Elite Mathematics
Edexcel International A Level

IAL PURE 3 · WMA13

WMA13/01 May/June 2021 Worked Solutions

May/June 2021 paper rebuilt with the worked solution after each question.

9	2020–2026	Single-paper solutions
paper questions	19 exam sessions	checked by Dr Eslam



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PAST PAPER

WMA13/01 May/June 2021

May/June 2021

9	75	1 h 30
questions	marks	exam time

Each question is followed by its worked solution and final answer.

WMA13/01 MAY/JUNE 2021

Question 1

7 marks | Numerical Methods

1. The curve C has equation

$$y = x^2 \cos\left(\frac{1}{2}x\right) \qquad 0 < x \leqslant \pi$$

The curve has a stationary point at the point P.

(a) Show, using calculus, that the x coordinate of P is a solution of the equation

$$x = 2\arctan\left(\frac{4}{x}\right)$$

(4)

Using the iteration formula

$$x_{n+1} = 2\arctan\left(\frac{4}{x_n}\right) \qquad x_1 = 2$$

(b) find the value of x_2 and the value of x_6, giving your answers to 3 decimal places.

(3)

WORKED SOLUTION

Question 1 · WMA13/01 May/June 2021

1 **Part (a)**

Using the product and chain rules,

$$y' = 2x\cos(x/2) - \frac{x^2}{2}\sin(x/2).$$

At a stationary point, divide by $x/2$, since $x > 0$, to obtain $4\cos(x/2) = x\sin(x/2)$. The endpoint $x = \pi$ does not satisfy this equation, so at P we have $0 < x < \pi$ and $\cos(x/2) > 0$. Hence $\tan(x/2) = 4/x$. The principal arctangent is valid because $0 < x/2 < \pi/2$:

$$x = 2\arctan(4/x).$$

$$\boxed{x = 2\arctan(4/x)}$$

2 Part (b)

Use radian mode and retain full calculator precision between iterations:

n	1	2	3	4	5	6
x_n	2	2.214297436	2.130426058	2.162810067	2.150238622	2.155108774

Thus the requested second and sixth terms, not the limiting value, are as below.

$$\boxed{x_2 = 2.214, \qquad x_6 = 2.155}$$

FINAL ANSWER

(a) $x = 2\arctan(4/x)$ (b) $x_2 = 2.214, \qquad x_6 = 2.155$

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Question 2

9 marks | Trig Identities & Exact Values

2. (a) Show that

$$\frac{1-\cos 2x}{2\sin 2x} \equiv k\tan x \qquad x \neq (90n)^\circ \qquad n \in \mathbb{Z}$$

where k is a constant to be found.

(3)

(b) Hence solve, for $0 < \theta < 90^\circ$

$$\frac{9(1-\cos 2\theta)}{2\sin 2\theta} = 2\sec^2\theta$$

giving your answers to one decimal place.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(6)

WORKED SOLUTION

Question 2 · WMA13/01 May/June 2021

1 Part (a)

Use $1-\cos 2x = 2\sin^2 x$ and $\sin 2x = 2\sin x\cos x$:

$$\frac{1-\cos 2x}{2\sin 2x} = \frac{2\sin^2 x}{4\sin x\cos x} = \frac{1}{2}\tan x.$$

The source excludes multiples of 90°, so the cancelled sine and denominator cosine are nonzero.

$$\boxed{k = \frac{1}{2}}$$

2 Part (b)

Part (a) reduces the equation to $(9/2)\tan\theta = 2\sec^2\theta$. Set $u = \tan\theta$ and use $\sec^2\theta = 1+\tan^2\theta$:

$$4u^2 - 9u + 4 = 0, \qquad u = \frac{9 \pm \sqrt{17}}{8}.$$

Both roots are positive. On $0 < \theta < 90^\circ$, tangent is one-to-one, so each gives exactly one angle: $31.36697777\ldots^\circ$ and $58.63302223\ldots^\circ$.

$$\boxed{\theta = 31.4^\circ, \quad 58.6^\circ}$$

FINAL ANSWER

(a) $k = \frac{1}{2}$ (b) $\theta = 31.4°, \quad 58.6°$

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Question 3

8 marks | Integration

3. (i) Find

$$\int \frac{12}{(2x-1)^2}\,\mathrm{d}x$$

giving your answer in simplest form.

(2)

(ii) (a) Write $\dfrac{4x+3}{x+2}$ in the form

$$A + \frac{B}{x+2}$$ where A and B are constants to be found

(b) Hence find, using algebraic integration, the exact value of

$$\int_{-8}^{-5} \frac{4x+3}{x+2}\,\mathrm{d}x$$

giving your answer in simplest form.

(6)

WORKED SOLUTION

Question 3 · WMA13/01 May/June 2021

1 **Part (i)**

Write the integrand as $12(2x-1)^{-2}$. Reverse the chain rule, dividing by the inner derivative 2:

$$\int 12(2x-1)^{-2}\,dx = -6(2x-1)^{-1} + C.$$

$$\boxed{-\frac{6}{2x-1} + C}$$

2 **Part (ii)**

(a) Since $4x+3 = 4(x+2) - 5$,

$$\frac{4x+3}{x+2} = 4 - \frac{5}{x+2}, \qquad A = 4,\ B = -5.$$

(b) A primitive is $4x - 5\ln|x+2|$. The absolute value is essential here: $x+2$ is negative throughout $[-8,-5]$.

$$\int_{-8}^{-5} \frac{4x+3}{x+2}\,dx = (-20 - 5\ln 3) - (-32 - 5\ln 6) = 12 + 5\ln 2.$$

$$\boxed{A = 4,\ B = -5; \qquad I = 12 + 5\ln 2}$$

FINAL ANSWER

(i) $-\frac{6}{2x-1} + C$ (ii) $A = 4,\ B = -5; \qquad I = 12 + 5\ln 2$

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Question 4

10 marks | Functions, Composition & Inverses

4. The functions f and g are defined by

$$\mathrm{f}(x) = \frac{4x+6}{x-5} \qquad x \in \mathbb{R},\ x \neq 5$$

$$\mathrm{g}(x) = 5 - 2x^2 \qquad x \in \mathbb{R},\ x \leqslant 0$$

(a) Solve the equation

$$\mathrm{fg}(x) = 3$$

(4)

(b) Find f^{-1}

(3)

(c) Sketch and label, on the same axes, the curve with equation $y = \mathrm{g}(x)$ and the curve with equation $y = \mathrm{g}^{-1}(x)$. Show on your sketch the coordinates of the points where each curve meets or cuts the coordinate axes.

(3)

WORKED SOLUTION

Question 4 · WMA13/01 May/June 2021

1 Part (a)

Substitute $g(x) = 5 - 2x^2$ into f:

$$fg(x) = \frac{4(5-2x^2)+6}{5-2x^2-5} = \frac{26-8x^2}{-2x^2} = 4 - \frac{13}{x^2}.$$

The composite requires $x \leq 0$ from g and $g(x) \neq 5$ from f; therefore $x < 0$. Now $4 - 13/x^2 = 3$ gives $x^2 = 13$. Only the negative root belongs to the composite domain.

$$\boxed{x = -\sqrt{13}}$$

2 Part (b)

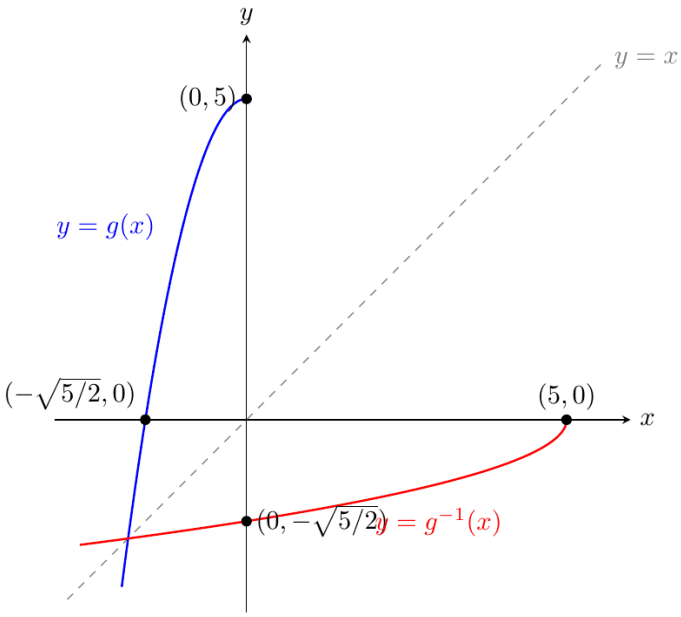
Let $y = (4x+6)/(x-5)$. Then $yx - 5y = 4x + 6$, so $x(y-4) = 5y + 6$ and $x = (5y+6)/(y-4)$. Exchange the variable names. The original function is $4 + 26/(x-5)$, whose range excludes 4.

$$\boxed{f^{-1}(x) = \frac{5x+6}{x-4}, \quad x \neq 4}$$

3 Part (c)

The graph of g is only the left half of the downward parabola: $y = 5 - 2x^2$, $x \le 0$. Its intercepts are $(0, 5)$ and $(-\sqrt{5/2}, 0)$. For the inverse, $x = 5 - 2y^2$ and $y \le 0$, hence $g^{-1}(x) = -\sqrt{(5-x)/2}$, $x \le 5$. Reflect the original branch in $y = x$. The inverse intercepts are $(5, 0)$ and $(0, -\sqrt{5/2})$.

$$\boxed{g^{-1}(x) = -\sqrt{(5-x)/2},\ x \le 5; \quad \text{graphs as shown}}$$



FINAL ANSWER

(a) $x = -\sqrt{13}$ (b) $f^{-1}(x) = \frac{5x+6}{x-4}$, $x \neq 4$ (c)
$g^{-1}(x) = -\sqrt{(5-x)/2},\ x \le 5;$ graphs as shown

Question 5

7 marks | Log Graphs & Exponential Models

5.

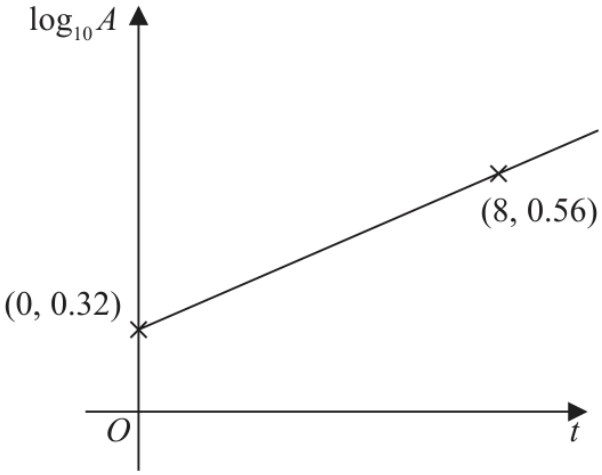


Figure 1

The growth of duckweed on a pond is being studied.

The surface area of the pond covered by duckweed, $A\text{m}^2$, at a time t days after the start of the study is modelled by the equation

$$A = pq^t \qquad \text{where } p \text{ and } q \text{ are positive constants}$$

Figure 1 shows the linear relationship between $\log_{10} A$ and t.

The points (0, 0.32) and (8, 0.56) lie on the line as shown.

(a) Find, to 3 decimal places, the value of p and the value of q.

(4)

Using the model with the values of p and q found in part (a),

(b) find the rate of increase of the surface area of the pond covered by duckweed, in m^2/day, exactly 6 days after the start of the study.
Give your answer to 2 decimal places.

(3)

1 Part (a)

Taking base-ten logarithms gives $\log_{10} A = \log_{10} p + t \log_{10} q$. The intercept is 0.32 and the gradient is $(0.56 - 0.32)/8 = 0.03$. Thus $p = 10^{0.32} = 2.08929613\ldots$ and $q = 10^{0.03} = 1.071519305\ldots$.

$$\boxed{p = 2.089, \qquad q = 1.072}$$

2 Part (b)

Using the parameter values requested in (a), $A = 2.089(1.072)^t$. For an arbitrary positive base, $d(q^t)/dt = q^t \ln q$, so

$$A'(6) = 2.089(1.072)^6 \ln(1.072) = 0.2204219233\ldots.$$

This is a positive area increase per day. Retaining unrounded parameters also gives 0.22 to two decimal places.

$$\boxed{0.22\ \text{m}^2/\text{day}}$$

FINAL ANSWER

(a) $p = 2.089, \qquad q = 1.072$ (b) $0.22\ \text{m}^2/\text{day}$

Question 6

8 marks | Modulus

6. Given that k is a positive constant,

(a) on separate diagrams, sketch the graph with equation

(i) $y = k - 2|x|$

(ii) $y = \left|2x - \frac{k}{3}\right|$

Show on each sketch the coordinates, in terms of k, of each point where the graph meets or cuts the axes.

(4)

(b) Hence find, in terms of k, the values of x for which

$$\left|2x - \frac{k}{3}\right| = k - 2|x|$$

giving your answers in simplest form.

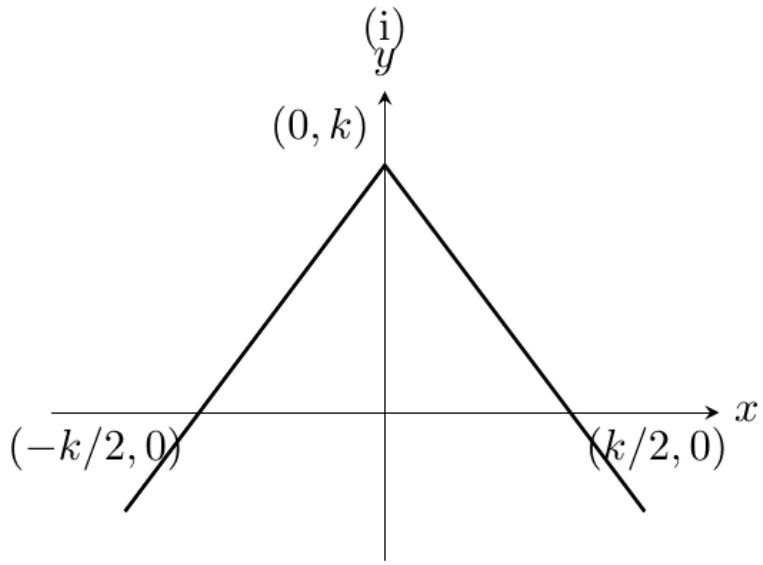
(4)

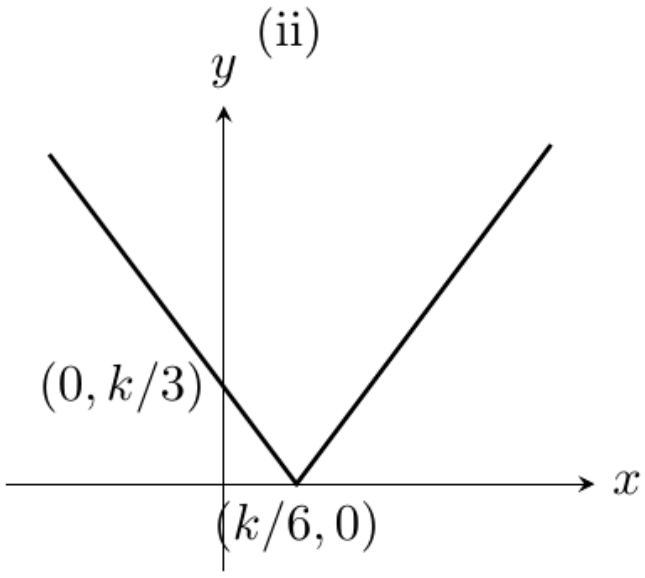
1 **Part (a)**

(i) $y = k - 2|x|$ is an inverted V with vertex $(0, k)$ and roots $x = \pm k/2$. (ii) $y = |2x - k/3|$ is an upright V with vertex $(k/6, 0)$ and y-intercept $(0, k/3)$. Because $k > 0$, both sketches have the ordering shown. Axes below use coordinates normalized by k, while labels give the original coordinates.

\qquad

$$\boxed{\text{(i) } (0,k),(\pm k/2,0); \quad \text{(ii) } (0,k/3),(k/6,0)}$$





2 **Part (b)**

Consider the breakpoints 0 and $k/6$. For $x < 0$, the equation is $k/3 - 2x = k + 2x$, giving $x = -k/6$, valid in this branch. For $0 \le x < k/6$, it is $k/3 - 2x = k - 2x$, impossible because $k > 0$. For $x \ge k/6$, it is $2x - k/3 = k - 2x$, giving $x = k/3$, valid in this branch. Both have nonnegative right side and satisfy the original equation.

$$\boxed{x = -\frac{k}{6}, \qquad x = \frac{k}{3}}$$

✓ **FINAL ANSWER**

(a) $\textbf{(i)}\ (0, k), (\pm k/2, 0); \quad \textbf{(ii)}\ (0, k/3), (k/6, 0)$ (b) $x = -\frac{k}{6}, \qquad x = \frac{k}{3}$

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Question 7

5 marks | Differentiation Techniques

7. Given that

$$x = 6\sin^2 2y \qquad 0 < y < \frac{\pi}{4}$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{A\sqrt{(Bx - x^2)}}$$

where A and B are integers to be found.

(5)

WORKED SOLUTION

Question 7 · WMA13/01 May/June 2021

1 Part (main)

Differentiate x with respect to y:

$$\frac{dx}{dy} = 6 \cdot 2\sin 2y\cos 2y \cdot 2 = 24\sin 2y\cos 2y.$$

Since $0 < 2y < \pi/2$, both sine and cosine are positive. Using $\sin^2 2y = x/6$ and $\cos^2 2y = 1 - x/6$,

$$\frac{dx}{dy} = 24\sqrt{x/6}\sqrt{1 - x/6} = 4\sqrt{6x - x^2}.$$

On $0 < x < 6$ this is nonzero, so take its reciprocal.

$$\boxed{\frac{dy}{dx} = \frac{1}{4\sqrt{6x - x^2}}, \qquad A = 4,\ B = 6}$$

FINAL ANSWER

(main) $\frac{dy}{dx} = \frac{1}{4\sqrt{6x-x^2}}$, $A = 4,\ B = 6$

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Question 8

13 marks | Log Graphs & Exponential Models

8. A scientist is studying a population of fish in a lake. The number of fish, N, in the population, t years after the start of the study, is modelled by the equation

$$N = \frac{600e^{0.3t}}{2 + e^{0.3t}} \qquad t \geqslant 0$$

Use the equation of the model to answer parts (a), (b), (c), (d) and (e).

(a) Find the number of fish in the lake at the start of the study.

(1)

(b) Find the upper limit to the number of fish in the lake.

(1)

(c) Find the time, after the start of the study, when there are predicted to be 500 fish in the lake. Give your answer in years and months to the nearest month.

(4)

(d) Show that

$$\frac{dN}{dt} = \frac{Ae^{0.3t}}{(2 + e^{0.3t})^2}$$

where A is a constant to be found.

(3)

Given that when $t = T$, $\frac{dN}{dt} = 8$

(e) find the value of T to one decimal place.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(4)

WORKED SOLUTION

Question 8 · WMA13/01 May/June 2021

1 **Part (a)**

At the start, $t = 0$ and $e^{0.3t} = 1$. Thus $N(0) = 600/(2 + 1) = 200$.

$$\boxed{200 \text{ fish}}$$

2 **Part (b)**

Divide numerator and denominator by $e^{0.3t}$ to write $N = 600/(1 + 2e^{-0.3t})$. As $t \to \infty$, the denominator tends to 1, so the upper limit is 600 fish.

$$\boxed{600 \text{ fish}}$$

---

3 **Part (c)**

Set $N = 500$ and multiply through: $1000 + 500e^{0.3t} = 600e^{0.3t}$, so $e^{0.3t} = 10$. Hence $t = \ln 10/0.3 = 7.675283643\ldots$ years. The fractional year is $12(0.675283643\ldots) = 8.10340372\ldots$ months. Round only the months to the nearest whole month.

$$\boxed{7 \text{ years } 8 \text{ months}}$$

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4 **Part (d)**

The quotient rule gives

$$\frac{dN}{dt} = \frac{180e^{0.3t}(2 + e^{0.3t}) - 600e^{0.3t}(0.3e^{0.3t})}{(2 + e^{0.3t})^2} = \frac{360e^{0.3t}}{(2 + e^{0.3t})^2}.$$

The two $e^{0.6t}$ terms cancel exactly.

$$\boxed{A = 360}$$

---

5 **Part (e)**

Let $z = e^{0.3T}$. Since $T \geq 0$, require $z \geq 1$. Then $360z/(2 + z)^2 = 8$ gives $45z = (2 + z)^2$, hence

$$z^2 - 41z + 4 = 0, \qquad z = \frac{41 \pm \sqrt{1665}}{2}.$$

The smaller root lies between 0 and 1 and would give $T < 0$, so reject it. For the larger root,

$$T = \frac{1}{0.3}\ln\left(\frac{41 + \sqrt{1665}}{2}\right) = 12.37061331\ldots.$$

$$\boxed{T = 12.4 \text{ years}}$$

FINAL ANSWER

(a) 200 fish (b) 600 fish (c) 7 years 8 months (d) $A = 360$ (e) $T = 12.4$ years

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Question 9

8 marks | Trig Equations & R-Form

9. (a) Express $12\sin x - 5\cos x$ in the form $R\sin(x - \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the exact value of R and give the value of α in radians, to 3 decimal places.

(3)

The function g is defined by

$$g(\theta) = 10 + 12\sin\left(2\theta - \frac{\pi}{6}\right) - 5\cos\left(2\theta - \frac{\pi}{6}\right) \qquad \theta > 0$$

Find

(b) (i) the minimum value of $g(\theta)$

(ii) the smallest value of θ at which the minimum value occurs.

(3)

The function h is defined by

$$h(\beta) = 10 - (12\sin\beta - 5\cos\beta)^2$$

(c) Find the range of h.

(2)

WORKED SOLUTION

Question 9 · WMA13/01 May/June 2021

1 **Part (a)**

Expand $R\sin(x-\alpha) = R\cos\alpha\sin x - R\sin\alpha\cos x$. Therefore $R\cos\alpha = 12$, $R\sin\alpha = 5$. Thus $R = \sqrt{12^2 + 5^2} = 13$ and $\alpha = \arctan(5/12) = 0.3947911197\ldots$ radians, in the required first quadrant.

$$\boxed{13\sin(x - 0.395) \quad (R = 13,\ \alpha = 0.395\ \text{rad})}$$

2 Part (b)

Write $g(\theta) = 10 + 13\sin(2\theta - \pi/6 - \alpha)$. (i) The minimum is $10 - 13 = -3$. (ii) For sine to equal -1, $2\theta - \pi/6 - \alpha = -\pi/2 + 2\pi n$. Hence $\theta = -\pi/6 + \alpha/2 + \pi n$. For $n = 0$ this is negative, so the smallest permitted value uses $n = 1$:

$$\theta = \frac{5\pi}{6} + \frac{1}{2}\arctan(5/12) = 2.815389438\ldots \text{ radians}.$$

Use the unrounded phase angle for the calculation.

$$\boxed{\min g = -3; \qquad \theta_{\min} = \frac{5\pi}{6} + \frac{1}{2}\arctan(5/12)}$$

3 Part (c)

As β varies over the reals, $12\sin\beta - 5\cos\beta$ covers $[-13, 13]$. Its square therefore covers $[0, 169]$, not $[169, 169]$. Subtracting from 10 reverses the bounds, giving $[-159, 10]$; both endpoints are attained.

$$\boxed{-159 \le h(\beta) \le 10}$$

FINAL ANSWER

(a) $13\sin(x - 0.395) \quad (R = 13,\ \alpha = 0.395\ \text{rad})$ (b) $\min g = -3; \qquad \theta_{\min} = \frac{5\pi}{6} + \frac{1}{2}\arctan(5/12)$ (c) $-159 \le h(\beta) \le 10$