (-7/2, 10); \quad \text{minima } (-27/2, 0), (13/2, 0)}$$



FINAL ANSWER

(a) $V = (-7/2, -10)$ (b) $x \leq -\frac{87}{8}$ or $x \geq \frac{45}{4}$ (c)
$\text{Maximum } (-7/2, 10); \quad \text{minima } (-27/2, 0), (13/2, 0)$

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Question 8

8 marks | Log Graphs & Exponential Models

8. A dose of antibiotics is given to a patient.

The amount of the antibiotic, x milligrams, in the patient's bloodstream t hours after the dose was given, is found to satisfy the equation

$$\log_{10} x = 2.74 - 0.079t$$

(a) Show that this equation can be written in the form

$$x = pq^{-t}$$

where p and q are constants to be found. Give the value of p to the nearest whole number and the value of q to 2 significant figures.

(4)

(b) With reference to the equation in part (a), interpret the value of the constant p.

(1)

When a different dose of the antibiotic is given to another patient, the values of x and t satisfy the equation

$$x = 400 \times 1.4^{-t}$$

(c) Use calculus to find, to 2 significant figures, the value of $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ when $t = 5$

(3)

WORKED SOLUTION

Question 8 · WMA13/01 January 2022

1 **Part (a)**

Exponentiate base 10:

$$x = 10^{2.74-0.079t} = 10^{2.74}(10^{0.079})^{-t}.$$

Thus $p = 10^{2.74} = 549.5408739\ldots$ and $q = 10^{0.079} = 1.199499303\ldots$. Round p to the nearest integer and q to two significant figures.

$$\boxed{p = 550, \qquad q = 1.2}$$

2 Part (b)

At $t = 0$, $x = pq^0 = p$. Thus p is the initial amount of antibiotic in the patient's bloodstream, approximately 550 milligrams immediately after the dose in this model.

$$\boxed{p \text{ represents the initial amount, approximately } 550 \text{ mg}}$$

3 Part (c)

For the second patient, write $x = 400e^{-t\ln 1.4}$. Differentiate to obtain $dx/dt = -400\ln(1.4)\,1.4^{-t}$. At $t = 5$, the derivative is $-25.02470969\ldots \text{ mg h}^{-1}$. The negative sign indicates the amount is decreasing.

$$\boxed{\left.\frac{dx}{dt}\right|_{t=5} = -25 \text{ mg h}^{-1}}$$

FINAL ANSWER

(a) $p = 550, \quad q = 1.2$ (b) p represents the initial amount, approximately 550 mg (c) $\left.\frac{dx}{dt}\right|_{t=5} = -25 \text{ mg h}^{-1}$

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Question 9

8 marks | Trig Identities & Exact Values

9. **In this question you must show detailed reasoning.**

Solutions relying entirely on calculator technology are not acceptable.

(i) Solve, for $0 < x \leqslant \pi$, the equation

$$2\sec^2 x - 3\tan x = 2$$

giving the answers, as appropriate, to 3 significant figures.

(4)

(ii) Prove that

$$\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} \equiv 2$$

(4)

WORKED SOLUTION

Question 9 · WMA13/01 January 2022

1 **Part (i)**

Using $\sec^2 x = 1 + \tan^2 x$, the equation becomes $2\tan^2 x - 3\tan x = 0$, or $\tan x(2\tan x - 3) = 0$. If $\tan x = 0$, the given interval $0 < x \leq \pi$ contains only $x = \pi$. If $\tan x = 3/2$, the sole root in this interval is $x = \arctan(3/2) = 0.982793723\ldots$ radians. The excluded pole $x = \pi/2$ supplies no solution.

$$\boxed{x = 0.983 \text{ radians}, \quad \pi}$$

2 **Part (ii)**

Use angle addition to expand:

$$\sin 3\theta = \sin 2\theta\cos\theta + \cos 2\theta\sin\theta = 2\sin\theta\cos^2\theta + (1 - 2\sin^2\theta)\sin\theta = 3\sin\theta - 4\sin^3\theta,$$

$$\cos 3\theta = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta = (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta = 4\cos^3\theta - 3\cos\theta.$$

Where $\sin\theta\cos\theta \neq 0$, divide by the stated denominators:

$$\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = 3 - 4\sin^2\theta - (4\cos^2\theta - 3) = 6 - 4(\sin^2\theta + \cos^2\theta) = 2.$$

$$\boxed{\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = 2}$$

FINAL ANSWER

(i) $x = 0.983 \text{ radians}, \quad \pi$ (ii) $\frac{\sin 3\theta}{\sin \theta} - \frac{\cos 3\theta}{\cos \theta} = 2$

Question 10

7 marks | Differentiation Techniques

10.

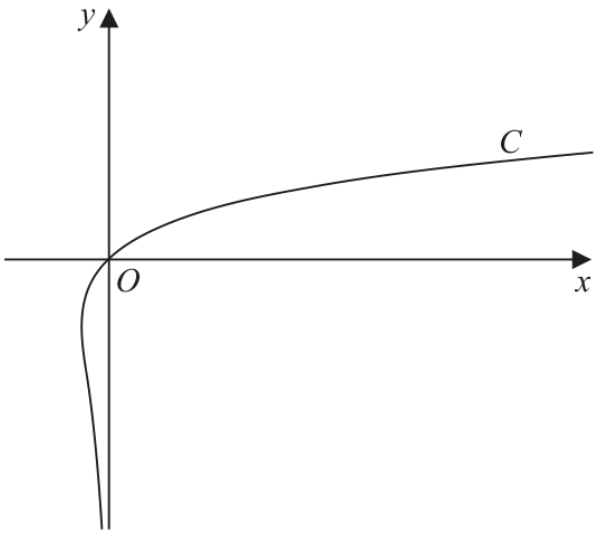


Figure 3

Figure 3 shows a sketch of the curve C with equation

$$x = ye^{2y} \qquad y \in \mathbb{R}$$

(a) Show that

$$\frac{dy}{dx} = \frac{y}{x(1+2y)}$$

(4)

Given that the straight line with equation $x = k$, where k is a constant, cuts C at exactly two points,

(b) find the range of possible values for k.

(3)

1 Part (a)

Differentiate x with respect to y by the product rule:

$$\frac{dx}{dy} = e^{2y} + 2ye^{2y} = e^{2y}(1+2y).$$

Where $dx/dy \neq 0$, take the reciprocal:

$$\frac{dy}{dx} = \frac{e^{-2y}}{1+2y}.$$

For $y \neq 0$, the original relation gives $e^{2y} = x/y$, hence the required form $dy/dx = y/[x(1+2y)]$. At the origin, that last quotient is $0/0$; the preceding uncancelled derivative gives $dy/dx = 1$. At $y = -1/2$ there is no finite dy/dx.

$$\boxed{\frac{dy}{dx} = \frac{y}{x(1+2y)} \quad \text{where defined}}$$

2 Part (b)

Consider $x = h(y) = ye^{2y}$. Its derivative $h'(y) = e^{2y}(1+2y)$ vanishes at $y = -1/2$. It is negative before this y value and positive after it, so the minimum x value is $h(-1/2) = -1/(2e)$. As $y \to -\infty$, $h(y) \to 0$ from below; also $h(0) = 0$ and h then increases without bound. Thus a level $x = k$ meets both negative-y branches precisely between the minimum and zero. At the minimum the two intersections coincide; at zero there is just the finite point $y = 0$.

$$\boxed{-\frac{1}{2e} < k < 0}$$

FINAL ANSWER

(a) $\frac{dy}{dx} = \frac{y}{x(1+2y)}$ where defined (b) $-\frac{1}{2e} < k < 0$