Elite Mathematics
Edexcel International A Level

IAL PURE 3 · WMA13

WMA13/01 January 2021 Worked Solutions

January 2021 paper rebuilt with the worked solution after each question.

10	2020–2026	Single-paper solutions
paper questions	19 exam sessions	checked by Dr Eslam



Dr Eslam Ahmed
Assistant Lecturer, Faculty of Engineering, Cairo University
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PAST PAPER

WMA13/01 January 2021

January 2021

10	75	1 h 30
questions	marks	exam time

Each question is followed by its worked solution and final answer.

WMA13/01 JANUARY 2021

Question 1

3 marks | Integration

1. Find

$$\int \frac{x^2 - 5}{2x^3} \mathrm{d}x \qquad x > 0$$

giving your answer in simplest form.

(3)

WORKED SOLUTION

Question 1 · WMA13/01 January 2021

1 **Part (main)**

First split the fraction into powers:

$$\frac{x^2 - 5}{2x^3} = \frac{1}{2}x^{-1} - \frac{5}{2}x^{-3}.$$

Integrate term by term. Since $x > 0$, use $\ln x$ without absolute-value notation:

$$\int \frac{x^2 - 5}{2x^3}\, dx = \frac{1}{2}\ln x + \frac{5}{4}x^{-2} + C.$$

$$\boxed{\frac{1}{2}\ln x + \frac{5}{4x^2} + C}$$

✓ **FINAL ANSWER**

(main) $\frac{1}{2}\ln x + \frac{5}{4x^2} + C$

Question 2

6 marks | Graph Transformations

2.

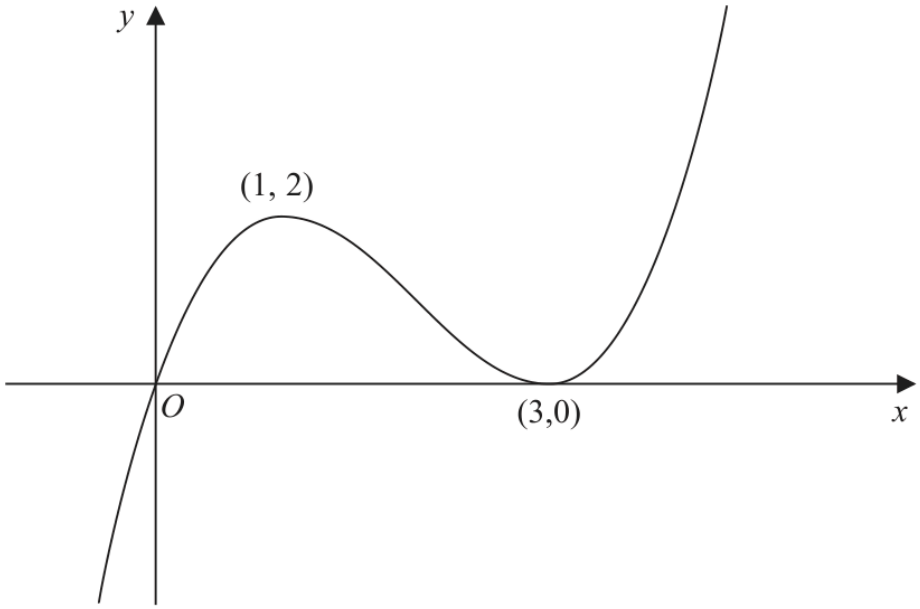


Figure 1

Figure 1 shows a sketch of the curve with equation $y = f(x)$, where $x \in \mathbb{R}$ and $f(x)$ is a polynomial.

The curve passes through the origin and touches the x-axis at the point (3, 0)

There is a maximum turning point at (1, 2) and a minimum turning point at (3, 0)

On separate diagrams, sketch the curve with equation

(i) $y = 3f(2x)$

(3)

(ii) $y = f(-x) - 1$

(3)

On each sketch, show clearly the coordinates of

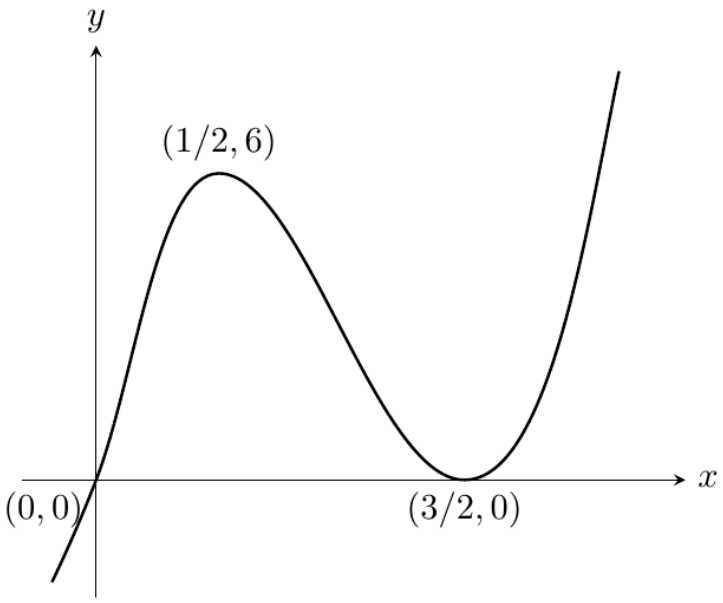
- the point where the curve crosses the y-axis
- any maximum or minimum turning points

1

Part (i)

For $y = 3f(2x)$, a source point (u, v) maps to $(u/2, 3v)$. Thus the y-intercept stays $(0, 0)$, the maximum becomes $(1/2, 6)$ and the minimum becomes $(3/2, 0)$. The horizontal compression and positive vertical stretch preserve the turning-point types. The sketch below records the given shape and mapped features; it does not assume an additional formula for f.

$$\boxed{(0,0); \quad \max(1/2, 6); \quad \min(3/2, 0)}$$

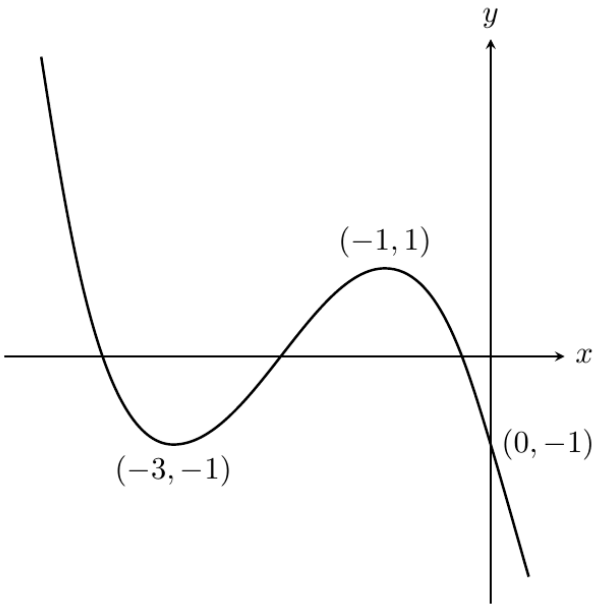


2

Part (ii)

For $y = f(-x) - 1$, map (u, v) to $(-u, v - 1)$: reflect in the y-axis, then shift down 1. The y-intercept is $(0, -1)$, the maximum is $(-1, 1)$ and the minimum is $(-3, -1)$. The left end now rises and the right end falls.

$$\boxed{(0,-1); \quad \max(-1, 1); \quad \min(-3, -1)}$$



FINAL ANSWER

(i) $(0,0)$; $\max(1/2, 6)$; $\min(3/2, 0)$ (ii) $(0,-1)$; $\max(-1,1)$; $\min(-3,-1)$

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Question 3

8 marks | Rational Algebra

3.

$$\mathrm{f}(x) = 3 - \frac{x-2}{x+1} + \frac{5x+26}{2x^2-3x-5} \qquad x > 4$$

(a) Show that

$$\mathrm{f}(x) = \frac{ax+b}{cx+d} \qquad x > 4$$

where a, b, c and d are integers to be found.

(4)

(b) Hence find $\mathrm{f}^{-1}(x)$

(2)

(c) Find the domain of f^{-1}

(2)

WORKED SOLUTION

Question 3 · WMA13/01 January 2021

1 **Part (a)**

Factor $2x^2 - 3x - 5 = (2x-5)(x+1)$. Then

$$3 - \frac{x-2}{x+1} = \frac{2x+5}{x+1},$$

so

$$f(x) = \frac{(2x+5)(2x-5)+5x+26}{(x+1)(2x-5)} = \frac{4x^2+5x+1}{(x+1)(2x-5)} = \frac{(4x+1)(x+1)}{(x+1)(2x-5)} = \frac{4x+1}{2x-5}.$$

On $x > 4$ the cancelled factor is nonzero. One integer choice is $a = 4, b = 1, c = 2, d = -5$.

$$\boxed{f(x) = \frac{4x+1}{2x-5},\ x > 4}$$

2 **Part (b)**

Let $y = (4x+1)/(2x-5)$. Rearranging gives $2xy - 5y = 4x + 1$, hence $x(2y-4) = 5y+1$ and $x = (5y+1)/(2y-4)$. Swap variable names to obtain the inverse rule; its restricted domain is determined in (c).

$$\boxed{f^{-1}(x) = \frac{5x+1}{2x-4}}$$

3 Part (c)

Rewrite $f(x) = 2 + 11/(2x - 5)$. For $x > 4$, the positive denominator is greater than 3 and grows without bound. Thus $0 < 11/(2x - 5) < 11/3$, so $2 < f(x) < 17/3$. Neither bound is attained: $x = 4$ is excluded and $f(x) = 2$ is impossible. The range of f is the domain of its inverse.

$$\boxed{2 < x < \frac{17}{3}}$$

FINAL ANSWER

(a) $f(x) = \frac{4x+1}{2x-5},\ x > 4$ (b) $f^{-1}(x) = \frac{5x+1}{2x-4}$ (c) $2 < x < \frac{17}{3}$

Question 4

9 marks | Modulus

4.

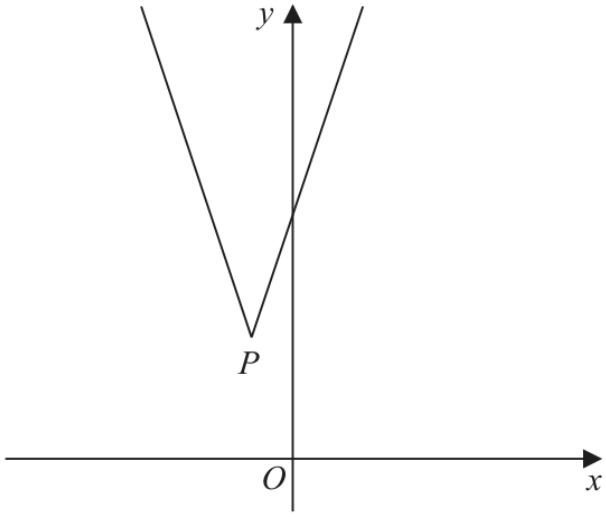


Figure 2

Figure 2 shows a sketch of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = |3x + a| + a$$

and where a is a positive constant.

The graph has a vertex at the point P, as shown in Figure 2.

(a) Find, in terms of a, the coordinates of P.

(2)

(b) Sketch the graph with equation $y = \mathrm{g}(x)$, where

$$\mathrm{g}(x) = |x + 5a|$$

On your sketch, show the coordinates, in terms of a, of each point where the graph cuts or meets the coordinate axes.

(2)

The graph with equation $y = \mathrm{g}(x)$ intersects the graph with equation $y = \mathrm{f}(x)$ at two points.

(c) Find, in terms of a, the coordinates of the two points.

(5)

1 Part (a)

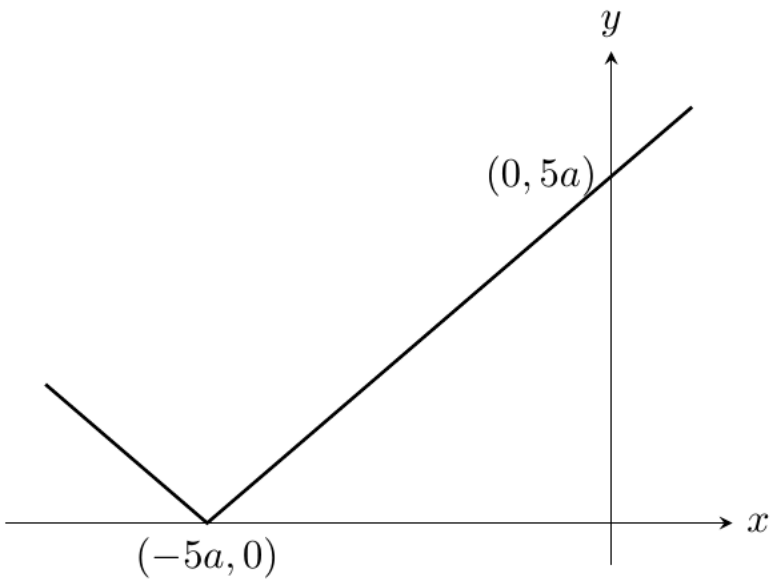
At the vertex the inner expression vanishes: $3x + a = 0$, giving $x = -a/3$ and $f(x) = a$.

$$\boxed{P = (-a/3, a)}$$

2 Part (b)

$g(x) = |x + 5a|$ is a V translated left by $5a$, with vertex $(-5a, 0)$ and y-intercept $(0, 5a)$. Since $a > 0$, the vertex lies strictly left of the y-axis. Coordinates in the drawing are normalized by a; labels give the actual intercepts.

$$\boxed{(-5a, 0), \quad (0, 5a)}$$



3 Part (c)

Solve $|x + 5a| = |3x + a| + a$ using breakpoints $-5a < -a/3$. For $x < -5a$: $-x - 5a = -3x$, giving $x = 5a/2$, outside this branch, so reject it. For $-5a \le x < -a/3$: $x + 5a = -3x$, giving $x = -5a/4$ and $y = 15a/4$. For $x \ge -a/3$: $x + 5a = 3x + 2a$, giving $x = 3a/2$ and $y = 13a/2$. The last two values belong to their respective intervals and satisfy both original functions.

$$\boxed{\left(-\frac{5a}{4}, \frac{15a}{4}\right), \quad \left(\frac{3a}{2}, \frac{13a}{2}\right)}$$

FINAL ANSWER

(a) $P = (-a/3, a)$ (b) $(-5a, 0), \quad (0, 5a)$ (c) $\left(-\frac{5a}{4}, \frac{15a}{4}\right), \quad \left(\frac{3a}{2}, \frac{13a}{2}\right)$

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Question 5

11 marks | Log Graphs & Exponential Models

5. The temperature, θ°C, inside an oven, t minutes after the oven is switched on, is given by

$$\theta = A - 180\mathrm{e}^{-kt}$$

where A and k are positive constants.

Given that the temperature inside the oven is initially 18 °C,

(a) find the value of A.

(2)

The temperature inside the oven, 5 minutes after the oven is switched on, is 90 °C.

(b) Show that $k = p\ln q$ where p and q are rational numbers to be found.

(4)

Hence find

(c) the temperature inside the oven 9 minutes after the oven is switched on, giving your answer to 3 significant figures,

(2)

(d) the rate of increase of the temperature inside the oven 9 minutes after the oven is switched on. Give your answer in °C min^{-1} to 3 significant figures.

(3)

WORKED SOLUTION

Question 5 · WMA13/01 January 2021

1 **Part (a)**

At $t = 0$, $18 = A - 180e^{0} = A - 180$. Hence $A = 198$.

$$\boxed{A = 198}$$

2 **Part (b)**

At $t = 5$, $90 = 198 - 180e^{-5k}$, so $180e^{-5k} = 108$ and $e^{-5k} = 3/5$. Taking logarithms gives $-5k = \ln(3/5)$, hence $k = (1/5)\ln(5/3)$. Thus one suitable rational pair is $p = 1/5$, $q = 5/3$.

$$\boxed{k = \frac{1}{5}\ln\frac{5}{3};\quad p = \frac{1}{5},\ q = \frac{5}{3}}$$

3 Part (c)

Use the unrounded k:

$$\theta(9) = 198 - 180e^{-(9/5)\ln(5/3)} = 126.22970096\ldots\,^{\circ}\mathrm{C}.$$

Round the temperature to three significant figures.

$$\boxed{126^{\circ}\mathrm{C}}$$

4 Part (d)

Differentiate with respect to time: $d\theta/dt = 180ke^{-kt}$. At $t = 9$ this is $180[\ln(5/3)/5]e^{-(9/5)\ln(5/3)} = 7.332421555\ldots$. The sign is positive because the oven is warming.

$$\boxed{7.33^{\circ}\mathrm{C\,min^{-1}}}$$

FINAL ANSWER

(a) $A = 198$ (b) $k = \frac{1}{5}\ln\frac{5}{3}$; $\quad p = \frac{1}{5}$, $q = \frac{5}{3}$ (c) $126^{\circ}\mathrm{C}$ (d) $7.33^{\circ}\mathrm{C\,min^{-1}}$

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Question 6

8 marks | Numerical Methods

6.

$$\mathrm{f}(x) = x\cos\left(\frac{x}{3}\right) \qquad x > 0$$

(a) Find $\mathrm{f}'(x)$

(2)

(b) Show that the equation $\mathrm{f}'(x) = 0$ can be written as

$$x = k\arctan\left(\frac{k}{x}\right)$$

where k is an integer to be found.

(2)

(c) Starting with $x_1 = 2.5$ use the iteration formula

$$x_{n+1} = k\arctan\left(\frac{k}{x_n}\right)$$

with the value of k found in part (b), to calculate the values of x_2 and x_6 giving your answers to 3 decimal places.

(2)

(d) Using a suitable interval and a suitable function that should be stated, show that a root of $\mathrm{f}'(x) = 0$ is 2.581 correct to 3 decimal places.

(2)

WORKED SOLUTION

Question 6 · WMA13/01 January 2021

1 **Part (a)**

By the product and chain rules, $f'(x) = \cos(x/3) - (x/3)\sin(x/3)$.

$$\boxed{f'(x) = \cos(x/3) - \frac{x}{3}\sin(x/3)}$$

2 Part (b)

Set $f'(x) = 0$: $\cos(x/3) = (x/3)\sin(x/3)$. At a solution, cosine cannot vanish (that would leave a nonzero right side), so $\tan(x/3) = 3/x$. On the principal positive branch $0 < x/3 < \pi/2$, this gives

$$x = 3\arctan(3/x), \qquad k = 3.$$

This is the branch used by the supplied iteration for the root near 2.581. The original domain $x > 0$ also admits later tangent branches: in general $x = 3\arctan(3/x) + 3n\pi$. The principal-branch rearrangement must not be interpreted as listing every positive stationary point.

$$\boxed{k = 3; \quad x = 3\arctan(3/x) \text{ on the selected branch}}$$

3 Part (c)

Using radians and $x_1 = 2.5$, the successive values are

n	1	2	3	4	5	6
x_n	2.5	2.628174152	2.554102174	2.596526871	2.572101407	2.586122337

Keep full precision between steps and round the requested terms only.

$$\boxed{x_2 = 2.628, \qquad x_6 = 2.586}$$

4 Part (d)

Define $F(x) = \cos(x/3) - (x/3)\sin(x/3)$, which is continuous near 2.581. The rounding interval endpoints give

$$F(2.5805) = 0.0003467157\ldots > 0, \qquad F(2.5815) = -0.0003456897\ldots < 0.$$

By continuity a root lies strictly between these endpoints; every number in this open interval rounds to 2.581 to three decimal places.

$$\boxed{x = 2.581 \text{ correct to 3 d.p.}}$$

✓ FINAL ANSWER

(a) $f'(x) = \cos(x/3) - \frac{x}{3}\sin(x/3)$ (b) $k = 3; \quad x = 3\arctan(3/x)$ on the selected branch (c) $x_2 = 2.628, \qquad x_6 = 2.586$ (d) $x = 2.581$ correct to 3 d.p.

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Question 7

9 marks | Trig Identities & Exact Values

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

7. (a) Prove that

$$\frac{\sin 2x}{\cos x} + \frac{\cos 2x}{\sin x} \equiv \operatorname{cosec} x \qquad x \neq \frac{n\pi}{2} \quad n \in \mathbb{Z}$$

(3)

(b) Hence solve, for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$7 + \frac{\sin 4\theta}{\cos 2\theta} + \frac{\cos 4\theta}{\sin 2\theta} = 3\cot^2 2\theta$$

giving your answers in radians to 3 significant figures where appropriate.

(6)

WORKED SOLUTION

Question 7 · WMA13/01 January 2021

1 Part (a)

Apply the double-angle identities to the left side:

$$\frac{2\sin x\cos x}{\cos x} + \frac{1-2\sin^2 x}{\sin x} = 2\sin x + \frac{1}{\sin x} - 2\sin x = \csc x.$$

All divisions and cancellations are valid under the given exclusion $x \neq n\pi/2$.

$$\boxed{\frac{\sin 2x}{\cos x} + \frac{\cos 2x}{\sin x} = \csc x}$$

2 Part (b)

Using (a) with $x = 2\theta$ gives $7 + \csc 2\theta = 3\cot^2 2\theta$. Set $u = \csc 2\theta$ and use $\cot^2 2\theta = u^2 - 1$: $3u^2 - u - 10 = (3u+5)(u-2) = 0$. Thus $\sin 2\theta = 1/2$ or $\sin 2\theta = -3/5$. As $-\pi < 2\theta < \pi$, the first gives $2\theta = \pi/6, 5\pi/6$. Write $\gamma = \arcsin(3/5)$; the second gives $2\theta = -\gamma, -\pi + \gamma$. Therefore the four values are $\pi/12, 5\pi/12, -0.3217505544\ldots, -1.249045772\ldots$. None makes an original sine or cosine denominator zero.

$$\boxed{\theta = \frac{\pi}{12},\ \frac{5\pi}{12},\ -0.322,\ -1.25}$$

FINAL ANSWER

(a) $\frac{\sin 2x}{\cos x} + \frac{\cos 2x}{\sin x} = \csc x$ (b) $\theta = \frac{\pi}{12},\ \frac{5\pi}{12},\ -0.322,\ -1.25$

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Question 8

7 marks | Log Graphs & Exponential Models

8. The percentage, P, of the population of a small country who have access to the internet, is modelled by the equation

$$P = ab^t$$

where a and b are constants and t is the number of years after the start of 2005

Using the data for the years between the start of 2005 and the start of 2010, a graph is plotted of $\log_{10} P$ against t.

The points are found to lie approximately on a straight line with gradient 0.09 and intercept 0.68 on the $\log_{10} P$ axis.

(a) Find, according to the model, the value of a and the value of b, giving your answers to 2 decimal places.

(4)

(b) In the context of the model, give a practical interpretation of the constant a.

(1)

(c) Use the model to estimate the percentage of the population who had access to the internet at the start of 2015

(2)

WORKED SOLUTION

Question 8 · WMA13/01 January 2021

1 Part (a)

Taking logarithms gives $\log_{10} P = \log_{10} a + t \log_{10} b$. Match the intercept and gradient: $\log_{10} a = 0.68$, $\log_{10} b = 0.09$. Thus $a = 10^{0.68} = 4.786300923\ldots$ and $b = 10^{0.09} = 1.230268771\ldots$.

$$\boxed{a = 4.79, \qquad b = 1.23}$$

2 Part (b)

At $t = 0$, $P = a$. Thus a is the percentage of the country's population with internet access at the start of 2005, approximately 4.79%.

$$\boxed{\text{Initial percentage with internet access at the start of 2005}}$$

3 Part (c)

The start of 2015 is $t = 10$ years after the start of 2005. Using the model parameters, $P = 4.79(1.23)^{10} = 37.9652818\ldots\%$, giving 38.0% to three significant figures. Using unrounded parameters gives $10^{0.68+0.09(10)} = 38.0189396\ldots\%$, the same rounded answer.

$$\boxed{38.0\%}$$

FINAL ANSWER

(a) $a = 4.79, \quad b = 1.23$ (b) $\textbf{Initial percentage with internet access at the start of 2005}$ (c) 38.0%

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Question 9

4 marks | Integration

9. Find

(i) $\displaystyle\int \frac{3x-2}{3x^2-4x+5}\,\mathrm{d}x$

(2)

(ii) $\displaystyle\int \frac{\mathrm{e}^{2x}}{(\mathrm{e}^{2x}-1)^3}\,\mathrm{d}x \qquad x \neq 0$

(2)

WORKED SOLUTION

Question 9 · WMA13/01 January 2021

1 Part (i)

The denominator derivative is $6x-4=2(3x-2)$. Hence reverse the logarithmic chain rule:

$$\int \frac{3x-2}{3x^2-4x+5}\,dx=\frac{1}{2}\ln(3x^2-4x+5)+C.$$

The log argument is always positive because $3x^2-4x+5=3(x-2/3)^2+11/3$.

$$\boxed{\frac{1}{2}\ln(3x^2-4x+5)+C}$$

2 Part (ii)

Put $u=e^{2x}-1$, so $du=2e^{2x}\,dx$. Then $\int e^{2x}(e^{2x}-1)^{-3}\,dx=(1/2)\int u^{-3}\,du=-1/(4u^2)+C$. The source excludes $x=0$, where the integrand is undefined.

$$\boxed{-\frac{1}{4(e^{2x}-1)^2}+C}$$

FINAL ANSWER

(i) $\frac{1}{2}\ln(3x^2-4x+5)+C$ (ii) $-\frac{1}{4(e^{2x}-1)^2}+C$

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Question 10

10 marks | Differentiation Techniques

10. The curve C has equation

$$x = 3\sec^2 2y \qquad x > 3 \qquad 0 < y < \frac{\pi}{4}$$

(a) Find $\dfrac{\mathrm{d}x}{\mathrm{d}y}$ in terms of y.

(2)

(b) Hence show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{p}{qx\sqrt{x-3}}$$

where p is irrational and q is an integer, stating the values of p and q.

(3)

(c) Find the equation of the normal to C at the point where $y = \dfrac{\pi}{12}$, giving your answer in the form $y = mx + c$, giving m and c as exact irrational numbers.

(5)

WORKED SOLUTION

Question 10 · WMA13/01 January 2021

1 **Part (a)**

Apply the chain rule twice:

$$\frac{dx}{dy} = 3 \cdot 2\sec 2y\,(2\sec 2y\tan 2y) = 12\sec^2 2y\tan 2y.$$

$$\boxed{\frac{dx}{dy} = 12\sec^2 2y\tan 2y}$$

2 **Part (b)**

From $x = 3\sec^2 2y$, $\sec^2 2y = x/3$ and $\tan^2 2y = x/3 - 1$. Since $0 < 2y < \pi/2$, tangent is positive, so $\tan 2y = \sqrt{(x-3)/3}$. Thus $dx/dy = 4x\sqrt{(x-3)/3} > 0$. Taking the reciprocal gives

$$\frac{dy}{dx} = \frac{\sqrt{3}}{4x\sqrt{x-3}}, \qquad p = \sqrt{3}, \quad q = 4.$$

$$\boxed{p = \sqrt{3}, \quad q = 4}$$

3

Part (c)

At $y = \pi/12$, $2y = \pi/6$ and $x = 3(4/3) = 4$. The tangent gradient is $\sqrt{3}/(4 \cdot 4\sqrt{1}) = \sqrt{3}/16$. Therefore the normal gradient is $-16/\sqrt{3} = -16\sqrt{3}/3$. Using the point $(4, \pi/12)$,

$$y - \frac{\pi}{12} = -\frac{16\sqrt{3}}{3}(x - 4) \Longrightarrow y = -\frac{16\sqrt{3}}{3}x + \frac{64\sqrt{3}}{3} + \frac{\pi}{12}.$$

$$\boxed{y = -\frac{16\sqrt{3}}{3}x + \frac{64\sqrt{3}}{3} + \frac{\pi}{12}}$$

✓ **FINAL ANSWER**

(a) $\frac{dx}{dy} = 12\sec^2 2y \tan 2y$ (b) $p = \sqrt{3}, \quad q = 4$ (c) $y = -\frac{16\sqrt{3}}{3}x + \frac{64\sqrt{3}}{3} + \frac{\pi}{12}$