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CHAPTER 13

Numerical Root-Finding

When an equation cannot be solved exactly, trap the root between two values and close in on it with an iteration. Calculator work, but every step must be shown.

3	7	19
CONCEPTS	VARIANTS	EXAM QUESTIONS

YOU WILL MASTER

- Show a root lies in an interval using a change of sign, and know when that fails
- Rearrange $f(x) = 0$ into the iteration form $x = g(x)$ that is asked for
- Use $x_{n+1} = g(x_n)$ to find terms and a root to the stated accuracy
- Prove a root is correct to a given number of decimal places

CHAPTER 13 AT A GLANCE

Two note pages, seven ways to be asked

Your notes

CHAPTER 13 MAP

Numerical Methods

01 **13.1 Change of sign**

- root in an interval
- when it fails: asymptotes
- root to *n* d.p.

02 **13.2 Iteration $x = g(x)$**

- rearrange to the given form
- start value and index
- successive iterates
- staircase and cobweb

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13.2 Iteration: $x = g(x)$ 6

Change of Sign NOTE 13.1

Your notes

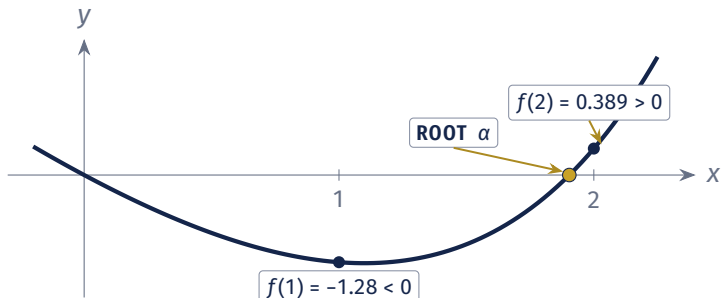
VISUAL NOTES · P3.13

Change of Sign

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How does a change of sign show there is a root?

- Write the equation as $f(x) = 0$.
- If f is **continuous** on $[a, b]$ and $f(a)$, $f(b)$ have **opposite signs**, the curve must cross the x-axis between a and b: there is a root α with $a < \alpha < b$.



$f(x) = e^x - 3x - 1$: negative at $x = 1$, positive at $x = 2$, so a root lies between.

When does the change of sign method fail?

- If f is **not continuous** (an asymptote) in the interval, the sign can change **without** a root: $\tan 1 > 0$ and $\tan 2 < 0$, but $\tan x$ has an asymptote at $\frac{\pi}{2}$, not a root.
- If there are **two roots** close together, the sign may not change at all.

How do I show a root is correct to n decimal places?

Your notes

- To show $\alpha = 1.904$ to 3 d.p., test the **bounds** 1.9035 and 1.9045.
- A change of sign between them (with continuity) proves $1.9035 < \alpha < 1.9045$, so α rounds to 1.904.

Examiner Tips and Tricks

- Full marks need three things: both values (with their signs), the words "**change of sign**" and "**continuous**", and a conclusion: "hence there is a root in $[1, 2]$".
- Give the function values to at least 2 significant figures, or just their signs if they are clearly positive or negative.

WHEN YOU SEE → YOU DO

"Show that the equation $f(x) = 0$ has a root α in the interval $[6, 7]$"	→ $f(6)$, $f(7)$, opposite signs, continuous, conclusion
"Hence show that α lies between 1 and 2"	→ the "hence" equation is $f(x) = 0$: test $f(1)$ and $f(2)$
"By choosing a suitable interval, show that $\alpha = 1.904$ to 3 decimal places"	→ test 1.9035 and 1.9045

Your notes

Worked Example

? $f(x) = e^x - 3x - 1$. (a) Show that $f(x) = 0$ has a root α in the interval $[1, 2]$. (b) Show that $\alpha = 1.904$ correct to 3 decimal places.

$f(1) = -1.28 < 0$, $f(2) = 0.389 > 0$ ← (A) EVALUATE BOTH ENDS

change of sign and f continuous ← BOTH CONDITIONS STATED

$f(1.9035) = -0.0012 < 0$, $f(1.9045) = 0.0025 > 0$ ← (B) THE ROUNDING BOUNDS

ANSWER (a) root in $[1, 2]$ (b) change of sign, continuous, so $1.9035 < \alpha < 1.9045$: $\alpha = 1.904$ (3 d.p.)

HOW THIS IS EXAMINED · past WMA13 papers

change of sign SEEN 14× IN EXAMS failure at a pole NOT SEEN YET • BE READY
stated accuracy NOT SEEN YET • BE READY

Iteration: $x = g(x)$ NOTE 13.2

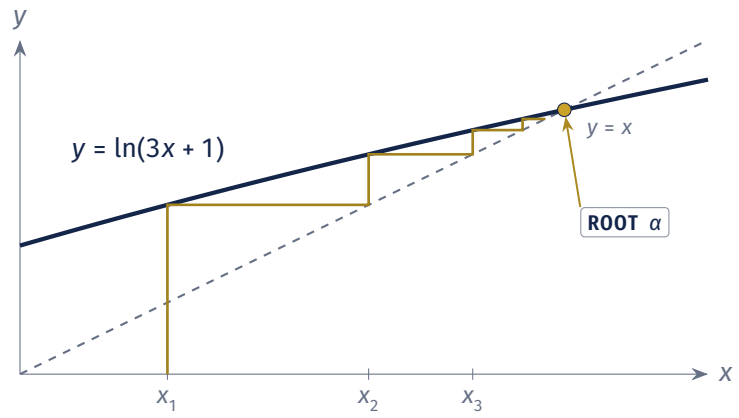
Your notes

Iteration: $x = g(x)$

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What is fixed-point iteration?

- Rearrange $f(x) = 0$ into the form $x = g(x)$.
- Use the **iteration formula** $x_{n+1} = g(x_n)$: put a starting value x_1 in, get x_2, put x_2 in, get x_3, and so on.
- If the values settle down (**converge**), they approach a root α, where the line $y = x$ meets $y = g(x)$.



Staircase diagram for $x_{n+1} = \ln(3x_n + 1)$ starting at $x_1 = 1.5$: the steps climb towards $\alpha \approx 1.904$.

How do I show the rearrangement?

- Start from $f(x) = 0$ and rearrange **step by step** until you reach exactly the given form. Do not start from the answer.
- Typical moves: isolate an exponential and take ln; isolate a power and take a root; divide by x.

How do I use the iteration formula?

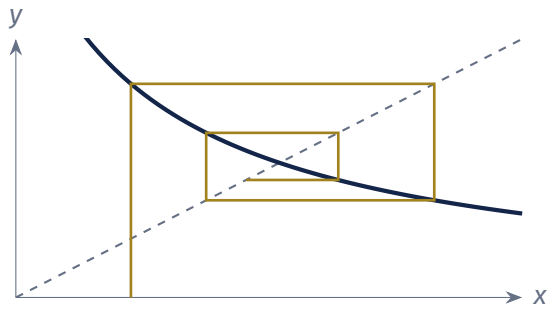
STEP 1 Type the starting value, e.g. 1.5, and press **=**.

STEP 2 Type the formula with **ANS** in place of x_n: ln(3 ANS + 1).

STEP 3 Press **=** repeatedly. Each press gives the next term.

STEP 4 Write down each requested term to the stated accuracy, but keep the **full value** in the calculator.

Your notes



A different g can give a **cobweb** instead: the values alternate either side of α while converging.

Examiner Tips and Tricks

- Check the **index**: if $x_1 = 1.5$, then "x_2" is the first value you calculate.
- "Find, to 4 decimal places, the value of x_3 and the value of α": give x_3 from the calculator, and α when the values stop changing at that accuracy.
- Radians! Iterations with trig functions need radian mode.
- Iteration questions are often the last part of a longer question (a tangent through the origin, a model with ln): the "show that" form is given, so you can always do the iteration even if earlier parts went wrong.

WHEN YOU SEE → YOU DO

"Show that the equation can be written in the form $x = \ln(3x + 1)$**"**	→ start from $f(x) = 0$; rearrange to exactly that form
"Using the iteration formula $x_{n+1} = \ldots$ **with** $x_1 = 1.5$**, find** x_2 **and** x_3**"**	→ ANS method; check the index; 4 d.p.
"find, to 4 decimal places, the value of α**"**	→ iterate until the 4 d.p. value is stable

Your notes

Worked Example

? (a) Show that $e^x - 3x - 1 = 0$ can be written as $x = \ln(3x + 1)$. (b) Use $x_{n+1} = \ln(3x_n + 1)$ with $x_1 = 1.5$ to find x_2 and x_3 to 4 d.p. (c) Find α to 3 d.p.

$e^x = 3x + 1 \Rightarrow x = \ln(3x + 1)$ ← **(A) ISOLATE e^x, TAKE ln**

$x_2 = \ln(5.5) = 1.7047$ ← **(B) FIRST PRESS**

$x_3 = \ln(3 \times 1.70475\cdots + 1) = 1.8106$ ← **USE THE FULL VALUE OF x_2**

$x_4 = 1.8613, \ldots, x_9 = 1.9030, \ldots \to 1.9038$ ← **(C) KEEP GOING UNTIL STABLE**

ANSWER (b) $x_2 = 1.7047$, $x_3 = 1.8106$ (c) $\alpha = 1.904$ (3 d.p.)

HOW THIS IS EXAMINED · past WMA13 papers

rearrangement SEEN 13× IN EXAMS start value and index SEEN 8× IN EXAMS
successive iterates SEEN 17× IN EXAMS

MISTAKE GALLERY

Where marks are lost

Your notes

× WRONG

"$f(1) < 0$ and $f(2) > 0$ so there is a root."

✓ RIGHT

Add "change of sign **and** f is continuous".

Both conditions are needed for the mark.

× WRONG

Claiming a root of $\tan x$ in $[1, 2]$ from a sign change.

✓ RIGHT

There is an asymptote at $\frac{\pi}{2}$, not a root.

Check continuity first.

× WRONG

Using the rounded $x_2 = 1.7047$ to get x_3.

✓ RIGHT

Use the full calculator value (ANS).

Rounding errors build up.

× WRONG

Calling the first calculated value x_1.

✓ RIGHT

If x_1 is given, the first calculated value is x_2.

Read the starting index.

× WRONG

Testing 1.903 and 1.905 to show $\alpha = 1.904$.

✓ RIGHT

Test 1.9035 and 1.9045.

The interval must be the rounding bounds.

THE CHAPTER ON ONE PAGE

Numerical Root-Finding

Your notes

13.1

Change of sign

$f(a)$ and $f(b)$ opposite signs, f continuous: root in (a, b).
Fails at asymptotes, or with two close roots.

13.1

Accuracy

To show $\alpha = 1.904$: test 1.9035 and 1.9045.

13.2

Rearrange

From $f(x) = 0$ to the exact form given, step by step.

13.2

Iterate

$x_{n+1} = g(x_n)$ with ANS.
Check the index; keep full values; round only answers.
Staircase or cobweb towards α.

BOTTOM LINE Trap the root with a sign change, close in with an iteration, and prove the final digits with the rounding bounds.