Elite Mathematics
Edexcel International A Level

IAL PURE 3 · WMA13

WMA13/01 October 2020 Worked Solutions

October 2020 paper rebuilt with the worked solution after each question.

9	2020–2026	Single-paper solutions
paper questions	19 exam sessions	checked by Dr Eslam



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PAST PAPER

WMA13/01 October 2020

October 2020

9	75	1 h 30
questions	marks	exam time

Each question is followed by its worked solution and final answer.

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Question 1

5 marks | Trig Equations & R-Form

1. Solve, for $0 \leqslant x < 360°$, the equation

$$2\cos 2x = 7\cos x$$

giving your solutions to one decimal place.

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(5)

WORKED SOLUTION

Question 1 · WMA13/01 October 2020

1 Part (main)

Use $\cos 2x = 2\cos^2 x - 1$ and put $u = \cos x$:

$$2(2u^2 - 1) = 7u \quad \Longrightarrow \quad 4u^2 - 7u - 2 = (4u + 1)(u - 2) = 0.$$

The value $u = 2$ is impossible since $-1 \leq \cos x \leq 1$. Thus $\cos x = -1/4$. In $0 \leq x < 360°$, the solutions lie in quadrants II and III:

$$x = \arccos(-1/4) = 104.477512\ldots°, \qquad x = 360° - 104.477512\ldots° = 255.522487\ldots°.$$

$$\boxed{x = 104.5°, \quad 255.5°}$$

FINAL ANSWER

(main) $x = 104.5°, \quad 255.5°$

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Question 2

6 marks | Log Graphs & Exponential Models

2. A scientist monitored the growth of bacteria on a dish over a 30-day period.

The area, $N\text{mm}^2$, of the dish covered by bacteria, t days after monitoring began, is modelled by the equation

$$\log_{10} N = 0.0646t + 1.478 \qquad 0 \leqslant t \leqslant 30$$

(a) Show that this equation may be written in the form

$$N = ab^t$$

where a and b are constants to be found. Give the value of a to the nearest integer and give the value of b to 3 significant figures.

(4)

(b) Use the model to find the area of the dish covered by bacteria 30 days after monitoring began. Give your answer, in mm^2, to 2 significant figures.

(2)

WORKED SOLUTION

Question 2 · WMA13/01 October 2020

1 Part (a)

Exponentiate in base ten and use index laws:

$$N = 10^{0.0646t+1.478} = 10^{1.478}\left(10^{0.0646}\right)^t = ab^t.$$

Therefore $a = 10^{1.478} = 30.060763\ldots$, which rounds to 30 to the nearest integer, and $b = 10^{0.0646} = 1.16037937\ldots$, which rounds to 1.16 to three significant figures.

$$\boxed{a = 30, \qquad b = 1.16}$$

2 Part (b)

At $t = 30$, use the original model to retain precision:

$$N = 10^{0.0646(30)+1.478} = 10^{3.416} = 2606.15355\ldots \text{ mm}^2.$$

To two significant figures the covered area is $2.6 \times 10^3 \text{ mm}^2$. Using the rounded model $30(1.16)^{30}$ gives the same two-significant-figure answer.

$$\boxed{2.6 \times 10^3 \text{ mm}^2}$$

FINAL ANSWER

(a) $a = 30, \qquad b = 1.16$ (b) $2.6 \times 10^3\ \text{mm}^2$

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Question 3

7 marks | Differentiation Techniques

3.

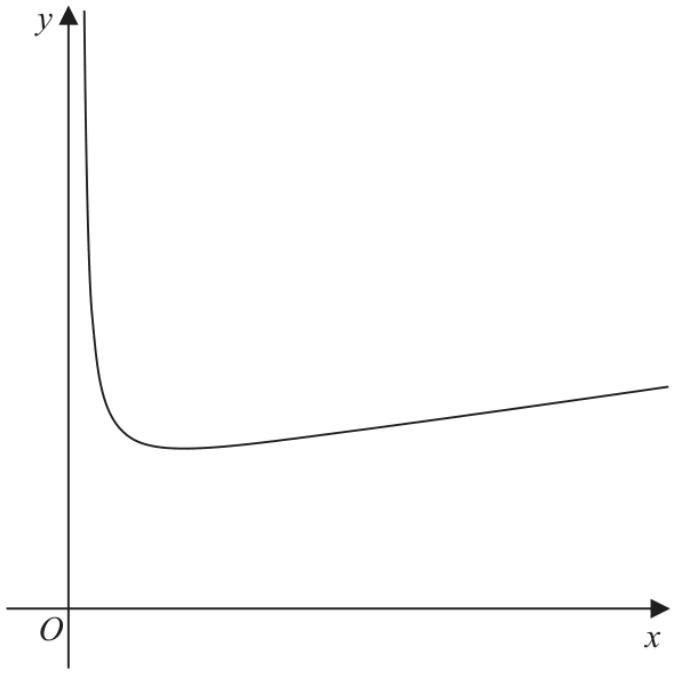


Figure 1

Figure 1 shows a sketch of a curve with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{2x+3}{\sqrt{4x-1}} \qquad x > \frac{1}{4}$$

(a) Find, in simplest form, $\mathrm{f}'(x)$.

(4)

(b) Hence find the range of f.

(3)

WORKED SOLUTION

Question 3 · WMA13/01 October 2020

1 **Part (a)**

Write $f(x) = (2x+3)(4x-1)^{-1/2}$. The product and chain rules give

$$f'(x) = 2(4x-1)^{-1/2} - 2(2x+3)(4x-1)^{-3/2} = \frac{2(4x-1)-2(2x+3)}{(4x-1)^{3/2}} = \frac{4(x-2)}{(4x-1)^{3/2}}, \quad x > \tfrac{1}{4}.$$

$$\boxed{f'(x) = \frac{4(x-2)}{(4x-1)^{3/2}}}$$

2 Part (b)

The denominator of $f'(x)$ is positive throughout the domain, so the derivative is negative for $1/4 < x < 2$ and positive for $x > 2$. Consequently $x = 2$ is the global minimum, with $f(2) = 7/\sqrt{7} = \sqrt{7}$. Also $f(x) \to \infty$ as $x \to 1/4^{+}$ and as $x \to \infty$. By continuity all values at least $\sqrt{7}$ are attained, including the minimum.

$$\boxed{\text{range}(f) = [\sqrt{7}, \infty)}$$

FINAL ANSWER

(a) $f'(x) = \frac{4(x-2)}{(4x-1)^{3/2}}$ (b) $\text{range}(f) = [\sqrt{7}, \infty)$

Question 4

8 marks | Modulus

4.

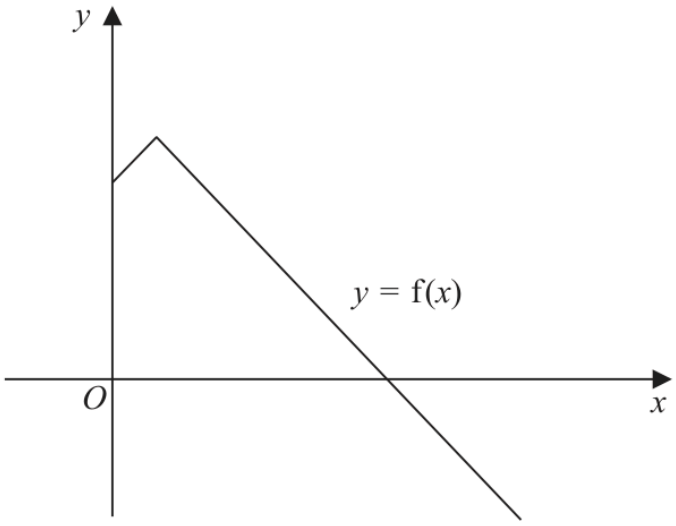


Figure 2

Figure 2 shows a sketch of part of the graph with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 21 - 2|2 - x| \qquad x \geqslant 0$$

(a) Find $\mathrm{ff}(6)$

(2)

(b) Solve the equation $\mathrm{f}(x) = 5x$

(2)

Given that the equation $\mathrm{f}(x) = k$, where k is a constant, has exactly two roots,

(c) state the set of possible values of k.

(2)

The graph with equation $y = \mathrm{f}(x)$ is transformed onto the graph with equation $y = a\mathrm{f}(x - b)$

The vertex of the graph with equation $y = a\mathrm{f}(x - b)$ is $(6, 3)$.

Given that a and b are constants,

(d) find the value of a and the value of b.

(2)

1 **Part (a)**

Apply f twice, in order: $f(6) = 21 - 2|2 - 6| = 13$. Then $ff(6) = f(13) = 21 - 2|2 - 13| = 21 - 22 = -1$. The intermediate input 13 lies in the domain $x \geq 0$.

$$\boxed{ff(6) = -1}$$

2 **Part (b)**

For $0 \leq x \leq 2$, $f(x) = 17 + 2x$, so $17 + 2x = 5x$ gives $x = 17/3$, outside that branch; reject it. For $x \geq 2$, $f(x) = 25 - 2x$. Then $25 - 2x = 5x$, giving $x = 25/7$. This belongs to the second branch and satisfies the original modulus equation.

$$\boxed{x = \frac{25}{7}}$$

3 **Part (c)**

On the left branch $0 \leq x \leq 2$, f rises from $f(0) = 17$ to $f(2) = 21$. The right branch decreases from 21 without lower bound. A horizontal line therefore meets both branches at distinct points exactly when $17 \leq k < 21$. At $k = 17$, one root is the permitted endpoint $x = 0$; at $k = 21$, the two intersections merge at the vertex, so 21 is excluded.

$$\boxed{17 \leq k < 21}$$

4 **Part (d)**

The original vertex is $(2, 21)$. Under $y = af(x - b)$ it maps to $(2 + b, 21a)$. Comparing with $(6, 3)$ gives $2 + b = 6$ and $21a = 3$.

$$\boxed{a = \frac{1}{7}, \qquad b = 4}$$

✓ **FINAL ANSWER**

(a) $ff(6) = -1$ (b) $x = \frac{25}{7}$ (c) $17 \leq k < 21$ (d) $a = \frac{1}{7}, \qquad b = 4$

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Question 5

8 marks | Trig Identities & Exact Values

5. (a) Show that

$$\sin 3x \equiv 3\sin x - 4\sin^3 x$$

(4)

(b) Hence find, using algebraic integration,

$$\int_0^{\frac{\pi}{3}} \sin^3 x \, dx$$

(4)

WORKED SOLUTION

Question 5 · WMA13/01 October 2020

1 Part (a)

Start with the angle-addition identity:

$$\sin 3x = \sin(2x + x) = \sin 2x \cos x + \cos 2x \sin x.$$

Using $\sin 2x = 2\sin x \cos x$ and $\cos 2x = 1 - 2\sin^2 x$,

$$\sin 3x = 2\sin x \cos^2 x + (1 - 2\sin^2 x)\sin x = 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x = 3\sin x - 4\sin^3 x.$$

$$\boxed{\sin 3x \equiv 3\sin x - 4\sin^3 x}$$

2 Part (b)

From (a), $\sin^3 x = (3\sin x - \sin 3x)/4$. Therefore, using algebraic integration,

$$\int_0^{\pi/3} \sin^3 x \, dx = \left[-\frac{3}{4}\cos x + \frac{1}{12}\cos 3x\right]_0^{\pi/3} = \left(-\frac{3}{8} - \frac{1}{12}\right) - \left(-\frac{3}{4} + \frac{1}{12}\right) = \frac{5}{24}.$$

$$\boxed{\frac{5}{24}}$$

FINAL ANSWER

(a) $\sin 3x \equiv 3\sin x - 4\sin^3 x$ (b) $\frac{5}{24}$

Question 6

9 marks | Numerical Methods

6.

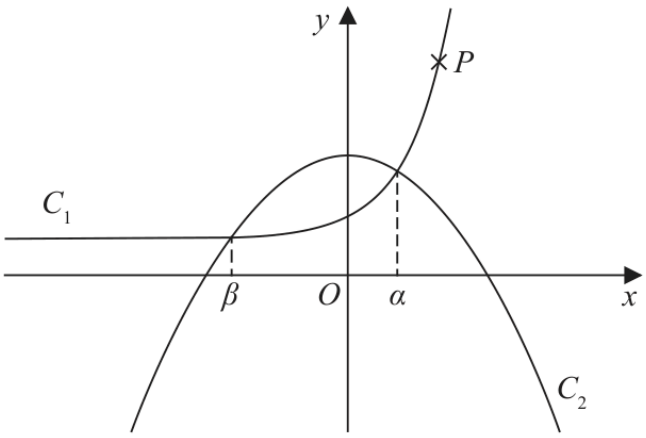


Figure 3

Figure 3 shows a sketch of curve C_1 with equation $y = 5e^{x-1} + 3$

and curve C_2 with equation $y = 10 - x^2$

The point P lies on C_1 and has y coordinate 18

(a) Find the x coordinate of P, writing your answer in the form $\ln k$, where k is a constant to be found.

(3)

The curve C_1 meets the curve C_2 at $x = \alpha$ and at $x = \beta$, as shown in Figure 3.

(b) Using a suitable interval and a suitable function that should be stated, show that to 3 decimal places $\alpha = 1.134$

(3)

The iterative equation

$$x_{n+1} = -\sqrt{7 - 5e^{x_n - 1}}$$

is used to find an approximation to β.

Using this iterative formula with $x_1 = -3$

(c) find the value of x_2 and the value of β, giving each answer to 6 decimal places.

(3)

1 Part (a)

At P, $18 = 5e^{x-1} + 3$, so $e^{x-1} = 3$. Taking natural logarithms gives $x - 1 = \ln 3$, hence $x = 1 + \ln 3 = \ln e + \ln 3 = \ln(3e)$.

$$\boxed{x = \ln(3e), \qquad k = 3e}$$

2 Part (b)

Define $F(x) = 5e^{x-1} + x^2 - 7$; its zeros are the intersection abscissae, and F is continuous. At the endpoints of the three-decimal rounding interval,

$$F(1.1335) = -0.0010714193\ldots < 0, \qquad F(1.1345) = 0.0069135450\ldots > 0.$$

Continuity guarantees a root between them. Moreover $F'(x) = 5e^{x-1} + 2x > 0$ on this positive interval, so that root is unique there. It is the positive intersection α, and rounds to 1.134 to three decimal places.

$$\boxed{\alpha = 1.134 \text{ to 3 d.p.}}$$

3 Part (c)

Start at $x_1 = -3$ and keep full precision in $x_{n+1} = -\sqrt{7 - 5e^{x_n - 1}}$:

n	x_n
1	-3
2	-2.628387681746
3	-2.620535277934
4	-2.620335527988
5	-2.620330426051
6	-2.620330295726
7	-2.620330292397

The later iterates agree to six decimal places. The negative root corresponds to β in the figure.

$$\boxed{x_2 = -2.628388, \qquad \beta = -2.620330}$$

FINAL ANSWER

(a) $x = \ln(3e), \qquad k = 3e$ (b) $\alpha = 1.134$ to 3 d.p. (c) $x_2 = -2.628388, \qquad \beta = -2.620330$

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Question 7

9 marks | Trig Equations & R-Form

7. (a) Express $\cos x + 4\sin x$ in the form $R\cos(x - \alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α, in radians, to 3 decimal places.

(3)

A scientist is studying the behaviour of seabirds in a colony.

She models the height above sea level, H metres, of one of the birds in the colony by the equation

$$H = \frac{24}{3 + \cos\left(\frac{1}{2}t\right) + 4\sin\left(\frac{1}{2}t\right)} \qquad 0 \leqslant t \leqslant 6.5$$

where t seconds is the time after it leaves the nest.

Find, according to the model,

(b) the minimum height of the seabird above sea level, giving your answer to the nearest cm,

(2)

(c) the value of t, to 2 decimal places, when $H = 10$

(4)

WORKED SOLUTION

Question 7 · WMA13/01 October 2020

1 **Part (a)**

Expand $R\cos(x-\alpha) = R\cos\alpha\cos x + R\sin\alpha\sin x$. Matching coefficients gives $R\cos\alpha = 1$, $R\sin\alpha = 4$. Thus $R = \sqrt{1^2 + 4^2} = \sqrt{17}$ and $\tan\alpha = 4$. Since $0 < \alpha < \pi/2$, $\alpha = \arctan 4 = 1.32581766\ldots$ radians.

$$\boxed{R = \sqrt{17}, \qquad \alpha = 1.326 \text{ radians}}$$

2 Part (b)

The model becomes $H = 24/[3 + \sqrt{17}\cos(t/2 - \alpha)]$. For $0 \le t \le 6.5$, the phase lies between $-\alpha$ and $3.25 - \alpha$; throughout this interval the denominator remains positive. Its maximum $3 + \sqrt{17}$ is attained at $t = 2\alpha = 2.651635\ldots$, inside the time interval. The height is then smallest:

$$H_{\min} = \frac{24}{3 + \sqrt{17}} = 3.369316876\ldots \text{ m}.$$

To the nearest centimetre this is 3.37 m (337 cm).

$$\boxed{H_{\min} = 3.37 \text{ m}}$$

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3 Part (c)

For $H = 10$, $3 + \sqrt{17}\cos(t/2 - \alpha) = 24/10$, so

$$\cos(t/2 - \alpha) = -\frac{3}{5\sqrt{17}}.$$

Let $\gamma = \arccos[-3/(5\sqrt{17})]$. In the permitted phase interval $[-\alpha, 3.25 - \alpha]$, only $+\gamma$ occurs: $-\gamma < -\alpha$, and shifts by 2π are outside it. Using the unrounded $\alpha = \arctan 4$,

$$t = 2(\alpha + \gamma) = 6.085307854\ldots \text{ s}.$$

$$\boxed{t = 6.09 \text{ s}}$$

FINAL ANSWER

(a) $R = \sqrt{17}, \qquad \alpha = 1.326$ radians (b) $H_{\min} = 3.37$ m (c) $t = 6.09$ s

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Question 8

9 marks | Differentiation Techniques

8. (i) The curve C has equation $y = \mathrm{g}(x)$ where

$$\mathrm{g}(x) = \mathrm{e}^{3x} \sec 2x \qquad -\frac{\pi}{4} < x < \frac{\pi}{4}$$

(a) Find $\mathrm{g}'(x)$

(2)

(b) Hence find the x coordinate of the stationary point of C.

(3)

(ii) A different curve has equation

$$x = \ln(\sin y) \qquad 0 < y < \frac{\pi}{2}$$

Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^x}{\mathrm{f}(x)}$$

where $\mathrm{f}(x)$ is a function of e^x that should be found.

(4)

WORKED SOLUTION

Question 8 · WMA13/01 October 2020

1 **Part (i)(a)**

By the product and chain rules,

$$g'(x) = 3e^{3x}\sec 2x + 2e^{3x}\sec 2x \tan 2x = e^{3x}\sec 2x(3 + 2\tan 2x).$$

$$\boxed{g'(x) = e^{3x}\sec 2x(3 + 2\tan 2x)}$$

2 Part (i)(b)

For $-\pi/4 < x < \pi/4$, $e^{3x}\sec 2x$ is positive and nonzero. Hence $g'(x) = 0$ precisely when $3 + 2\tan 2x = 0$, or $\tan 2x = -3/2$. Since $-\pi/2 < 2x < \pi/2$, tangent is one-to-one there:

$$x = -\frac{1}{2}\arctan\frac{3}{2} = -0.4913968616\ldots.$$

$$\boxed{x = -\frac{1}{2}\arctan\frac{3}{2} \approx -0.491}$$

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3 Part (ii)

Differentiate $x = \ln(\sin y)$ with respect to y:

$$\frac{dx}{dy} = \frac{\cos y}{\sin y}, \qquad \frac{dy}{dx} = \frac{\sin y}{\cos y}.$$

The original equation gives $\sin y = e^{x}$. Since $0 < y < \pi/2$, cosine is positive, so $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - e^{2x}}$. Therefore

$$\frac{dy}{dx} = \frac{e^{x}}{\sqrt{1 - e^{2x}}}, \qquad f(x) = \sqrt{1 - e^{2x}}.$$

Here $x < 0$ from the original interval, so the square root is real and nonzero.

$$\boxed{f(x) = \sqrt{1 - e^{2x}}, \qquad x < 0}$$

FINAL ANSWER

(i)(a) $g'(x) = e^{3x}\sec 2x(3 + 2\tan 2x)$ (i)(b) $x = -\frac{1}{2}\arctan\frac{3}{2} \approx -0.491$ (ii) $f(x) = \sqrt{1 - e^{2x}}, \qquad x < 0$

Question 9

14 marks | Applications of Differentiation

9. (a) Given that

$$\frac{x^4 - x^3 - 10x^2 + 3x - 9}{x^2 - x - 12} \equiv x^2 + P + \frac{Q}{x - 4} \qquad x > -3$$

find the value of the constant P and show that $Q = 5$

(4)

The curve C has equation $y = \mathrm{g}(x)$, where

$$\mathrm{g}(x) = \frac{x^4 - x^3 - 10x^2 + 3x - 9}{x^2 - x - 12} \qquad -3 < x < 3.5 \qquad x \in \mathbb{R}$$

(b) Find the equation of the tangent to C at the point where $x = 2$
Give your answer in the form $y = mx + c$, where m and c are constants to be found.

(5)

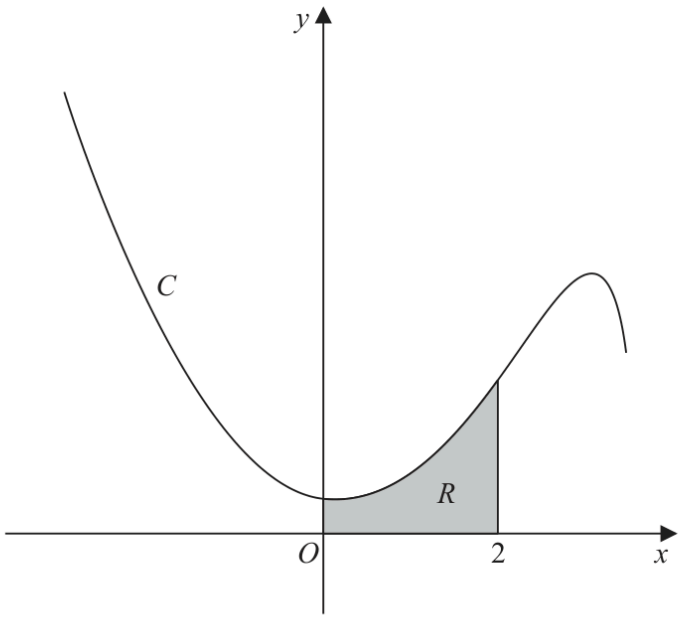


Figure 4

Figure 4 shows a sketch of the curve C.
The region R, shown shaded in Figure 4, is bounded by C, the y-axis, the x-axis and the line with equation $x = 2$

(c) Find the exact area of R, writing your answer in the form $a + b\ln 2$, where a and b are constants to be found.

(5)

1 Part (a)

Factor the denominator as $(x-4)(x+3)$ and multiply the proposed identity through by it:

$$x^4 - x^3 - 10x^2 + 3x - 9 = (x^2 + P)(x^2 - x - 12) + Q(x+3).$$

The right side expands to $x^4 - x^3 + (P-12)x^2 + (Q-P)x + 3Q - 12P$. Comparing coefficients gives $P - 12 = -10$, so $P = 2$, and $Q - P = 3$, so $Q = 5$. The constant check is $3(5) - 12(2) = -9$, as required. The identity holds where the original fraction is defined: the stated $x > -3$ still excludes $x = 4$.

$$\boxed{P = 2, \qquad Q = 5}$$

2 Part (b)

From (a), $g(x) = x^2 + 2 + 5/(x-4)$ on the curve domain. Thus $g'(x) = 2x - 5/(x-4)^2$. At $x = 2$, $g(2) = 4 + 2 - 5/2 = 7/2$ and $g'(2) = 4 - 5/4 = 11/4$. The tangent through $(2, 7/2)$ is

$$y - \frac{7}{2} = \frac{11}{4}(x-2) \quad \Longrightarrow \quad y = \frac{11}{4}x - 2.$$

$$\boxed{y = \frac{11}{4}x - 2}$$

3 Part (c)

The source shading is above the x-axis from $x = 0$ to $x = 2$, so integrate g over those bounds.

$$\text{Area}(R) = \int_0^2 \left(x^2 + 2 + \frac{5}{x-4}\right) dx = \left[\frac{x^3}{3} + 2x + 5\ln|x-4|\right]_0^2.$$

The absolute value is essential because $x - 4 < 0$ throughout this interval. Evaluating gives

$$\frac{8}{3} + 4 + 5\ln 2 - 5\ln 4 = \frac{20}{3} - 5\ln 2.$$

Thus $a = 20/3$ and $b = -5$ in the requested form.

$$\boxed{\text{Area}(R) = \frac{20}{3} - 5\ln 2}$$

FINAL ANSWER

(a) $P = 2, \quad Q = 5$ (b) $y = \frac{11}{4}x - 2$ (c) $\text{Area}(R) = \frac{20}{3} - 5\ln 2$