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P3

COMPLETE BOOK P3

Pure 3 Complete Visual Notes

Every concept, every way it is examined, with diagrams, worked examples and the exam's own trigger words. Thirteen chapters, one system.

13

CHAPTERS

141

VARIANTS

314

EXAM QUESTIONS

YOU WILL MASTER

- ▶ 38 note pages in Save My Exams format, with your notes margin on every page
- ▶ Every variant from past WMA13 papers, marked with how often it has been seen
- ▶ Worked examples with step-by-step call-outs, all answers checked
- ▶ A mistake gallery and a one-page summary at the end of every chapter

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Thirteen chapters



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Rational Algebra

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1.1 Rational Expressions, 1.2 Improper Algebraic Fractions • 6 variants • 12 exam questions

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Your notes



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01

CHAPTER 01

Rational Algebra

Algebraic fractions behave exactly like number fractions. Factorise first, and every cancel, add and divide becomes routine.

3

CONCEPTS

6

VARIANTS

12

EXAM QUESTIONS

YOU WILL MASTER

- ▶ Factorise and cancel without losing the values x is not allowed to take
- ▶ Add, subtract, multiply and divide algebraic fractions into one simplest fraction
- ▶ Divide an improper fraction to get $Ax + B + \frac{C}{\dots}$ and check it
- ▶ Finish every “Show that” in the exact form the question asks for

CHAPTER 01 AT A GLANCE

Two note pages, six ways to be asked



CHAPTER 01 MAP

Rational Algebra

01

1.1 Rational expressions

cancel: linear denominator

cancel: quadratic denominator

+ and -: common denominator

× and ÷

02

1.2 Improper algebraic fractions

divide by a linear divisor

divide by a quadratic divisor

$$Ax + B + \frac{C}{(x - 1)^2}$$

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Rational Expressions

NOTE 1.1



Rational Expressions

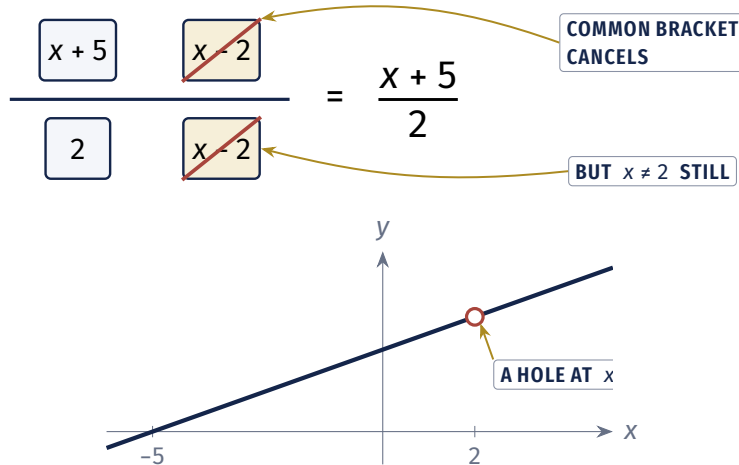
PURE 3 › ALGEBRA AND FUNCTIONS › RATIONAL EXPRESSIONS • Exam code WMA13 • Dr Eslam Ahmed

What is a rational expression?

- A **rational expression** is a fraction whose numerator and denominator are polynomials, e.g. $\frac{x^2 + 3x - 10}{2x - 4}$.
- It follows exactly the same rules as number fractions.
- It is **undefined** wherever the denominator is zero.
 - $\frac{x^2 + 3x - 10}{2x - 4}$ is undefined at $x = 2$.

How do I simplify a rational expression?

- **Factorise** the numerator and the denominator completely.
 - common factor: $2x^2 - 6x = 2x(x - 3)$
 - difference of two squares: $x^2 - 9 = (x - 3)(x + 3)$
 - quadratic: $2x^2 + 5x - 3 = (2x - 1)(x + 3)$
- **Cancel** any bracket that is a factor of the whole numerator **and** the whole denominator.
- Never cancel **terms** that are added or subtracted.
- The values that made the **original** denominator zero are still not allowed.



Only whole brackets cancel, and the forbidden value stays forbidden.

e.g. SIMPLIFY $\frac{x^2 - 9}{x^2 + 3x}$

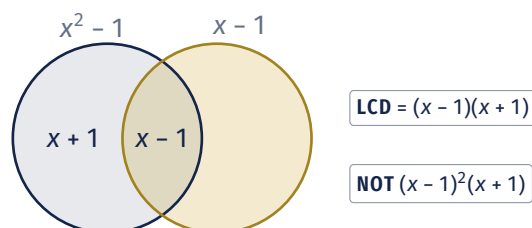
$$\frac{(x-3)(x+3)}{x(x+3)}$$

← FACTORISE TOP AND BOTTOM

ANSWER $\frac{x-3}{x}, \quad x \neq 0, x \neq -3$

How do I add or subtract rational expressions?

- STEP 1** Factorise every denominator.
- STEP 2** Build the **lowest common denominator** (LCD): every different factor, once each, at its highest power.
- STEP 3** Rewrite each fraction over the LCD and combine the numerators. Put each numerator in a **bracket**.
- STEP 4** Expand, collect, then factorise the new numerator and cancel if you can.



Building the LCD of $\frac{\dots}{x^2 - 1}$ and $\frac{\dots}{x - 1}$: the shared factor is used once.

How do I multiply or divide rational expressions?



- **Multiply:** multiply the numerators together and the denominators together, then cancel.
- **Divide: flip the second fraction** and multiply.

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \times \frac{D}{C}$$
- Factorise **before** multiplying out, so that cancelling is easy.
- Exclude the zeros of every original denominator **and** of the numerator you divided by.



Examiner Tips and Tricks

- The most common form of this question is “Show that $f(x) = \frac{ax+b}{cx+d}$, where a, b, c and d are integers to be found”. The final line must be in exactly that form.
- The answer is usually used again: to find $f^{-1}(x)$ (C02), to differentiate (C10) or to integrate (C12). A slip here costs marks later.
- Check your simplified answer by substituting a simple value such as $x = 3$ into the original and into your answer.

WHEN YOU SEE → YOU DO

“Simplify fully” / “in its simplest form”

→ factorise top and bottom completely, then cancel common **brackets**

“Show that $f(x) = \frac{ax+b}{cx+d}$, where a, b, c and d are integers”

→ combine over the LCD, factorise the new top, cancel

“Express as a single fraction”

→ one fraction, factorised, nothing left to cancel

“ $x \in \mathbb{R}, x > 2$ ” written after $f(x)$

→ the domain is part of the question: keep it with your answer

**Worked Example**

Simplify fully $\frac{2x^2 + 5x - 3}{x^2 + x - 6}$.

$$\frac{(2x - 1)(x + 3)}{(x + 3)(x - 2)}$$

← FACTORISE BOTH QUADRATICS

$$x \neq -3, \quad x \neq 2$$

← EXCLUSIONS FROM THE ORIGINAL DENOMINATOR

$$x = 0 : \quad \frac{-3}{-6} = \frac{1}{2} = \frac{-1}{-2}$$

← QUICK CHECK WITH A SIMPLE VALUE

ANSWER $\frac{2x - 1}{x - 2}, \quad x \neq -3, 2$

**Worked Example**

$f(x) = \frac{2x+5}{x-2} - \frac{3x+9}{x^2+x-6}, \quad x > 2$. Show that $f(x) = \frac{ax+b}{cx+d}$, where a, b, c and d are integers to be found.

$$x^2 + x - 6 = (x + 3)(x - 2)$$

← FACTORISE THE DENOMINATOR: LCD IS $(x + 3)(x - 2)$

$$\frac{(2x + 5)(x + 3) - (3x + 9)}{(x + 3)(x - 2)}$$

← ONE FRACTION; BRACKET THE SECOND NUMERATOR

$$= \frac{2x^2 + 8x + 6}{(x + 3)(x - 2)} = \frac{2(x + 1)(x + 3)}{(x + 3)(x - 2)}$$

← EXPAND, COLLECT, FACTORISE

$$x = 3 : \quad \frac{11}{1} - \frac{18}{6} = 8 = \frac{2(3) + 2}{3 - 2}$$

← CHECK

ANSWER $f(x) = \frac{2x+2}{x-2}; \quad a = 2, b = 2, c = 1, d = -2$



Worked Example



Simplify fully $\frac{x^2 - 9}{x^2 + 2x} \div \frac{x - 3}{x^2 - 4}$.

$$\frac{(x - 3)(x + 3)}{x(x + 2)} \times \frac{(x - 2)(x + 2)}{x - 3}$$

$$\frac{(x + 3)(x - 2)}{x}$$

$$x \neq 0, \pm 2, 3$$

← FLIP THE SECOND FRACTION AND FACTORISE

← CANCEL $(x - 3)$ AND $(x + 2)$

← BOTH DENOMINATORS AND THE DIVISOR'S NUMERATOR

ANSWER $\frac{(x + 3)(x - 2)}{x}, \quad x \neq 0, \pm 2, 3$

HOW THIS IS EXAMINED • past WMA13 papers

cancel: linear denominator NOT SEEN YET • BE READY

cancel: quadratic denominator NOT SEEN YET • BE READY

common denominator SEEN 4× IN EXAMS product or quotient NOT SEEN YET • BE READY



Your notes

Improper Algebraic Fractions

NOTE 1.2



Improper Algebraic Fractions

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What is an improper algebraic fraction?

- A fraction is **improper** when the degree of the numerator is **greater than or equal to** the degree of the denominator.
 - $\frac{2x^2 + 5x - 8}{x + 2}$ (degree 2 over degree 1) is improper.
 - $\frac{3}{x + 2}$ is proper.
- Like $\frac{17}{5} = 3 + \frac{2}{5}$, an improper algebraic fraction can be written as
 - **quotient** + $\frac{\text{remainder}}{\text{divisor}}$
- The **remainder** always has a lower degree than the divisor.

THE DIVISION IDENTITY

LEARN THIS: NOT GIVEN

$$F(x) \equiv Q(x) D(x) + R(x)$$

How do I divide using long division?

- STEP 1** Write both polynomials in descending powers, using 0 for any missing power.
- STEP 2** Divide the leading term of what is left by the leading term of the divisor.
- STEP 3** Multiply the divisor by that term and **subtract** the whole line.
- STEP 4** Repeat until the degree of what is left is less than the degree of the divisor.



$$\begin{array}{r}
 (2x^2 + 5x - 8) : (x + 2) = 2x + 1 + \frac{-10}{x + 2} \\
 \underline{-2x^2 - 4x} \\
 x - 8 \\
 \underline{-x - 2} \\
 -10
 \end{array}$$

Long division of $2x^2 + 5x - 8$ by $x + 2$: quotient $2x + 1$, remainder -10 .

Is there a quicker way?

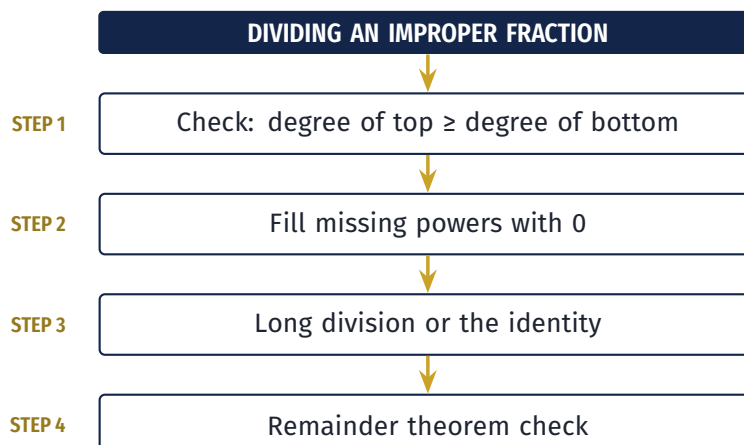
- Write the **identity** with unknown constants and compare.
 - $2x^2 + 5x - 8 \equiv (Ax + B)(x + 2) + C$
- Substitute the root of the divisor ($x = -2$) to find C immediately: this is the **remainder theorem**.
- Compare coefficients (highest power, constant term) to find the rest.
- This method is especially quick when the divisor is a **repeated factor** such as $(x - 1)^2$.

e.g. FIND THE REMAINDER WHEN $x^3 - 2x + 7$ IS DIVIDED BY $x + 2$

$$F(-2) = -8 + 4 + 7$$

← SUBSTITUTE THE ROOT OF THE DIVISOR

ANSWER remainder = 3





Examiner Tips and Tricks

- When the question gives the form, e.g. " $Ax + B + \frac{C}{x+2}$ ", the identity method is usually fastest and needs no long division layout.
- Divide by the **expanded** quadratic: $(x-1)^2 = x^2 - 2x + 1$.
- This division is almost always followed by "Hence" integration (C12):

$$\int \frac{C}{x+2} dx = C \ln |x+2|.$$

 Your notes

WHEN YOU SEE → YOU DO

"Write $g(x)$ in the form $Ax + B + \frac{C}{x+2}$, where A, B and C are integers to be found"

→ long division by $(x+2)$, or the identity and comparing coefficients

"Given $h(x) = Ax + B + \frac{C}{(x-1)^2}$... find A, B and C "

→ identity; put $x = 1$ first

"Hence use algebraic integration" / "Hence find $\int h(x) dx$ "

→ integrate the divided form term by term



Worked Example



$g(x) = \frac{2x^2 + 5x - 8}{x+2}$, $x > -2$. Write $g(x)$ in the form $Ax + B + \frac{C}{x+2}$, where A, B and C are integers to be found.

$$2x(x+2) = 2x^2 + 4x; \quad \text{subtract: } x - 8$$

← **DIVIDE LEADING TERMS:** $2x^2 \div x = 2x$

$$1(x+2) = x+2; \quad \text{subtract: } -10$$

← **NEXT TERM OF THE QUOTIENT:** $+1$

$$2(-2)^2 + 5(-2) - 8 = -10$$

← **REMAINDER THEOREM CHECK**

ANSWER $g(x) = 2x + 1 - \frac{10}{x+2}$; $A = 2$, $B = 1$, $C = -10$



Worked Example



$h(x) = \frac{2x^3 - 7x^2 + 8x + 1}{(x-1)^2}$, $x > 1$. Given $h(x) = Ax + B + \frac{C}{(x-1)^2}$, find A , B and C .

$$2x^3 - 7x^2 + 8x + 1 \equiv (Ax + B)(x-1)^2 + C$$

← MULTIPLY THROUGH BY $(x-1)^2$

$$x = 1 : 2 - 7 + 8 + 1 = C \Rightarrow C = 4$$

← THE BRACKET VANISHES

$$x^3 : A = 2$$

← COMPARE THE HIGHEST POWER

$$x = 0 : 1 = B + C \Rightarrow B = -3$$

← COMPARE CONSTANTS

ANSWER $h(x) = 2x - 3 + \frac{4}{(x-1)^2}$

HOW THIS IS EXAMINED • past WMA13 papers

linear divisor **SEEN 2× IN EXAMS** quadratic divisor **SEEN 6× IN EXAMS**



Your notes

MISTAKE GALLERY

Where marks are lost

× **WRONG**

$$\frac{x+6}{x} = 6$$

✓ **RIGHT**No cancelling: $\frac{x+6}{x} = 1 + \frac{6}{x}$.

Only a bracket that multiplies everything can be cancelled.

× **WRONG**

$$\frac{(x+3)(2x-1)}{(x+3)(x-2)} = \frac{2x-1}{x-2} \text{ (nothing else).}$$

✓ **RIGHT**Add $x \neq -3, x \neq 2$.

Exclusions come from the original denominator.

× **WRONG**

$$\frac{A}{B} - \frac{3x+9}{B} = \frac{A-3x+9}{B}$$

✓ **RIGHT**

$$\frac{A-(3x+9)}{B} = \frac{A-3x-9}{B}$$

The minus sign applies to the whole numerator.

× **WRONG**Dividing $x^3 - 2x + 1$ with no gap for x^2 .✓ **RIGHT**Write $x^3 + 0x^2 - 2x + 1$ first.

Missing powers shift every column.

× **WRONG**Stopping at $\frac{2(x+1)(x+3)}{(x+3)(x-2)}$.✓ **RIGHT**Cancel and match $\frac{ax+b}{cx+d}$.

"Show that" wants the exact final form.

THE CHAPTER ON ONE PAGE

Rational Algebra



1.1

Cancel

Factorise top and bottom completely.
 Cancel whole brackets only, never terms.
 Keep the **original** exclusions.

1.1

Add and subtract

LCD: every factor once.
 Bracket the second numerator after a minus.
 Factorise the new top and cancel.

1.1

Multiply and divide

$\times$: straight across.
 $\div$: flip the **second** fraction.
 Exclude zeros of the divisor's top too.

1.2

Divide

Top degree $\geq$ bottom: divide.
 $F \equiv QD + R$, degree $R <$ degree D .
 Put the root of D in to get R fast.

BOTTOM LINE Factorise first. Cancel only whole brackets, keep the forbidden values, and finish in the exact form the question asks for.



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02

CHAPTER 02

Functions, Composition and Inverses

A function is a machine: know what it accepts (domain), what it can produce (range), how to chain two machines, and how to run one backwards.

4

CONCEPTS

13

VARIANTS

26

EXAM QUESTIONS

YOU WILL MASTER

- ▶ State domains and ranges, including the range on an interval with a turning point
- ▶ Evaluate and form composite functions in the right order: fg means g first
- ▶ Solve equations such as $gg(x) = 78$ and $f\left(\frac{1}{a}\right) = g(a + 3)$
- ▶ Find f^{-1} with its domain, sketch it, and find where f meets f^{-1}

CHAPTER 02 AT A GLANCE

Three note pages, thirteen ways to be asked



CHAPTER 02 MAP

Functions

01

2.1 Language of functions

domain and range

one-one and many-one

range on an interval

restricted domain

02

2.2 Composite functions

evaluate $fg(5)$ form $fg(x)$ domain of fg solve $gg(x) = k$ $f\left(\frac{1}{a}\right) = g(a + 3)$

03

2.3 Inverse functions

find $f^{-1}(x)$

restrict, then invert

graph of f^{-1} $f^{-1}(k)$ without the formula $f(x) = f^{-1}(x)$

Contents

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Language of Functions

NOTE 2.1



Language of Functions

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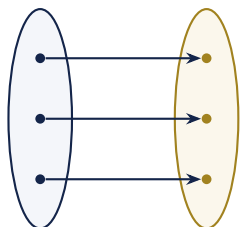
What is a function?

- A **function** is a rule that gives **exactly one output** for every allowed input.
- The **domain** is the set of allowed inputs (x-values). It is written after the rule: $f(x) = 2x^2 - 5, x \geq 0$.
- The **range** is the set of outputs it actually produces (y-values). Write it using $f(x)$, e.g. $f(x) \geq -5$, **not** using x .
- Notation: $f(x) = \dots$ or $f : x \mapsto \dots$ mean the same thing.

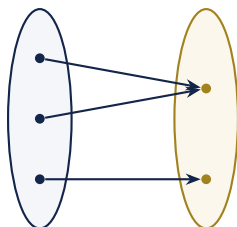
What are one-one and many-one functions?

- **One-one**: every output comes from only one input. A horizontal line meets the graph at most once.
- **Many-one**: some output comes from two or more inputs, e.g. x^2 gives 4 from $x = 2$ and $x = -2$.
- **One-many** rules (e.g. $\pm\sqrt{x}$) are **not functions** at all.
- Only a **one-one** function has an inverse.

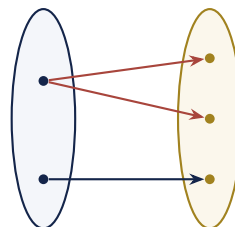
ONE-ONE



MANY-ONE



NOT A FUNCTION

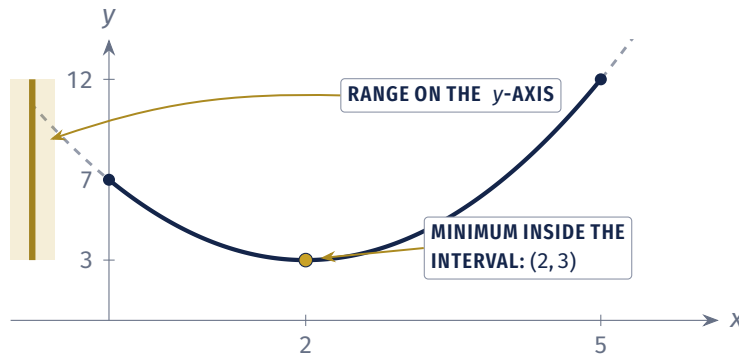


Mapping diagrams. Each input may send only one arrow.

How do I find the range on an interval?

- Sketch the graph **only over the given domain**.
- Find the value at **each end** of the interval.

- Find any **turning point inside** the interval: it may be the smallest or largest value.
- The range runs from the lowest to the highest of these values. Use $\leq$ for values that are reached, $<$ for values only approached.



$f(x) = x^2 - 4x + 7, 0 \leq x \leq 5$: the minimum is **inside** the interval, so the range is $3 \leq f(x) \leq 12$, not $7 \leq f(x) \leq 12$.

e.g. STATE THE RANGE OF $f(x) = 2x^2 - 5, x \geq 0$

$f(0) = -5$ and f increases for $x \geq 0$

THE LOWEST POINT IS AT THE START OF THE DOMAIN

ANSWER $f(x) \geq -5$



Examiner Tips and Tricks

- The range must be written in terms of $f(x)$ (or y): " $f(x) \geq -5$ ", never " $x \geq -5$ ".
- For a rational function such as $\frac{2x+2}{x-2}, x > 2$, think about what happens as $x \rightarrow 2$ and as $x \rightarrow \infty$: the outputs approach (but never reach) 2, so the range is $f(x) > 2$.
- Always check whether the domain contains a turning point. Using only the end values is the most common error in this topic.

WHEN YOU SEE → YOU DO

"State the range of f "→ sketch over the domain; end values and turning points; answer in $f(x)$ **" $0 \leq x \leq 5$ " with a quadratic**

→ is the vertex inside? then it gives the min or max

"Given that f is one-one / has an inverse, find the smallest k " with domain $x \geq k$ → k is the x -coordinate of the turning point**Worked Example**

? $f(x) = x^2 - 4x + 7$, $0 \leq x \leq 5$. Find the range of f .

$$f(x) = (x - 2)^2 + 3$$

← COMPLETE THE SQUARE: VERTEX (2, 3)

$$x = 2 \text{ is inside } [0, 5] \Rightarrow \text{min} = 3$$

← TURNING POINT INSIDE THE INTERVAL

$$f(0) = 7, \quad f(5) = 12$$

← BOTH ENDS: THE LARGER GIVES THE MAXIMUM

ANSWER $3 \leq f(x) \leq 12$

**Worked Example**

? $f(x) = (x - 3)^2 + 1$, $x \geq k$. Given that f is one-one, find the least value of k and state the range of f for this k .

vertex at $x = 3$

← LEFT OF THE VERTEX f DECREASES, RIGHT OF IT f INCREASES

$$k = 3$$

← START THE DOMAIN AT THE TURNING POINT SO NO OUTPUT REPEATS

ANSWER least $k = 3$; range $f(x) \geq 1$

HOW THIS IS EXAMINED · past WMA13 papers

algebraic restrictions **SEEN 1× IN EXAMS** restricted branch **SEEN 3× IN EXAMS**
range on an interval **SEEN 10× IN EXAMS**



Your notes

Composite Functions

NOTE 2.2

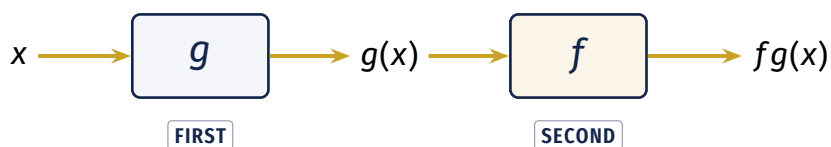


Composite Functions

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What is a composite function?

- A **composite function** applies one function and then another to the result.
- $fg(x)$ means $f(g(x))$: apply g **first**, then f . Read it from the **inside out** (right to left).
- In general $fg(x) \neq gf(x)$: the order matters.
- $f^2(x)$ or $ff(x)$ means apply f twice.



$fg(x)$: the input goes into g first, and its output goes into f .

How do I find $fg(x)$ or evaluate $fg(5)$?

- **A value**: work out $g(5)$ first, then put that number into f .
- **An expression**: replace every x in $f(x)$ by the whole expression $g(x)$, in brackets, then simplify.
- Keep exact values (fractions, surds, $\ln$, e) all the way through.

What is the domain of a composite function?

- x must be allowed in g , **and** the output $g(x)$ must be allowed in f .
- So the domain of fg is the set of x in the domain of g for which $g(x)$ lies in the domain of f .

How do I solve equations such as $gg(x) = k$ or $f\left(\frac{1}{a}\right) = g(a + 3)$?

- Form the expression first, then solve: $gg(x) = g(g(x))$.

- Look for a substitution: in $(x^2 - 3)^2 - 3 = 78$ treat $x^2 - 3$ as one block.
- $f\left(\frac{1}{a}\right) = g(a + 3)$ is **not** a composite: put $\frac{1}{a}$ into f and $a + 3$ into g separately.
- Reject any answer that makes an input illegal (e.g. a zero denominator).



e.g. $f(x) = 3x - 1$, $g(x) = x^2$. **FIND** $fg(2)$ **AND** $gf(2)$

$$g(2) = 4 \Rightarrow f(4) = 11$$

← fg : g **FIRST**

$$f(2) = 5 \Rightarrow g(5) = 25$$

← gf : f **FIRST**

ANSWER $fg(2) = 11$, $gf(2) = 25$: the order matters



Examiner Tips and Tricks

- Write the middle step: " $fg(5) = f(7) = \dots$ ". It earns the method mark even if the arithmetic slips.
- When forming $fg(x)$ put $g(x)$ in a **bracket** before you substitute.
- "Solve $gg(x) = 126$ ": after expanding, a quartic in x is usually a quadratic in x^2 . Remember $x^2 = \text{negative}$ has no real solutions.

WHEN YOU SEE → YOU DO

"Find the value of $fg(5)$ " / "Find $gf(2)$ "

→ inside function first; write the middle value

"Find $fg(x)$, giving your answer in its simplest form"

→ substitute $g(x)$ in brackets into f , then simplify

"Solve the equation $gg(x) = 126$ "

→ form $g(g(x))$, then treat the inner bracket as one block

"Solve the equation $f\left(\frac{1}{a}\right) = g(a + 3)$ "

→ two separate substitutions, then solve; check denominators

**Worked Example**

$f(x) = \frac{x+7}{x-2}$, $x \neq 2$ and $g(x) = 2x - 1$. (a) Find $fg(4)$. (b) Find $fg(x)$ in its simplest form.

$$g(4) = 7$$

← (A) g FIRST

$$f(7) = \frac{14}{5}$$

← THEN f

$$fg(x) = \frac{(2x-1)+7}{(2x-1)-2}$$

← (B) REPLACE x IN f BY $(2x-1)$

ANSWER (a) $fg(4) = \frac{14}{5}$ (b) $fg(x) = \frac{2x+6}{2x-3}$, $x \neq \frac{3}{2}$

**Worked Example**

$g(x) = x^2 - 3$, $x \in \mathbb{R}$. Solve $gg(x) = 78$.

$$(x^2 - 3)^2 - 3 = 78$$

← FORM $g(g(x))$

$$(x^2 - 3)^2 = 81 \Rightarrow x^2 - 3 = \pm 9$$

← TREAT $x^2 - 3$ AS ONE BLOCK

$$x^2 = 12 \quad \text{or} \quad x^2 = -6 \text{ (no real roots)}$$

← REJECT THE NEGATIVE SQUARE

ANSWER $x = \pm 2\sqrt{3}$

**Worked Example**

? $f(x) = \frac{x+7}{x-2}$, $x \neq 2$ and $g(x) = 2x - 1$. Solve $f\left(\frac{1}{a}\right) = g(a+3)$.

$$f\left(\frac{1}{a}\right) = \frac{\frac{1}{a} + 7}{\frac{1}{a} - 2} = \frac{1 + 7a}{1 - 2a}$$

← MULTIPLY TOP AND BOTTOM BY a

$$g(a+3) = 2a + 5$$

← SEPARATE SUBSTITUTION

$$1 + 7a = (2a + 5)(1 - 2a) \Rightarrow 4a^2 + 15a - 4 = 0$$

← CROSS-MULTIPLY AND COLLECT

$$(4a - 1)(a + 4) = 0$$

← CHECK: $\frac{1}{a} \neq 2$ FOR BOTH

ANSWER $a = \frac{1}{4}$ or $a = -4$

**Worked Example**

? $f(x) = \ln(x - 1)$, $x > 1$ and $g(x) = 3 - x^2$, $x \in \mathbb{R}$. Find the domain of fg .

$$g(x) > 1$$

← THE OUTPUT OF g MUST BE A VALID INPUT FOR f

$$3 - x^2 > 1 \Rightarrow x^2 < 2$$

← SOLVE THE INEQUALITY

ANSWER $-\sqrt{2} < x < \sqrt{2}$

HOW THIS IS EXAMINED • past WMA13 papers

form $fg(x)$ **SEEN 3× IN EXAMS** domain of fg **NOT SEEN YET • BE READY**

evaluate $fg(5)$ **SEEN 11× IN EXAMS** solve a composite equation **SEEN 6× IN EXAMS**

$f(u) = g(v)$ **SEEN 1× IN EXAMS**

Inverse Functions

NOTE 2.3



Inverse Functions

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What is an inverse function?

- The **inverse** f^{-1} reverses f : if $f(3) = 10$ then $f^{-1}(10) = 3$.
- $ff^{-1}(x) = f^{-1}f(x) = x$.
- Only **one-one** functions have an inverse. A many-one function must first be **restricted** to a one-one part.
- $f^{-1}(x)$ is **not** $\frac{1}{f(x)}$.

 f

- domain: inputs of f
- range: outputs of f

 f^{-1}

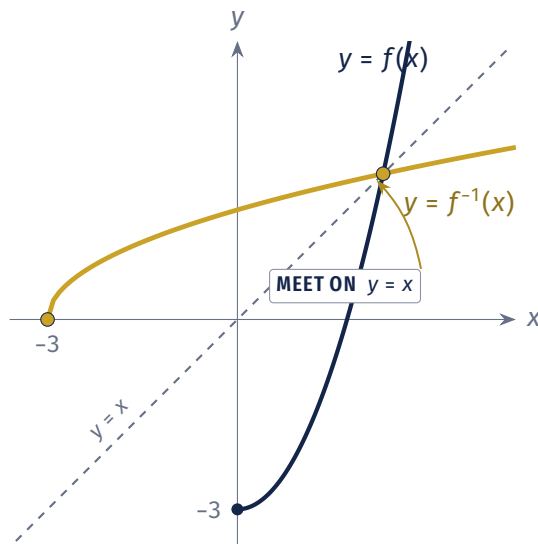
- domain = **range of f**
- range = **domain of f**

How do I find $f^{-1}(x)$?

- STEP 1** Write $y = f(x)$.
- STEP 2** Rearrange to make x the subject.
- STEP 3** Swap: replace y by x , and write $f^{-1}(x) = \dots$
- STEP 4** State the domain of f^{-1} : it is the **range of f** .

What does the graph of $y = f^{-1}(x)$ look like?

- It is the **reflection** of $y = f(x)$ in the line $y = x$.
- Every point (a, b) on f becomes (b, a) on f^{-1} ; intercepts swap axes; asymptotes swap ($y = 3$ becomes $x = 3$).
- If f is **increasing**, f and f^{-1} can only meet **on** the line $y = x$, so solve $f(x) = x$.



$f(x) = x^2 - 3, x \geq 0$ and its inverse: mirror images in $y = x$, meeting on the line.

How do I find $f^{-1}(k)$ without the formula?

- $f^{-1}(k)$ is the input that makes f give k . Solve $f(x) = k$, keeping only the solution in the domain of f .

e.g. $f(x) = x^3 + 2x - 1$. **FIND** $f^{-1}(11)$

$$x^3 + 2x - 1 = 11 \Rightarrow x = 2$$

SOLVE $f(x) = 11$ (TRY SMALL INTEGERS)

ANSWER $f^{-1}(11) = 2$



Examiner Tips and Tricks

- The answer " $f^{-1}(x) = \dots$ " needs its domain to score full marks: " $x > 3$ ", taken from the range of f .
- With a quadratic, choose the $\pm$ root that matches the domain of f : for $x \geq 3$ use $3 + \sqrt{\dots}$.
- For a **decreasing** function, $f(x) = f^{-1}(x)$ is not guaranteed to happen only on $y = x$. Solve $f(x) = f^{-1}(x)$ directly and check the answer is in both domains.

 Your notes

WHEN YOU SEE → YOU DO

"Find $f^{-1}(x)$ " / "Find f^{-1} "

→ $y = f(x)$, make x the subject, swap, state the domain

"State the domain of f^{-1} "

→ it is the range of f

"On Diagram 1, sketch the curve with equation $y = f^{-1}(x)$ "

→ reflect in $y = x$; swap intercepts and asymptotes

"find the exact x coordinate of P " where $y = f(x)$ meets $y = f^{-1}(x)$

→ increasing f : solve $f(x) = x$; otherwise solve $f(x) = f^{-1}(x)$



Worked Example



$f(x) = \frac{x+7}{x-2}$, $x > 2$. Find $f^{-1}(x)$ and state its domain.

$$y(x-2) = x+7 \Rightarrow xy - 2y = x+7$$

← MULTIPLY OUT

$$x(y-1) = 2y+7 \Rightarrow x = \frac{2y+7}{y-1}$$

← COLLECT THE x TERMS, FACTORISE

$$x > 2 \Rightarrow f(x) > 1$$

← RANGE OF f : AS $x \rightarrow \infty$, $f \rightarrow 1$;
AS $x \rightarrow 2^+$, $f \rightarrow \infty$

ANSWER $f^{-1}(x) = \frac{2x+7}{x-1}$, $x > 1$

**Worked Example**

? $f(x) = 4 + 3 \ln(x - 1)$, $x > 1$ and $g(x) = 2e^x + 3$, $x \in \mathbb{R}$. Find $f^{-1}(x)$ and $g^{-1}(x)$, each with its domain.

$$\ln(x - 1) = \frac{y - 4}{3} \Rightarrow x = 1 + e^{\frac{y-4}{3}}$$

← UNDO +4, $\times 3$, THEN $\ln$

range of f is $\mathbb{R}$

← SO f^{-1} ACCEPTS EVERY x

$$e^x = \frac{y - 3}{2} \Rightarrow x = \ln\left(\frac{y - 3}{2}\right)$$

← UNDO +3, $\times 2$, THEN e

ANSWER $f^{-1}(x) = 1 + e^{\frac{x-4}{3}}$, $x \in \mathbb{R}$ $g^{-1}(x) = \ln\left(\frac{x-3}{2}\right)$, $x > 3$

**Worked Example**

? $f(x) = x^2 - 6x + 5$, $x \geq 3$. Find $f^{-1}(x)$, stating its domain.

$$y = (x - 3)^2 - 4$$

← COMPLETE THE SQUARE

$$x - 3 = \pm\sqrt{y + 4} \Rightarrow x = 3 + \sqrt{y + 4}$$

← TAKE + BECAUSE $x \geq 3$

ANSWER $f^{-1}(x) = 3 + \sqrt{x + 4}$, $x \geq -4$

**Worked Example**

$f(x) = x^2 - 3$, $x \geq 0$. The curves $y = f(x)$ and $y = f^{-1}(x)$ meet at P . Find the exact x -coordinate of P .

f is increasing, so P lies on $y = x$

$$x^2 - 3 = x \Rightarrow x^2 - x - 3 = 0$$

$$x = \frac{1 \pm \sqrt{13}}{2}$$

← THE CURVES ARE MIRROR IMAGES IN $y = x$

← SOLVE $f(x) = x$

← REJECT THE NEGATIVE ROOT: $x \geq 0$

ANSWER $x = \frac{1 + \sqrt{13}}{2}$

**Worked Example**

$f(x) = \frac{6}{x+1}$, $x > 0$. Find $f^{-1}(x)$ and solve $f^{-1}(x) = f(x)$.

$$f^{-1}(x) = \frac{6}{x} - 1, \quad 0 < x < 6$$

$$\frac{6}{x+1} = \frac{6}{x} - 1 \Rightarrow 6x = 6(x+1) - x(x+1)$$

$$x^2 + x - 6 = 0 \Rightarrow x = 2 \quad (x = -3 \text{ rejected})$$

← RANGE OF f IS $0 < f(x) < 6$

← f IS DECREASING: SOLVE THE ACTUAL EQUATION

← CHECK 2 IS IN BOTH DOMAINS

ANSWER $x = 2$

HOW THIS IS EXAMINED • past WMA13 papers

find $f^{-1}(x)$ **SEEN 16× IN EXAMS** restrict then invert **NOT SEEN YET • BE READY**

graph of f^{-1} **SEEN 2× IN EXAMS** $f^{-1}(k)$ directly **SEEN 2× IN EXAMS** $f = f^{-1}$ **SEEN 3× IN EXAMS**

MISTAKE GALLERY

Where marks are lost

× **WRONG**

Range of $f(x) = 2x^2 - 5$, $x \geq 0$: “ $x \geq -5$ ”.

✓ **RIGHT**

$f(x) \geq -5$.

The range describes outputs.

× **WRONG**

Range of $x^2 - 4x + 7$ on $0 \leq x \leq 5$ from the ends only: $7 \leq f(x) \leq 12$.

✓ **RIGHT**

$3 \leq f(x) \leq 12$: the vertex (2,3) is inside.

Check for a turning point in the interval.

× **WRONG**

$fg(4) = g(f(4))$.

✓ **RIGHT**

$fg(4) = f(g(4))$: g first.

Read composites from the inside out.

× **WRONG**

$f^{-1}(x) = \frac{1}{f(x)}$.

✓ **RIGHT**

f^{-1} undoes f ; it is not the reciprocal.

The -1 is notation, not a power.

× **WRONG**

$f^{-1}(x) = 3 \pm \sqrt{x+4}$.

✓ **RIGHT**

$3 + \sqrt{x+4}$ only, because $x \geq 3$ in f .

An inverse must be a function: one output.

THE CHAPTER ON ONE PAGE

Functions**2.1****Domain and range**

Domain: allowed x . Range: outputs, written with $f(x)$.

On an interval: both ends **and** any turning point inside.

2.2**Composites**

$fg(x) = f(g(x))$: g first.

Substitute $g(x)$ in brackets.

Domain of fg : $g(x)$ must be allowed in f .

2.3**Inverses**

$y = f(x)$, make x the subject, swap.

Domain of f^{-1} = range of f .

Graph: reflect in $y = x$.

2.3 **f meets f^{-1}**

Increasing f : solve $f(x) = x$.

Otherwise solve $f(x) = f^{-1}(x)$ and check both domains.

BOTTOM LINE Know what goes in, what comes out, which machine runs first, and how to run it backwards. Always state the domain.



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03

CHAPTER 03

Graph Transformations

Inside the bracket moves x the “wrong” way; outside moves y the way it looks. Track a few key points and the whole graph follows.

2

CONCEPTS

4

VARIANTS

13

EXAM QUESTIONS

YOU WILL MASTER

- ▶ Map any point under $f(x) + a$, $f(x + a)$, $af(x)$, $f(ax)$, $-f(x)$ and $f(-x)$
- ▶ Combine an inside change and an outside change, e.g. $3f(2x)$ or $f(-x) - 1$
- ▶ Sketch transformed graphs with every turning point and intercept labelled
- ▶ Work backwards: find a and b from a transformed vertex or maximum

CHAPTER 03 AT A GLANCE

Two note pages, four ways to be asked



CHAPTER 03 MAP

Graph Transformations

01

3.1 Single transformations

outside: $f(x) + a$, $af(x)$, $-f(x)$ inside: $f(x + a)$, $f(ax)$, $f(-x)$

02

3.2 Combinations of transformations

inside and outside: $3f(2x)$ reflection and shift: $f(-x) - 1$ find a , b from a vertex

max and min values

Contents

✧ 3.1	Single Transformations	3
✧ 3.2	Combinations of Transformations	7

Single Transformations

NOTE 3.1

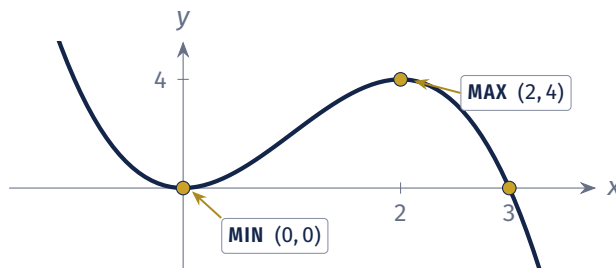


Single Transformations

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What is a graph transformation?

- A **transformation** changes the equation $y = f(x)$ and so moves, stretches or reflects its graph.
- Every transformation is either **outside** the function or **inside** the bracket:
 - **outside** changes act on the y -coordinates, and do what they look like
 - **inside** changes act on the x -coordinates, and do the **opposite** of what they look like
- To sketch a transformed graph, move the **key points** (turning points, intercepts) and keep the shape.



The base curve for this page: $y = f(x)$ with a minimum at $(0,0)$, a maximum at $(2,4)$ and a root at $(3,0)$.

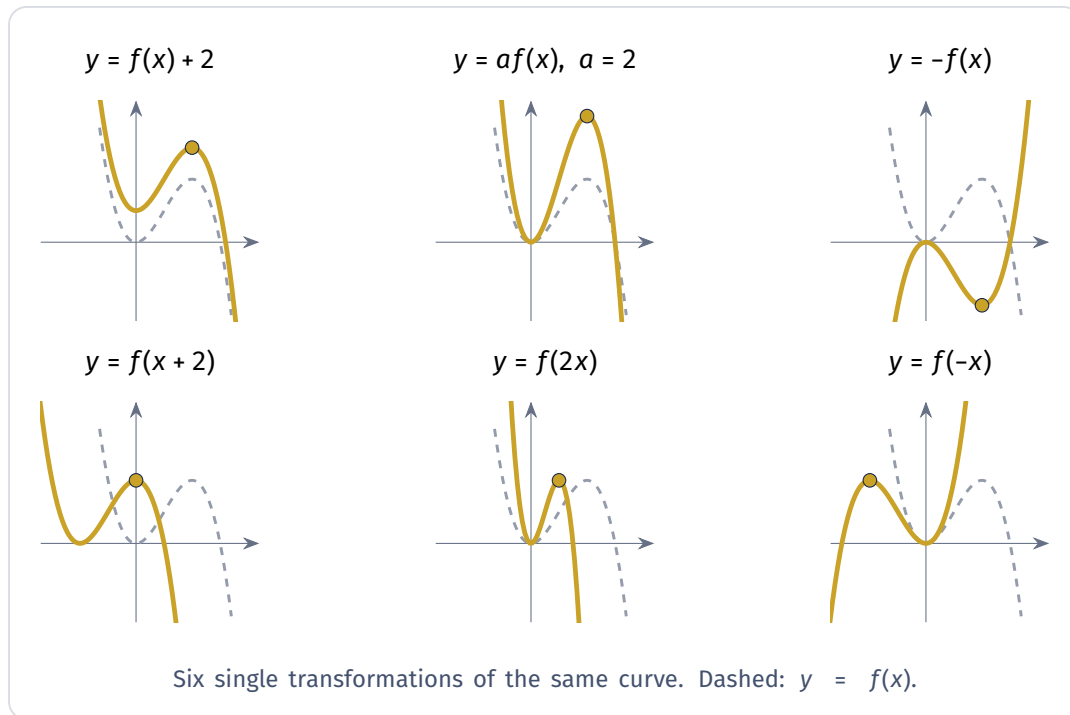
How do outside changes work?

- $y = f(x) + a$: **translation** by $\begin{pmatrix} 0 \\ a \end{pmatrix}$. $(x, y) \rightarrow (x, y + a)$
- $y = af(x)$: **vertical stretch**, scale factor a . $(x, y) \rightarrow (x, ay)$
- $y = -f(x)$: **reflection** in the x -axis. $(x, y) \rightarrow (x, -y)$
- x -coordinates never change. Roots of $af(x)$ stay where they are.

How do inside changes work?

 Your notes

- $y = f(x + a)$: **translation** by $\begin{pmatrix} -a \\ 0 \end{pmatrix}$. $(x, y) \rightarrow (x - a, y)$ ($+a$ moves **left**)
- $y = f(ax)$: **horizontal stretch**, scale factor $\frac{1}{a}$. $(x, y) \rightarrow \left(\frac{x}{a}, y\right)$ ($f(2x)$ **squashes**)
- $y = f(-x)$: **reflection** in the y -axis. $(x, y) \rightarrow (-x, y)$
- y -coordinates never change. The y -intercept of $f(ax)$ stays the same.



EQUATION	TRANSFORMATION	(2, 4) MAPS TO
$f(x) + 2$	translation $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$	(2, 6)
$2f(x)$	vertical stretch, scale factor 2	(2, 8)
$-f(x)$	reflection in the x -axis	(2, -4)
$f(x + 2)$	translation $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$	(0, 4)
$f(2x)$	horizontal stretch, scale factor $\frac{1}{2}$	(1, 4)
$f(-x)$	reflection in the y -axis	(-2, 4)

e.g. WHERE DOES (2, 4) GO UNDER $y = f(x - 3)$?

$$x - 3 = 2 \Rightarrow x = 5$$

INSIDE: SOLVE NEW INSIDE =
OLD x

ANSWER (5, 4)



Examiner Tips and Tricks

- For an inside change, solve “new inside = old x-coordinate”. For $f(2x)$ and the point $(2, 4)$: $2x = 2$, so $x = 1$. This never goes the wrong way.
- Describe a stretch with its **direction** and **scale factor**: “a stretch parallel to the x-axis, scale factor $\frac{1}{2}$ ”.
- Asymptotes move too: $y = f(x) + 2$ moves a horizontal asymptote up by 2.

 Your notes

WHEN YOU SEE → YOU DO

“Find the point to which P is mapped when ... is transformed to ...”

→ inside: solve for x ; outside: apply to y

“ $y = f(2x)$ ”

→ halve the x-coordinates; y unchanged

“ $y = 3f(x)$ ”

→ treble the y-coordinates; roots unchanged

“ $y = f(-x)$ ” / “ $y = -f(x)$ ”

→ inside minus: mirror in y-axis; outside minus: mirror in x-axis



Worked Example



The point $P(6, -2)$ lies on the curve $y = f(x)$. Find the point to which P is mapped when the curve is transformed to (a) $y = f(3x)$ (b) $y = 2f(x + 1)$ (c) $y = |f(x)|$.

$$3x = 6 \Rightarrow x = 2$$

← (A) INSIDE: NEW INSIDE = OLD x ; y UNCHANGED

$$x + 1 = 6 \Rightarrow x = 5; \quad y = 2(-2) = -4$$

← (B) INSIDE SHIFT, THEN OUTSIDE STRETCH

$$y = |-2| = 2$$

← (C) MODULUS OUTSIDE: NEGATIVE y IS REFLECTED UP

ANSWER (a) $(2, -2)$ (b) $(5, -4)$ (c) $(6, 2)$

HOW THIS IS EXAMINED · past WMA13 papers

vertical transformation **SEEN 1× IN EXAMS** horizontal transformation **SEEN 1× IN EXAMS**



Your notes

Combinations of Transformations

NOTE 3.2



Combinations of Transformations

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How do I apply two transformations at once?

- Separate the equation into its **inside** part and its **outside** part.
 - $y = 3f(2x)$: inside $2x$ (horizontal stretch $\frac{1}{2}$), outside $\times 3$ (vertical stretch 3)
 - $y = f(-x) - 1$: inside $-x$ (reflect in y -axis), outside -1 (down 1)
- An inside change and an outside change **do not affect each other**: apply the inside one to x and the outside one to y .
- Two **outside** changes follow the normal order of operations: $2f(x) - 5$ means stretch first, then subtract 5.

MAPPING A POINT UNDER A COMBINED TRANSFORMATION

STEP 1

Write the key point (p, q)

STEP 2

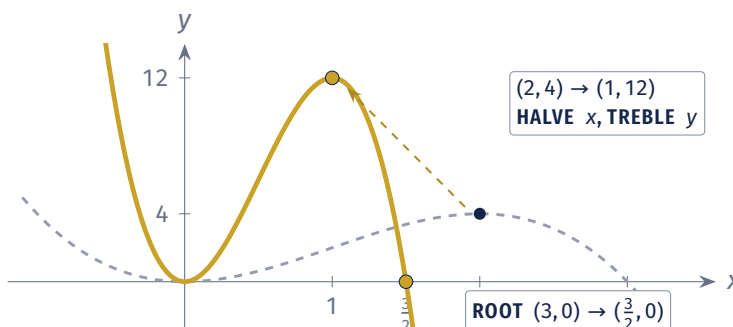
Inside: solve "new inside = p " for the new x

STEP 3

Outside: apply the outside operations to q in order

STEP 4

Plot and label the new point



$y = 3f(2x)$: squash towards the y -axis, then stretch away from the x -axis.

How do I find a and b from a transformed point?

- Use the same rule backwards: **inside** gives an equation in the x -coordinates, **outside** gives an equation in the y -coordinates.
- For $y = af(x + b)$: new $x = \text{old } x - b$ and new $y = a \times \text{old } y$.
- A vertex, maximum or minimum is the best point to use: it is easy to read from the question.

How do transformations change maximum and minimum values?

- Outside changes alter the **values**: if f has minimum -3 , then $2f(x) - 5$ has minimum $2(-3) - 5 = -11$.
- A **negative** outside factor swaps maxima and minima: $-f$ turns a minimum of -3 into a maximum of 3 .
- Inside changes alter **where** it happens: if f is minimum at $x = 1.2$, then $f(2x)$ is minimum at $x = 0.6$.



Examiner Tips and Tricks

- “On separate diagrams”: one clean sketch per part, with every turning point and the y -intercept labelled with both coordinates.
- The y -intercept is found by putting $x = 0$ into the **new** equation: for $y = f(-x) - 1$ it is $f(0) - 1$.
- This topic often appears inside an R-form question (C07): “State the y coordinate of the minimum points of $y = 2f(x) - 5$ ”. Only one line of work is needed.

WHEN YOU SEE → YOU DO

“On separate diagrams, sketch $y = 3f(2x)$ and $y = f(-x) - 1$ ”

→ move every key point: inside to x , outside to y

“the vertex of $y = af(x + b)$ is $(-2, 5)$ ”

→ equation in x gives b ; equation in y gives a

“State the y coordinate of the minimum points on $y = 2f(x) - 5$ ”

→ apply the outside operations to the minimum value

“the smallest value of x at which a maximum occurs for $y = -f(2x)$ ”

→ $-$ swaps max and min; $2x$ halves the x -value





Worked Example



The curve $y = f(x)$ has a minimum at $(0, 0)$, a maximum at $(2, 4)$ and crosses the x -axis at $(3, 0)$. On separate diagrams sketch (i) $y = 3f(2x)$ (ii) $y = f(-x) - 1$, showing the turning points and the y -intercept.

$$(0, 0) \rightarrow (0, 0), \quad (2, 4) \rightarrow (1, 12)$$

(I) HALVE EVERY x , TREBLE EVERY y

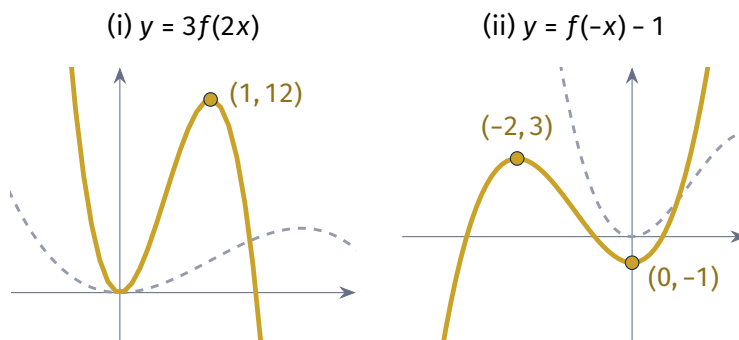
$$(3, 0) \rightarrow \left(\frac{3}{2}, 0\right)$$

(I) THE ROOT MOVES TOO

$$(0, 0) \rightarrow (0, -1), \quad (2, 4) \rightarrow (-2, 3)$$

(II) MIRROR IN THE y -AXIS, THEN DOWN 1

ANSWER (i) min $(0, 0)$, max $(1, 12)$ (ii) min $(0, -1)$ (also the y -intercept), max $(-2, 3)$



The two answers. Dashed: $y = f(x)$.

Your notes

**Worked Example**

? $f(x) = 2|x - 3| + 1$. The graph of $y = f(x)$ has its vertex at P . The graph is transformed to $y = af(x + b)$, whose vertex is $(-2, 5)$. Find P , a and b .

$$x - 3 = 0 \Rightarrow P(3, 1)$$

← VERTEX: THE MODULUS IS ZERO

$$x + b = 3 \text{ when } x = -2 \Rightarrow b = 5$$

← INSIDE: NEW INSIDE = OLD x

$$a \times 1 = 5 \Rightarrow a = 5$$

← OUTSIDE: NEW $y = a \times$ OLD y

ANSWER $P(3, 1)$, $a = 5$, $b = 5$

**Worked Example**

? $f(x)$, $x > 0$, has minimum value -3 , first occurring at $x = 1.2$. (a) State the minimum value of $y = 2f(x) - 5$. (b) State the smallest x at which a maximum occurs for $y = -f(2x)$, and the maximum value.

$$2(-3) - 5 = -11$$

← (A) OUTSIDE OPERATIONS ON THE MINIMUM VALUE

$$2x = 1.2 \Rightarrow x = 0.6$$

← (B) THE MINIMUM OF f BECOMES A MAXIMUM OF $-f$; INSIDE HALVES x

$$-(-3) = 3$$

← THE MAXIMUM VALUE

ANSWER (a) -11 (b) $x = 0.6$, maximum value 3

HOW THIS IS EXAMINED • past WMA13 papers

inside and outside **SEEN 12× IN EXAMS** reflection and shift **SEEN 2× IN EXAMS**

MISTAKE GALLERY

Where marks are lost

× **WRONG**

$y = f(x + 2)$ moves the graph 2 to the right.

✓ **RIGHT**

It moves 2 to the **left**: $(x, y) \rightarrow (x - 2, y)$.

Inside changes do the opposite.

× **WRONG**

$y = f(2x)$ doubles the x -coordinates.

✓ **RIGHT**

It **halves** them: scale factor $\frac{1}{2}$.

Solve $2x = \text{old } x$.

× **WRONG**

$y = 2f(x)$ moves the roots.

✓ **RIGHT**

Roots stay: $2 \times 0 = 0$.

Outside changes never touch x .

× **WRONG**

For $y = f(-x) - 1$, writing the y -intercept as $(0, 0)$.

✓ **RIGHT**

$f(0) - 1 = -1$, so $(0, -1)$.

Use the new equation at $x = 0$.

× **WRONG**

$y = -f(2x)$: maximum at the same place as f 's maximum.

✓ **RIGHT**

The **minimum** of f becomes the maximum.

A negative outside factor swaps max and min.

THE CHAPTER ON ONE PAGE

Graph Transformations

**3.1****Outside: y changes** $f(x) + a$: up a . $af(x)$: $y \times a$. $-f(x)$: mirror in the x -axis.

Roots never move.

3.1**Inside: x changes** $f(x + a)$: left a . $f(ax)$: $x \div a$. $f(-x)$: mirror in the y -axis. y -intercept never moves.**3.2****Combined**Inside to x , outside to y , independently.Solve "new inside = old x ".

Two outside steps: usual order.

3.2**Backwards and values**Vertex of $af(x + b)$: x -equation gives b , y -equation gives a .

Max and min values follow the outside operations.

BOTTOM LINE Move the key points, not the whole curve: inside changes act on x and go the opposite way, outside changes act on y and do what they say.



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04

CHAPTER 04

Modulus

Every modulus question is a graph question in disguise: find the corner, split into two straight branches, and keep only what is allowed.

3

CONCEPTS

12

VARIANTS

20

EXAM QUESTIONS

YOU WILL MASTER

- ▶ Sketch $|f(x)|$, $f(|x|)$ and every $a|x + p| + q$ graph with exact corners and intercepts
- ▶ Solve modulus equations by branches, and reject the answers that are not allowed
- ▶ Count solutions and find the parameter range for “exactly two roots”
- ▶ Solve modulus inequalities and write the final interval correctly

CHAPTER 04 AT A GLANCE

Three note pages, twelve ways to be asked



CHAPTER 04 MAP

Modulus

01 — 4.1 Sketching modulus graphs

$$y = |f(x)| \quad y = f(|x|) \quad y = a|x + p| + q$$

find unknown constants

02 — 4.2 Solving modulus equations

$$|f(x)| = g(x) \quad |f(x)| = |g(x)| \quad \text{exactly } n \text{ roots}$$

modulus = a curve $f(|x|) = c$

03 — 4.3 Solving modulus inequalities

$$|f(x)| > g(x) \quad < \text{ or } \leq: \text{ the ends} \quad \text{In or quadratic inside}$$

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✧ 4.2 Solving Modulus Equations	8
✧ 4.3 Solving Modulus Inequalities	14

Sketching Graphs of Modulus Functions

NOTE 4.1

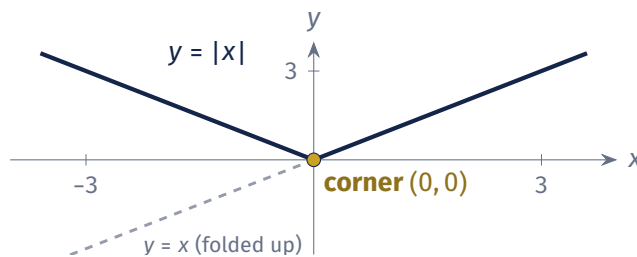


Sketching Graphs of Modulus Functions

PURE 3 > ALGEBRA AND FUNCTIONS > MODULUS • Exam code WMA13 • Dr Eslam Ahmed

What is the modulus function?

- The **modulus** of a number is its **size**: how far it is from 0, ignoring the sign.
 - $|5| = 5$ and $|-5| = 5$
 - The modulus is **never negative**: $|x| \geq 0$ for every x .
- The modulus function $y = |x|$ is defined in two **branches**:
 - $|x| = x$ when $x \geq 0$ (leave it alone)
 - $|x| = -x$ when $x < 0$ (change its sign)
- Its graph is a **V-shape** with its **corner** (vertex) at the origin.



The graph of $y = |x|$: the part of $y = x$ below the axis is folded up.

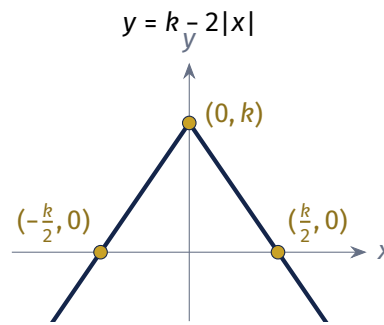
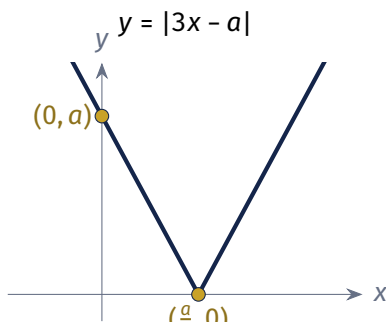
How do I sketch $y = a|x + p| + q$?

- Every graph of this type is a **V-shape**: two straight arms that meet at a **corner**.
- The corner is where the inside of the modulus equals **zero**.

- STEP 1** Find the corner: solve $x + p = 0$, so $x = -p$. Its height is $y = q$.
- STEP 2** Decide the direction: $a > 0$ opens **upwards** (V); $a < 0$ opens **downwards** ($\wedge$).
- STEP 3** Find the y-intercept: put $x = 0$ **inside** the modulus first, then take the size.
- STEP 4** Find any x-intercepts: solve $a|x + p| + q = 0$ by branches (see 4.2).



STEP 5 Draw two straight arms through these points and label every coordinate.



Two of the most common exam graphs. Label the corner and every intercept.

e.g. **SKETCH** $y = |2x - 6|$, **SHOWING WHERE THE GRAPH MEETS THE AXES**

$$2x - 6 = 0 \Rightarrow x = 3$$

← **CORNER ON THE x-AXIS**

$$x = 0 : y = |-6| = 6$$

← **y-INTERCEPT: INSIDE FIRST**

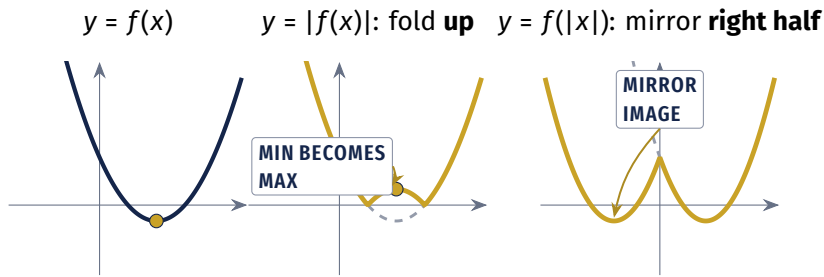
ANSWER V-shape, corner (3, 0), y-intercept (0, 6)

How do I sketch $y = |f(x)|$?

- The modulus acts on the **output** (y-values), so no part of the graph can be below the x-axis.
- Keep every part of $y = f(x)$ that is **on or above** the x-axis.
- **Reflect** every part that is **below** the x-axis in the x-axis: $(x, y) \rightarrow (x, -y)$.
 - x-intercepts do not move.
 - A minimum below the axis becomes a **maximum** above it.

How do I sketch $y = f(|x|)$?

- The modulus acts on the **input** (x-values), so $f(|x|)$ gives the same output for x and $-x$.
- **Delete** the part of the graph where $x < 0$.
- **Reflect** the part where $x \geq 0$ in the y-axis. The result is symmetric about the y-axis.
 - Negative y-values **stay negative**.
 - A turning point (p, q) with $p > 0$ gives a second one at $(-p, q)$.



The same f , two different modulus graphs. Dashed: the part that moved.

What is the difference between $y = |f(x)|$ and $y = f(|x|)$?

$y = |f(x)|$ (modulus outside)

- changes the y -values
- nothing below the x -axis
- fold **up** in the x -axis

$y = f(|x|)$ (modulus inside)

- changes the x -values
- symmetric in the y -axis
- mirror the **right half** to the left



Examiner Tips and Tricks

- “On separate diagrams” means one clean sketch for each part. Do not draw them on top of each other.
- “Showing the coordinates of the points where the graph meets the axes”: every intercept needs both coordinates, e.g. $(0, a)$, not just a .
- The y -intercept of $y = |3x - a|$ is $(0, a)$, never $(0, -a)$.
- Your sketch in part (a) is the tool for parts (b) and (c). Draw it carefully.

WHEN YOU SEE → YOU DO

“sketch the graph of $y = |f(x)|$ ”

→ keep above the axis, **re-reflect** the parts below

“sketch the graph of $y = f(|x|)$ ”

→ keep $x \geq 0$, **copy** it into $x < 0$ by reflecting in the y-axis

“ $y = |3x - a|$... where a is a positive constant”

→ corner $(\frac{a}{3}, 0)$; y-intercept $(0, a)$: answers in terms of a

“The graph has a maximum point at $(3, 7)$ ” with unknowns

→ corner $\Rightarrow$ inside = 0; height $\Rightarrow$ outside constant



Worked Example



Sketch the graph of $y = 5 - |3x - 22|$, showing the coordinates of the maximum point and of any points where the graph meets the coordinate axes.

$$3x - 22 = 0 \Rightarrow x = \frac{22}{3}$$

STEP 1: THE CORNER IS WHERE THE INSIDE EQUALS ZERO

$$y = 5 - 0 = 5 \Rightarrow \left(\frac{22}{3}, 5\right)$$

AT THE CORNER THE MODULUS IS 0; MINUS SIGN MEANS THE CORNER IS A MAXIMUM

$$|3x - 22| = 5 \Rightarrow 3x - 22 = 5 \text{ or } 3x - 22 = -5$$

x-INTERCEPTS: SET $y = 0$ AND SPLIT INTO TWO BRANCHES

$$x = 9 \text{ or } x = \frac{17}{3}$$

SOLVE EACH BRANCH

$$x = 0 : y = 5 - |-22| = -17$$

y-INTERCEPT: INSIDE FIRST, THEN TAKE THE SIZE

ANSWER Λ -shape, maximum $(\frac{22}{3}, 5)$, crosses the x-axis at $(\frac{17}{3}, 0)$ and $(9, 0)$, the y-axis at $(0, -17)$

**Worked Example**

The graph of $y = a - |2x - b|$, where a and b are constants, has a maximum point at $(3, 7)$. Find a and b , and the coordinates of the point where the graph crosses the y -axis.

$$2(3) - b = 0 \Rightarrow b = 6$$

← THE CORNER IS AT $x = 3$, SO THE INSIDE IS ZERO THERE

$$7 = a - 0 \Rightarrow a = 7$$

← AT THE CORNER THE MODULUS IS 0

$$y = 7 - |0 - 6| = 1$$

← PUT $x = 0$

ANSWER $a = 7$, $b = 6$, crosses the y -axis at $(0, 1)$

**Worked Example**

$f(x) = x^2 - 4x + 3$. On separate diagrams sketch (a) $y = |f(x)|$ (b) $y = f(|x|)$, giving the coordinates of any turning points.

$$f(x) = (x - 1)(x - 3), \quad \min(2, -1), \quad (0, 3)$$

← KEY POINTS OF $y = f(x)$ FIRST

$$(2, -1) \rightarrow (2, 1)$$

← (A) THE PART BETWEEN $x = 1$ AND $x = 3$ IS FOLDED UP: THE MINIMUM BECOMES A MAXIMUM

$$(2, -1) \rightarrow (-2, -1)$$

← (B) THE RIGHT HALF IS MIRRORRED: A SECOND MINIMUM APPEARS; y STAYS NEGATIVE

ANSWER (a) maximum $(2, 1)$ (b) minima $(2, -1)$ and $(-2, -1)$, maximum $(0, 3)$

HOW THIS IS EXAMINED • past WMA13 papers

$|f(x)|$ fold **SEEN 3× IN EXAMS** $f(|x|)$ mirror **NOT SEEN YET • BE READY**

$a|x + p| + q$ V-shape **SEEN 16× IN EXAMS** find constants **SEEN 3× IN EXAMS**

Solving Modulus Equations

NOTE 4.2



Solving Modulus Equations

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How do I solve $|f(x)| = g(x)$?

- A modulus has two **branches**, so the equation splits into two ordinary equations:
 - $f(x) = g(x)$ (where $f(x) \geq 0$)
 - $-f(x) = g(x)$ (where $f(x) < 0$)
- Solving both always gives two candidate answers, but **not all of them are real solutions**.
- **Sketch both graphs** first: the number of crossings tells you how many answers to keep.

STEP 1 Sketch $y = |f(x)|$ and $y = g(x)$ on the same axes and count the intersections.

STEP 2 Find the corner and write the two branch equations.

STEP 3 Solve each one.

STEP 4 **Reject** any answer that is not on its own branch, or that makes $g(x) < 0$.

e.g. SOLVE $|x + 1| = 3$

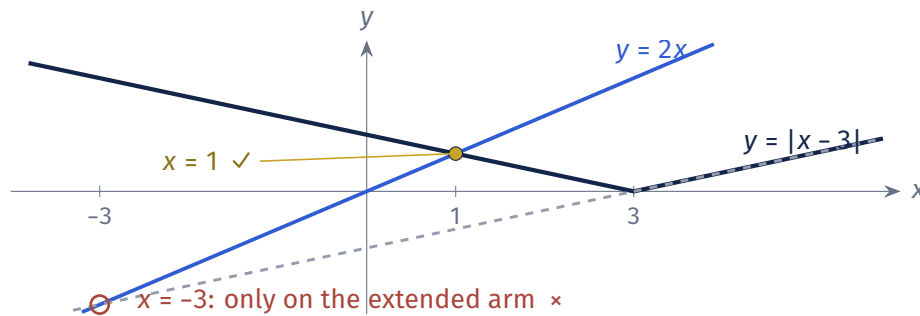
$$x + 1 = 3 \quad \text{or} \quad x + 1 = -3$$

RIGHT SIDE IS A POSITIVE CONSTANT: BOTH BRANCHES ARE FINE

ANSWER $x = 2$ or $x = -4$

Why must I check my answers?

- Each branch equation is only true on its own side of the corner.
- An answer from the branch $x \geq 3$ that turns out to be $x = -3$ is **not** a solution: the line never meets that arm.
- A modulus cannot equal a negative number, so any answer that makes the right-hand side negative must go.



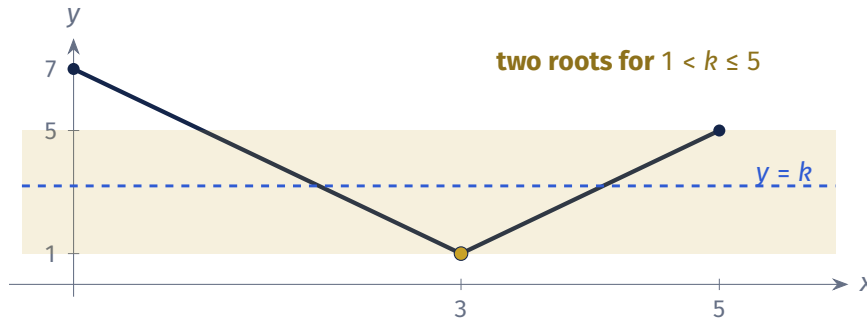
$|x - 3| = 2x$ has one solution. The algebra also produces $x = -3$, which must be rejected.

How do I solve $|f(x)| = |g(x)|$?

- Two numbers with the same size are either **equal** or **negatives** of each other:
 - $f(x) = g(x)$ or $f(x) = -g(x)$
- Alternatively **square both sides**: $(f(x))^2 = (g(x))^2$. This is safe because both sides are ≥ 0 .
- Both answers are usually valid here, but substitute to check.

How do I find the values of k for “exactly two roots”?

- The roots of $f(x) = k$ are where the horizontal line $y = k$ meets $y = f(x)$.
- Slide the line** up and down the sketch and count the crossings in each band of heights.
- Check the boundary heights one by one:
 - at the **corner** height the line touches once (one root)
 - at a **domain end** the point counts only if that end is included ($\leq$)

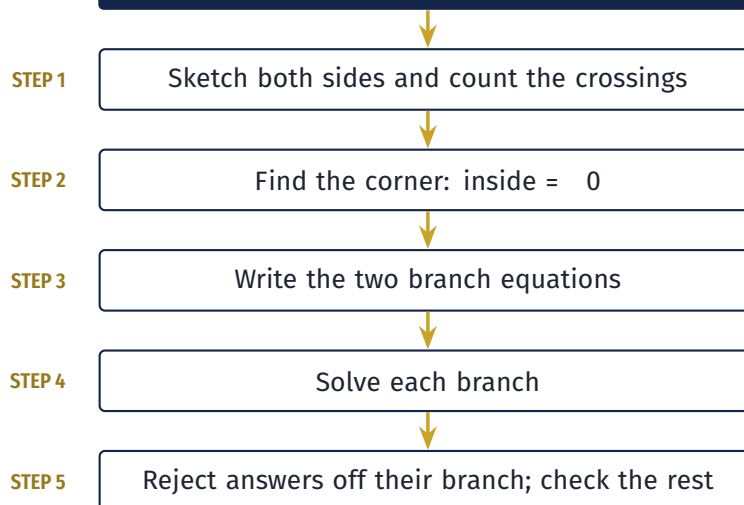


$$f(x) = |2x - 6| + 1, 0 \leq x \leq 5: \text{two roots only for } 1 < k \leq 5.$$

How do I solve $f(|x|) = c$?

- Substitute $u = |x|$. The equation becomes $f(u) = c$, with the rule $u \geq 0$.
- Solve for u and **reject** any negative value.
- Each positive u gives **two** answers, $x = u$ and $x = -u$; $u = 0$ gives only $x = 0$.

SOLVING ANY MODULUS EQUATION



Examiner Tips and Tricks

- If the question says "Solve", write the final answers clearly and **cross out or state** the rejected one with a reason: " $x = -3$ is not valid because $x \geq 3$ on this branch".
- With a parameter, e.g. $|3x - a| = |x + a|$ with $a > 0$, keep a in every answer and use $a > 0$ when you check.
- A root exactly at the corner comes out of **both** branches. List it once.

WHEN YOU SEE → YOU DO

“Solve the equation $|ax + b| = \dots$ ”

→ two branches, then **check** each answer against its branch

“ $|\dots| = |\dots|$ ”

→ inside = inside **or** inside = -inside (or square both sides)

“Given that the equation $f(x) = k$ has exactly two roots, find the range of values of k ”

→ sketch, slide $y = k$, check each boundary

“ $f(|x|) = 48$ ”

→ put $u = |x| \geq 0$, solve, then $x = \pm u$

 Your notes



Worked Example



Solve the equation $|x - 3| = 2x$.

$$x \geq 3 : x - 3 = 2x \Rightarrow x = -3$$

← RIGHT BRANCH; BUT -3 IS NOT ≥ 3 , SO REJECT

$$x < 3 : -(x - 3) = 2x \Rightarrow 3 = 3x \Rightarrow x = 1$$

← LEFT BRANCH; $1 < 3$ SO IT IS VALID

$$|1 - 3| = 2 = 2(1)$$

← CHECK IN THE ORIGINAL EQUATION

ANSWER $x = 1$

**Worked Example**

? Solve $|2x - 1| = |x + 4|$.

$$2x - 1 = x + 4 \Rightarrow x = 5$$

← SAME SIGN

$$2x - 1 = -(x + 4) \Rightarrow 3x = -3 \Rightarrow x = -1$$

← OPPOSITE SIGNS

$$|9| = |9|, \quad |-3| = |3|$$

← BOTH CHECK

ANSWER $x = 5$ or $x = -1$

**Worked Example**

? $f(x) = |2x - 6| + 1$, $0 \leq x \leq 5$. Given that the equation $f(x) = k$ has exactly two roots, find the range of values of k .

$$\text{corner } (3, 1), \quad f(0) = 7, \quad f(5) = 5$$

← THE KEY HEIGHTS: CORNER AND BOTH ENDS OF THE DOMAIN

$k = 1$: one root

← THE LINE JUST TOUCHES THE CORNER

$1 < k \leq 5$: two roots

← THE LINE CROSSES BOTH ARMS; $x = 5$ IS INCLUDED

$5 < k \leq 7$: one root

← THE RIGHT ARM STOPS AT HEIGHT 5

ANSWER $1 < k \leq 5$

**Worked Example**

? Solve $x^2 - 4 = |x - 2|$.

$$x \geq 2 : \quad x^2 - 4 = x - 2 \Rightarrow (x - 2)(x + 1) = 0$$

← $x = 2$ **VALID**; $x = -1$ **REJECTED**
(**NOT** ≥ 2)

$$x < 2 : \quad x^2 - 4 = 2 - x \Rightarrow (x + 3)(x - 2) = 0$$

← $x = -3$ **VALID**; $x = 2$ **ALREADY**
FOUND

ANSWER $x = 2$ or $x = -3$

**Worked Example**

? $f(x) = x^2 + 2x - 8$. Solve $f(|x|) = 0$.

$$u^2 + 2u - 8 = 0 \Rightarrow (u + 4)(u - 2) = 0$$

← **LET** $u = |x|$

$$u = -4 \text{ (reject),} \quad u = 2$$

← $|x|$ **CAN NEVER BE NEGATIVE**

ANSWER $x = 2$ or $x = -2$

HOW THIS IS EXAMINED · past WMA13 papers

$|ax + b| = a$ line **SEEN 5× IN EXAMS** $|f| = |g|$ **SEEN 3× IN EXAMS**

exactly n roots **SEEN 5× IN EXAMS** modulus = a curve **SEEN 2× IN EXAMS**

$f(|x|) = c$ **SEEN 2× IN EXAMS**

Solving Modulus Inequalities

NOTE 4.3

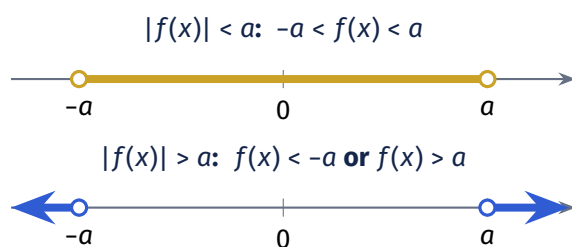


Solving Modulus Inequalities

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How do I solve $|f(x)| < a$ and $|f(x)| > a$?

- Think of the modulus as a **distance from 0**.
- $|f(x)| < a$ means $f(x)$ is within a of 0: one **sandwich** interval
 - $-a < f(x) < a$
- $|f(x)| > a$ means $f(x)$ is further than a from 0: **two separate pieces**
 - $f(x) < -a$ or $f(x) > a$
- With $\leq$ or $\geq$ the end values are **included**.



“Less than” stays inside; “greater than” goes outside.

e.g. **SOLVE** $|x - 2| < 5$

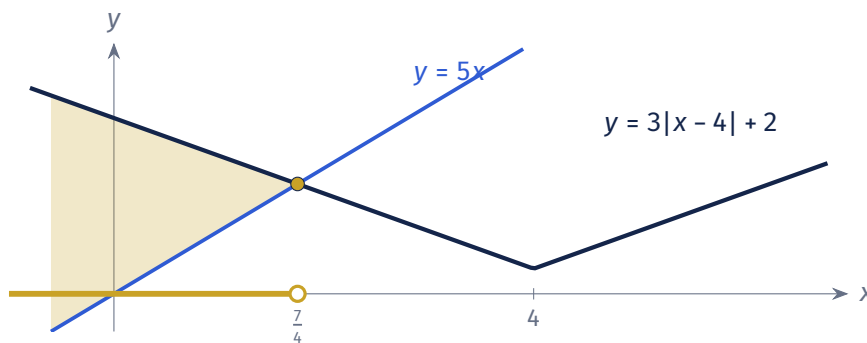
$$-5 < x - 2 < 5$$

← LESS THAN: ONE SANDWICH

ANSWER $-3 < x < 7$

How do I solve $|f(x)| > g(x)$ when $g(x)$ is not a constant?

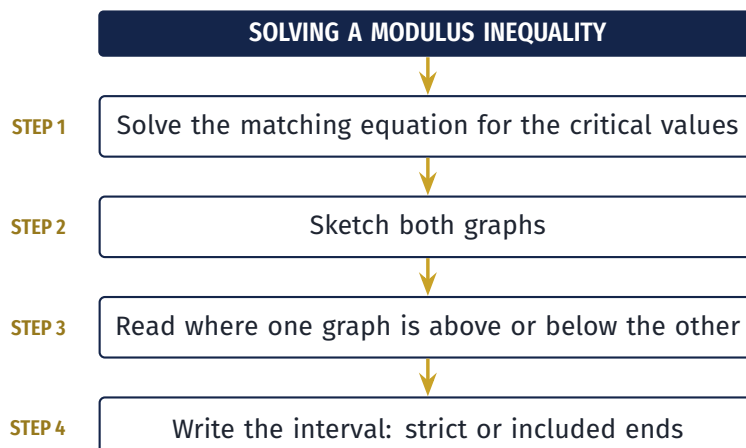
- Solve the matching **equation** $|f(x)| = g(x)$ first (see 4.2). Its solutions are the **critical values**.
- Sketch both graphs and read off where the modulus graph is **above** ($>$) or **below** ($<$) the other graph.
- Write the interval using the critical values: strict signs exclude them, $\leq$ and $\geq$ include them.



$3|x - 4| + 2 > 5x$ holds where the V is above the line: $x < \frac{7}{4}$.

What if the inside is a logarithm or a quadratic?

- Write down the **domain** first, e.g. $\ln x$ needs $x > 0$; $\ln(x + 1)$ needs $x > -1$.
- Remove the modulus with a sandwich ($<$) or two pieces ($>$), then solve each part.
- Dividing or multiplying by a **negative** number **reverses** every inequality sign.
- Finally intersect your answer with the domain.





Examiner Tips and Tricks

- Two separate pieces must be joined by “or”: write $x \leq -2$ or $x \geq 4$, never $-2 \geq x \geq 4$.
- “Find, in terms of k , the range of values of x ”: the critical values contain k , but the sketch still decides which side to take.
- Always sketch, even roughly. Choosing the side from algebra alone is the most common lost mark.



WHEN YOU SEE → YOU DO

“Solve the inequality $| \dots | > \dots$ ”

→ solve the **equation** first, then use the sketch to choose the side

“Find, in terms of k , the range of values of x that satisfy ...”

→ critical values in k ; the sketch decides the side

$|g(x)| < c$ ($c > 0$)

→ one **sandwich**: $-c < g(x) < c$

$|g(x)| > c$

→ **two pieces**: $g(x) < -c$ or $g(x) > c$



Worked Example



Solve the inequality $3|x - 4| + 2 > 5x$.

$$x \geq 4 : 3x - 12 + 2 = 5x \Rightarrow x = -5$$

← EQUATION FIRST; -5 IS NOT ≥ 4 , REJECT

$$x < 4 : 12 - 3x + 2 = 5x \Rightarrow x = \frac{7}{4}$$

← THE ONLY CRITICAL VALUE

V above the line for $x < \frac{7}{4}$

← READ THE SIDE FROM THE SKETCH

ANSWER $x < \frac{7}{4}$

**Worked Example**

? Solve (a) $|2x + 1| \leq 5$ (b) $|x - 1| \geq 3$.

$$-5 \leq 2x + 1 \leq 5 \Rightarrow -6 \leq 2x \leq 4$$

← (A) "LESS THAN": ONE SANDWICH; SUBTRACT 1 FROM ALL THREE PARTS

$$x - 1 \leq -3 \quad \text{or} \quad x - 1 \geq 3$$

← (B) "GREATER THAN": TWO PIECES

ANSWER (a) $-3 \leq x \leq 2$ (b) $x \leq -2$ or $x \geq 4$

**Worked Example**

? Solve $|3 - 2 \ln x| < 1$, $x > 0$.

$$-1 < 3 - 2 \ln x < 1$$

← SANDWICH

$$-4 < -2 \ln x < -2$$

← SUBTRACT 3 FROM ALL THREE PARTS

$$2 > \ln x > 1 \Rightarrow 1 < \ln x < 2$$

← DIVIDE BY -2: BOTH SIGNS REVERSE

$$e < x < e^2$$

← APPLY $e^{(\cdot)}$; THIS IS INSIDE $x > 0$

ANSWER $e < x < e^2$

HOW THIS IS EXAMINED • past WMA13 papers

$|ax + b|$ vs a line **SEEN 6× IN EXAMS** $<$ or $\leq$ ends **NOT SEEN YET • BE READY**

$\ln$ or quadratic inside **SEEN 2× IN EXAMS**

MISTAKE GALLERY

Where marks are lost

× **WRONG**

For $y = |f(x)|$, reflecting the left half of the graph.

✓ **RIGHT**

Only the parts **below** the x-axis are folded up.

$|f(x)|$ changes y-values, not x-values.

× **WRONG**

$|x - 3| = 2x \Rightarrow x = -3 \text{ or } x = 1.$

✓ **RIGHT**

$x = -3$ came from the branch $x \geq 3$, so it is rejected: $x = 1$ only.

Every branch answer must lie on its own branch.

× **WRONG**

y-intercept of $y = |3x - a|$ is $(0, -a).$

✓ **RIGHT**

It is $(0, a)$: a modulus is never negative.

Put $x = 0$ *inside* the modulus, then take the size.

× **WRONG**

$|x - 1| \geq 3 \Rightarrow -2 \geq x \geq 4.$

✓ **RIGHT**

$x \leq -2 \text{ or } x \geq 4.$

"Greater than" always splits into two pieces.

× **WRONG**

$|a + b| = |a| + |b|.$

✓ **RIGHT**

Only true when a and b have the same sign.

Use branches instead.

THE CHAPTER ON ONE PAGE

Modulus**4.1****Sketch**

$|f(x)|$: fold below-axis parts up.
 $f(|x|)$: mirror the right half.
 $a|x+p|+q$: corner $(-p, q)$; y-intercept: inside first.
 Label every corner and intercept.

4.2**Equations**

Split at the corner. Solve each branch.
Reject answers off their branch or giving a negative modulus.
 $|A| = |B| \Rightarrow A = \pm B$.
 $f(|x|) = c$: $u = |x| \geq 0$, then $x = \pm u$.

4.2**Exactly n roots**

Sketch on the domain, slide $y = k$.
 Corner height: 1 root. Check each end point.

4.3**Inequalities**

Solve the equation first, then read the side from the graph.
 $|g| < c$: $-c < g < c$. $|g| > c$: $g < -c$ or $g > c$.
 Strict sign: ends not included.

BOTTOM LINE Find the corner, split into two straight branches, and keep only the answers the graph allows.



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05

CHAPTER 05

Trigonometric Functions and Graphs

Three new functions are just reciprocals, and three more are inverses on a restricted piece of the curve. Know their graphs and you know their values.

2

CONCEPTS

6

VARIANTS

1

EXAM QUESTIONS

YOU WILL MASTER

- ▶ Define and evaluate sec, cosec and cot exactly, in degrees and radians
- ▶ Sketch their graphs with asymptotes, turning points and range
- ▶ Use arcsin, arccos and arctan with their restricted domains and ranges
- ▶ Find exact values such as $\tan(\arcsin \frac{3}{5})$ using a right-angled triangle

CHAPTER 05 AT A GLANCE

Two note pages, six ways to be asked



CHAPTER 05 MAP

Trigonometric Functions

01 — 5.1 Reciprocal trig functions

$$\sec x = \frac{1}{\cos x}$$

$$\operatorname{cosec} x = \frac{1}{\sin x}$$

$$\cot x = \frac{1}{\tan x}$$

graphs and asymptotes

02 — 5.2 Inverse trig functions

arcsin x

arccos x

arctan x

exact values via triangles

Contents

✱ 5.1	Reciprocal Trigonometric Functions	3
✱ 5.2	Inverse Trigonometric Functions	7

Reciprocal Trigonometric Functions

NOTE 5.1



Reciprocal Trigonometric Functions

PURE 3 > TRIGONOMETRY > RECIPROCAL AND INVERSE FUNCTIONS • Exam code WMA13 • Dr Eslam Ahmed

What are sec, cosec and cot?

- They are the **reciprocals** of the three basic trig functions:

RECIPROCAL TRIG FUNCTIONS

LEARN THIS: NOT GIVEN

$$\sec x = \frac{1}{\cos x} \quad \operatorname{cosec} x = \frac{1}{\sin x} \quad \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

- Memory hook: look at the **third letter**. **sec** ↔ **cos**, **cosec** ↔ **sin**, **cot** ↔ **tan**.
- Your calculator has no sec, cosec or cot button: type $1 \div \cos$, $1 \div \sin$ or $1 \div \tan$.
- Each one is **undefined** wherever its denominator is zero.
 - $\sec x$ is undefined when $\cos x = 0$: $x = 90^\circ, 270^\circ, \dots \left(\frac{\pi}{2}, \frac{3\pi}{2}, \dots\right)$
 - $\operatorname{cosec} x$ and $\cot x$ are undefined when $\sin x = 0$: $x = 0^\circ, 180^\circ, 360^\circ, \dots$

How do I find exact values?

- Find the exact value of $\sin$, $\cos$ or $\tan$ first (with its quadrant sign), then take the **reciprocal**.
- Rationalise if needed: $\frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$.

e.g. **FIND THE EXACT VALUE OF** $\operatorname{cosec} 240^\circ$

$$\sin 240^\circ = -\sin 60^\circ = -\frac{\sqrt{3}}{2}$$

← THIRD QUADRANT: SINE IS NEGATIVE

$$\operatorname{cosec} 240^\circ = -\frac{2}{\sqrt{3}}$$

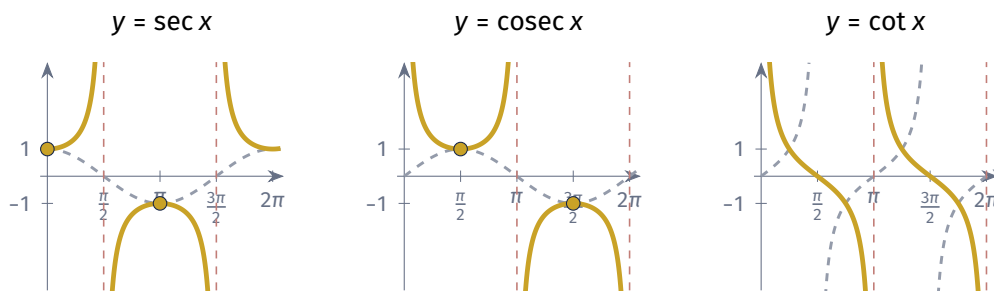
← TAKE THE RECIPROCAL

ANSWER $-\frac{2\sqrt{3}}{3}$

What do their graphs look like?



- Where $\cos x = \pm 1$, $\sec x = \pm 1$: these are the **turning points** of $y = \sec x$.
- Where $\cos x = 0$, $\sec x$ has a vertical **asymptote**.
- As $\cos x$ gets small, $\sec x$ gets large: the curve **bends away** from the axis.
- Each graph has the **same period** as its basic function: 360° for $\sec$ and cosec , 180° for $\cot$.



Dashed: the basic function. Red dashed lines: asymptotes where the basic function is zero.

GRAPH	ASYMPTOTES	PERIOD	RANGE
$\sec x$	$x = 90^\circ, 270^\circ, \dots$	360°	$y \leq -1$ or $y \geq 1$
$\operatorname{cosec} x$	$x = 0^\circ, 180^\circ, 360^\circ, \dots$	360°	$y \leq -1$ or $y \geq 1$
$\cot x$	$x = 0^\circ, 180^\circ, 360^\circ, \dots$	180°	all real y



Examiner Tips and Tricks

- On a sketch, draw the asymptotes as dashed lines and label them with their equations, e.g. $x = \frac{\pi}{2}$.
- $\sec x$ and $\operatorname{cosec} x$ can never be between -1 and 1 . An equation such as $\sec x = 0.5$ has **no solutions**.
- Transformations still work: $y = 3 + 2 \operatorname{cosec} x$ has range $y \leq 1$ or $y \geq 5$.

WHEN YOU SEE → YOU DO

“Find the exact value of $\sec \frac{5\pi}{6}$ ”

→ exact cos with its quadrant sign, then the reciprocal; rationalise

“Sketch $y = \operatorname{cosec} x$... state the equations of the asymptotes”→ asymptotes where $\sin x = 0$; turning points at ± 1 **“Solve $\operatorname{cosec} x = 3$ ”**→ turn it over: $\sin x = \frac{1}{3}$, then solve as usual**“State the range of $y = a + b \sec x$ ”**→ start from $\sec x \leq -1$ or ≥ 1 , then transform**Worked Example**Find the exact values of (a) $\sec \frac{5\pi}{6}$ (b) $\cot \frac{\pi}{3}$ (c) $\operatorname{cosec} \frac{7\pi}{4}$.

$$\cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2} \Rightarrow \sec \frac{5\pi}{6} = -\frac{2}{\sqrt{3}}$$

← (A) SECOND QUADRANT: COS NEGATIVE

$$\tan \frac{\pi}{3} = \sqrt{3} \Rightarrow \cot \frac{\pi}{3} = \frac{1}{\sqrt{3}}$$

← (B) RECIPROCAL OF TAN

$$\sin \frac{7\pi}{4} = -\frac{\sqrt{2}}{2} \Rightarrow \operatorname{cosec} \frac{7\pi}{4} = -\frac{2}{\sqrt{2}}$$

← (C) FOURTH QUADRANT: SIN NEGATIVE

ANSWER (a) $-\frac{2\sqrt{3}}{3}$ (b) $\frac{\sqrt{3}}{3}$ (c) $-\sqrt{2}$



Worked Example



Sketch $y = 3 + 2 \sec x$ for $0 \leq x \leq 2\pi$. State the equations of the asymptotes, the coordinates of the turning points and the range.

$$\cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$$

← ASYMPTOTES DO NOT MOVE UNDER OUTSIDE CHANGES

$$\sec 0 = 1 \rightarrow y = 5; \quad \sec \pi = -1 \rightarrow y = 1$$

← TURNING POINTS: APPLY $3 + 2(\)$ TO ± 1

$$\sec x \geq 1 \Rightarrow y \geq 5; \quad \sec x \leq -1 \Rightarrow y \leq 1$$

← TRANSFORM THE RANGE OF $\sec$

ANSWER asymptotes $x = \frac{\pi}{2}, x = \frac{3\pi}{2}$; minima $(0, 5), (2\pi, 5)$; maximum $(\pi, 1)$; range $y \leq 1$ or $y \geq 5$

HOW THIS IS EXAMINED • past WMA13 papers

$\sec$ NOT SEEN YET • BE READY cosec NOT SEEN YET • BE READY $\cot$ NOT SEEN YET • BE READY



Your notes

Inverse Trigonometric Functions

NOTE 5.2



Inverse Trigonometric Functions

PURE 3 > TRIGONOMETRY > RECIPROCAL AND INVERSE FUNCTIONS • Exam code WMA13 • Dr Eslam Ahmed

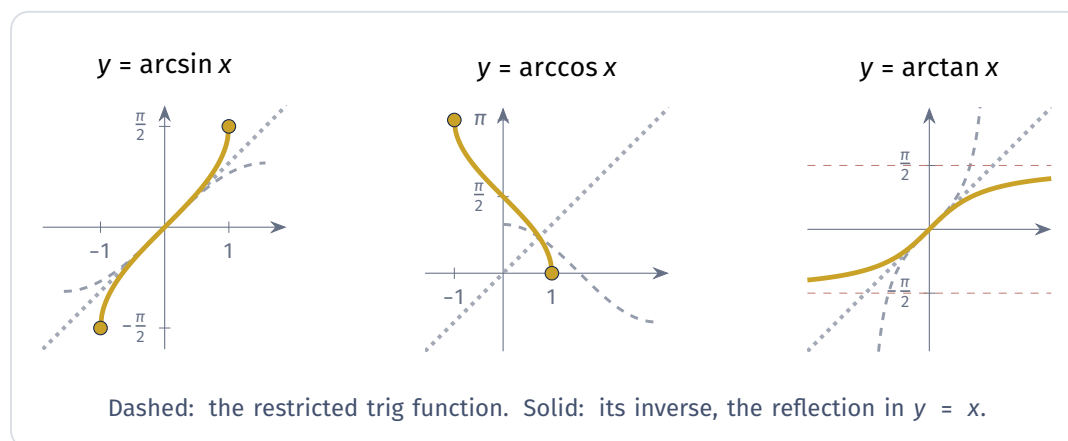
What are arcsin, arccos and arctan?

- They are the **inverse functions** of sin, cos and tan: $\arcsin x$ is the angle whose sine is x .
- $\sin x$ is **many-one**, so it has no inverse on its whole domain. It is **restricted** to a one-one piece first (see 2.3).
- The output is always in **radians** and always in the **principal range**:

FUNCTION	DOMAIN (INPUTS)	RANGE (OUTPUT ANGLES)
$\arcsin x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq \arcsin x \leq \frac{\pi}{2}$
$\arccos x$	$-1 \leq x \leq 1$	$0 \leq \arccos x \leq \pi$
$\arctan x$	all real x	$-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$

What do their graphs look like?

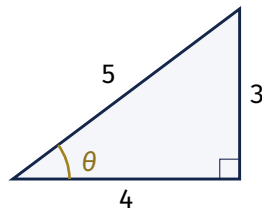
- Each graph is the **reflection in $y = x$** of the restricted trig graph.
- $\arcsin$ and $\arctan$ pass through the **origin** and are increasing; $\arccos$ is decreasing and passes through $(0, \frac{\pi}{2})$.
- $\arctan x$ has horizontal asymptotes $y = \pm \frac{\pi}{2}$.



How do I find exact values?



- Use the exact-value table, but choose the angle **inside the principal range**.
 - $\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$ (in $[0, \pi]$), $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$ (not $\frac{7\pi}{6}$)
- For expressions such as $\tan\left(\arcsin\frac{3}{5}\right)$: let $\theta = \arcsin\frac{3}{5}$, draw a **right-angled triangle** with $\sin\theta = \frac{3}{5}$, and read off the side you need.



$$\sin\theta = \frac{3}{5} \text{ (GIVEN)}$$

$$\cos\theta = \frac{4}{5}, \tan\theta = \frac{3}{4}$$

MISSING SIDE BY PYTHAGORAS

$\theta = \arcsin\frac{3}{5}$: the 3-4-5 triangle gives every other ratio of θ .



Examiner Tips and Tricks

- Your calculator must be in **radian** mode for these questions; answers are angles in radians unless stated otherwise.
- “Sketch C” for $y = \arcsin\frac{x}{2}$: mark both end points, $(-2, -\frac{\pi}{2})$ and $(2, \frac{\pi}{2})$. The graph **stops** there.
- $\arcsin x$ is often written $\sin^{-1} x$. It is **not** $\frac{1}{\sin x}$, which is $\operatorname{cosec} x$.

WHEN YOU SEE → YOU DO

“ $f(x) = \arcsin\frac{x}{2}$, $-2 \leq x \leq 2$... Sketch C”

→ stretched $\arcsin$: end points $(\pm 2, \pm \frac{\pi}{2})$, through 0

“Given $x = 2 \sin y$, show that $\frac{dy}{dx} = \dots$ ”

→ rewrite $y = \arcsin\frac{x}{2}$ as $x = 2 \sin y$ and differentiate (C10)

“Find the exact value of $\cos\left(\arctan\frac{3}{4}\right)$ ”

→ right-angled triangle; read the ratio

“Solve $\arcsin(2x - 1) = \frac{\pi}{6}$ ”

→ take $\sin$ of both sides

**Worked Example**

The curve C has equation $y = \arcsin \frac{x}{2}$, $-2 \leq x \leq 2$. (a) Sketch C .
 (b) Find the exact x -coordinate of the point on C with $y = \frac{\pi}{4}$.

$$x = \pm 2 \Rightarrow y = \arcsin(\pm 1) = \pm \frac{\pi}{2}$$

← (A) END POINTS OF THE GRAPH

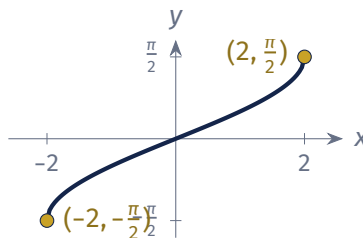
$$x = 0 \Rightarrow y = 0$$

← PASSES THROUGH THE ORIGIN, INCREASING

$$\sin \frac{\pi}{4} = \frac{x}{2} \Rightarrow x = 2 \times \frac{\sqrt{2}}{2}$$

← (B) TAKE sin OF BOTH SIDES

ANSWER (a) increasing curve from $(-2, -\frac{\pi}{2})$ through O to $(2, \frac{\pi}{2})$ (b)
 $x = \sqrt{2}$

**Worked Example**

Find the exact values of (a) $\arccos\left(-\frac{\sqrt{3}}{2}\right)$ (b) $\sin\left(2 \arctan \frac{3}{4}\right)$.

$$\cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2} \text{ and } \frac{5\pi}{6} \in [0, \pi]$$

← (A) CHOOSE THE ANGLE IN THE PRINCIPAL RANGE

$$\theta = \arctan \frac{3}{4} : \sin \theta = \frac{3}{5}, \cos \theta = \frac{4}{5}$$

← (B) 3--4--5 TRIANGLE

$$\sin 2\theta = 2 \sin \theta \cos \theta = 2 \times \frac{3}{5} \times \frac{4}{5}$$

← DOUBLE ANGLE FORMULA (C06)

ANSWER (a) $\frac{5\pi}{6}$ (b) $\frac{24}{25}$

HOW THIS IS EXAMINED • past WMA13 papers $\arcsin$ **SEEN 1× IN EXAMS** $\arccos$ **NOT SEEN YET • BE READY** $\arctan$ **NOT SEEN YET • BE READY**

Your notes

MISTAKE GALLERY

Where marks are lost

× **WRONG**

$$\sec x = \frac{1}{\sin x}.$$

✓ **RIGHT**

$$\sec x = \frac{1}{\cos x}.$$

Third letter: sec pairs with cos.

× **WRONG**Solving $\sec x = 0.5$ and getting answers.✓ **RIGHT**No solutions: $|\sec x| \geq 1$.
 $\cos x = 2$ is impossible.× **WRONG**

$$\arcsin\left(-\frac{1}{2}\right) = \frac{7\pi}{6}.$$

✓ **RIGHT** $-\frac{\pi}{6}$: the answer must be in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
Inverse trig gives the principal value only.× **WRONG**

$$\sin^{-1} x = \frac{1}{\sin x}.$$

✓ **RIGHT**

$$\sin^{-1} x = \arcsin x; \quad \frac{1}{\sin x} = \operatorname{cosec} x.$$

The -1 means inverse function.

× **WRONG**Sketching $y = \arcsin x$ beyond $x = \pm 1$.✓ **RIGHT**The graph stops at $(\pm 1, \pm \frac{\pi}{2})$.
The domain is $-1 \leq x \leq 1$.

THE CHAPTER ON ONE PAGE

Trigonometric Functions



5.1

Reciprocals

$$\sec = \frac{1}{\cos}, \operatorname{cosec} = \frac{1}{\sin}, \cot = \frac{1}{\tan}.$$

Undefined where the denominator is 0.

Exact value: basic ratio with sign, then flip.

5.1

Graphs

Asymptotes where the basic function is 0.

Turning points at ± 1 .

$|\sec|, |\operatorname{cosec}| \geq 1$; $\cot$ takes every value.

5.2

Inverses

$\arcsin, \arccos$: inputs $[-1, 1]$; $\arctan$: all inputs.

Outputs: $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $[0, \pi]$, $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Graph: reflect in $y = x$.

5.2

Exact values

Pick the angle in the principal range.
 $\operatorname{trig}(\arcsin \dots)$: draw the right-angled triangle.

BOTTOM LINE Reciprocal functions flip the values and add asymptotes; inverse functions reverse a restricted piece and always answer in the principal range.



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06

CHAPTER 06

Trigonometric Identities and Exact Values

Every identity in this chapter comes from two facts: $\sin^2 + \cos^2 = 1$ and the addition formulae. Learn where they come from and you can rebuild any of them in the exam.

3

CONCEPTS

11

VARIANTS

22

EXAM QUESTIONS

YOU WILL MASTER

- ▶ Use $\sec^2 x = 1 + \tan^2 x$ and $\operatorname{cosec}^2 x = 1 + \cot^2 x$ to simplify and substitute
- ▶ Expand $\sin(A \pm B)$, $\cos(A \pm B)$, $\tan(A \pm B)$ and find exact values with the right signs
- ▶ Choose the right form of $\cos 2A$ every time
- ▶ Write clear, one-directional proofs of trigonometric identities

CHAPTER 06 AT A GLANCE

Four note pages, eleven ways to be asked



CHAPTER 06 MAP

Trigonometric Identities

01

6.1 Further identities

$$\sec^2 = 1 + \tan^2$$

$$\operatorname{cosec}^2 = 1 + \cot^2$$

 $t = \tan x$ substitution

02

6.2 Compound angle formulae

$$\sin(A \pm B), \cos(A \pm B)$$

$$\tan(A \pm B)$$

exact values with quadrant signs

03

6.3 Double angle formulae

$$\sin 2A, \tan 2A$$

three forms of $\cos 2A$

half angles

04

6.4 Trigonometric proof

compound and double angle proofs

reciprocal identity proofs

Contents

✱ 6.1	Further Trigonometric Identities	3
✱ 6.2	Compound Angle Formulae	7
✱ 6.3	Double Angle Formulae	10
✱ 6.4	Trigonometric Proof	13

Further Trigonometric Identities

NOTE 6.1

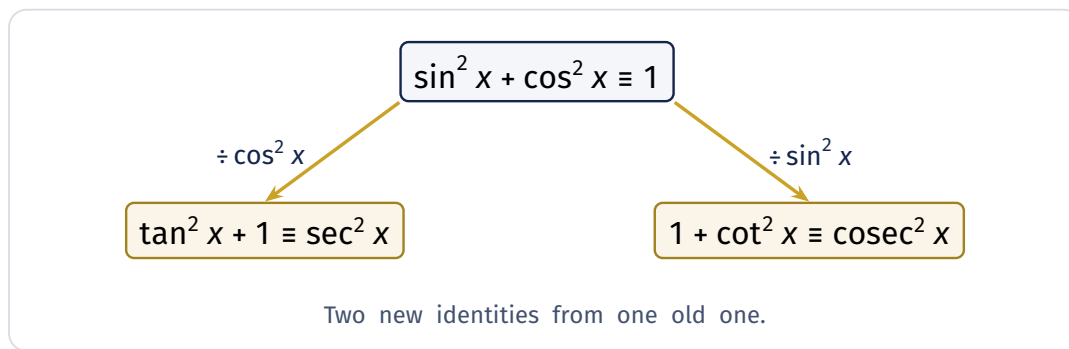


Further Trigonometric Identities

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Where do the new Pythagorean identities come from?

- Start from the identity you already know: $\sin^2 x + \cos^2 x \equiv 1$.
- Divide every term** by $\cos^2 x$ or by $\sin^2 x$:



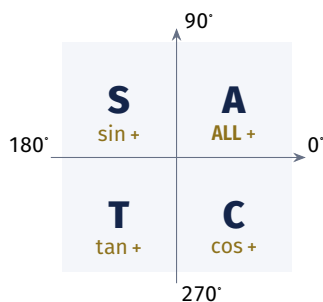
PYTHAGOREAN IDENTITIES

LEARN THIS: NOT GIVEN

$$\sec^2 x \equiv 1 + \tan^2 x \quad \text{cosec}^2 x \equiv 1 + \cot^2 x$$

How do I use them?

- To **simplify**: replace $\sec^2 x - 1$ by $\tan^2 x$, $\text{cosec}^2 x - 1$ by $\cot^2 x$.
- To make an equation contain **one** trig function: an equation with $\sec^2 x$ and $\tan x$ becomes a quadratic in $\tan x$.
- To find one ratio from another: from $\tan x$ you get $\sec^2 x$, then $\sec x$ with the **correct sign** for the quadrant.
- The substitution $t = \tan x$ turns many mixed expressions into algebra: $\sec^2 x = 1 + t^2$, $\cot x = \frac{1}{t}$, $\tan 2x = \frac{2t}{1 - t^2}$.



Which ratios are positive in each quadrant (CAST). Use it whenever a square root appears.



e.g. SIMPLIFY $\frac{\sec^2 x - 1}{\operatorname{cosec}^2 x - 1}$

$$\frac{\tan^2 x}{\cot^2 x}$$

$$\tan^2 x \times \tan^2 x$$

USE BOTH PYTHAGOREAN IDENTITIES

DIVIDING BY $\cot^2 x$ IS MULTIPLYING BY $\tan^2 x$

ANSWER $\tan^4 x$



Examiner Tips and Tricks

- These two identities are **not** in the formula booklet: learn them, or rebuild them by dividing $\sin^2 + \cos^2 = 1$.
- “Use the substitution $t = \tan x$ to show that ...” means: write **every** term in terms of t , then clear the fractions.
- After squaring or using $\sec^2$, always choose the sign of the square root using the quadrant.

WHEN YOU SEE → YOU DO

$\sec^2 x$ together with $\tan x$ in one expression

→ $\sec^2 x = 1 + \tan^2 x$: everything in $\tan x$

$\operatorname{cosec}^2 x$ together with $\cot x$

→ $\operatorname{cosec}^2 x = 1 + \cot^2 x$: everything in $\cot x$

“Use the substitution $t = \tan x$ to show that the equation ... can be written in the form ...”

→ replace each ratio by its t -form, multiply out the denominators

“Given $\tan x = -\frac{3}{4}$ and x is obtuse, find the exact value of $\sec x$ ”

→ $\sec^2 = 1 + \tan^2$, then the sign from CAST



Worked Example



Given that $\tan x = -\frac{3}{4}$ and $90^\circ < x < 180^\circ$, find the exact value of $\sec x$ and of $\operatorname{cosec} x$.

$$\sec^2 x = 1 + \frac{9}{16} = \frac{25}{16}$$

← PYTHAGOREAN IDENTITY

$$\sec x = -\frac{5}{4}$$

← SECOND QUADRANT: $\cos$ IS NEGATIVE, SO $\sec$ IS TOO

$$\operatorname{cosec}^2 x = 1 + \cot^2 x = 1 + \frac{16}{9} = \frac{25}{9} \Rightarrow$$

$$\operatorname{cosec} x = \frac{5}{3}$$

← SECOND QUADRANT: $\sin$ IS POSITIVE

ANSWER $\sec x = -\frac{5}{4}, \operatorname{cosec} x = \frac{5}{3}$



Worked Example



(a) Use the substitution $t = \tan x$ to show that the equation $2 \tan 2x + \cot x \sec^2 x = 0$ can be written as $t^4 - 4t^2 - 1 = 0$. (b) Hence solve the equation for $0 \leq x < 360^\circ$, to one decimal place.

$$\frac{4t}{1-t^2} + \frac{1+t^2}{t} = 0$$

$$4t^2 + (1+t^2)(1-t^2) = 0 \Rightarrow t^4 - 4t^2 - 1 = 0$$

$$t^2 = 2 + \sqrt{5} \quad (t^2 = 2 - \sqrt{5} < 0 \text{ rejected})$$

$$\tan x = \pm 2.058 \dots$$

← (A) $\tan 2x = \frac{2t}{1-t^2}$, $\cot x = \frac{1}{t}$,
 $\sec^2 x = 1 + t^2$

← MULTIPLY BY $t(1-t^2)$ AND REARRANGE

← (B) QUADRATIC IN t^2

← BOTH SIGNS: FOUR ANGLES IN 0° TO 360°

ANSWER $x = 64.1^\circ, 115.9^\circ, 244.1^\circ, 295.9^\circ$



Your notes

HOW THIS IS EXAMINED • past WMA13 papers

reciprocal Pythagorean **SEEN 1× IN EXAMS**

mixed reciprocal and double angle **SEEN 1× IN EXAMS**

Compound Angle Formulae

NOTE 6.2



Compound Angle Formulae

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What are the compound angle formulae?

- They expand the trig of a **sum or difference** of two angles. They are in the formula booklet:

COMPOUND ANGLE FORMULAE

IN THE FORMULA BOOKLET

$$\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

- $\sin(A + B) \neq \sin A + \sin B$. You must use the formula.
- Sign pattern: sin keeps the **same sign**; cos **changes the sign** ($\cos(A + B)$ has a minus).

$$\text{sin: } \sin A \cos B \pm \cos A \sin B$$

SAME SIGN AS THE BRACKET

$$\text{cos: } \cos A \cos B \mp \sin A \sin B$$

OPPOSITE SIGN TO THE BRACKET

Read the signs from the formula, not from memory.

How do I find exact values with them?

- Split the angle into two exact angles: $75^\circ = 45^\circ + 30^\circ$, $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$.
- If you are given $\sin A$ only, find $\cos A$ with a **right-angled triangle** (or $\sin^2 + \cos^2 = 1$), then fix its **sign** from the quadrant.



e.g. **FIND THE EXACT VALUE OF** $\sin 75^\circ$

$$\sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$

← SPLIT INTO EXACT ANGLES

← EXACT VALUES

ANSWER $\sin 75^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}$



Examiner Tips and Tricks

- In equations such as $\sin(x + 30^\circ) = 2 \cos x$: expand, then divide by $\cos x$ to get an equation in $\tan x$.
- $\tan(x + 45^\circ) = \frac{\tan x + 1}{1 - \tan x}$, because $\tan 45^\circ = 1$. This appears often.
- Write the expanded formula with the actual angles **before** substituting numbers: it is the method mark.

WHEN YOU SEE → YOU DO

“Find the exact value of $\cos 15^\circ$ ” / “of $\tan \frac{7\pi}{12}$ ”

→ split into $45^\circ, 30^\circ, 60^\circ$ (or $\frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{3}$)

“Given $\sin A = \frac{3}{5}$... B is obtuse ... find $\cos(A + B)$ ”

→ triangle for the missing ratios; CAST for their signs

“Using the identity for $\cos(A + B)$, prove that ...”

→ put $B = A$ (see 6.4)

$\sin(x + \alpha)$ or $\cos(x - \alpha)$ inside an equation

→ expand and collect $\sin x$ and $\cos x$

**Worked Example**

$\sin A = \frac{3}{5}$ where A is acute, and $\cos B = -\frac{5}{13}$ where B is obtuse.
Find the exact value of $\cos(A + B)$.

$$\cos A = \frac{4}{5}$$

← 3--4--5 TRIANGLE; A ACUTE SO POSITIVE

$$\sin B = \frac{12}{13}$$

← 5--12--13 TRIANGLE; B OBTUSE SO $\sin B > 0$

$$\cos(A + B) = \frac{4}{5} \left(-\frac{5}{13}\right) - \frac{3}{5} \cdot \frac{12}{13}$$

← $\cos A \cos B - \sin A \sin B$

ANSWER $\cos(A + B) = -\frac{56}{65}$

**Worked Example**

Given that $\tan(x + 45^\circ) = 5$, find the exact value of $\tan x$.

$$\frac{\tan x + 1}{1 - \tan x} = 5$$

← $\tan(A + B)$ WITH $\tan 45^\circ = 1$

$$\tan x + 1 = 5 - 5 \tan x \Rightarrow 6 \tan x = 4$$

← CROSS-MULTIPLY AND COLLECT

ANSWER $\tan x = \frac{2}{3}$

HOW THIS IS EXAMINED • past WMA13 papers

compound angle identity **SEEN 2× IN EXAMS**

exact compound value **NOT SEEN YET • BE READY**

standard exact angle **SEEN 1× IN EXAMS**

Double Angle Formulae

NOTE 6.3



Double Angle Formulae

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What are the double angle formulae?

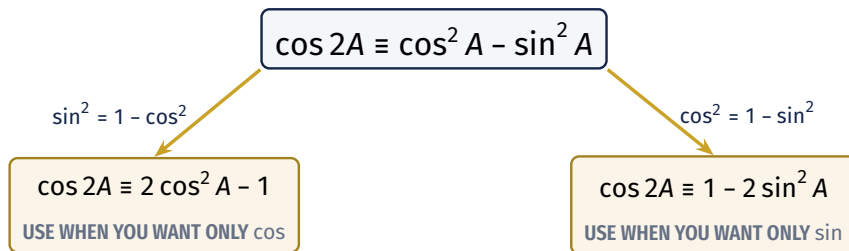
- Put $B = A$ in the compound angle formulae:

DOUBLE ANGLE FORMULAE

LEARN THIS: NOT GIVEN

$$\sin 2A \equiv 2 \sin A \cos A \quad \tan 2A \equiv \frac{2 \tan A}{1 - \tan^2 A}$$

- $\cos 2A$ has **three forms**. Choose the one that leaves only one trig function in your expression:

The three forms of $\cos 2A$ and when to use each.

How do I use double angles backwards (half angles)?

- Rearranging the $\cos 2A$ forms gives the **power-reduction** identities, used for integration (C12):
 - $\cos^2 A \equiv \frac{1 + \cos 2A}{2} \quad \sin^2 A \equiv \frac{1 - \cos 2A}{2}$
- With $A = \frac{\theta}{2}$ they give **half-angle** values: $\cos^2 \frac{\theta}{2} = \frac{1 + \cos \theta}{2}$.
- Choose the sign of the square root from the quadrant of the **half** angle.
- Also: $2 \sin A \cos A = \sin 2A$, so $\sin A \cos A = \frac{1}{2} \sin 2A$.



e.g. FIND THE EXACT VALUE OF $\cos 15^\circ$ USING A DOUBLE ANGLE FORMULA

$$\cos 30^\circ = 2 \cos^2 15^\circ - 1$$

←

$$\cos^2 15^\circ = \frac{1 + \frac{\sqrt{3}}{2}}{2} = \frac{2 + \sqrt{3}}{4}$$

←

ANSWER $\cos 15^\circ = \frac{\sqrt{2 + \sqrt{3}}}{2}$ (positive: 15° is acute)



Examiner Tips and Tricks

- An equation containing both $\cos 2x$ and $\sin x$: use $1 - 2 \sin^2 x$ to get a quadratic in $\sin x$. With $\cos x$: use $2 \cos^2 x - 1$.
- An equation containing $\sin 2x$ and $\sin x$ (or $\cos x$): write $\sin 2x = 2 \sin x \cos x$ and **factorise**. Never divide by $\sin x$ or $\cos x$: you lose solutions.
- $\cos 2A \equiv 2 \cos^2 A - 1$ is **not** in the formula booklet in this form; it is a common “prove” question (6.4).

WHEN YOU SEE → YOU DO

an expression with $2x$ and x together

→ double angle formula to make every angle x

“ $\cos 2x$ ” with $\sin x$ (or $\cos x$)

→ choose $1 - 2 \sin^2 x$ (or $2 \cos^2 x - 1$)

“Given $\cos A = \frac{1}{3}$, $\frac{3\pi}{2} < A < 2\pi$, find the exact value of $\sin 2A$ ”

→ missing ratio with its quadrant sign, then the formula

$\int \cos^2 x \, dx$ or $\int \sin^2 x \, dx$

→ power-reduction identity first (C12)



Worked Example



Given that $\cos A = \frac{1}{3}$ and $\frac{3\pi}{2} < A < 2\pi$, find the exact values of $\sin 2A$, $\cos 2A$ and $\tan 2A$.

$$\sin A = -\sqrt{1 - \frac{1}{9}} = -\frac{2\sqrt{2}}{3}$$

← FOURTH QUADRANT: sin **NEGA-TIVE**

$$\sin 2A = 2 \left(-\frac{2\sqrt{2}}{3} \right) \left(\frac{1}{3} \right) = -\frac{4\sqrt{2}}{9}$$

← $2 \sin A \cos A$

$$\cos 2A = 2 \left(\frac{1}{9} \right) - 1 = -\frac{7}{9}$$

← $2 \cos^2 A - 1$: **ONLY** cos **NEEDED**

$$\tan 2A = \frac{-4\sqrt{2}/9}{-7/9}$$

← $\frac{\sin 2A}{\cos 2A}$

ANSWER $\sin 2A = -\frac{4\sqrt{2}}{9}$, $\cos 2A = -\frac{7}{9}$, $\tan 2A = \frac{4\sqrt{2}}{7}$



Your notes

HOW THIS IS EXAMINED • past WMA13 papers

double angle identity **SEEN 5× IN EXAMS** half angle **NOT SEEN YET • BE READY**

exact double angle value **NOT SEEN YET • BE READY** half angle sign **NOT SEEN YET • BE READY**

Trigonometric Proof

NOTE 6.4

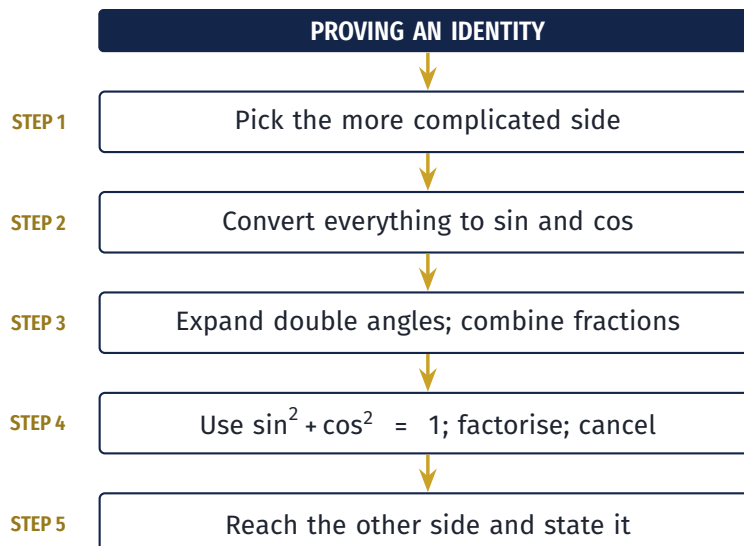


Trigonometric Proof

PURE 3 > TRIGONOMETRY > IDENTITIES • Exam code WMA13 • Dr Eslam Ahmed

How do I prove a trigonometric identity?

- Start with **one side** (usually the more complicated one) and transform it, step by step, until it becomes the other side.
- **Never** work on both sides at once, and never start from the identity you are proving.
- Useful moves:
 - write sec, cosec, cot, tan in terms of sin and cos
 - expand double or compound angles
 - combine fractions over a common denominator
 - use $\sin^2 + \cos^2 = 1$ (and its rearrangements) to replace squares
 - factorise, especially a difference of two squares
- Finish by writing the target expression, then “ $\equiv$ RHS” or “as required”.





Examiner Tips and Tricks

- Show **every** step: a proof that jumps two lines loses the final mark.
- Use the identity sign $\equiv$ in your working, or “LHS = ...”.
- “Hence” after a proof: the next part is usually an equation that becomes easy using what you proved (C07).

 Your notes

WHEN YOU SEE → YOU DO

“Prove that ...” / “Show that ... $\equiv$...”

→ one side only; convert to sin and cos

“Using the identity for $\cos(A + B)$, prove that $\cos 2A \equiv 2 \cos^2 A - 1$ ”

→ put $B = A$, then replace $\sin^2 A$

$1 - \cos 2x$ or $1 + \cos 2x$ in a proof

→ $2 \sin^2 x$ or $2 \cos^2 x$

“(a) Prove ... (b) Hence solve ...”

→ use (a) to replace the left side in (b)



Worked Example



Using the identity for $\cos(A + B)$, prove that $\cos 2A \equiv 2 \cos^2 A - 1$.

$$\cos 2A = \cos(A + A) = \cos A \cos A - \sin A \sin A$$

← PUT $B = A$

$$= \cos^2 A - (1 - \cos^2 A)$$

← $\sin^2 A = 1 - \cos^2 A$

ANSWER $= 2 \cos^2 A - 1$ as required

**Worked Example**

? Prove that $\operatorname{cosec} 2x - \cot 2x \equiv \tan x$.

$$\text{LHS} = \frac{1}{\sin 2x} - \frac{\cos 2x}{\sin 2x} = \frac{1 - \cos 2x}{\sin 2x}$$

← WRITE IN sin AND cos; ONE FRACTION

$$= \frac{2 \sin^2 x}{2 \sin x \cos x}$$

← $1 - \cos 2x = 2 \sin^2 x$; DOUBLE ANGLE FOR sin

$$= \frac{\sin x}{\cos x}$$

← CANCEL $2 \sin x$

ANSWER = $\tan x = \text{RHS}$

**Worked Example**

? Prove that $\sec x - \cos x \equiv \sin x \tan x$.

$$\text{LHS} = \frac{1}{\cos x} - \cos x = \frac{1 - \cos^2 x}{\cos x}$$

← COMMON DENOMINATOR

$$= \frac{\sin^2 x}{\cos x} = \sin x \cdot \frac{\sin x}{\cos x}$$

← PYTHAGOREAN IDENTITY

ANSWER = $\sin x \tan x = \text{RHS}$

**Worked Example**

? Prove that $\cos 3x \equiv 4 \cos^3 x - 3 \cos x$.

$$\cos(2x + x) = \cos 2x \cos x - \sin 2x \sin x$$

← COMPOUND ANGLE

$$= (2 \cos^2 x - 1) \cos x - 2 \sin^2 x \cos x$$

← DOUBLE ANGLES

$$= 2 \cos^3 x - \cos x - 2(1 - \cos^2 x) \cos x$$

← $\sin^2 x = 1 - \cos^2 x$

ANSWER = $4 \cos^3 x - 3 \cos x = \text{RHS}$

HOW THIS IS EXAMINED · past WMA13 papersprove a compound or double angle identity **SEEN 9× IN EXAMS**prove a reciprocal identity **SEEN 4× IN EXAMS**

Your notes

MISTAKE GALLERY

Where marks are lost

× **WRONG**

$$\sin(A + B) = \sin A + \sin B.$$

✓ **RIGHT**

$$\sin A \cos B + \cos A \sin B.$$

Trig functions do not distribute over +.

× **WRONG**

$$\cos(A + B) = \cos A \cos B + \sin A \sin B.$$

✓ **RIGHT**

$$\cos A \cos B - \sin A \sin B.$$

cos swaps the sign.

× **WRONG**

$$\tan x = -\frac{3}{4}, \text{ obtuse, so } \sec x = +\frac{5}{4}.$$

✓ **RIGHT**
 $\sec x = -\frac{5}{4}$: cos is negative in the second quadrant.

Square roots need a sign from CAST.

× **WRONG**Dividing $\sin 2x = \sin x$ by $\sin x$.✓ **RIGHT**

$$\text{Factorise: } \sin x(2 \cos x - 1) = 0.$$

Dividing loses the solutions $\sin x = 0$.× **WRONG**

Proving by working on both sides until they “meet”.

✓ **RIGHT**

Transform one side into the other.

A proof must flow in one direction.

THE CHAPTER ON ONE PAGE

Trigonometric Identities



6.1

Pythagorean

$$\sin^2 + \cos^2 = 1; \quad \sec^2 = 1 + \tan^2;$$

$$\operatorname{cosec}^2 = 1 + \cot^2.$$

$t = \tan x$ turns mixed ratios into algebra.

6.2

Compound

$\sin(A \pm B)$: same sign. $\cos(A \pm B)$: opposite sign.

Missing ratio: triangle, then CAST sign.

6.3

Double

$$\sin 2A = 2 \sin A \cos A.$$

$$\cos 2A = \cos^2 - \sin^2 = 2 \cos^2 - 1 = 1 - 2 \sin^2.$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}.$$

6.4

Proof

One side only. Convert to sin, cos.

Combine, use Pythagoras, factorise, cancel.

End with “= RHS”.

BOTTOM LINE Two roots grow the whole tree: $\sin^2 + \cos^2 = 1$ and the addition formulae. Pick the form that leaves one trig function, and keep every sign honest.



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07

CHAPTER 07

Trigonometric Equations and R-Form

Turn the equation into one trig function of one angle, transform the interval to match, and collect every solution before shifting back.

3

CONCEPTS

11

VARIANTS

35

EXAM QUESTIONS

YOU WILL MASTER

- ▶ Write $a \sin x + b \cos x$ as $R \sin(x \pm \alpha)$ or $R \cos(x \pm \alpha)$
- ▶ Read maximum and minimum values, including reciprocals and squares, from the R-form
- ▶ Solve equations with identities, factorising and the R-form, in degrees or radians
- ▶ Transform the interval for $2x$, $x + \alpha$ and find every solution

CHAPTER 07 AT A GLANCE

Three note pages, eleven ways to be asked



CHAPTER 07 MAP

Trigonometric Equations

01

7.1 Harmonic form

$R \sin(x \pm \alpha)$

$R \cos(x \pm \alpha)$

exact R and α

02

7.2 Maximum and minimum

max and min values

where they occur

$$\frac{k}{c + f(x)}$$

$$c - (f(x))^2$$

03

7.3 Solving trig equations

identity reduction

$R\text{-form: } a \sin x + b \cos x = c$

zero products

even powers

axis intercepts

Contents

✱ 7.1	Harmonic Form (R-Form)	3
✱ 7.2	Maximum and Minimum Values	7
✱ 7.3	Solving Trigonometric Equations	10

Harmonic Form (R-Form)

NOTE 7.1

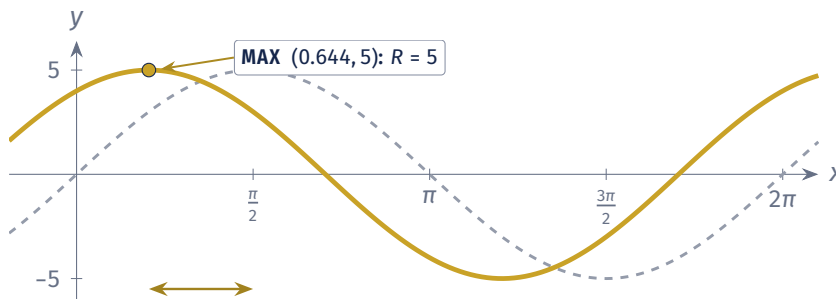


Harmonic Form (R-Form)

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What is the harmonic form?

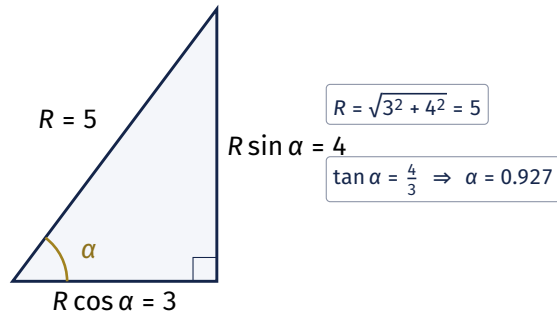
- Any expression $a \sin x + b \cos x$ is a single **shifted sine wave**. It can be written as
 - $R \sin(x \pm \alpha)$ or $R \cos(x \pm \alpha)$, with $R > 0$
- R is the **amplitude**: $R = \sqrt{a^2 + b^2}$.
- α is the **phase shift**, an acute angle (unless the question says otherwise).



$y = 3 \sin x + 4 \cos x$ is the wave $5 \sin x$ shifted left by $\alpha \approx 0.927$.

How do I find R and α ?

- STEP 1** **Expand** the form the question asks for, e.g. $R \sin(x + \alpha) = R \sin x \cos \alpha + R \cos x \sin \alpha$.
- STEP 2** **Compare coefficients** with $a \sin x + b \cos x$: $R \cos \alpha = a$, $R \sin \alpha = b$.
- STEP 3** $R = \sqrt{a^2 + b^2}$ (square and add).
- STEP 4** $\tan \alpha = \frac{R \sin \alpha}{R \cos \alpha} = \frac{b}{a}$ (divide). Use the **acute** angle with the correct signs.



The coefficients form a right-angled triangle with hypotenuse R .

FORM ASKED FOR	EXPANSION TO COMPARE
$R \sin(x + \alpha)$	$R \cos \alpha \sin x + R \sin \alpha \cos x$
$R \sin(x - \alpha)$	$R \cos \alpha \sin x - R \sin \alpha \cos x$
$R \cos(x + \alpha)$	$R \cos \alpha \cos x - R \sin \alpha \sin x$
$R \cos(x - \alpha)$	$R \cos \alpha \cos x + R \sin \alpha \sin x$



Examiner Tips and Tricks

- Give R exactly ($\sqrt{29}$, not 5.39) and α to the accuracy asked, usually 3 or 4 significant figures, in the units asked (radians unless degrees are stated).
- Choose the form so that $R \cos \alpha$ and $R \sin \alpha$ are both **positive**: then α is acute and $\tan \alpha = \frac{b}{a}$ with positive numbers.
- If the angle is $2x$, keep it: $\sqrt{3} \sin 2x - \cos 2x = 2 \sin\left(2x - \frac{\pi}{6}\right)$.

WHEN YOU SEE → YOU DO

“Express $8 \sin x - 15 \cos x$ in the form $R \sin(x - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$ ”

→ expand, compare, $R = \sqrt{a^2 + b^2}$, $\tan \alpha = \dots$

“Give the exact value of R ”

→ leave the surd

“Give the value of α , in radians, to 3 decimal places”

→ calculator in radian mode

**Worked Example**

Express $3 \sin x + 4 \cos x$ in the form $R \sin(x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give α to 3 decimal places.

$$R \sin x \cos \alpha + R \cos x \sin \alpha$$

← EXPAND THE FORM ASKED FOR

$$R \cos \alpha = 3, \quad R \sin \alpha = 4$$

← COMPARE COEFFICIENTS OF $\sin x$ AND $\cos x$

$$R = \sqrt{9 + 16} = 5, \quad \tan \alpha = \frac{4}{3}$$

← SQUARE AND ADD; DIVIDE

ANSWER $3 \sin x + 4 \cos x = 5 \sin(x + 0.927)$

**Worked Example**

Express $2 \cos \theta - 5 \sin \theta$ in the form $R \cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

$$R \cos \theta \cos \alpha - R \sin \theta \sin \alpha$$

← EXPAND $R \cos(\theta + \alpha)$

$$R \cos \alpha = 2, \quad R \sin \alpha = 5$$

← COMPARE: THE MINUS SIGNS MATCH

$$R = \sqrt{29}, \quad \tan \alpha = \frac{5}{2}$$

← EXACT R

ANSWER $2 \cos \theta - 5 \sin \theta = \sqrt{29} \cos(\theta + 1.190)$



Worked Example



$f(x) = \sqrt{3} \sin 2x - \cos 2x$. Express $f(x)$ in the form $R \sin(2x - \alpha)$, $R > 0$, $0 < \alpha < \frac{\pi}{2}$, giving the exact values of R and α .

$$R \cos \alpha = \sqrt{3}, \quad R \sin \alpha = 1$$

← COMPARE WITH $R \sin 2x \cos \alpha - R \cos 2x \sin \alpha$

$$R = \sqrt{3 + 1} = 2, \quad \tan \alpha = \frac{1}{\sqrt{3}}$$

← EXACT VALUES

ANSWER $f(x) = 2 \sin\left(2x - \frac{\pi}{6}\right)$

HOW THIS IS EXAMINED • past WMA13 papers

sine form **SEEN 5× IN EXAMS** cosine form **SEEN 4× IN EXAMS**



Your notes

Maximum and Minimum Values

NOTE 7.2



Maximum and Minimum Values

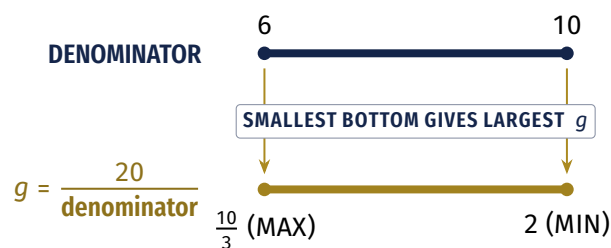
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How do I find the maximum and minimum of $a \sin x + b \cos x$?

- Write it as $R \sin(x + \alpha)$. Since $-1 \leq \sin(\dots) \leq 1$:
 - maximum value = R , when $\sin(x + \alpha) = 1$
 - minimum value = $-R$, when $\sin(x + \alpha) = -1$
- Where** it happens: solve $x + \alpha = \frac{\pi}{2}$ (for the max) or $x + \alpha = \frac{3\pi}{2}$ (for the min), and keep the answer that the question asks for ("smallest positive value of x ").
- For $R \cos(x + \alpha)$: maximum when $x + \alpha = 0$ (or 2π), minimum when $x + \alpha = \pi$.

What about expressions built from it?

- Outside changes** (C03): $c + kf(x)$ has max $c + kR$ (if $k > 0$).
- Reciprocals** $\frac{k}{c + f(x)}$ (with $c > R$ so the bottom is always positive): the fraction is **largest** when the denominator is **smallest**.
- Squares** $c - (f(x))^2$: $(f(x))^2$ runs from 0 to R^2 , because $f(x)$ passes through 0.



For $g(x) = \frac{20}{8 + 2 \sin(2x - \frac{\pi}{6})}$: the denominator runs from 6 to 10, so g runs from 2 to $\frac{10}{3}$, the other way round.



Examiner Tips and Tricks

- “Write down the exact maximum value”: one line from R, no calculus needed.
- “Find the smallest positive value of x at which this occurs”: if the solution of $x + \alpha = \frac{\pi}{2}$ is negative, add 2π (or π for a $2x$ angle).
- For a reciprocal, check that the denominator never reaches 0; for a square, check that the inside can reach 0.

 Your notes

WHEN YOU SEE → YOU DO

“write down the exact maximum value of $g(x)$ ”

→ use $\pm R$ in place of the trig part

“the smallest positive value of x for which this maximum value occurs”

→ solve $\sin(\dots) = 1$ (or -1), shift back

“ $g(x) = \frac{18}{34 - 3f(x)}$... minimum value of g ”

→ smallest g when the denominator is largest

“ $h(\beta) = 10 - (12 \sin \beta - 5 \cos \beta)^2$... Find the range of h ”

→ square runs from 0 to R^2



Worked Example

? $f(x) = 3 \sin x + 4 \cos x = 5 \sin(x + 0.927)$. (a) Find the maximum value of $g(x) = 3 + f(x)$ and the smallest positive x at which it occurs. (b) Find the range of $h(x) = 10 - (f(x))^2$.

$$\max g = 3 + 5 = 8$$

← (A) $\sin(\dots) = 1$

$$x + 0.927 = \frac{\pi}{2} \Rightarrow x = 0.644$$

← POSITIVE, SO THIS IS THE SMALLEST

$$0 \leq (f(x))^2 \leq 25$$

← (B) f RUNS FROM -5 TO 5 AND PASSES THROUGH 0

ANSWER (a) max 8 at $x = 0.644$ (b) $-15 \leq h(x) \leq 10$



Worked Example



$g(x) = \frac{20}{8 + \sqrt{3} \sin 2x - \cos 2x}$, $0 < x < \pi$. Using $\sqrt{3} \sin 2x - \cos 2x = 2 \sin\left(2x - \frac{\pi}{6}\right)$, find the maximum and minimum values of g , and the value of x at which the minimum occurs.

$$6 \leq 8 + 2 \sin\left(2x - \frac{\pi}{6}\right) \leq 10$$

$$\max g = \frac{20}{6} = \frac{10}{3}, \quad \min g = \frac{20}{10} = 2$$

$$2x - \frac{\pi}{6} = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{3}$$

← RANGE OF THE DENOMINATOR:
ALWAYS POSITIVE

← LARGEST g FROM THE SMALLEST
DENOMINATOR

← MINIMUM OF g : MAXIMUM OF
THE DENOMINATOR

ANSWER $\max \frac{10}{3}$, $\min 2$ at $x = \frac{\pi}{3}$

HOW THIS IS EXAMINED • past WMA13 papers

max or min **SEEN 8× IN EXAMS** squared expression **SEEN 1× IN EXAMS**

reciprocal **SEEN 1× IN EXAMS** angle of the extremum **SEEN 3× IN EXAMS**



Your notes

Solving Trigonometric Equations

NOTE 7.3

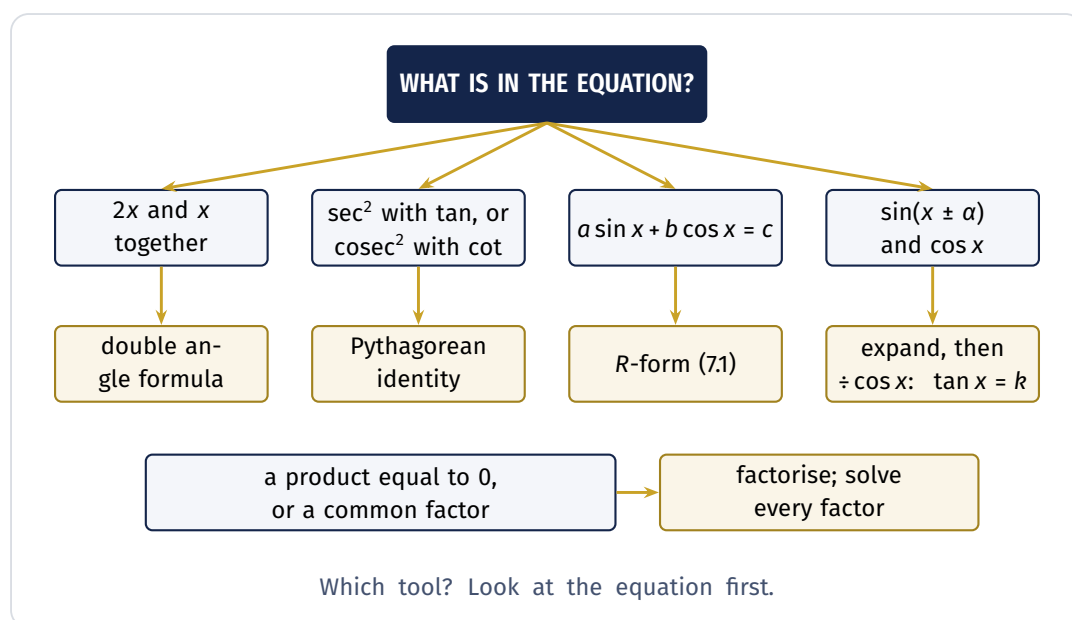


Solving Trigonometric Equations

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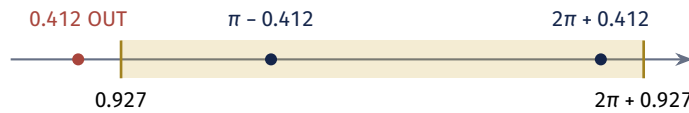
What is the strategy for any trig equation?

- Aim for **one trig function** of **one angle**: $\sin(\text{angle}) = k$, $\tan(\text{angle}) = k$, ...
- Look at what is in the equation and choose the tool:



How do I find every solution in the interval?

- STEP 1** If the angle is $2x$, $x + \alpha$, $\frac{x}{2}$, ..., **transform the interval** first: $0 \leq x < 360^\circ \Rightarrow 0 \leq 2x < 720^\circ$.
- STEP 2** Find the principal value with the calculator.
- STEP 3** Find the **second** value in one period: $\sin$: $180^\circ - \text{PV}$; $\cos$: $360^\circ - \text{PV}$; $\tan$: $\text{PV} + 180^\circ$.
- STEP 4** Add or subtract whole periods until you leave the transformed interval.
- STEP 5** **Shift back** to x (halve, subtract α , ...) and only then round.



VALUES OF $x + 0.927$ THAT LIE IN THE SHADED INTERVAL

Transforming the interval for $\sin(x + 0.927) = 0.4$, $0 \leq x < 2\pi$.



Examiner Tips and Tricks

- “Show all stages of your working. Solutions relying entirely on calculator technology are not acceptable”: write the identity you use, the factorised form and the principal value.
- Never divide by $\sin x$ or $\cos x$ when it could be 0: **factorise** instead.
- For $\sin^2 x = k$ or $\tan^2 x = k$, take **both** square roots: $\tan x = \pm\sqrt{k}$.
- Check the interval: strict ($0 < x < \pi$) or inclusive ($0 \leq x \leq 2\pi$), degrees or radians, and the accuracy asked.
- A labelled point on a graph (“the curve meets the x -axis at R ”) means: set $y = 0$, solve, and pick the root that matches the picture.

WHEN YOU SEE → YOU DO

“Show that the equation $8 \cos \theta = 3 \operatorname{cosec} \theta$ can be written in the form $\sin 2\theta = k$ ”

→ $\operatorname{cosec} = \frac{1}{\sin}$, multiply up,
 $2 \sin \theta \cos \theta = \sin 2\theta$

“Given $2 \sin(x + 45^\circ) = \cos(x - 60^\circ)$, show that $\tan x = \dots$ ”

→ expand both, collect
 $\sin x$ and $\cos x$, divide by
 $\cos x$

“Hence solve ... $2 \sin(2\theta) = \cos(2\theta - 105^\circ)$ ”

→ same equation with a
new angle: transform
the interval

“Hence solve $3 \sin x + 4 \cos x = 2$ ”

→ R-form, then $\sin(x + \alpha) = \frac{2}{R}$
on the shifted interval

$(2x - 1)(\sec x + 2) = 0$

→ each factor separately;
keep the algebraic root

**Worked Example**Solve, for $0 \leq \theta < 360^\circ$, $3 \cos 2\theta + 5 \sin \theta = 4$.

$$3(1 - 2 \sin^2 \theta) + 5 \sin \theta - 4 = 0$$

← $\cos 2\theta = 1 - 2 \sin^2 \theta$: **ONLY sin LEFT**

$$6 \sin^2 \theta - 5 \sin \theta + 1 = 0 \Rightarrow \\ (3 \sin \theta - 1)(2 \sin \theta - 1) = 0$$

← **QUADRATIC IN sin θ**

$$\sin \theta = \frac{1}{3} : 19.5^\circ, 160.5^\circ$$

← **PV AND $180^\circ - \text{PV}$**

$$\sin \theta = \frac{1}{2} : 30^\circ, 150^\circ$$

← **EXACT VALUES**

ANSWER $\theta = 19.5^\circ, 30^\circ, 150^\circ, 160.5^\circ$

**Worked Example**

(a) Show that $2 \sin(\theta + 30^\circ) = 3 \cos \theta$ can be written as $\tan \theta = \frac{2}{\sqrt{3}}$.
 (b) Hence solve, for $0 \leq x < 180^\circ$, $2 \sin(2x + 30^\circ) = 3 \cos 2x$.

$$2 \left(\frac{\sqrt{3}}{2} \sin \theta + \frac{1}{2} \cos \theta \right) = 3 \cos \theta$$

← **(A) EXPAND $\sin(\theta + 30^\circ)$**

$$\sqrt{3} \sin \theta = 2 \cos \theta \Rightarrow \tan \theta = \frac{2}{\sqrt{3}}$$

← **COLLECT, DIVIDE BY $\cos \theta$**

$$\theta = 2x, \quad 0 \leq 2x < 360^\circ : \quad 2x = 49.1^\circ, 229.1^\circ$$

← **(B) TRANSFORM THE INTERVAL;
tan: ADD 180°**

ANSWER $x = 24.6^\circ, 114.6^\circ$

**Worked Example**

Using $3 \sin x + 4 \cos x = 5 \sin(x + 0.927)$, solve $3 \sin x + 4 \cos x = 2$ for $0 \leq x < 2\pi$, to 3 decimal places.

$$\sin(x + 0.927) = 0.4$$

← DIVIDE BY R

$$0.927 \leq x + 0.927 < 7.210$$

← TRANSFORM THE INTERVAL

$$x + 0.927 = \pi - 0.412, 2\pi + 0.412$$

← 0.412 ITSELF IS BELOW THE INTERVAL

ANSWER $x = 1.803, 5.767$

**Worked Example**

(i) Solve, for $0 < x < \pi$, $(2x - 1)(\sec x + 2) = 0$. (ii) Solve, for $0 < \theta < 180^\circ$, $8 \cos \theta = 3 \operatorname{cosec} \theta$.

$$2x - 1 = 0 \Rightarrow x = \frac{1}{2}$$

← (I) ALGEBRAIC FACTOR: IN THE INTERVAL, KEEP IT

$$\sec x = -2 \Rightarrow \cos x = -\frac{1}{2} \Rightarrow x = \frac{2\pi}{3}$$

← TRIG FACTOR

$$8 \sin \theta \cos \theta = 3 \Rightarrow 4 \sin 2\theta = 3$$

← (II) MULTIPLY BY $\sin \theta$; DOUBLE ANGLE

$$0 < 2\theta < 360^\circ : 2\theta = 48.6^\circ, 131.4^\circ$$

← TRANSFORM THE INTERVAL

ANSWER (i) $x = \frac{1}{2}, \frac{2\pi}{3}$ (ii) $\theta = 24.3^\circ, 65.7^\circ$

HOW THIS IS EXAMINED • past WMA13 papers

phase shift **SEEN 7× IN EXAMS** identity reduction **SEEN 18× IN EXAMS**

even power **SEEN 1× IN EXAMS** zero product **SEEN 1× IN EXAMS**

axis intercept **SEEN 2× IN EXAMS**

MISTAKE GALLERY

Where marks are lost

× **WRONG**

$R = a + b$, or $R = 5.39$ when “exact” is asked.

✓ **RIGHT**

$R = \sqrt{a^2 + b^2}$, left as a surd.
Square and add the coefficients.

× **WRONG**

$\tan \alpha = \frac{a}{b}$ for $R \sin(x + \alpha)$ with $a \sin x + b \cos x$.

✓ **RIGHT**

$\tan \alpha = \frac{R \sin \alpha}{R \cos \alpha} = \frac{b}{a}$.
Compare coefficients carefully.

× **WRONG**

Solving $\sin 2x = k$ only on $0 \leq 2x < 360^\circ$ when $0 \leq x < 360^\circ$.

✓ **RIGHT**

Use $0 \leq 2x < 720^\circ$: twice as many solutions.
Transform the interval first.

× **WRONG**

$\sin 2x = \sin x \Rightarrow 2 \cos x = 1$ only.

✓ **RIGHT**

$\sin x(2 \cos x - 1) = 0$: also $\sin x = 0$.
Dividing by $\sin x$ loses solutions.

× **WRONG**

$\tan^2 x = 3 \Rightarrow \tan x = \sqrt{3}$ only.

✓ **RIGHT**

$\tan x = \pm\sqrt{3}$.
Both square roots.

THE CHAPTER ON ONE PAGE

Trig Equations and R-Form



7.1

R-form

Expand the form asked, compare coefficients.

$$R = \sqrt{a^2 + b^2}; \tan \alpha = \frac{R \sin \alpha}{R \cos \alpha}$$

Exact R ; α in the units asked.

7.2

Max and min

Trig part = ± 1 gives $\pm R$.

Reciprocal: order reverses. Square: from 0 to R^2 .

Where: solve $\sin(\dots) = \pm 1$.

7.3

Strategy

One function, one angle: double angle, Pythagoras, R -form, expand-and-divide, or factorise.

7.3

Every solution

Transform the interval, PV, second value, add periods, shift back, then round.

BOTTOM LINE One trig function, one angle, the right interval: get those three and every trig equation becomes routine.



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08

CHAPTER 08

Exponential and Logarithmic Functions

e^x and $\ln x$ undo each other. Isolate the exponential, take logs, and always check that what you take the log of is positive.

3

CONCEPTS

8

VARIANTS

12

EXAM QUESTIONS

YOU WILL MASTER

- ▶ Sketch $y = e^{ax+b} + c$ and $y = \ln(x + a) + b$ with asymptotes and intercepts
- ▶ Solve $e^{ax+b} = p$ and $\ln(ax + b) = q$ exactly
- ▶ Solve quadratics in e^x and in $\ln x$, rejecting impossible values
- ▶ Use the laws of logarithms and check every log argument is positive

CHAPTER 08 AT A GLANCE

Two note pages, eight ways to be asked



CHAPTER 08 MAP

Exponentials and Logarithms

01

8.1 e and $\ln$ graph of $e^{ax+b} + c$ graph of $\ln(x + a) + b$

inverse functions

02

8.2 Exponential and log equations

 $e^{ax+b} = p$ fraction with e^{kt} quadratic in e^x $\ln(ax + b) = q$

log laws

quadratic in $\ln x$

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The Functions e^x and $\ln x$

NOTE 8.1

The Functions e^x and $\ln x$

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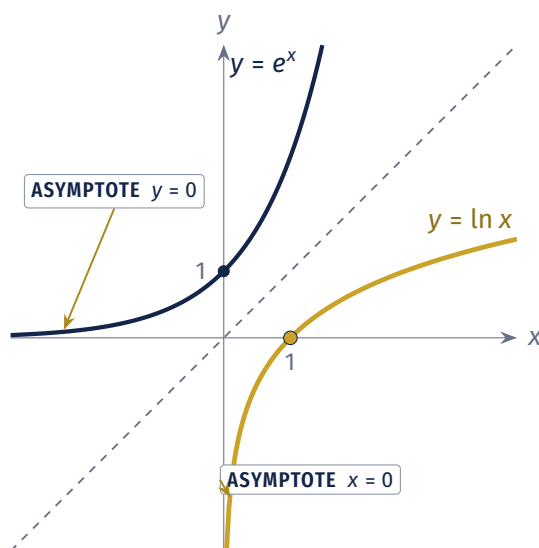
What are e^x and $\ln x$?

- $e \approx 2.718$ is the special number for which the gradient of $y = e^x$ equals its height at every point.
- $\ln x$ is the logarithm to base e : $\ln x = \log_e x$.
- They are **inverse functions**: $e^{\ln x} = x$ ($x > 0$) and $\ln(e^x) = x$.
- So their graphs are reflections of each other in $y = x$.

KEY FACTS

LEARN THIS: NOT GIVEN

$$e^0 = 1 \quad \ln 1 = 0 \quad \ln e = 1 \quad y = e^x \iff x = \ln y$$



$y = e^x$ and $y = \ln x$: mirror images in $y = x$.

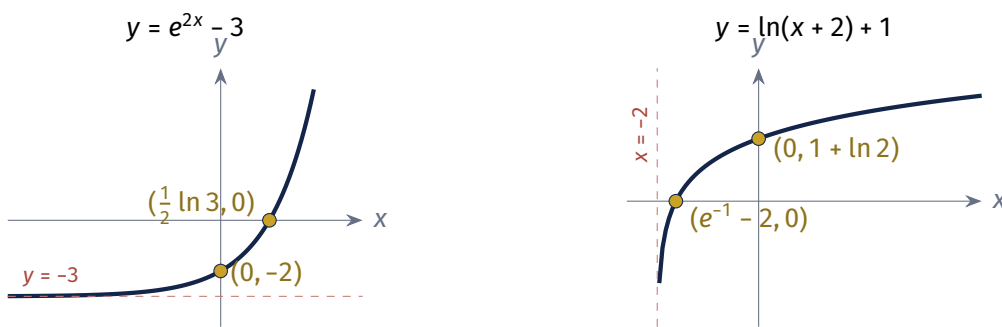
	$y = e^x$	$y = \ln x$
domain	all real x	$x > 0$
range	$y > 0$	all real y
intercept	$(0, 1)$	$(1, 0)$
asymptote	$y = 0$	$x = 0$

How do I sketch $y = e^{ax+b} + c$?

- $e^{(\dots)}$ is always **positive**, so the curve stays on one side of the **horizontal asymptote** $y = c$.
- $a > 0$: increasing (growth). $a < 0$: decreasing (decay), approaching $y = c$ as $x \rightarrow \infty$.
- y-intercept: put $x = 0$: $y = e^b + c$.
- x-intercept (if $c < 0$): solve $e^{ax+b} = -c$.

How do I sketch $y = \ln(x + a) + b$?

- The inside must be **positive**: $x > -a$. The **vertical asymptote** is $x = -a$.
- x-intercept: solve $\ln(x + a) = -b$, i.e. $x = e^{-b} - a$.



Exam-style sketches: label the asymptote with its equation and give every intercept exactly.



Examiner Tips and Tricks

- “On your sketch, state the equation of the asymptote”: write $y = 24$ (or $x = -2$) next to a dashed line. A number alone is not an equation.
- Give intercepts **exactly**: $(\frac{1}{2} \ln 3, 0)$, not $(0.549, 0)$, unless a decimal is asked for.
- e^x can never be 0 or negative. A curve $y = e^{(\dots)} + c$ never crosses its asymptote.



WHEN YOU SEE → YOU DO

“Sketch the graph of H against t ... state the equation of the asymptote”

→ $H = Ae^{-kt} + c$: start at $A + c$, decay towards $H = c$

“Find the coordinates of the points where the curve meets the axes” for $e^{(\dots)}$ or $\ln(\dots)$

→ put $x = 0$; put $y = 0$ and solve with $\ln$ or e

“State the range of f ” with $f(x) = e^x + c$

→ $f(x) > c$



Worked Example



Sketch $y = e^{2x} - 3$, giving the equation of the asymptote and the exact coordinates of the points where the curve meets the axes.

$$x \rightarrow -\infty : e^{2x} \rightarrow 0 \Rightarrow y \rightarrow -3$$

← HORIZONTAL ASYMPTOTE $y = -3$

$$x = 0 : y = 1 - 3 = -2$$

← y-INTERCEPT

$$e^{2x} = 3 \Rightarrow 2x = \ln 3$$

← x-INTERCEPT: TAKE $\ln$

ANSWER asymptote $y = -3$; $(0, -2)$ and $(\frac{1}{2} \ln 3, 0)$



Worked Example



Sketch $y = \ln(x + 2) + 1$, giving the asymptote and the exact intercepts.

$$x + 2 > 0 \Rightarrow \text{asymptote } x = -2$$

← THE INSIDE OF $\ln$ MUST BE POSITIVE

$$x = 0 : y = \ln 2 + 1$$

← y-INTERCEPT

$$\ln(x + 2) = -1 \Rightarrow x = e^{-1} - 2$$

← x-INTERCEPT: APPLY $e^{(\)}$

ANSWER asymptote $x = -2$; $(0, 1 + \ln 2)$ and $(e^{-1} - 2, 0)$

HOW THIS IS EXAMINED • past WMA13 papersexponential sketch **SEEN 1× IN EXAMS** logarithm sketch **SEEN 1× IN EXAMS**

Your notes

Solving Exponential and Logarithmic Equations

NOTE 8.2



Solving Exponential and Logarithmic Equations

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How do I solve $e^{ax+b} = p$?

- STEP 1** **Isolate** the exponential: get $e^{(\dots)}$ alone on one side.
- STEP 2** Check the other side is **positive**. If not, there is no solution.
- STEP 3** Take $\ln$ of both sides: $ax + b = \ln p$.
- STEP 4** Solve, and keep the answer exact (in $\ln$) unless a decimal is asked for.

e.g. **SOLVE** $3e^{2x-1} + 5 = 17$

$$e^{2x-1} = 4$$

← **ISOLATE: SUBTRACT 5, DIVIDE BY 3**

$$2x - 1 = \ln 4 = 2 \ln 2$$

← **TAKE $\ln$**

ANSWER $x = \frac{1}{2} + \ln 2$

What if the exponential appears more than once?

- **Fraction** such as $N = \frac{600e^{0.1t}}{2 + e^{0.1t}}$: multiply up, collect the $e^{0.1t}$ terms, factorise, divide.
- **Hidden quadratic**: $e^{2x} = (e^x)^2$. Let $u = e^x$, solve the quadratic in u , **reject** $u \leq 0$, then $x = \ln u$.
- $e^{-x} = \frac{1}{e^x} = \frac{1}{u}$: multiply through by u to clear it.

How do I solve log equations?

- One log: $\ln(ax + b) = q \Rightarrow ax + b = e^q$.
- Several logs: combine them into **one** log using the laws, then remove it.
- Hidden quadratic in $\ln x$: let $u = \ln x$, then $x = e^u$.

- **Check** every answer: the inside of every original log must be positive.

**LAWS OF LOGARITHMS (ANY BASE)****LEARN THIS: NOT GIVEN**

$$\ln a + \ln b = \ln(ab) \quad \ln a - \ln b = \ln \frac{a}{b} \quad k \ln a = \ln(a^k)$$

**Examiner Tips and Tricks**

- “Solutions based entirely on calculator technology are not acceptable”: show the isolated exponential and the line where you take logs.
- $\ln(a + b) \neq \ln a + \ln b$. Only products and quotients split.
- When $u = e^x$ gives a negative root, write “ $e^x = -2$ has no solution” to show you rejected it deliberately.

WHEN YOU SEE → YOU DO**“Find the value of t for which $H = 144$ ”**→ substitute, isolate $e^{-0.05t}$, take $\ln$ **“Find the exact value of x ”**→ leave the answer with $\ln$ (or e) **e^{2x} with e^x , or e^x with e^{-x}** → $u = e^x$: quadratic; reject $u \leq 0$ **“ $f(x) = \ln(\dots)$, $x > 0$... Find the exact values of a such that $fg(a) = g(a)$ ”**

→ form the equation, combine logs, then check every argument is positive

**Worked Example**

The number of animals in a reserve is modelled by $N = \frac{600e^{0.1t}}{2 + e^{0.1t}}$, $t \geq 0$. (a) Find the initial number. (b) Find t when $N = 400$. (c) Explain why N can never reach 600.

$$t = 0 : N = \frac{600}{3} = 200$$

← (A) $e^0 = 1$

$$400(2 + e^{0.1t}) = 600e^{0.1t} \Rightarrow 800 = 200e^{0.1t}$$

← (B) MULTIPLY UP AND COLLECT

$$e^{0.1t} = 4 \Rightarrow t = 10 \ln 4$$

← TAKE $\ln$

$$N = \frac{600}{2e^{-0.1t} + 1} < 600$$

← (C) DIVIDE TOP AND BOTTOM BY $e^{0.1t}$; THE DENOMINATOR IS ALWAYS GREATER THAN 1

ANSWER (a) 200 (b) $t = 10 \ln 4 = 13.86$ (c) $N \rightarrow 600$ but $N < 600$ for every t

**Worked Example**

Solve $2e^x - 7 + 3e^{-x} = 0$, giving exact answers.

$$2u - 7 + \frac{3}{u} = 0 \Rightarrow 2u^2 - 7u + 3 = 0$$

← LET $u = e^x$; MULTIPLY BY u

$$(2u - 1)(u - 3) = 0 \Rightarrow u = \frac{1}{2}, 3$$

← BOTH POSITIVE: BOTH VALID

ANSWER $x = \ln \frac{1}{2} = -\ln 2$ or $x = \ln 3$



Worked Example



Solve (a) $\ln(3x - 2) = 4$ (b) $\ln x + \ln(x - 2) = \ln 15$ (c) $(\ln x)^2 - 3 \ln x + 2 = 0$.

$$3x - 2 = e^4 \Rightarrow x = \frac{e^4 + 2}{3}$$

← (A) APPLY $e^{(\)}$ TO BOTH SIDES

$$\ln(x(x - 2)) = \ln 15 \Rightarrow x^2 - 2x - 15 = 0$$

← (B) PRODUCT LAW, THEN COMPARE

$$x = 5 \quad (x = -3 \text{ rejected: } \ln(-3) \text{ undefined})$$

← CHECK THE ARGUMENTS

$$(\ln x - 1)(\ln x - 2) = 0$$

← (C) QUADRATIC IN $\ln x$

ANSWER (a) $x = \frac{e^4 + 2}{3}$ (b) $x = 5$ (c) $x = e$ or $x = e^2$



Your notes

HOW THIS IS EXAMINED • past WMA13 papers

$e^{ax+b} = p$ **SEEN 7× IN EXAMS** fraction with e^{kt} **SEEN 1× IN EXAMS**

quadratic in e^x **SEEN 1× IN EXAMS** $\ln(ax + b) = q$ **SEEN 3× IN EXAMS**

log laws **NOT SEEN YET • BE READY** quadratic in $\ln x$ **SEEN 1× IN EXAMS**

MISTAKE GALLERY

Where marks are lost

× **WRONG**

$$3e^{2x} = 12 \Rightarrow \ln(3e^{2x}) = 6x.$$

✓ **RIGHT**

Divide first: $e^{2x} = 4$, then $2x = \ln 4$.
Isolate the exponential before taking logs.

× **WRONG**

$$\ln(x + 3) = \ln x + \ln 3.$$

✓ **RIGHT**

No law for the log of a sum.
Only $\ln(ab)$ and $\ln \frac{a}{b}$ split.

× **WRONG**

Keeping $x = -3$ from $x(x - 2) = 15$.

✓ **RIGHT**

Reject it: $\ln(-3)$ is undefined.
Check every original log argument.

× **WRONG**

$$u = e^x = -2 \Rightarrow x = \ln(-2).$$

✓ **RIGHT**

No solution from $u = -2$.
 $e^x > 0$ always.

× **WRONG**

Asymptote of $y = 280e^{-0.05t} + 24$ is $y = 0$.

✓ **RIGHT**

$H = 24$.
The added constant moves the asymptote.

THE CHAPTER ON ONE PAGE

Exponentials and Logarithms



8.1

Graphs

e^x : through $(0, 1)$, asymptote $y = 0$, range $y > 0$.

$\ln x$: through $(1, 0)$, asymptote $x = 0$, domain $x > 0$.

Mirror images in $y = x$.

8.1

Transformed

$e^{ax+b} + c$: asymptote $y = c$.

$\ln(x + a) + b$: asymptote $x = -a$.

Exact intercepts.

8.2

Exponential equations

Isolate, check positive, take $\ln$.

e^{2x} with e^x : $u = e^x$, reject $u \leq 0$.

8.2

Log equations

Combine to one log, then apply e .

Check every argument is positive.

BOTTOM LINE Isolate, then undo: $\ln$ removes e , e removes $\ln$. And nothing inside a log, or equal to an exponential, may be zero or negative.



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09

CHAPTER 09

Logarithmic Graphs and Exponential Models

Logs turn curves into straight lines, and exponentials describe anything that grows or decays in proportion to its size. Read the line, then read the story.

4

CONCEPTS

16

VARIANTS

34

EXAM QUESTIONS

YOU WILL MASTER

- ▶ Turn $y = ax^n$ and $y = kb^x$ into straight lines and read a , n , k , b from a graph
- ▶ Use initial values and two observations to find the constants of a model
- ▶ Find when a model reaches a target value, and its long-term value
- ▶ Judge and explain whether a model is realistic

CHAPTER 09 AT A GLANCE

Three note pages, sixteen ways to be asked



CHAPTER 09 MAP

Log Graphs and Models

01

9.1 Logarithmic graphs

log y against log x: $y = ax^n$ log y against x: $y = kb^x$ meaning of a and k

use the line

02

9.2 Growth and decay

initial value

two observations

when does it reach ...?

predict at a time

03

9.3 Interpreting models

long-term value

can it reach ...?

percentage multiplier

compare models

complete months

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✱ 9.2	Exponential Growth and Decay	7
✱ 9.3	Interpreting Exponential Models	10

Logarithmic Graphs

NOTE 9.1



Logarithmic Graphs

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Why take logs of data?

- Curves are hard to read; **straight lines** are easy: $Y = mX + c$ gives two numbers, the gradient and the intercept.
- Taking logs turns the two important curved models into straight lines:

Power model $y = ax^n$
 $\log y = \log a + n \log x$

plot $\log y$ against $\log x$
 gradient = n , intercept = $\log a$

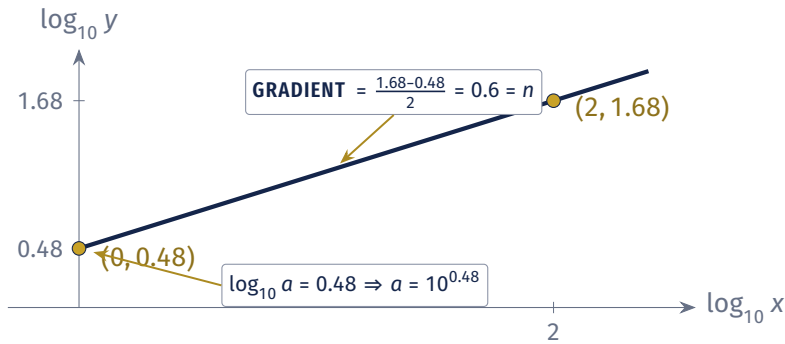
Exponential model $y = kb^x$
 $\log y = \log k + x \log b$

plot $\log y$ against x
 gradient = $\log b$, intercept = $\log k$

What is on the axes tells you which model it is.

How do I find the constants from the line?

- STEP 1** Take logs of the model and use the log laws to write it as $\log y = (\text{gradient}) \times (\dots) + (\text{intercept})$.
- STEP 2** Find the gradient and intercept of the line from the two points given.
- STEP 3** Match: the gradient and the intercept are **logs** of the constants (except n).
- STEP 4** Undo the log: if the log is $\log_{10}$, then $a = 10^{\text{intercept}}$; if it is $\ln$, then $a = e^{\text{intercept}}$.



$\log_{10} y$ against $\log_{10} x$: the gradient is the power n ; the intercept is $\log_{10} a$, not a .

e.g. A LINE OF $\ln y$ AGAINST x HAS GRADIENT 0.3 AND INTERCEPT 1.2. FIND y IN TERMS OF x

$$\ln y = 1.2 + 0.3x$$

← WRITE THE LINE

$$y = e^{1.2+0.3x} = e^{1.2}e^{0.3x}$$

← APPLY $e^{(\)}$; SPLIT THE POWER

ANSWER $y = 3.32e^{0.3x}$



Examiner Tips and Tricks

- The most common error: saying the intercept **is** a (or k). It is $\log a$. Undo it with the **same base** as the axes.
- In $y = kb^x$, the gradient is $\log b$, not b : $b = 10^{\text{gradient}}$.
- In $y = ax^n$, a is the value of y when $x = 1$ (not when $x = 0$). Use this when asked to interpret a .

WHEN YOU SEE → YOU DO

axes labelled $\log_{10} y$ and $\log_{10} x$ → power model $y = ax^n$:
gradient n , intercept $\log a$ axes labelled $\log_{10} A$ and t → exponential model $A = pq^t$: gradient $\log q$, intercept $\log p$ “Find, to 3 decimal places, the value of p and the value of q ”→ $p = 10^{\text{intercept}}$, $q = 10^{\text{gradient}}$ “Interpret the value of a in the context of the model”→ a is y when $x = 1$ (one unit of the input)

Worked Example



The surface area $A \text{ m}^2$ covered by a plant t days after the start of a study is modelled by $A = pq^t$. A graph of $\log_{10} A$ against t is a straight line through $(0, 0.45)$ and $(10, 0.95)$. Find p and q to 3 decimal places.

$$\log_{10} A = \log_{10} p + t \log_{10} q$$

← TAKE LOGS; PRODUCT AND POWER LAWS

$$\log_{10} p = 0.45 \Rightarrow p = 10^{0.45}$$

← INTERCEPT

$$\log_{10} q = \frac{0.95 - 0.45}{10} = 0.05 \Rightarrow q = 10^{0.05}$$

← GRADIENT

ANSWER $p = 2.818$, $q = 1.122$



Worked Example



A graph of $\log_{10} y$ against $\log_{10} x$ is a straight line through $(0, 0.48)$ and $(2, 1.68)$. Find the equation connecting y and x , and interpret the constant a .

$$\log_{10} y = \log_{10} a + n \log_{10} x$$

← POWER MODEL $y = ax^n$

$$n = \frac{1.68 - 0.48}{2} = 0.6, \quad a = 10^{0.48} = 3.02$$

← GRADIENT AND INTERCEPT

$$x = 1 : y = a$$

← INTERPRETATION

ANSWER $y = 3.02x^{0.6}$; a is the value of y when $x = 1$

HOW THIS IS EXAMINED · past WMA13 papers

$\log y$ vs $\log x$ **SEEN 6× IN EXAMS** $\log y$ vs x **SEEN 13× IN EXAMS**

use a log model **SEEN 1× IN EXAMS** interpret a **SEEN 2× IN EXAMS**



Your notes

Exponential Growth and Decay

NOTE 9.2



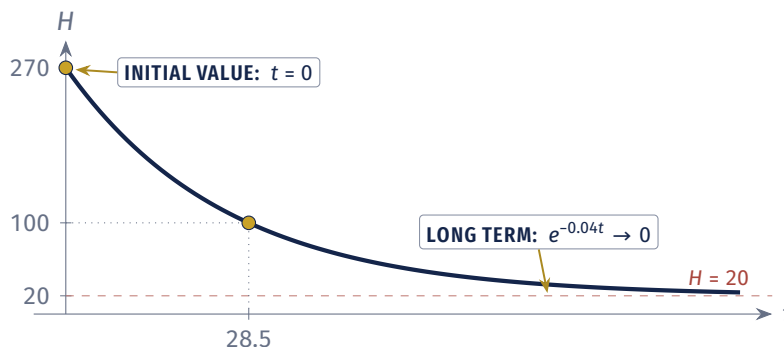
Exponential Growth and Decay

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What do exponential models look like?

- **Growth:** $N = Ae^{kt}$ with $k > 0$. **Decay:** $N = Ae^{-kt}$ with $k > 0$.
- With an added constant, $H = Ae^{-kt} + c$, the quantity decays **towards** c , not towards 0 (e.g. a cooling object approaches room temperature).
- The rate of change is proportional to the amount (or to the distance from c):

$$\frac{dH}{dt} = -k(H - c).$$



$$H = 250e^{-0.04t} + 20: \text{ starts at 270, falls towards 20.}$$

How do I find the constants?

- **Initial value:** put $t = 0$. Since $e^0 = 1$, $N(0) = A$ (or $A + c$).
- **Two observations:** substitute both, then **divide** one equation by the other to eliminate A and leave one exponential.

How do I find when a model reaches a value?

- STEP 1** Set the model equal to the target value.
- STEP 2** Isolate the exponential.
- STEP 3** Take $\ln$ and solve for t .



STEP 4 Convert to the units asked (minutes, years, the calendar year).



Examiner Tips and Tricks

- “Initial” always means $t = 0$, whatever the units of t .
- Keep unrounded values of constants in your calculator; round only the final answer.
- “Show by differentiation that $\frac{dH}{dt} = a + bH$ ”: differentiate, then replace $250e^{-0.04t}$ by $H - 20$.

WHEN YOU SEE → YOU DO

“Find the initial temperature”

→ put $t = 0$

“Find the value of t for which $H = 144$ ”

→ set equal, isolate $e^{(\dots)}$, take $\ln$

“Given that ... 30 m^2 at the start ... 50 m^2 after T days”

→ first fact gives a constant; second gives T

“Show by differentiation that $\frac{dH}{dt} = a + bH$ ”

→ differentiate, then eliminate the exponential using H



Worked Example



A hot object cools so that its temperature after t minutes is $H = 250e^{-0.04t} + 20$. (a) Find the initial temperature. (b) Find t when $H = 100$. (c) Show that $\frac{dH}{dt} = a + bH$, stating a and b .

$$H(0) = 250 + 20 = 270$$

← (A) $e^0 = 1$

$$250e^{-0.04t} = 80 \Rightarrow -0.04t = \ln 0.32$$

← (B) ISOLATE, TAKE $\ln$

$$\frac{dH}{dt} = -10e^{-0.04t} = -0.04(H - 20)$$

← (C) $250e^{-0.04t} = H - 20$

ANSWER (a) 270°C (b) $t = 28.49$ (c) $\frac{dH}{dt} = 0.8 - 0.04H$: $a = 0.8$, $b = -0.04$



Worked Example



The value of a machine is modelled by $V = Ae^{kt}$. After 2 years it is worth \$3200 and after 5 years \$1638.40. Find A and k , and the yearly percentage loss in value.

$$Ae^{2k} = 3200, \quad Ae^{5k} = 1638.4$$

← SUBSTITUTE BOTH OBSERVATIONS

$$e^{3k} = 0.512 \Rightarrow k = \frac{1}{3} \ln 0.512$$

← DIVIDE TO ELIMINATE A

$$A = \frac{3200}{e^{2k}} = \frac{3200}{0.64}$$

← SUBSTITUTE BACK: $e^k = 0.8$

ANSWER $A = 5000$, $k = \ln 0.8 = -0.223$; it keeps 80% each year, so loses 20% per year

HOW THIS IS EXAMINED • past WMA13 papers

initial value **SEEN 15× IN EXAMS** two observations **SEEN 6× IN EXAMS**

time to reach a target **SEEN 4× IN EXAMS** predict at a time **SEEN 4× IN EXAMS**



Your notes

Interpreting Exponential Models

NOTE 9.3

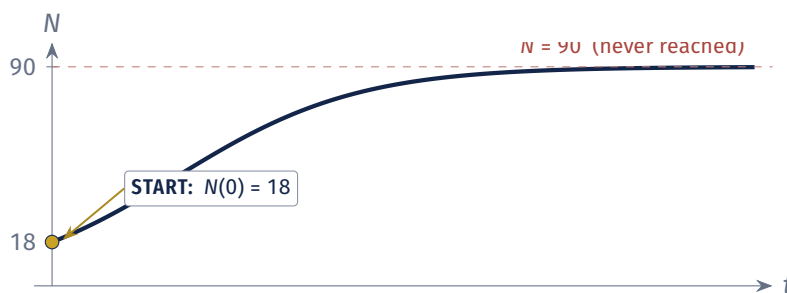


Interpreting Exponential Models

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How do I find the long-term value?

- As $t \rightarrow \infty$: $e^{-kt} \rightarrow 0$ and $e^{kt} \rightarrow \infty$.
- $H = Ae^{-kt} + c \rightarrow c$: the long-term value is the **added constant**.
- For a fraction such as $N = \frac{90pe^{0.2t}}{3 + pe^{0.2t}}$: divide top and bottom by $e^{0.2t}$ to get $\frac{90p}{3e^{-0.2t} + p} \rightarrow 90$.



A limited-growth model: fast at first, then levelling off below its limit.

How do I judge whether a model is realistic?

- Find the **range** of values the model can produce for the allowed t .
- Compare with the context: can the value really be that large, that small, or negative?
- Give a **specific** reason using numbers from the model, e.g. "the model gives $N < 90$ for all t , so it can never reach 100".
- When comparing two models, refer to what each predicts at the start, in the long term, or for the data given.

What does a multiplier mean?

- $V = Ab^t$ (or Ae^{kt} with $b = e^k$): each unit of time multiplies V by b .
- $b = 0.8$: keeps 80%, so **loses** 20% per unit of time. $b = 1.05$: **gains** 5%.

What about “complete months” or “the first year”?

- Solve for t exactly, then translate: if $t = 13.86$ months, the target is first passed during the 14th month, so the answer is 14 complete months.
- Read the inequality: “more than” and “at least” can change which whole number is correct. Check the integer values either side.



Examiner Tips and Tricks

- “Explain why ... can never reach 500”: show the limit (or maximum) from the model and compare: “ $N \rightarrow 450$ as $t \rightarrow \infty$ and $N < 450$, so N never reaches 500”.
- A negative output in a profit model means a **loss**: “a loss of about £830 000”. Mention the units.
- Never say only “the model is unrealistic”. Say **why**, with a number.

WHEN YOU SEE → YOU DO

“Find, according to the model, the maximum possible surface area”

→ let $t \rightarrow \infty$; divide by the largest exponential

“Explain why, according to this model, the number ... can never reach 500”

→ state the limit/bound and compare

“State, giving a reason, whether the model is suitable for ...”

→ evaluate the model at that value; compare with reality

“Interpret the value 0.8 in the context of the model”

→ multiplier per unit time: 20% lost each year





Worked Example



$N = \frac{90pe^{0.2t}}{3 + pe^{0.2t}}$, where N is the number of plants after t weeks. Initially there are 18 plants. (a) Show that $p = 0.75$. (b) Find the maximum possible number of plants according to the model. (c) Explain why the model predicts the number of plants will never reach 100.

$$t = 0 : \frac{90p}{3 + p} = 18$$

← (A) INITIAL VALUE

$$90p = 54 + 18p \Rightarrow 72p = 54$$

← SOLVE

$$N = \frac{90p}{3e^{-0.2t} + p} \rightarrow \frac{90p}{p} = 90$$

← (B) DIVIDE BY $e^{0.2t}$; $e^{-0.2t} \rightarrow 0$

ANSWER (a) $p = 0.75$ (b) 90 (c) $N < 90$ for all t , so N can never reach 100

HOW THIS IS EXAMINED · past WMA13 papers

long-term value **SEEN 4× IN EXAMS** range feasibility **SEEN 4× IN EXAMS**

compare models **NOT SEEN YET · BE READY** multiplier as a percentage **SEEN 2× IN EXAMS**

mixed log model **SEEN 1× IN EXAMS** complete time periods **SEEN 1× IN EXAMS**



Your notes

MISTAKE GALLERY

Where marks are lost

× **WRONG**

Intercept 0.45 on a $\log_{10} A$ graph,
so $p = 0.45$.

✓ **RIGHT**

$\log_{10} p = 0.45$, so $p = 10^{0.45} = 2.818$.

The axes are logs; undo them.

× **WRONG**

In $y = kb^x$, gradient = b .

✓ **RIGHT**

Gradient = $\log b$.

Take logs of the model before matching.

× **WRONG**

Long-term value of $H = 250e^{-0.04t} + 20$ is 0.

✓ **RIGHT**

It is 20: $e^{-0.04t} \rightarrow 0$ leaves +20.

Only the exponential part disappears.

× **WRONG**

$b = 0.8$ means “decreases by 80%”.

✓ **RIGHT**

It keeps 80%: decreases by 20%.

Multiplier = fraction retained.

× **WRONG**

Rounding k to -0.22 before the next part.

✓ **RIGHT**

Keep the exact $k = \ln 0.8$ (or full calculator value).

Early rounding changes later answers.

THE CHAPTER ON ONE PAGE

Log Graphs and Exponential Models



9.1

Straight lines

$y = ax^n$: $\log y$ vs $\log x$, gradient n , intercept $\log a$.

$y = kb^x$: $\log y$ vs x , gradient $\log b$, intercept $\log k$.

Undo with the same base.

9.2

Constants

Initial: $t = 0$.

Two observations: divide to eliminate A .

Reaching a value: isolate, take $\ln$.

9.3

Long term

$e^{-kt} \rightarrow 0$; the added constant is the limit.

Fraction: divide by the largest exponential.

9.3

Interpret

Range vs reality, with a number.

Multiplier b : keeps $100b\%$.

Complete periods: check the integers.

BOTTOM LINE Logs straighten curves, $e^0 = 1$ gives the start, $e^{-\infty} = 0$ gives the end, and every judgement about a model needs a number from the model.



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10

CHAPTER 10

Differentiation Techniques

Look at the structure first: a function of a function needs the chain rule, two functions multiplied need the product rule, one divided by another needs the quotient rule.

4

CONCEPTS

13

VARIANTS

40

EXAM QUESTIONS

YOU WILL MASTER

- ▶ Differentiate e^{kx} , $\ln kx$, a^x and every trig function, including sec, cosec, cot
- ▶ Use the chain rule, including two or more layers
- ▶ Use the product and quotient rules, combined with the chain rule, and simplify
- ▶ Find $\frac{dy}{dx}$ when x is given in terms of y

CHAPTER 10 AT A GLANCE

Four note pages, thirteen ways to be asked



CHAPTER 10 MAP

Differentiation

01

10.1 Standard derivatives

e^{kx}, a^x

$\ln kx$

$\sin, \cos, \tan$

$\sec, \operatorname{cosec}, \cot$

02

10.2 Chain rule

one layer

two or more layers

03

10.3 Product and quotient rules

product

quotient

with chain

simplify first

04

10.4 x in terms of y

flip $\frac{dx}{dy}$ to get $\frac{dy}{dx}$

Contents

✱ 10.1	Differentiating Standard Functions	3
✱ 10.2	The Chain Rule	6
✱ 10.3	Product and Quotient Rules	9
✱ 10.4	Differentiating x as a Function of y	13

Differentiating Standard Functions

NOTE 10.1



Differentiating Standard Functions

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What are the standard derivatives?

y	$\frac{dy}{dx}$	NOTE
e^{kx}	ke^{kx}	learn
a^x	$a^x \ln a$	learn; $a > 0$
$\ln kx$	$\frac{1}{x}$	learn: the k disappears
$\sin kx$	$k \cos kx$	learn
$\cos kx$	$-k \sin kx$	learn: note the minus
$\tan kx$	$k \sec^2 kx$	formula booklet
$\sec x$	$\sec x \tan x$	formula booklet
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$	formula booklet
$\cot x$	$-\operatorname{cosec}^2 x$	formula booklet

- Angles must be in **radians** for the trig derivatives.
- $\ln kx = \ln k + \ln x$, so its derivative is just $\frac{1}{x}$.
- Differentiate a sum or difference term by term; constants multiply through.

Should I simplify first?

- Yes, if it removes a product or quotient. Use the **log laws** or split the fraction.
 - $\ln(x^2(x+1)) = 2 \ln x + \ln(x+1)$
 - $\frac{x^3+2}{x} = x^2 + 2x^{-1}$



e.g. DIFFERENTIATE $y = 3e^{2x} - 4 \ln 5x + \sin 3x$

$$6e^{2x}$$

$$-\frac{4}{x}$$

← e^{kx} : MULTIPLY BY $k = 2$

← $\ln 5x$: THE 5 DISAPPEARS

ANSWER $\frac{dy}{dx} = 6e^{2x} - \frac{4}{x} + 3 \cos 3x$



Examiner Tips and Tricks

- The derivatives of tan, sec, cosec, cot are in the formula booklet; the others must be learnt.
- Every derivative of a “co” function (cos, cosec, cot) starts with a minus.

WHEN YOU SEE → YOU DO

e^{kx} , $\ln kx$, $\sin kx$ in a sum

→ term by term; keep the inner multiplier k

a^x , e.g. 2^x or 10^y

→ $a^x \ln a$

$\ln$ of a product, quotient or power

→ split with the log laws first

HOW THIS IS EXAMINED • past WMA13 papers

exponential NOT SEEN YET • BE READY natural log NOT SEEN YET • BE READY

trig SEEN 1x IN EXAMS reciprocal trig NOT SEEN YET • BE READY a^x NOT SEEN YET • BE READY



Worked Example



Find $\frac{dy}{dx}$ for (a) $y = 2^x$ (b) $y = \sec 2x$ (c) $y = \ln(x^2(x+1))$.

$$2^x \ln 2$$

← (A) a^x RULE

$$2 \sec 2x \tan 2x$$

← (B) BOOKLET RESULT WITH THE INNER MULTIPLIER 2

$$y = 2 \ln x + \ln(x+1) \Rightarrow \frac{2}{x} + \frac{1}{x+1}$$

← (C) LOG LAWS FIRST

ANSWER (a) $2^x \ln 2$ (b) $2 \sec 2x \tan 2x$ (c) $\frac{3x+2}{x(x+1)}$



Your notes

The Chain Rule

NOTE 10.2



The Chain Rule

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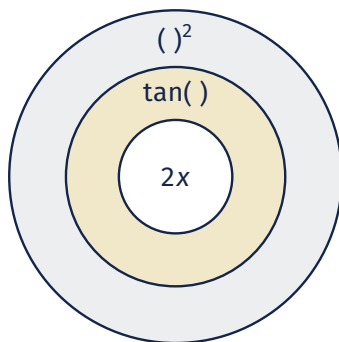
What is the chain rule?

- It differentiates a **function of a function**, such as $(3x^2 - 5)^4$, $e^{\sin 3x}$ or $\ln(\sin 2x)$.
- Differentiate the **outside**, keep the inside the same, then multiply by the derivative of the **inside**.

THE CHAIN RULE

LEARN THIS: NOT GIVEN

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

OUTER: $2 \tan 2x$ MIDDLE: $\sec^2 2x$

INNER: 2

$$\frac{d}{dx} \tan^2 2x = 4 \tan 2x \sec^2 2x$$

Peel the layers from the outside in, and multiply what you get.

What are the common chain rule patterns?

- $(f(x))^n \rightarrow n(f(x))^{n-1} f'(x)$
- $e^{f(x)} \rightarrow f'(x)e^{f(x)}$
- $\ln f(x) \rightarrow \frac{f'(x)}{f(x)}$
- $\sin f(x) \rightarrow f'(x) \cos f(x)$, and similarly for the other trig functions

e.g. DIFFERENTIATE $y = \sqrt{4x+1}$

$$y = (4x+1)^{\frac{1}{2}} \Rightarrow \frac{1}{2}(4x+1)^{-\frac{1}{2}} \times 4$$

← POWER OUTSIDE, $4x+1$ INSIDE

ANSWER $\frac{dy}{dx} = \frac{2}{\sqrt{4x+1}}$



Examiner Tips and Tricks

- Write the inside function as u if it helps, but give the final answer in x .
- $\sin^2 x$ means $(\sin x)^2$: the power is the outer layer.
- "Writing your answer in simplest form": $\frac{2 \cos 2x}{\sin 2x} = 2 \cot 2x$.

WHEN YOU SEE → YOU DO

a bracket raised to a power

→ power outside, bracket inside

e , $\ln$, $\sin$, ... of something more than kx

→ chain rule: multiply by the inner derivative

$\tan^2(2x)$, $e^{\sin 3x}$, $\ln(\cos^2 x)$

→ two or more layers: multiply all the layer derivatives



Worked Example



Differentiate (a) $(3x^2 - 5)^4$ (b) $\ln(\sin 2x)$ (c) $e^{\sin 3x}$ (d) $\tan^2 2x$.

$$4(3x^2 - 5)^3 \times 6x$$

← (A) POWER, THEN INNER $6x$

$$\frac{2 \cos 2x}{\sin 2x}$$

← (B) $\frac{f'}{f}$

$$3 \cos 3x e^{\sin 3x}$$

← (C) INNER DERIVATIVE TIMES THE SAME EXPONENTIAL

$$2 \tan 2x \times \sec^2 2x \times 2$$

← (D) THREE LAYERS

ANSWER (a) $24x(3x^2 - 5)^3$ (b) $2 \cot 2x$ (c) $3 \cos 3x e^{\sin 3x}$ (d) $4 \tan 2x \sec^2 2x$

HOW THIS IS EXAMINED • past WMA13 papers

chain **SEEN 5× IN EXAMS** nested chain **SEEN 4× IN EXAMS**



Your notes

Product and Quotient Rules

NOTE 10.3



Product and Quotient Rules

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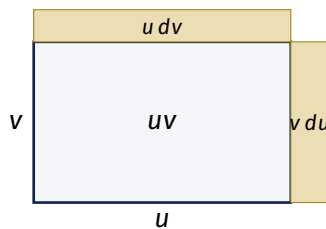
When do I use the product rule?

- When y is two functions of x **multiplied**: $y = uv$, e.g. $x^2 e^{3x}$, $x^2 \ln 2x$, $e^x \sin x$.

PRODUCT RULE

LEARN THIS: NOT GIVEN

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

CHANGE IN AREA $\approx u dv + v du$

Why it works: when a rectangle $u \times v$ grows, the new area is the two strips $u dv$ and $v du$.

When do I use the quotient rule?

- When y is one function **divided** by another: $y = \frac{u}{v}$.

QUOTIENT RULE

IN THE FORMULA BOOKLET

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

- Order matters in the numerator: “bottom times derivative of top, **minus** top times derivative of bottom”.
- Do not expand v^2 in the denominator; it often cancels with a factor in the numerator.

How do I simplify the answer?

- Take out **common factors**: e^{3x} , powers of x , powers of a bracket.
- In a quotient with $v = (4x + 3)^2$, the numerator has a factor $(4x + 3)$: cancel one power.
- “Show that $f'(x) = \frac{2e^{2x}(ax + b)}{(4x + 3)^3}$ ” tells you the factorised shape to aim for.



Examiner Tips and Tricks

- Write u, v, u', v' in a small list before you build the formula: it earns the method mark.
- A product can always be differentiated instead of using the quotient rule: $\frac{u}{v} = uv^{-1}$. Use whichever is cleaner.
- The quotient rule is in the booklet; the product rule is not.

WHEN YOU SEE → YOU DO

“ $y = x^2 \ln 2x$... **find, in simplest form, $\frac{dy}{dx}$** ”

→ product rule; factorise

“ $f(x) = \frac{e^{2x}}{(4x + 3)^2}$... **show that $f'(x) = \dots$** ”

→ quotient rule, chain on the bottom, cancel $(4x + 3)$

“**Show by calculus that f is a decreasing function**”

→ find $f'(x)$ and show it is negative for every x in the domain



**Worked Example**

Differentiate (a) $y = x^2 e^{3x}$ (b) $y = x^2 \ln 2x$, simplifying fully.

$$u = x^2, v = e^{3x}; \quad u' = 2x, v' = 3e^{3x}$$

← (A) LIST THE PARTS

$$x^2(3e^{3x}) + e^{3x}(2x)$$

← $uv' + vu'$

$$x^2 \cdot \frac{1}{x} + 2x \ln 2x$$

← (B) $\frac{d}{dx} \ln 2x = \frac{1}{x}$

ANSWER (a) $xe^{3x}(3x + 2)$ (b) $x(1 + 2 \ln 2x)$

**Worked Example**

$f(x) = \frac{e^{2x}}{(4x+3)^2}$. Show that $f'(x) = \frac{2e^{2x}(ax+b)}{(4x+3)^3}$, stating a and b .

$$u = e^{2x}, u' = 2e^{2x}; \quad v = (4x+3)^2, v' = 8(4x+3)$$

← CHAIN RULE FOR v'

$$f'(x) = \frac{(4x+3)^2 2e^{2x} - e^{2x} 8(4x+3)}{(4x+3)^4}$$

← QUOTIENT RULE

$$= \frac{2e^{2x}[(4x+3) - 4]}{(4x+3)^3}$$

← TAKE OUT $2e^{2x}(4x+3)$ AND CANCEL

ANSWER $f'(x) = \frac{2e^{2x}(4x-1)}{(4x+3)^3}$: $a = 4, b = -1$



Worked Example



Show that $\frac{d}{dx} \left(\frac{\sin x}{1 + \cos x} \right) = \frac{1}{1 + \cos x}$.

$$\frac{(1 + \cos x) \cos x - \sin x(-\sin x)}{(1 + \cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1 + \cos x)^2} = \frac{\cos x + 1}{(1 + \cos x)^2}$$

← QUOTIENT RULE

← $\sin^2 + \cos^2 = 1$

ANSWER $= \frac{1}{1 + \cos x}$ as required

HOW THIS IS EXAMINED • past WMA13 papers

product **SEEN 1× IN EXAMS** quotient **SEEN 5× IN EXAMS** simplify first **SEEN 1× IN EXAMS**

product with chain **SEEN 12× IN EXAMS** quotient with chain **SEEN 4× IN EXAMS**



Your notes

Differentiating x as a Function of y

NOTE 10.4

Differentiating x as a Function of y

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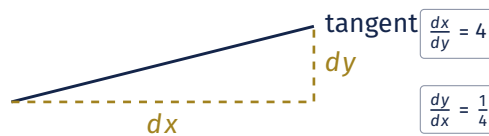
What if x is given in terms of y ?

- Equations such as $x = 4 \sin^2 y$ or $x = ye^{2y}$ cannot easily be rearranged to $y = \dots$
- Differentiate x **with respect to** y , then turn it upside down:

RECIPROCAL RULE

LEARN THIS: NOT GIVEN

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \quad \left(\frac{dx}{dy} \neq 0 \right)$$



Rise and run swap: if x changes by 4 when y changes by 1, then y changes by $\frac{1}{4}$ per unit of x .

How do I give the answer in terms of x ?

- The reciprocal is usually in terms of y . Use the **original equation** to replace the trig or exponential parts by x .
- Draw a right-angled triangle or use $\sin^2 + \cos^2 = 1$ to convert $\cos y$ when you know $\sin y$.
- Inverse functions use the same idea: $y = \arcsin x \Rightarrow x = \sin y \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$.



Examiner Tips and Tricks

- This is one of the most frequent P3 questions. The steps are always: $\frac{dx}{dy}$, reciprocal, then rewrite in x using the original equation.
- Check the sign of the square root with the given range of y (e.g. $0 < y < \frac{\pi}{2}$ makes $\cos y > 0$).
- A **vertical tangent** occurs where $\frac{dx}{dy} = 0$ (see C11).

Your notes

WHEN YOU SEE → YOU DO

“Given that $x = 6 \sin^2 2y$... show that

$$\frac{dy}{dx} = \frac{1}{A\sqrt{x(B-x)}}”$$

→ $\frac{dx}{dy}$, reciprocal, then replace $\sin$ and $\cos$ using x

**“ $y = \log_{10}(2x+1)$ (a) Express x in terms of y
(b) Hence find $\frac{dy}{dx}$ ”**

$$\rightarrow x = \frac{10^y - 1}{2}; \frac{dx}{dy} = \frac{10^y \ln 10}{2}$$

“Given $x = 2 \sin y$, show that $\frac{dy}{dx} = (A - x^2)^{-\frac{1}{2}}$ ”

→ $\cos y = \sqrt{1 - \sin^2 y}$, written in x



Worked Example



Given $x = 4 \sin^2 y$, $0 < y < \frac{\pi}{2}$, show that $\frac{dy}{dx} = \frac{1}{2\sqrt{4x-x^2}}$.

$$\frac{dx}{dy} = 8 \sin y \cos y$$

← CHAIN RULE ON $(\sin y)^2$

$$\sin y = \frac{\sqrt{x}}{2}, \quad \cos y = \sqrt{1 - \frac{x}{4}} = \frac{\sqrt{4-x}}{2}$$

← FROM THE EQUATION; $\cos y > 0$ IN THE RANGE

$$\frac{dx}{dy} = 8 \cdot \frac{\sqrt{x}}{2} \cdot \frac{\sqrt{4-x}}{2} = 2\sqrt{4x-x^2}$$

← SUBSTITUTE

ANSWER $\frac{dy}{dx} = \frac{1}{2\sqrt{4x-x^2}}$



Worked Example



$y = \log_{10}(2x + 1)$. (a) Express x in terms of y . (b) Hence find $\frac{dy}{dx}$ in terms of x .

$$2x + 1 = 10^y \Rightarrow x = \frac{10^y - 1}{2}$$

← (A) UNDO THE LOG

$$\frac{dx}{dy} = \frac{10^y \ln 10}{2}$$

← (B) a^y RULE

$$\frac{dy}{dx} = \frac{2}{10^y \ln 10} = \frac{2}{(2x + 1) \ln 10}$$

← RECIPROCAL, THEN $10^y = 2x + 1$

ANSWER $\frac{dy}{dx} = \frac{2}{(2x + 1) \ln 10}$

HOW THIS IS EXAMINED • past WMA13 papers

x in terms of y **SEEN 14× IN EXAMS**



Your notes

MISTAKE GALLERY

Where marks are lost

× **WRONG**

$$\frac{d}{dx} \ln 5x = \frac{5}{x}.$$

✓ **RIGHT**

$$\frac{1}{x}.$$

$$\ln 5x = \ln 5 + \ln x.$$

× **WRONG**

$$\frac{d}{dx} \cos 3x = 3 \sin 3x.$$

✓ **RIGHT**

$$-3 \sin 3x.$$

The derivative of cos is -sin.

× **WRONG**

$$\frac{d}{dx} (x^2 e^{3x}) = 2x \cdot 3e^{3x}.$$

✓ **RIGHT**

$$3x^2 e^{3x} + 2x e^{3x}.$$

A product needs the product rule.

× **WRONG**

Quotient rule with the numerator the wrong way round.

✓ **RIGHT** $vu' - uv'$ on top.

The minus makes order matter.

× **WRONG**

$$x = 4 \sin^2 y \Rightarrow \frac{dy}{dx} = 8 \sin y \cos y.$$

✓ **RIGHT**That is $\frac{dx}{dy}$; take its reciprocal.

Check which variable you differentiated.

THE CHAPTER ON ONE PAGE

Differentiation Techniques



10.1

Standard

$e^{kx} \rightarrow ke^{kx}$; $a^x \rightarrow a^x \ln a$; $\ln kx \rightarrow \frac{1}{x}$.
 $\sin kx \rightarrow k \cos kx$; $\cos kx \rightarrow -k \sin kx$.
 tan, sec, cosec, cot: booklet.

10.2

Chain

Outside, keep inside, times derivative of inside.
 Several layers: multiply every layer.

10.3

Product, quotient

$(uv)' = uv' + vu'$.
 $\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$.
 Factorise and cancel.

10.4

 $x = f(y)$

$\frac{dy}{dx} = 1 \div \frac{dx}{dy}$, then rewrite in x with the original equation.

BOTTOM LINE Read the structure: layers mean chain, a product means product rule, a fraction means quotient rule, and x in terms of y means flip $\frac{dx}{dy}$.



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CHAPTER 11

Applications of Differentiation

The derivative is a gradient: use it for tangents and normals, set it to zero for turning points, read its sign for increasing and decreasing, and read it as a rate in a model.

4

CONCEPTS

16

VARIANTS

54

EXAM QUESTIONS

YOU WILL MASTER

- ▶ Write the tangent or normal at a point, including tangents through an outside point and vertical tangents
- ▶ Find and classify stationary points, including on trig and exponential curves
- ▶ Use turning points to count solutions and restrict parameters
- ▶ Show a function is increasing or decreasing, and find and interpret rates in models

CHAPTER 11 AT A GLANCE

Four note pages, sixteen ways to be asked



CHAPTER 11 MAP

Applications of Differentiation

01

11.1 Tangents and normals

at a point through an outside point

axis intercepts and areas vertical tangents

02

11.2 Stationary points

find coordinates classify periodic curves

count solutions parameter conditions

03

11.3 Increasing and decreasing

sign of $f'(x)$ "show f is decreasing"

04

11.4 Rates of change

rate at a time time for a given rate

rate in terms of the quantity

Contents

✱ 11.1	Tangents and Normals	3
✱ 11.2	Stationary Points	7
✱ 11.3	Increasing and Decreasing Functions	10
✱ 11.4	Rates of Change in Models	12

Tangents and Normals

NOTE 11.1

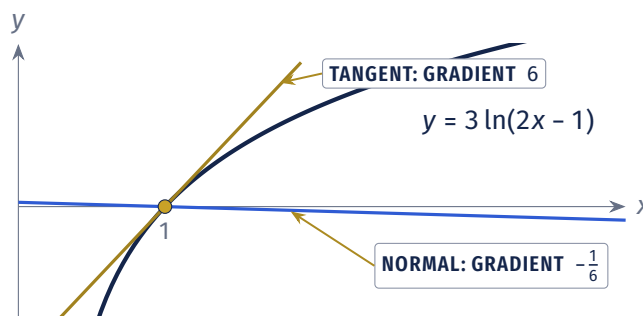


Tangents and Normals

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How do I find the equation of a tangent or normal?

- STEP 1** Find the coordinates of the point (x_1, y_1) : substitute the given coordinate into the curve.
- STEP 2** Differentiate and substitute x_1 to get the gradient of the **tangent**, m .
- STEP 3** For the **normal** use the negative reciprocal: $-\frac{1}{m}$.
- STEP 4** Use $y - y_1 = m(x - x_1)$ and rearrange to the form asked.



The tangent touches the curve; the normal is perpendicular to it at the same point.

What if the tangent must pass through a given point?

- Write the tangent at a **general** point $x = a$: $y - f(a) = f'(a)(x - a)$.
- Substitute the coordinates of the outside point (often the origin) and solve for a .

What about curves given as $x = f(y)$?

- The gradient is $\frac{dy}{dx} = 1 \div \frac{dx}{dy}$ (see 10.4).
- A **vertical tangent** ($x = \text{constant}$) occurs where $\frac{dx}{dy} = 0$. Find y there, then x from the curve; the tangent is $x = \dots$



Examiner Tips and Tricks

- Keep exact values: “ $y = mx + c$, where m and c are constants to be found” usually wants surds, π or $\ln$.
- For areas of triangles made with axes, find each line’s intercept, then sketch the triangle before computing “ $\frac{1}{2} \times \text{base} \times \text{height}$ ”.
- Normal gradient: $-\frac{1}{m}$, not $\frac{1}{m}$ and not $-m$.



WHEN YOU SEE → YOU DO

“Find the equation of the tangent to C at P.
Write your answer in the form $y = mx + c$ ”

→ coordinates of P, f' at P,
 $y - y_1 = m(x - x_1)$

“the tangent to C at P passes through the origin”

→ tangent at $x = \alpha$, then
put (0, 0) in

“The normal to the curve at P cuts the
 x -axis at N. Find the exact area of triangle
OPN”

→ normal, its x -intercept,
then $\frac{1}{2} \times \text{base} \times \text{height}$

“The tangents at P and Q are parallel to the
 y -axis”

→ solve $\frac{dx}{dy} = 0$



Worked Example



The curve C has equation $y = 3 \ln(2x - 1)$. Find the equations of the tangent and the normal to C at the point where $x = 1$.

$$y = 3 \ln 1 = 0 \Rightarrow P(1, 0)$$

← THE POINT

$$\frac{dy}{dx} = \frac{6}{2x-1} = 6 \text{ at } x = 1$$

← CHAIN RULE

$$\text{normal gradient} = -\frac{1}{6}$$

← NEGATIVE RECIPROCAL

ANSWER tangent $y = 6x - 6$; normal $y = -\frac{1}{6}x + \frac{1}{6}$

**Worked Example**

The tangent to $y = e^{2x}$ at the point P passes through the origin. Find the exact coordinates of P .

$$y - e^{2a} = 2e^{2a}(x - a)$$

TANGENT AT A GENERAL POINT
 $x = a$

$$0 - e^{2a} = 2e^{2a}(0 - a) \Rightarrow 1 = 2a$$

PUT (0, 0) IN; DIVIDE BY $e^{2a} > 0$

ANSWER $P\left(\frac{1}{2}, e\right)$

**Worked Example**

$P(e, 4)$ lies on $y = 4 \ln x$. The normal to the curve at P meets the x -axis at N . Find the exact area of triangle OPN .

$$\frac{dy}{dx} = \frac{4}{x} = \frac{4}{e} \Rightarrow \text{normal gradient} = -\frac{e}{4}$$

GRADIENT AT P

$$0 - 4 = -\frac{e}{4}(x - e) \Rightarrow x = e + \frac{16}{e}$$

NORMAL MEETS $y = 0$

$$\frac{1}{2} \times \left(e + \frac{16}{e}\right) \times 4$$

BASE ON , HEIGHT 4

ANSWER Area = $2e + \frac{32}{e}$



Worked Example



The curve C has equation $x = ye^{2y}$. (a) Find the equation of the vertical tangent to C . (b) The line $x = k$ meets C at exactly two points. Find the range of values of k .

$$\frac{dx}{dy} = e^{2y}(1 + 2y) = 0 \Rightarrow y = -\frac{1}{2}$$

← PRODUCT RULE; $e^{2y} \neq 0$

$$x = -\frac{1}{2}e^{-1}$$

← SUBSTITUTE INTO THE CURVE

$$\text{for } y < 0 : -\frac{1}{2e} \leq x < 0$$

← (B) x FALLS TO $-\frac{1}{2e}$ THEN RISES BACK TOWARDS 0

ANSWER (a) $x = -\frac{1}{2e}$ (b) $-\frac{1}{2e} < k < 0$

HOW THIS IS EXAMINED • past WMA13 papers

tangent or normal at a point **SEEN 7× IN EXAMS** intercepts and areas **SEEN 2× IN EXAMS**
 through an outside point **SEEN 1× IN EXAMS** vertical tangent **SEEN 2× IN EXAMS**
 signed gradients **SEEN 1× IN EXAMS**



Your notes

Stationary Points

NOTE 11.2



Stationary Points

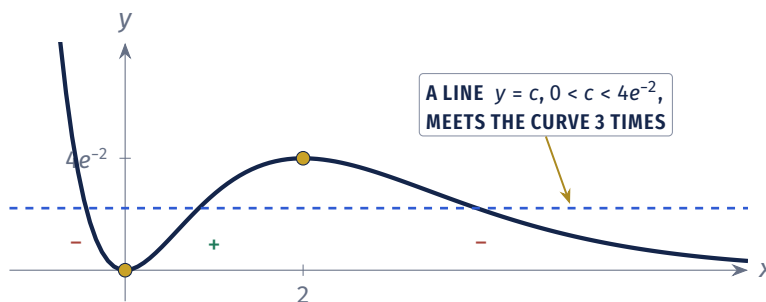
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How do I find stationary points?

- STEP 1** Differentiate, simplify and **factorise** $\frac{dy}{dx}$.
- STEP 2** Solve $\frac{dy}{dx} = 0$. Factors such as e^{-x} are never zero.
- STEP 3** Substitute each x into the **original** equation for y .

How do I decide if it is a maximum or a minimum?

- Second derivative:** $\frac{d^2y}{dx^2} < 0$ maximum; > 0 minimum; $= 0$ use another method.
- Sign of $\frac{dy}{dx}$** either side: $+ \rightarrow -$ maximum, $- \rightarrow +$ minimum.



$y = x^2e^{-x}$: a minimum at $(0, 0)$, a maximum at $(2, 4e^{-2})$, and the sign of $\frac{dy}{dx}$ between them.

How do I use turning points to count solutions?

- The number of solutions of $f(x) = c$ is the number of times the line $y = c$ meets the graph.
- Compare c with the **turning-point heights**, the values at the ends of the domain, and any asymptotes.
- To **force** a number of stationary points, look at the equation $f'(x) = 0$: e.g. a

quadratic factor has two distinct roots when its discriminant is positive.



Examiner Tips and Tricks

- On a trig curve, list **all** stationary angles in the interval, then decide which are maxima (the question may want only those).
- “Find the exact coordinates”: keep e^{-2} and $\ln$, do not use decimals.
- A factor e^{kx} in $f'(x)$ can be divided out because it is always positive; it never gives a stationary point.

WHEN YOU SEE → YOU DO

“Find the exact coordinates of the stationary points”

→ factorise f' , set each non-exponential factor to zero, use the original for y

“the curve has two turning points, at P and Q ... show that, at P and Q, ...”

→ $f'(x) = 0$ rearranged to the given form

“Given that the equation $f(x) = c$ has exactly three solutions”

→ compare c with turning heights and limits

“Find the range of values of k for which the curve has two stationary points”

→ discriminant of the factor in $f'(x)$



Worked Example



Find the exact coordinates of the stationary points of $y = x^2 e^{-x}$ and determine their nature.

$$\frac{dy}{dx} = 2xe^{-x} - x^2 e^{-x} = xe^{-x}(2 - x)$$

← PRODUCT RULE, THEN FACTORISE

$$x = 0 \text{ or } x = 2 \quad (e^{-x} \neq 0)$$

← SOLVE $\frac{dy}{dx} = 0$

signs: -, +, -

← LEFT OF 0, BETWEEN, RIGHT OF 2

ANSWER minimum (0, 0); maximum (2, $4e^{-2}$)

**Worked Example**

? $f(x) = e^x \sin x$, $0 \leq x \leq 2\pi$. Find the x -coordinates of the stationary points and identify the maximum.

$$f'(x) = e^x(\sin x + \cos x) = 0$$

← PRODUCT RULE; $e^x \neq 0$

$$\tan x = -1 \Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

← SECOND AND FOURTH QUADRANTS

$$f''(x) = 2e^x \cos x < 0 \text{ at } \frac{3\pi}{4}$$

← SECOND DERIVATIVE TEST

ANSWER maximum at $x = \frac{3\pi}{4}$; minimum at $x = \frac{7\pi}{4}$

**Worked Example**

? The curve $y = (x^2 + k)e^x$ has two distinct stationary points. Find the range of possible values of k .

$$\frac{dy}{dx} = e^x(x^2 + 2x + k)$$

← PRODUCT RULE AND FACTORISE

$$x^2 + 2x + k = 0 \text{ needs } b^2 - 4ac > 0$$

← TWO DISTINCT ROOTS

$$4 - 4k > 0$$

← DISCRIMINANT

ANSWER $k < 1$

HOW THIS IS EXAMINED • past WMA13 papers

stationary coordinates **SEEN 8× IN EXAMS** periodic curves **SEEN 2× IN EXAMS**

count intersections **SEEN 3× IN EXAMS** model extremum **SEEN 2× IN EXAMS**

optimise a gradient **SEEN 1× IN EXAMS**

parameter by number of stationary points **SEEN 2× IN EXAMS**

prescribed gradient **SEEN 2× IN EXAMS**

Increasing and Decreasing Functions

NOTE 11.3

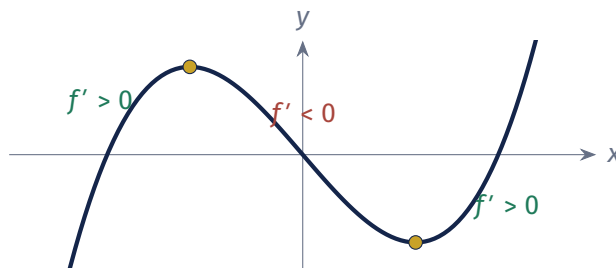


Increasing and Decreasing Functions

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How does the derivative tell me if a function is increasing?

- $f'(x) > 0$ on an interval: f is **increasing** there.
- $f'(x) < 0$ on an interval: f is **decreasing** there.
- To **show** a function is decreasing for its whole domain, show $f'(x)$ is negative for **every** x in the domain: squares, $e^{(\dots)}$ and positive constants help.



The sign of the gradient matches the direction of the curve.



Examiner Tips and Tricks

- “You must make your reasoning clear”: write a sentence, e.g. “ $(x-2)^2 > 0$ so $f'(x) = -\frac{6}{(x-2)^2} < 0$ for all $x > 2$, hence f is decreasing.”
- Increasing or decreasing functions are one-one, so they have inverses (C02).

WHEN YOU SEE → YOU DO

“Show, using calculus, that f is a decreasing function. You must make your reasoning clear”

→ $f'(x)$, then argue its sign for every x

“state whether f is an increasing or a decreasing function, giving a reason”

→ sign of $f'(x)$ with a reason

“Find the range of values of x for which f is increasing”

→ solve $f'(x) > 0$

 Your notes



Worked Example



$f(x) = \frac{2x+2}{x-2}$, $x > 2$. Show, using calculus, that f is a decreasing function.

$$f'(x) = \frac{2(x-2) - (2x+2)(1)}{(x-2)^2}$$

← QUOTIENT RULE

$$= \frac{-6}{(x-2)^2}$$

← SIMPLIFY THE NUMERATOR

$$(x-2)^2 > 0 \text{ for } x > 2$$

← REASONING

ANSWER $f'(x) < 0$ for all $x > 2$, so f is decreasing

HOW THIS IS EXAMINED • past WMA13 papers

sign of the first derivative **SEEN 11× IN EXAMS**

Rates of Change in Models

NOTE 11.4



Rates of Change in Models

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What does a derivative mean in a model?

- $\frac{dV}{dt}$ is the **rate of change** of V with respect to time: how fast V is changing at that instant.
- Its units are the units of V per unit of t , e.g. \$ per year, °C per minute.
- A **negative** rate means the quantity is decreasing. Say “decreasing at 571 dollars per year” rather than “-571”.

What are the three common questions?

- **Rate at a time**: differentiate first, then substitute t . Never substitute first.
- **Time for a given rate**: differentiate, set $\frac{dV}{dt}$ equal to the rate (with its sign), solve.
- **Rate in terms of the quantity**: differentiate, then replace the exponential using the model, e.g. $\frac{dH}{dt} = -0.04(H - 20)$.



Examiner Tips and Tricks

- “The rate of increase ... exactly 6 days after the start”: $t = 6$, not day 7.
- “Cooling at 5°C per minute” means $\frac{dH}{dt} = -5$.
- Give units in your final answer.

WHEN YOU SEE → YOU DO

“find the rate of increase of ... exactly 6 days after the start”

→ differentiate, then $t = 6$; units

“find the time at which the temperature is falling at 5°C per minute”

→ $\frac{dH}{dt} = -5$

“Show by differentiation that $\frac{dH}{dt} = a + bH$ ”

→ replace the exponential using H



Worked Example



The value of a car is $V = 5000(0.8)^t$ dollars after t years. Find the rate of change of V when $t = 3$, and interpret it.

$$\frac{dV}{dt} = 5000(0.8)^t \ln 0.8$$

← a^t RULE

$$t = 3 : 5000 \times 0.512 \times \ln 0.8 = -571.25$$

← SUBSTITUTE AFTER DIFFERENTIATING

ANSWER -571 (3 s.f.): the value is falling by about 571 dollars per year at $t = 3$



Worked Example



$H = 250e^{-0.04t} + 20$. Find the time at which the temperature is decreasing at 5°C per minute.

$$\frac{dH}{dt} = -10e^{-0.04t}$$

← DIFFERENTIATE

$$-10e^{-0.04t} = -5 \Rightarrow e^{-0.04t} = 0.5$$

← RATE WITH ITS SIGN

ANSWER $t = 25 \ln 2 = 17.33$ minutes

HOW THIS IS EXAMINED • past WMA13 papersrate at a time **SEEN 10× IN EXAMS** time for a given rate **SEEN 4× IN EXAMS**rate in terms of the quantity **SEEN 1× IN EXAMS**

Your notes

MISTAKE GALLERY

Where marks are lost

× **WRONG**Normal gradient = $\frac{1}{m}$.✓ **RIGHT**

$$-\frac{1}{m}$$

Perpendicular gradients multiply to -1.

× **WRONG**Substituting $t = 3$ into V and then “differentiating” a number.✓ **RIGHT**Differentiate V first, then substitute.

A number has derivative 0.

× **WRONG** $xe^{-x}(2-x) = 0 \Rightarrow x = 0, 2$ and $e^{-x} = 0$.✓ **RIGHT**Only $x = 0$, $x = 2$: $e^{-x} > 0$.

Exponentials are never zero.

× **WRONG**Using $\frac{dx}{dy} = 0$ for a horizontal tangent.✓ **RIGHT** $\frac{dx}{dy} = 0$ gives a **vertical** tangent.Horizontal means $\frac{dy}{dx} = 0$.× **WRONG**“ f is decreasing because $f'(3) < 0$ ”.✓ **RIGHT**Show $f'(x) < 0$ for **all** x in the domain.

One value proves nothing about the whole domain.

THE CHAPTER ON ONE PAGE

Applications of Differentiation



11.1

Lines

Tangent: $y - y_1 = m(x - x_1)$; normal: $-\frac{1}{m}$.

Through a point: tangent at $x = a$, substitute.

Vertical tangent: $\frac{dx}{dy} = 0$.

11.2

Turning points

Factorise f' , solve $= 0$, y from the original.

$f'' < 0$ max, > 0 min.

Count solutions with turning heights.

11.3

Direction

$f' > 0$ increasing, $f' < 0$ decreasing.

"Show": sign for every x , with a reason.

11.4

Rates

Differentiate, then substitute.

Sign and units in words.

Eliminate the exponential using the model.

BOTTOM LINE The derivative is the gradient: a line's slope, zero at a turn, positive going up, and a rate in a model. Always differentiate before you substitute.

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CHAPTER 12

Integration

Integration is differentiation run backwards. Spot the derivative you are looking at, divide by any inner multiplier, and use an identity when nothing matches.

5

CONCEPTS

18

VARIANTS

26

EXAM QUESTIONS

YOU WILL MASTER

- ▶ Integrate e^{kx} , $\frac{1}{ax+b}$, $\sin kx$, $\cos kx$, $\sec^2 kx$ and a^x
- ▶ Recognise $\frac{f'(x)}{f(x)}$ and $f'(x)(f(x))^n$ and integrate them at sight
- ▶ Use $\cos 2x$, $\tan^2 x = \sec^2 x - 1$ and other identities before integrating
- ▶ Evaluate definite integrals exactly and find areas, including regions made with lines

CHAPTER 12 AT A GLANCE

Four note pages, eighteen ways to be asked



CHAPTER 12 MAP

Integration

01 12.1 Standard integrals

e^{kx}, a^x

$\frac{1}{ax+b}$

$\sin kx, \cos kx, \sec^2 kx$

$(ax + b)^n$

find f from f'

02 12.2 Reverse chain rule

$\frac{f'(x)}{f(x)} \rightarrow \ln$

$f'(x)(f(x))^n$

03 12.3 Using trig identities

$\sin^2, \cos^2$

$\tan^2$

$\sin^3$ via a given identity

$(\sin x + \cos x)^2$

04 12.4 Definite integrals and areas

exact values

find a constant

areas above and below the axis

curve plus line regions

Contents

✱ 12.1	Integrating Standard Functions	3
✱ 12.2	Reverse Chain Rule	6
✱ 12.3	Integrating with Trigonometric Identities	9
✱ 12.4	Definite Integrals and Areas	12

Integrating Standard Functions

NOTE 12.1



Integrating Standard Functions

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What are the standard integrals?

$f(x)$	$\int f(x) dx$	REVERSE OF
e^{kx}	$\frac{1}{k}e^{kx} + c$	$\frac{d}{dx}e^{kx} = ke^{kx}$
a^x	$\frac{a^x}{\ln a} + c$	$\frac{d}{dx}a^x = a^x \ln a$
$\frac{1}{x}$	$\ln x + c$	$\frac{d}{dx} \ln x = \frac{1}{x}$
$\frac{1}{ax+b}$	$\frac{1}{a} \ln ax+b + c$	chain rule
$(ax+b)^n, n \neq -1$	$\frac{(ax+b)^{n+1}}{a(n+1)} + c$	chain rule
$\cos kx$	$\frac{1}{k} \sin kx + c$	$\frac{d}{dx} \sin kx$
$\sin kx$	$-\frac{1}{k} \cos kx + c$	$\frac{d}{dx} \cos kx$
$\sec^2 kx$	$\frac{1}{k} \tan kx + c$	$\frac{d}{dx} \tan kx$

- Pattern: when the inside is **linear** (kx or $ax+b$), integrate as usual and **divide by the coefficient of x** .
- Always add $+c$ to an indefinite integral.
- To find a function from its derivative, integrate, then use the given point to find c .

e.g. FIND $\int \frac{4}{(2x-1)^3} dx$

$$\int 4(2x-1)^{-3} dx = \frac{4(2x-1)^{-2}}{-2 \times 2}$$

POWER UP BY 1, DIVIDE BY NEW
POWER AND BY 2

ANSWER $-\frac{1}{(2x-1)^2} + c$



Examiner Tips and Tricks

- Check any answer by differentiating it: you should get back the integrand.
- $\int \frac{1}{3x} dx = \frac{1}{3} \ln |x| + c$. Writing $\frac{1}{3} \ln |3x| + c$ is also correct: the two differ only by a constant.
- Exact answers: keep $\ln$, e , π and surds.

 Your notes

WHEN YOU SEE → YOU DO

$$\frac{k}{ax+b}$$

$$\rightarrow \frac{k}{a} \ln |ax+b|$$

$$e^{kx}, \cos kx, \sin kx$$

$$\rightarrow \text{standard result } \div k$$

“Given $f'(x) = \dots$ and $f(1) = \dots$, find $f(x)$ ”

$\rightarrow$ integrate, then use the point to find c



Worked Example

? Find $\int \left(e^{3x} + \frac{4}{x} - \frac{2}{2x+1} + \cos 2x \right) dx$.

$$\frac{1}{3} e^{3x}$$

$$\leftarrow \div 3$$

$$4 \ln |x| - \ln |2x+1|$$

$$\leftarrow \frac{2}{2x+1} \cdot \frac{2}{2} \ln |2x+1|$$

$$\frac{1}{2} \sin 2x$$

$$\leftarrow \div 2$$

ANSWER $\frac{1}{3} e^{3x} + 4 \ln |x| - \ln |2x+1| + \frac{1}{2} \sin 2x + c$



Worked Example



$f'(x) = 6e^{2x} - \frac{1}{x}$, $x > 0$, and $f(1) = 3e^2 + 2$. Find $f(x)$.

$$f(x) = 3e^{2x} - \ln x + c$$

← INTEGRATE

$$3e^2 - 0 + c = 3e^2 + 2 \Rightarrow c = 2$$

← USE THE POINT

ANSWER $f(x) = 3e^{2x} - \ln x + 2$



Your notes

HOW THIS IS EXAMINED • past WMA13 papers

exponential NOT SEEN YET • BE READY trig NOT SEEN YET • BE READY $\frac{k}{ax+b}$ SEEN 5× IN EXAMS

power NOT SEEN YET • BE READY a^x NOT SEEN YET • BE READY

reciprocal powers SEEN 1× IN EXAMS

Reverse Chain Rule

NOTE 12.2



Reverse Chain Rule

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How do I recognise a reverse chain rule integral?

- Look for a function **and its derivative** in the same integrand (up to a constant multiple).
- Two main patterns:

RECOGNITION RESULTS

LEARN THIS: NOT GIVEN

$$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c \quad \int f'(x)(f(x))^n dx = \frac{(f(x))^{n+1}}{n+1} + c$$

- If the numerator is only a **multiple** of $f'(x)$, adjust with a constant: $\int \frac{9x}{3x^2 + k} dx = \frac{9}{6} \int \frac{6x}{3x^2 + k} dx = \frac{3}{2} \ln(3x^2 + k)$.
- The same idea works with e : $\int f'(x)e^{f(x)} dx = e^{f(x)} + c$.
- $\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\ln |\cos x| = \ln |\sec x| + c$.

$$\int \frac{e^{2x}}{e^{2x} + 3} dx$$

derivative of the
bottom is $2e^{2x}$
answer $\frac{1}{2} \ln(e^{2x} + 3)$

$$\int x(x^2 + 1)^4 dx$$

derivative of the
bracket is $2x$
answer $\frac{1}{10}(x^2 + 1)^5$

Spot the pair: the bottom (or the bracket) and its derivative.



Examiner Tips and Tricks

- Guess the answer, differentiate it, and adjust the constant. This is quicker and safer than a substitution.
- “Hence” after a differentiation part often means the integrand is (a multiple of) that derivative.



WHEN YOU SEE → YOU DO

a fraction whose top is (a multiple of) the derivative of its bottom

→ $k \ln |\text{bottom}|$

a bracket to a power, multiplied by (a multiple of) the bracket's derivative

→ raise the power, divide by the new power and the constant

“Given that $\int_2^5 \frac{9x}{3x^2+k} dx = \ln 8$, find k ”

→ integrate, apply limits, use log laws, solve



Worked Example



Find (a) $\int \frac{e^{2x}}{e^{2x}+3} dx$ (b) $\int x(x^2+1)^4 dx$ (c) $\int \cos x \sin^3 x dx$.

$$\frac{1}{2} \ln(e^{2x} + 3)$$

← (A) TOP IS HALF THE DERIVATIVE OF THE BOTTOM

$$\frac{1}{2} \cdot \frac{(x^2+1)^5}{5}$$

← (B) x IS HALF OF $2x$

$$\frac{1}{4} \sin^4 x$$

← (C) $\cos x$ IS THE DERIVATIVE OF $\sin x$

ANSWER (a) $\frac{1}{2} \ln(e^{2x} + 3) + c$ (b) $\frac{1}{10}(x^2 + 1)^5 + c$ (c) $\frac{1}{4} \sin^4 x + c$



Worked Example



Given that $k > 0$ and $\int_1^3 \frac{2x}{x^2 + k} dx = \ln 2$, find k .

$$\left[\ln(x^2 + k) \right]_1^3 = \ln(9 + k) - \ln(1 + k)$$

$$\ln \frac{9+k}{1+k} = \ln 2 \Rightarrow 9+k = 2+2k$$

← THE TOP IS EXACTLY THE DERIVATIVE OF THE BOTTOM

← LOG LAW, THEN COMPARE

ANSWER $k = 7$

HOW THIS IS EXAMINED • past WMA13 papers

$\frac{f'}{f}$ to $\ln$ **SEEN 6× IN EXAMS** $f' f^n$ **SEEN 8× IN EXAMS**



Your notes

Integrating with Trigonometric Identities

NOTE 12.3



Integrating with Trigonometric Identities

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How do I integrate $\sin^2 x$ and $\cos^2 x$?

- There is no direct rule. Use the double angle formula to remove the square (6.3):

POWER REDUCTION

LEARN THIS: NOT GIVEN

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

- So $\int \sin^2 x \, dx = \frac{x}{2} - \frac{1}{4} \sin 2x + c$ and $\int \cos^2 x \, dx = \frac{x}{2} + \frac{1}{4} \sin 2x + c$.

What about other trig integrals?

- $\tan^2 x = \sec^2 x - 1$, so $\int \tan^2 x \, dx = \tan x - x + c$.
- $\cot^2 x = \operatorname{cosec}^2 x - 1$, and $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$.
- $(\sin x + \cos x)^2 = 1 + 2 \sin x \cos x = 1 + \sin 2x$.
- $\sin^3 x$: use a supplied identity such as $\sin 3x = 3 \sin x - 4 \sin^3 x$, rearranged to $\sin^3 x = \frac{1}{4}(3 \sin x - \sin 3x)$.
- $\sin x \cos x = \frac{1}{2} \sin 2x$.

INTEGRATING A TRIG EXPRESSION

STEP 1

Is it a standard result or a $\frac{f'}{f}$ / $f'f^n$ pattern?

STEP 2

If not: is there a square or a product?

STEP 3

Rewrite with an identity into single sines and cosines

STEP 4

Integrate term by term; divide by inner multipliers



Examiner Tips and Tricks

- The question usually tells you the identity in part (a) ("Show that $(1 + 2 \cos 2x)^2 \equiv p + q \cos 2x + r \cos 4x$ "). Use it in part (b).
- Exact limits such as $\frac{\pi}{3}$ need exact trig values: $\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$.



WHEN YOU SEE → YOU DO

$$\int \sin^2 kx \text{ or } \int \cos^2 kx$$

→ power reduction with $\cos 2kx$

$$\int \tan^2 x$$

→ $\sec^2 x - 1$

"(a) Show that $\sin 3x \equiv 3 \sin x - 4 \sin^3 x$ (b) Hence find $\int_0^{\pi/3} \sin^3 x \, dx$ "

→ rearrange for $\sin^3 x$, integrate

$$(\sin x + \cos x)^2 \text{ or } (1 + 2 \cos 2x)^2$$

→ expand, then reduce squares



Worked Example



Find (a) $\int_0^{\pi/2} \cos^2 x \, dx$ (b) $\int \tan^2 x \, dx$ (c) $\int (\sin x + \cos x)^2 \, dx$.

$$\left[\frac{x}{2} + \frac{1}{4} \sin 2x \right]_0^{\pi/2} = \frac{\pi}{4}$$

← (A) POWER REDUCTION

$$\int (\sec^2 x - 1) \, dx = \tan x - x$$

← (B) PYTHAGOREAN IDENTITY

$$\int (1 + \sin 2x) \, dx = x - \frac{1}{2} \cos 2x$$

← (C) EXPAND; $\sin^2 + \cos^2 = 1$

ANSWER (a) $\frac{\pi}{4}$ (b) $\tan x - x + c$ (c) $x - \frac{1}{2} \cos 2x + c$



Worked Example



Using $\sin 3x \equiv 3 \sin x - 4 \sin^3 x$, find the exact value of $\int_0^{\pi/3} \sin^3 x \, dx$.

$$\sin^3 x = \frac{1}{4}(3 \sin x - \sin 3x)$$

← REARRANGE THE IDENTITY

$$\left[-\frac{3}{4} \cos x + \frac{1}{12} \cos 3x \right]_0^{\pi/3}$$

← INTEGRATE EACH TERM

$$\left(-\frac{3}{8} - \frac{1}{12} \right) - \left(-\frac{3}{4} + \frac{1}{12} \right)$$

← $\cos \frac{\pi}{3} = \frac{1}{2}, \cos \pi = -1$

ANSWER $\frac{5}{24}$

HOW THIS IS EXAMINED • past WMA13 papers

$\sin^2, \cos^2$ **SEEN 4× IN EXAMS** $\tan^2$ **NOT SEEN YET • BE READY**

reciprocal Pythagorean **NOT SEEN YET • BE READY** cubic via identity **SEEN 1× IN EXAMS**

proved identity **SEEN 1× IN EXAMS** square of a sum **SEEN 1× IN EXAMS**



Your notes

Definite Integrals and Areas

NOTE 12.4

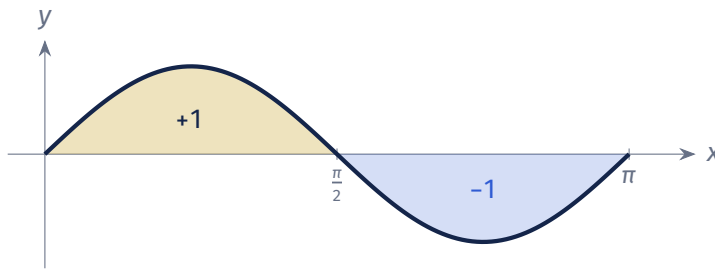


Definite Integrals and Areas

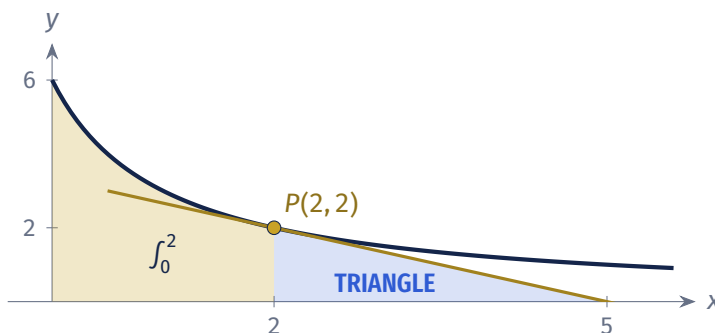
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How do I find an area under a curve?

- $\int_a^b f(x) dx$ gives the area between the curve and the x-axis **when the curve is above the axis**.
- Parts **below** the axis give **negative** values. Find where the curve crosses the axis, integrate each part separately, and add the sizes.
- A region bounded by a curve and a **straight line** is often “area under the curve” $\pm$ a **triangle** or trapezium. Sketch it and split it.



$$\int_0^{\pi} \sin 2x dx = 0, \text{ but the shaded area is } 1 + 1 = 2.$$



A region made of the area under a curve plus a triangle under a tangent.



Examiner Tips and Tricks

- “Find, using algebraic integration, the exact area”: show the integral with its limits and the antiderivative before substituting.
- Give the answer in the form asked, e.g. “ $P + Q \ln 3$, where P and Q are rational”.
- For a curve that touches the axis (a square such as $(1 - 2 \cos 2x)^2$), it never goes below, so a single integral gives the area.

 Your notes

WHEN YOU SEE → YOU DO

**“Find the exact area of the shaded region”
with a curve crossing the axis**

→ find the crossings, integrate each part, add sizes

**“The region R is bounded by C, the line l
and the x -axis”**

→ curve integral $\pm$ triangle;
find where l meets the axis

**“touches the positive x -axis for the second
time when $x = a$ ”**

→ solve curve = 0; take the second positive root as the limit

“giving your answer in the form $a\pi + b\sqrt{3}$ ”

→ exact trig values at the limits

**Worked Example**

The curve $y = (1 - 2 \cos 2x)^2$ touches the positive x -axis for the second time at $x = a$. (a) Show that $(1 - 2 \cos 2x)^2 \equiv 3 - 4 \cos 2x + 2 \cos 4x$. (b) Find the exact area between the curve, the axes and $x = a$.

$$1 - 4 \cos 2x + 4 \cos^2 2x = \\ 1 - 4 \cos 2x + 2(1 + \cos 4x)$$

← (A) EXPAND; POWER REDUCTION

$$\cos 2x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6} \Rightarrow a = \frac{5\pi}{6}$$

← (B) TOUCHES WHERE THE BRACKET IS 0

$$\left[3x - 2 \sin 2x + \frac{1}{2} \sin 4x \right]_0^{5\pi/6}$$

← THE CURVE IS NEVER BELOW THE AXIS

$$= \frac{5\pi}{2} - 2 \left(-\frac{\sqrt{3}}{2} \right) + \frac{1}{2} \left(-\frac{\sqrt{3}}{2} \right)$$

← EXACT VALUES AT $\frac{5\pi}{3}$ AND $\frac{10\pi}{3}$

ANSWER Area = $\frac{5\pi}{2} + \frac{3\sqrt{3}}{4}$

**Worked Example**

The curve C has equation $y = \frac{6}{x+1}$, $x \geq 0$. The tangent to C at $P(2, 2)$ meets the x -axis at Q . Find the exact area of the region bounded by C , the y -axis, the x -axis and the tangent.

$$\frac{dy}{dx} = -\frac{6}{(x+1)^2} = -\frac{2}{3} \Rightarrow y - 2 = -\frac{2}{3}(x - 2)$$

← TANGENT AT P

$$y = 0 : x = 5 \Rightarrow Q(5, 0)$$

← THE TRIANGLE HAS BASE 3, HEIGHT 2

$$\int_0^2 \frac{6}{x+1} dx = \left[6 \ln(x+1) \right]_0^2 = 6 \ln 3$$

← AREA UNDER THE CURVE

ANSWER Area = $6 \ln 3 + 3$

HOW THIS IS EXAMINED • past WMA13 papersexact definite integral **SEEN 1× IN EXAMS** find a constant **SEEN 1× IN EXAMS**area of a bounded region **SEEN 2× IN EXAMS** curve plus line geometry **SEEN 3× IN EXAMS**

Your notes

MISTAKE GALLERY

Where marks are lost

× **WRONG**

$$\int e^{3x} dx = 3e^{3x}.$$

✓ **RIGHT**

$$\frac{1}{3}e^{3x} + c.$$

Integrating divides by the inner multiplier.

× **WRONG**

$$\int \frac{1}{2x+1} dx = \ln |2x+1|.$$

✓ **RIGHT**

$$\frac{1}{2} \ln |2x+1| + c.$$

Check by differentiating.

× **WRONG**

$$\int \sin^2 x dx = \frac{\sin^3 x}{3}.$$

✓ **RIGHT**

$$\frac{x}{2} - \frac{1}{4} \sin 2x + c.$$

The reverse power rule needs the derivative $\cos x$ as a factor.

× **WRONG**

$$\text{Area} = \int_0^\pi \sin 2x dx = 0.$$

✓ **RIGHT**

Split at $\frac{\pi}{2}$: area = $1 + 1 = 2$.

Regions below the axis give negative integrals.

× **WRONG**

Forgetting $+c$.

✓ **RIGHT**

Every indefinite integral needs $+c$.

It is a mark on its own.

THE CHAPTER ON ONE PAGE

Integration



12.1

Standard

$$e^{kx} \rightarrow \frac{1}{k} e^{kx}; \quad \frac{1}{ax+b} \rightarrow \frac{1}{a} \ln |ax+b|.$$

$$\cos kx \rightarrow \frac{1}{k} \sin kx; \quad \sin kx \rightarrow -\frac{1}{k} \cos kx.$$

$$\sec^2 kx \rightarrow \frac{1}{k} \tan kx; + c.$$

12.2

Recognition

$$\frac{f'}{f} \rightarrow \ln |f|.$$

$$f' f^n \rightarrow \frac{f^{n+1}}{n+1}.$$

Adjust constants; check by differentiating.

12.3

Identities

$\sin^2, \cos^2$: use $\cos 2x$.
 $\tan^2 = \sec^2 - 1$.
 Expand squares; use given identities.

12.4

Areas

Split where the curve crosses the axis.
 Curve $\pm$ triangle for line regions.
 Exact values; form asked.

BOTTOM LINE Ask “what differentiates to this?”. Divide by the inner multiplier, spot f' next to f , and use an identity when there is a square.

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13

CHAPTER 13

Numerical Root-Finding

When an equation cannot be solved exactly, trap the root between two values and close in on it with an iteration. Calculator work, but every step must be shown.

3

CONCEPTS

7

VARIANTS

19

EXAM QUESTIONS

YOU WILL MASTER

- ▶ Show a root lies in an interval using a change of sign, and know when that fails
- ▶ Rearrange $f(x) = 0$ into the iteration form $x = g(x)$ that is asked for
- ▶ Use $x_{n+1} = g(x_n)$ to find terms and a root to the stated accuracy
- ▶ Prove a root is correct to a given number of decimal places

CHAPTER 13 AT A GLANCE

Two note pages, seven ways to be asked



CHAPTER 13 MAP

Numerical Methods

01

13.1 Change of sign

root in an interval

when it fails: asymptotes

root to n d.p.

02

13.2 Iteration $x = g(x)$

rearrange to the given form

start value and index

successive iterates

staircase and cobweb

Contents

✱ 13.1	Change of Sign	3
✱ 13.2	Iteration: $x = g(x)$	6

Change of Sign

NOTE 13.1

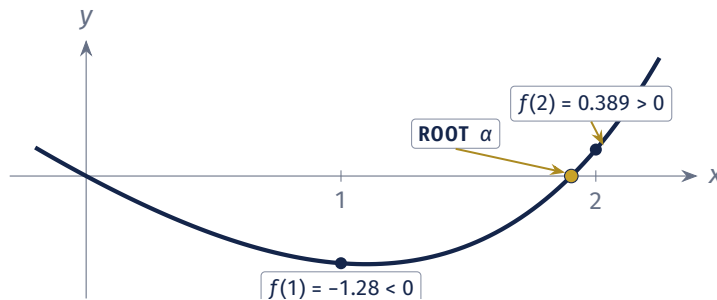


Change of Sign

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How does a change of sign show there is a root?

- Write the equation as $f(x) = 0$.
- If f is **continuous** on $[a, b]$ and $f(a), f(b)$ have **opposite signs**, the curve must cross the x -axis between a and b : there is a root α with $a < \alpha < b$.



$f(x) = e^x - 3x - 1$: negative at $x = 1$, positive at $x = 2$, so a root lies between.

When does the change of sign method fail?

- If f is **not continuous** (an asymptote) in the interval, the sign can change **without** a root: $\tan 1 > 0$ and $\tan 2 < 0$, but $\tan x$ has an asymptote at $\frac{\pi}{2}$, not a root.
- If there are **two roots** close together, the sign may not change at all.

How do I show a root is correct to n decimal places?



- To show $\alpha = 1.904$ to 3 d.p., test the **bounds** 1.9035 and 1.9045.
- A change of sign between them (with continuity) proves $1.9035 < \alpha < 1.9045$, so α rounds to 1.904.



Examiner Tips and Tricks

- Full marks need three things: both values (with their signs), the words “**change of sign**” and “**continuous**”, and a conclusion: “hence there is a root in $[1, 2]$ ”.
- Give the function values to at least 2 significant figures, or just their signs if they are clearly positive or negative.

WHEN YOU SEE → YOU DO

“Show that the equation $f(x) = 0$ has a root α in the interval $[6, 7]$ ”

→ $f(6), f(7)$, opposite signs, continuous, conclusion

“Hence show that α lies between 1 and 2”

→ the “hence” equation is $f(x) = 0$: test $f(1)$ and $f(2)$

“By choosing a suitable interval, show that $\alpha = 1.904$ to 3 decimal places”

→ test 1.9035 and 1.9045



Worked Example



$f(x) = e^x - 3x - 1$. (a) Show that $f(x) = 0$ has a root α in the interval $[1, 2]$. (b) Show that $\alpha = 1.904$ correct to 3 decimal places.

$$f(1) = -1.28 < 0, \quad f(2) = 0.389 > 0$$

← (A) EVALUATE BOTH ENDS

change of sign and f continuous

← BOTH CONDITIONS STATED

$$f(1.9035) = -0.0012 < 0, \quad f(1.9045) = 0.0025 > 0$$

← (B) THE ROUNDING BOUNDS

ANSWER (a) root in $[1, 2]$ (b) change of sign, continuous, so $1.9035 < \alpha < 1.9045$: $\alpha = 1.904$ (3 d.p.)

HOW THIS IS EXAMINED · past WMA13 papers

change of sign **SEEN 14× IN EXAMS** failure at a pole **NOT SEEN YET · BE READY**

stated accuracy **NOT SEEN YET · BE READY**



Your notes

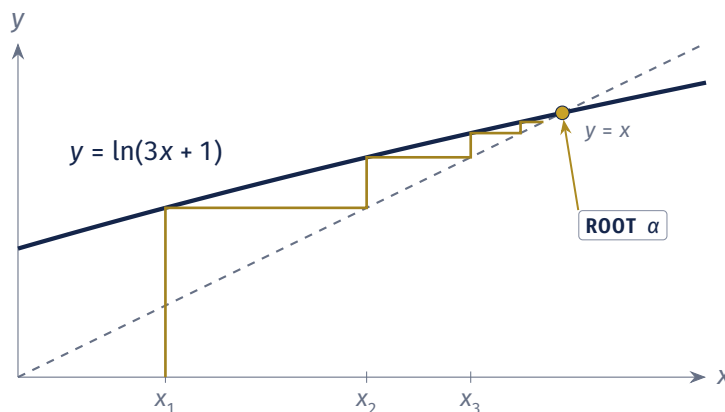
Iteration: $x = g(x)$ **NOTE 13.2**

Iteration: $x = g(x)$

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What is fixed-point iteration?

- Rearrange $f(x) = 0$ into the form $x = g(x)$.
- Use the **iteration formula** $x_{n+1} = g(x_n)$: put a starting value x_1 in, get x_2 , put x_2 in, get x_3 , and so on.
- If the values settle down (**converge**), they approach a root α , where the line $y = x$ meets $y = g(x)$.



Staircase diagram for $x_{n+1} = \ln(3x_n + 1)$ starting at $x_1 = 1.5$: the steps climb towards $\alpha \approx 1.904$.

How do I show the rearrangement?

- Start from $f(x) = 0$ and rearrange **step by step** until you reach exactly the given form. Do not start from the answer.
- Typical moves: isolate an exponential and take $\ln$; isolate a power and take a root; divide by x .

How do I use the iteration formula?

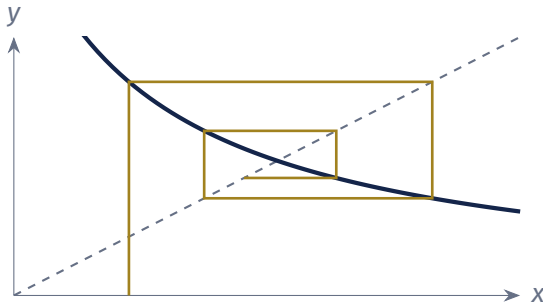
STEP 1 Type the starting value, e.g. 1.5, and press $=$.

STEP 2 Type the formula with **ANS** in place of x_n : $\ln(3 \text{ ANS} + 1)$.



STEP 3 Press $=$ repeatedly. Each press gives the next term.

STEP 4 Write down each requested term to the stated accuracy, but keep the **full value** in the calculator.



A different g can give a **cobweb** instead: the values alternate either side of α while converging.



Examiner Tips and Tricks

- Check the **index**: if $x_1 = 1.5$, then " x_2 " is the first value you calculate.
- "Find, to 4 decimal places, the value of x_3 and the value of α ": give x_3 from the calculator, and α when the values stop changing at that accuracy.
- Radians! Iterations with trig functions need radian mode.
- Iteration questions are often the last part of a longer question (a tangent through the origin, a model with $\ln$): the "show that" form is given, so you can always do the iteration even if earlier parts went wrong.

WHEN YOU SEE → YOU DO

"Show that the equation can be written in the form $x = \ln(3x + 1)$ "

→ start from $f(x) = 0$; rearrange to exactly that form

"Using the iteration formula $x_{n+1} = \dots$ with $x_1 = 1.5$, find x_2 and x_3 "

→ ANS method; check the index; 4 d.p.

"find, to 4 decimal places, the value of α "

→ iterate until the 4 d.p. value is stable



Worked Example



(a) Show that $e^x - 3x - 1 = 0$ can be written as $x = \ln(3x + 1)$. (b) Use $x_{n+1} = \ln(3x_n + 1)$ with $x_1 = 1.5$ to find x_2 and x_3 to 4 d.p. (c) Find α to 3 d.p.

$$e^x = 3x + 1 \Rightarrow x = \ln(3x + 1)$$

← (A) ISOLATE e^x , TAKE $\ln$

$$x_2 = \ln(5.5) = 1.7047$$

← (B) FIRST PRESS

$$x_3 = \ln(3 \times 1.70475 \dots + 1) = 1.8106$$

← USE THE FULL VALUE OF x_2

$$x_4 = 1.8613, \dots, x_9 = 1.9030, \dots \rightarrow 1.9038$$

← (C) KEEP GOING UNTIL STABLE

ANSWER (b) $x_2 = 1.7047$, $x_3 = 1.8106$ (c) $\alpha = 1.904$ (3 d.p.)

HOW THIS IS EXAMINED · past WMA13 papers

rearrangement **SEEN 13× IN EXAMS** start value and index **SEEN 8× IN EXAMS**

successive iterates **SEEN 17× IN EXAMS**



Your notes

MISTAKE GALLERY

Where marks are lost

× **WRONG**

" $f(1) < 0$ and $f(2) > 0$ so there is a root."

✓ **RIGHT**

Add "change of sign **and** f is continuous".

Both conditions are needed for the mark.

× **WRONG**

Claiming a root of $\tan x$ in $[1, 2]$ from a sign change.

✓ **RIGHT**

There is an asymptote at $\frac{\pi}{2}$, not a root.

Check continuity first.

× **WRONG**

Using the rounded $x_2 = 1.7047$ to get x_3 .

✓ **RIGHT**

Use the full calculator value (ANS).

Rounding errors build up.

× **WRONG**

Calling the first calculated value x_1 .

✓ **RIGHT**

If x_1 is given, the first calculated value is x_2 .

Read the starting index.

× **WRONG**

Testing 1.903 and 1.905 to show $\alpha = 1.904$.

✓ **RIGHT**

Test 1.9035 and 1.9045.

The interval must be the rounding bounds.

THE CHAPTER ON ONE PAGE

Numerical Root-Finding

**13.1****Change of sign**

$f(a)$ and $f(b)$ opposite signs, f continuous: root in (a, b) .

Fails at asymptotes, or with two close roots.

13.1**Accuracy**

To show $\alpha = 1.904$: test 1.9035 and 1.9045.

13.2**Rearrange**

From $f(x) = 0$ to the exact form given, step by step.

13.2**Iterate**

$x_{n+1} = g(x_n)$ with ANS.

Check the index; keep full values; round only answers.

Staircase or cobweb towards α .

BOTTOM LINE Trap the root with a sign change, close in with an iteration, and prove the final digits with the rounding bounds.