Elite Mathematics
Edexcel International A Level

IAL PURE 3 · WMA13

WMA13/01 October 2021 Worked Solutions

October 2021 paper rebuilt with the worked solution after each question.

10	2020–2026	Single-paper solutions
paper questions	19 exam sessions	checked by Dr Eslam



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PAST PAPER

WMA13/01 October 2021

October 2021

10	75	1 h 30
questions	marks	exam time

Each question is followed by its worked solution and final answer.

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Question 1

9 marks | Rational Algebra

1. The function f is defined by

$$\mathrm{f}(x) = \frac{5x}{x^2 + 7x + 12} + \frac{5x}{x + 4} \qquad x > 0$$

(a) Show that $\mathrm{f}(x) = \dfrac{5x}{x + 3}$

(3)

(b) Find f^{-1}

(3)

(c) (i) Find, in simplest form, $\mathrm{f}'(x)$.

(ii) Hence, state whether f is an increasing or a decreasing function, giving a reason for your answer.

(3)

WORKED SOLUTION

Question 1 · WMA13/01 October 2021

1 **Part (a)**

Factor $x^2 + 7x + 12 = (x + 3)(x + 4)$. A common denominator gives

$$f(x) = \frac{5x + 5x(x + 3)}{(x + 3)(x + 4)} = \frac{5x(x + 4)}{(x + 3)(x + 4)} = \frac{5x}{x + 3}.$$

The cancelled factor is nonzero throughout the original domain $x > 0$.

$$\boxed{f(x) = \frac{5x}{x + 3}, \quad x > 0}$$

2 **Part (b)**

Let $y = 5x/(x + 3)$. Then $yx + 3y = 5x$, so $x = 3y/(5 - y)$. Exchange variable names. Since $f(x) = 5 - 15/(x + 3)$ for $x > 0$, its range is $0 < f(x) < 5$. This becomes the inverse domain.

$$\boxed{f^{-1}(x) = \frac{3x}{5 - x}, \quad 0 < x < 5}$$

3 Part (c)

(i) By the quotient rule, $f'(x) = [5(x+3) - 5x]/(x+3)^2 = 15/(x+3)^2$. (ii) The numerator and denominator are positive on $x > 0$, so $f'(x) > 0$ throughout its domain. Therefore f is increasing.

$$\boxed{f'(x) = \frac{15}{(x+3)^2}; \quad f \text{ is increasing}}$$

FINAL ANSWER

(a) $f(x) = \frac{5x}{x+3}, \quad x > 0$ (b) $f^{-1}(x) = \frac{3x}{5-x}, \quad 0 < x < 5$ (c) $f'(x) = \frac{15}{(x+3)^2}; \quad f \text{ is increasing}$

Question 2

10 marks | Modulus

2.

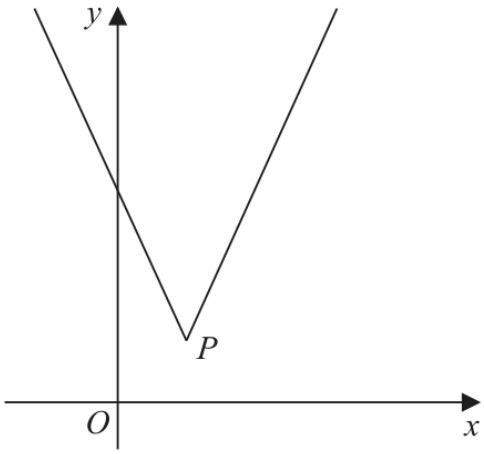


Figure 1

Figure 1 shows a sketch of part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = |3x - 13| + 5 \qquad x \in \mathbb{R}$$

The vertex of the graph is at point P, as shown in Figure 1.

(a) State the coordinates of P.

(2)

(b) (i) State the range of f.

(ii) Find the value of ff(4)

(2)

(c) Solve, using algebra and showing your working,

$$16 - 2x > |3x - 13| + 5$$

(4)

The graph with equation $y = \mathrm{f}(x)$ is transformed onto the graph with equation $y = a\mathrm{f}(x + b)$

The vertex of the graph with equation $y = a\mathrm{f}(x + b)$ is (4, 20)

Given that a and b are constants,

(d) find the value of a and the value of b.

(2)

1 Part (a)

The modulus vanishes at $3x - 13 = 0$, giving $x = 13/3$ and $y = 5$.

$$\boxed{P = (13/3, 5)}$$

2 Part (b)

(i) The modulus is nonnegative and attains zero, so the range is $[5, \infty)$. (ii) $f(4) = |12 - 13| + 5 = 6$, then $ff(4) = f(6) = |18 - 13| + 5 = 10$.

$$\boxed{\text{Range } [5, \infty); \qquad ff(4) = 10}$$

3 Part (c)

For $x < 13/3$, $|3x - 13| + 5 = 18 - 3x$. Thus $16 - 2x > 18 - 3x$ gives $x > 2$, so this branch contributes $2 < x < 13/3$. For $x \geq 13/3$, the right side is $3x - 8$. Then $16 - 2x > 3x - 8$ gives $x < 24/5$, so this branch contributes $13/3 \leq x < 24/5$. Combine the adjacent intervals.

$$\boxed{2 < x < \frac{24}{5}}$$

4 Part (d)

The new vertex is at $(13/3 - b, 5a)$. Therefore $5a = 20$ and $13/3 - b = 4$, giving $a = 4$ and $b = 1/3$.

$$\boxed{a = 4, \qquad b = \frac{1}{3}}$$

FINAL ANSWER

(a) $P = (13/3, 5)$ (b) $\text{Range } [5, \infty); \quad ff(4) = 10$ (c) $2 < x < \frac{24}{5}$ (d) $a = 4, \quad b = \frac{1}{3}$

Question 3

6 marks | Log Graphs & Exponential Models

3.

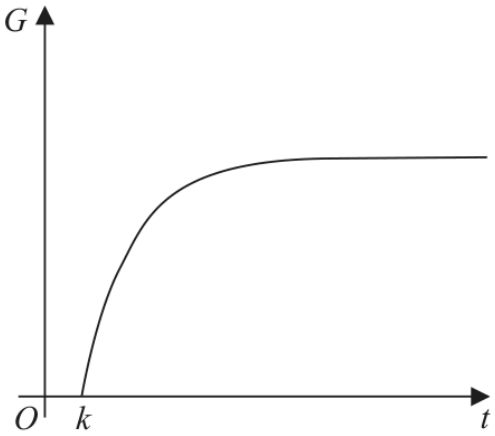


Figure 2

The total mass of gold, G tonnes, extracted from a mine is modelled by the equation

$$G = 40 - 30e^{1-0.05t} \qquad t \geqslant k \qquad G \geqslant 0$$

where t is the number of years after 1st January 1800.

Figure 2 shows a sketch of G against t.

Use the equation of the model to answer parts (a), (b) and (c).

(a) (i) Find the value of k.

(ii) Hence find the year and month in which gold started being extracted from the mine.

(3)

(b) Find the total mass of gold extracted from the mine up to 1st January 1870.

(2)

There is a limit to the mass of gold that can be extracted from the mine.

(c) State the value of this limit.

(1)

1 Part (a)

(i) Extraction starts at $G = 0$: $30e^{1-0.05k} = 40$, so $1 - 0.05k = \ln(4/3)$ and $k = 20(1 - \ln(4/3)) = 14.24635855\ldots$ years. (ii) This is 14 years and $12(0.24635855\ldots) = 2.95630261\ldots$ months after 1 January 1800. Two full months have elapsed from January 1814, so the date is in March 1814. Use the unrounded fractional year to identify the month.

$$\boxed{k = 14.2 \text{ years (3 s.f.)}; \quad \text{March 1814}}$$

2 Part (b)

On 1 January 1870, $t = 70$. Substitute:
$G(70) = 40 - 30e^{1-0.05(70)} = 40 - 30e^{-2.5} = 37.53745004\ldots$ tonnes.

$$\boxed{37.5 \text{ tonnes}}$$

3 Part (c)

As $t \to \infty$, $e^{1-0.05t} \to 0$, so $G \to 40$ tonnes.

$$\boxed{40 \text{ tonnes}}$$

FINAL ANSWER

(a) $k = 14.2$ years (3 s.f.); March 1814 (b) 37.5 tonnes (c) 40 tonnes

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Question 4

7 marks | Trig Identities & Exact Values

4. **In this question you should show detailed reasoning.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that the equation

$$2\sin(\theta - 30°) = 5\cos\theta$$

can be written in the form

$$\tan\theta = 2\sqrt{3}$$

(4)

(b) Hence, or otherwise, solve for $0 \leqslant x \leqslant 360°$

$$2\sin(x - 10°) = 5\cos(x + 20°)$$

giving your answers to one decimal place.

(3)

WORKED SOLUTION

Question 4 · WMA13/01 October 2021

1 Part (a)

Expand the compound angle using exact values:

$$2(\sin\theta\cos 30° - \cos\theta\sin 30°) = 5\cos\theta \Longrightarrow \sqrt{3}\sin\theta - \cos\theta = 5\cos\theta.$$

Thus $\sqrt{3}\sin\theta = 6\cos\theta$. If $\cos\theta = 0$, this equation is impossible, so divide by $\sqrt{3}\cos\theta$: $\tan\theta = 6/\sqrt{3} = 2\sqrt{3}$.

$$\boxed{\tan\theta = 2\sqrt{3}}$$

2 Part (b)

Put $\theta = x + 20°$, so $x - 10° = \theta - 30°$. Part (a) gives $\tan(x + 20°) = 2\sqrt{3}$. Since $20° \leq \theta \leq 380°$, the two permitted values are $\theta = 73.89788625\ldots°$ and $253.89788625\ldots°$. Subtract $20°$ from both and round last.

$$\boxed{x = 53.9°, \quad 233.9°}$$

FINAL ANSWER

(a) $\tan\theta = 2\sqrt{3}$ (b) $x = 53.9°, \quad 233.9°$

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Question 5

6 marks | Integration

5. (i) Find, by algebraic integration, the exact value of

$$\int_2^4 \frac{8}{(2x-3)^3}\,\mathrm{d}x$$

(4)

(ii) Find, in simplest form,

$$\int x\left(x^2+3\right)^7\,\mathrm{d}x$$

(2)

WORKED SOLUTION

Question 5 · WMA13/01 October 2021

1 Part (i)

Write the integrand as $8(2x-3)^{-3}$. A primitive is $-2(2x-3)^{-2}$, because its derivative is $8(2x-3)^{-3}$. Thus

$$\int_2^4 \frac{8}{(2x-3)^3}\,dx = \left[-\frac{2}{(2x-3)^2}\right]_2^4 = -\frac{2}{25} - (-2) = \frac{48}{25}.$$

$$\boxed{\frac{48}{25}}$$

2 Part (ii)

Since $d(x^2+3)/dx = 2x$, reverse the chain rule:

$$\int x(x^2+3)^7\,dx = \frac{1}{2}\cdot\frac{(x^2+3)^8}{8} + C.$$

$$\boxed{\frac{(x^2+3)^8}{16} + C}$$

FINAL ANSWER

(i) $\frac{48}{25}$ (ii) $\frac{(x^2+3)^8}{16} + C$

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Question 6

8 marks | Differentiation Techniques

6. (i) The curve C_1 has equation

$$y = 3\ln(x^2 - 5) - 4x^2 + 15 \qquad x > \sqrt{5}$$

Show that C_1 has a stationary point at $x = \dfrac{\sqrt{p}}{2}$ where p is a constant to be found.

(4)

(ii) A different curve C_2 has equation

$$y = 4x - 12\sin^2 x$$

(a) Show that, for this curve,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = A + B\sin 2x$$

where A and B are constants to be found.

(b) Hence, state the maximum gradient of this curve.

(4)

WORKED SOLUTION

Question 6 · WMA13/01 October 2021

1 **Part (i)**

Differentiate to obtain $y' = 6x/(x^2 - 5) - 8x$. At a stationary point in $x > \sqrt{5}$, x is nonzero, so $6/(x^2 - 5) = 8$, giving $6 = 8x^2 - 40$ and $x^2 = 23/4$. The original positive domain selects $x = \sqrt{23}/2$, which exceeds $\sqrt{5}$. Hence $p = 23$.

$$\boxed{p = 23, \qquad x = \frac{\sqrt{23}}{2}}$$

2 **Part (ii)**

(a) Differentiate the squared sine with the chain rule: $y' = 4 - 24\sin x\cos x = 4 - 12\sin 2x$. Thus $A = 4$, $B = -12$. (b) Since $-1 \le \sin 2x \le 1$, the largest derivative is attained when $\sin 2x = -1$: $4 - 12(-1) = 16$.

$$\boxed{A = 4, \quad B = -12; \qquad \text{maximum gradient} = 16}$$

FINAL ANSWER

(i) $p = 23, \quad x = \frac{\sqrt{23}}{2}$ (ii) $A = 4, \quad B = -12; \quad$ maximum gradient $= 16$

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Question 7

6 marks | Log Graphs & Exponential Models

7 The mass, M kg, of a species of tree can be modelled by the equation

$$\log_{10} M = 1.93 \log_{10} r + 0.684$$

where r cm is the base radius of the tree.

The base radius of a particular tree of this species is 45 cm.

According to the model,

(a) find the mass of this tree, giving your answer to 2 significant figures.

(2)

(b) Show that the equation of the model can be written in the form

$$M = pr^q$$

giving the values of the constants p and q to 3 significant figures.

(3)

(c) With reference to the model, interpret the value of the constant p.

(1)

WORKED SOLUTION

Question 7 · WMA13/01 October 2021

1 **Part (a)**

At $r = 45$, $\log_{10} M = 1.93 \log_{10} 45 + 0.684$. Exponentiate to obtain $M = 10^{0.684} 45^{1.93} = 7493.766412\ldots$ kg.

$$\boxed{M = 7.5 \times 10^3 \text{ kg}}$$

2 **Part (b)**

Exponentiate the full equation:

$$M = 10^{1.93 \log_{10} r + 0.684} = 10^{0.684} r^{1.93}.$$

Therefore $p = 10^{0.684} = 4.83058802\ldots$ and $q = 1.93$.

$$\boxed{p = 4.83, \qquad q = 1.93}$$

3

Part (c)

If the base radius is $r = 1$ cm, the model gives $M = p \cdot 1^q = p$ kg. Thus p is the predicted mass in kilograms of a tree of base radius 1 cm.

$$\boxed{\text{Predicted mass (kg) when base radius is 1 cm}}$$

✓ **FINAL ANSWER**

(a) $M = 7.5 \times 10^3$ kg (b) $p = 4.83, \qquad q = 1.93$ (c)

$\text{Predicted mass (kg) when base radius is 1 cm}$

Question 8

7 marks | Differentiation Techniques

8. A curve C has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \arcsin\left(\frac{1}{2}x\right) \qquad -2 \leqslant x \leqslant 2 \qquad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

(a) Sketch C.

(1)

(b) Given $x = 2\sin y$, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{A - x^2}}$$

where A is a constant to be found.

(3)

The point P lies on C and has y coordinate $\frac{\pi}{4}$

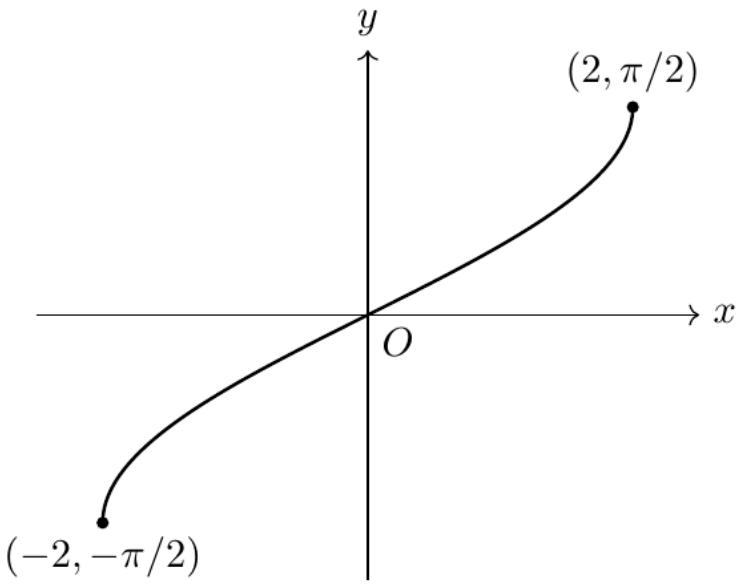
(c) Find the equation of the tangent to C at P. Write your answer in the form $y = mx + c$, where m and c are constants to be found.

(3)

1 Part (a)

The principal arcsine curve increases from $(-2, -\pi/2)$ through $(0, 0)$ to $(2, \pi/2)$. It is concave down on the negative-x half, concave up on the positive-x half, and vertical at both ends.

$$\boxed{y = \arcsin(x/2), \quad -2 \le x \le 2}$$



2 Part (b)

Differentiate the supplied inverse relation: $dx/dy = 2\cos y$. For $-\pi/2 < y < \pi/2$, cosine is positive, so $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2/4}$. Taking the reciprocal gives

$$\frac{dy}{dx} = \frac{1}{2\cos y} = \frac{1}{\sqrt{4 - x^2}}, \qquad -2 < x < 2.$$

Thus $A = 4$; the endpoints have vertical tangents, not finite derivatives.

$$\boxed{A = 4, \qquad \frac{dy}{dx} = \frac{1}{\sqrt{4 - x^2}}}$$

3 Part (c)

At $y = \pi/4$, $x = 2\sin(\pi/4) = \sqrt{2}$. The gradient is $1/\sqrt{4 - 2} = 1/\sqrt{2}$. Hence $y - \pi/4 = (x - \sqrt{2})/\sqrt{2}$, so $y = x/\sqrt{2} + \pi/4 - 1$.

$$\boxed{y = \frac{x}{\sqrt{2}} + \frac{\pi}{4} - 1}$$

FINAL ANSWER

(a) $y = \arcsin(x/2), \quad -2 \le x \le 2$ (b) $A = 4, \quad \frac{dy}{dx} = \frac{1}{\sqrt{4-x^2}}$ (c) $y = \frac{x}{\sqrt{2}} + \frac{\pi}{4} - 1$

Question 9

9 marks | Numerical Methods

9.

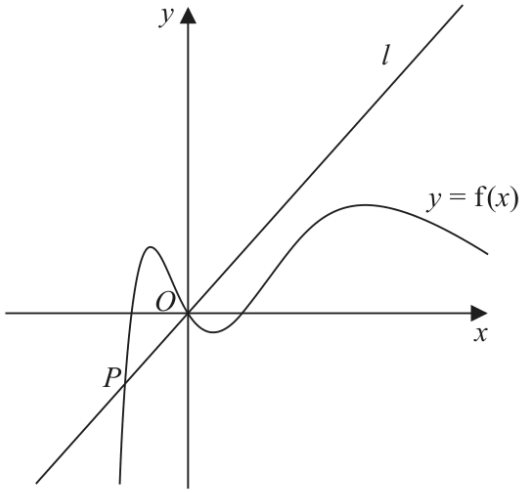


Figure 3

Figure 3 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x(x^2 - 4)\mathrm{e}^{-\frac{1}{2}x}$$

(a) Find $\mathrm{f}'(x)$.

(2)

The line l is the normal to the curve at O and meets the curve again at the point P.

The point P lies in the 3rd quadrant, as shown in Figure 3.

(b) Show that the x coordinate of P is a solution of the equation

$$x = -\frac{1}{2}\sqrt{16 + \mathrm{e}^{\frac{1}{2}x}}$$

(4)

(c) Using the iterative formula

$$x_{n+1} = -\frac{1}{2}\sqrt{16 + \mathrm{e}^{\frac{1}{2}x_n}} \quad \text{with } x_1 = -2$$

find, to 4 decimal places,

(i) the value of x_2

(ii) the x coordinate of P.

(3)

1 Part (a)

Write $f(x) = (x^3 - 4x)e^{-x/2}$. By the product rule,

$$f'(x) = (3x^2 - 4)e^{-x/2} - \frac{1}{2}(x^3 - 4x)e^{-x/2} = e^{-x/2}\left(-\frac{x^3}{2} + 3x^2 + 2x - 4\right).$$

$$\boxed{f'(x) = e^{-x/2}(-x^3/2 + 3x^2 + 2x - 4)}$$

2 Part (b)

At O, $f'(0) = -4$, so the normal gradient is $1/4$ and its equation is $y = x/4$. At the other intersection P, x is nonzero; equating curve and normal and dividing by x gives $(x^2 - 4)e^{-x/2} = 1/4$. Thus $x^2 = 4 + e^{x/2}/4 = (16 + e^{x/2})/4$. Since P lies in the third quadrant, select the negative square root.

$$\boxed{x = -\frac{1}{2}\sqrt{16 + e^{x/2}}}$$

3 Part (c)

Starting from $x_1 = -2$, $x_2 = -\frac{1}{2}\sqrt{16 + e^{-1}} = -2.0228617996\ldots$. Further unrounded iterates are $x_3 = -2.0226034084\ldots$, $x_4 = -2.0226063126\ldots$ and $x_5 = -2.0226062799\ldots$. The continuing sequence settles at $-2.0226062803\ldots$.

$$\boxed{\text{(i) } x_2 = -2.0229; \qquad \text{(ii) } x_P = -2.0226}$$

✓ FINAL ANSWER

(a) $f'(x) = e^{-x/2}(-x^3/2 + 3x^2 + 2x - 4)$ (b) $x = -\frac{1}{2}\sqrt{16 + e^{x/2}}$ (c) $\text{(i) } x_2 = -2.0229; \qquad \text{(ii) } x_P = -2.0226$

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Question 10

7 marks | Integration

10.

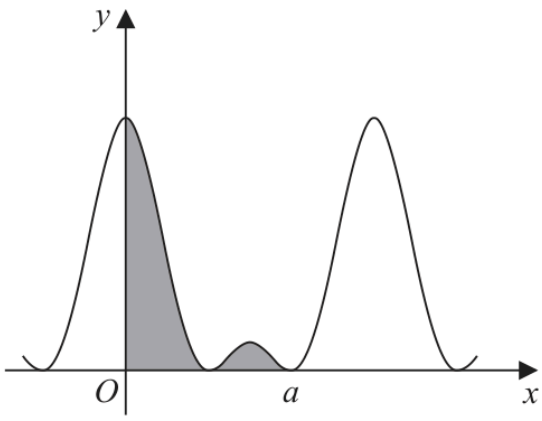


Figure 4

Figure 4 shows a sketch of part of the curve with equation

$$y = (1 + 2\cos 2x)^2$$

(a) Show that

$$(1 + 2\cos 2x)^2 \equiv p + q\cos 2x + r\cos 4x$$

where p, q and r are constants to be found.

(2)

The curve touches the positive x-axis for the second time when $x = a$, as shown in Figure 4.

The regions bounded by the curve, the y-axis and the x-axis up to $x = a$ are shown shaded in Figure 4.

(b) Find, using algebraic integration and making your method clear, the exact total area of the shaded regions. Write your answer in simplest form.

(5)

WORKED SOLUTION

Question 10 · WMA13/01 October 2021

1 **Part (a)**

Expand and use $\cos^2 2x = (1 + \cos 4x)/2$:

$$(1 + 2\cos 2x)^2 = 1 + 4\cos 2x + 4\cos^2 2x = 3 + 4\cos 2x + 2\cos 4x.$$

$$\boxed{p = 3, \quad q = 4, \quad r = 2}$$

2 **Part (b)**

The zeros satisfy $\cos 2x = -1/2$. The first two positive values are $x = \pi/3$ and $x = 2\pi/3$, so $a = 2\pi/3$. The squared curve is nonnegative, so integrate directly for the total shaded area:

$$\mathcal{A} = \int_0^{2\pi/3} (3 + 4\cos 2x + 2\cos 4x)\,dx = \left[3x + 2\sin 2x + \tfrac{1}{2}\sin 4x\right]_0^{2\pi/3}.$$

Use $\sin(4\pi/3) = -\sqrt{3}/2$ and $\sin(8\pi/3) = \sqrt{3}/2$ to obtain $\mathcal{A} = 2\pi - \sqrt{3} + \sqrt{3}/4 = 2\pi - 3\sqrt{3}/4$.

$$\boxed{2\pi - \frac{3\sqrt{3}}{4}}$$

FINAL ANSWER

(a) $p = 3, \quad q = 4, \quad r = 2$ (b) $2\pi - \frac{3\sqrt{3}}{4}$