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CHAPTER 12

Integration

Integration is differentiation run backwards. Spot the derivative you are looking at, divide by any inner multiplier, and use an identity when nothing matches.

5 CONCEPTS

18 VARIANTS

26 EXAM QUESTIONS

YOU WILL MASTER

- Integrate e^{kx}, $\frac{1}{ax+b}$, $\sin kx$, $\cos kx$, $\sec^2 kx$ and a^x
- Recognise $\frac{f'(x)}{f(x)}$ and $f'(x)\big(f(x)\big)^n$ and integrate them at sight
- Use $\cos 2x$, $\tan^2 x = \sec^2 x - 1$ and other identities before integrating
- Evaluate definite integrals exactly and find areas, including regions made with lines

CHAPTER 12 AT A GLANCE

Four note pages, eighteen ways to be asked

Your notes

CHAPTER 12 MAP

Integration

01 12.1 Standard integrals

e^{kx}, a^x | $\frac{1}{ax+b}$ | $\sin kx$, $\cos kx$, $\sec^2 kx$ | $(ax+b)^n$

find f from f'

02 12.2 Reverse chain rule

$\frac{f'(x)}{f(x)} \to \ln$ | $f'(x)\big(f(x)\big)^n$

03 12.3 Using trig identities

$\sin^2$, $\cos^2$ | $\tan^2$ | $\sin^3$ via a given identity | $(\sin x + \cos x)^2$

04 12.4 Definite integrals and areas

exact values | find a constant

areas above and below the axis | curve plus line regions

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Integrating Standard Functions NOTE 12.1

Your notes

Integrating Standard Functions

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What are the standard integrals?

$f(x)$	$\int f(x)\,dx$	REVERSE OF
e^{kx}	$\frac{1}{k}e^{kx} + c$	$\frac{d}{dx}e^{kx} = ke^{kx}$
a^x	$\frac{a^x}{\ln a} + c$	$\frac{d}{dx}a^x = a^x \ln a$
$\frac{1}{x}$	$\ln \lvert x \rvert + c$	$\frac{d}{dx}\ln x = \frac{1}{x}$
$\frac{1}{ax+b}$	$\frac{1}{a}\ln \lvert ax+b \rvert + c$	chain rule
$(ax+b)^n,\ n \neq -1$	$\frac{(ax+b)^{n+1}}{a(n+1)} + c$	chain rule
$\cos kx$	$\frac{1}{k}\sin kx + c$	$\frac{d}{dx}\sin kx$
$\sin kx$	$-\frac{1}{k}\cos kx + c$	$\frac{d}{dx}\cos kx$
$\sec^2 kx$	$\frac{1}{k}\tan kx + c$	$\frac{d}{dx}\tan kx$

- Pattern: when the inside is **linear** (kx or $ax+b$), integrate as usual and **divide by the coefficient of** x.
- Always add $+c$ to an indefinite integral.
- To find a function from its derivative, integrate, then use the given point to find c.

e.g. **FIND** $\int \frac{4}{(2x-1)^3}\,dx$

$$\int 4(2x-1)^{-3}\,dx = \frac{4(2x-1)^{-2}}{-2 \times 2}$$

← POWER UP BY 1, DIVIDE BY NEW POWER AND BY 2

ANSWER $-\frac{1}{(2x-1)^2} + c$

Your notes

Examiner Tips and Tricks

- Check any answer by differentiating it: you should get back the integrand.
- $\int \frac{1}{3x}\,dx = \frac{1}{3}\ln|x| + c$. Writing $\frac{1}{3}\ln|3x| + c$ is also correct: the two differ only by a constant.
- Exact answers: keep ln, e, π and surds.

WHEN YOU SEE → YOU DO

$\frac{k}{ax+b}$	$\longrightarrow \frac{k}{a}\ln\lvert ax+b\rvert$
e^{kx}, $\cos kx$, $\sin kx$	$\longrightarrow$ standard result $\div k$
"Given $f'(x) = \ldots$ and $f(1) = \ldots$, find $f(x)$"	$\longrightarrow$ integrate, then use the point to find c

Worked Example

? Find $\int \left(e^{3x} + \frac{4}{x} - \frac{2}{2x+1} + \cos 2x\right) dx$.

$\frac{1}{3}e^{3x}$ ← $\div 3$

$4\ln|x| - \ln|2x+1|$ ← $\frac{2}{2x+1}\colon \frac{2}{2}\ln|2x+1|$

$\frac{1}{2}\sin 2x$ ← $\div 2$

ANSWER $\frac{1}{3}e^{3x} + 4\ln|x| - \ln|2x+1| + \frac{1}{2}\sin 2x + c$

Your notes

Worked Example

? $f'(x) = 6e^{2x} - \frac{1}{x}$, $x > 0$, and $f(1) = 3e^2 + 2$. Find $f(x)$.

$f(x) = 3e^{2x} - \ln x + c$ ← INTEGRATE

$3e^2 - 0 + c = 3e^2 + 2 \Rightarrow c = 2$ ← USE THE POINT

ANSWER $f(x) = 3e^{2x} - \ln x + 2$

HOW THIS IS EXAMINED · past WMA13 papers

exponential NOT SEEN YET • BE READY trig NOT SEEN YET • BE READY $\frac{k}{ax+b}$ SEEN 5× IN EXAMS
power NOT SEEN YET • BE READY a^x NOT SEEN YET • BE READY
reciprocal powers SEEN 1× IN EXAMS

Reverse Chain Rule NOTE 12.2

Your notes

Reverse Chain Rule

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How do I recognise a reverse chain rule integral?

- Look for a function **and its derivative** in the same integrand (up to a constant multiple).
- Two main patterns:

RECOGNITION RESULTS LEARN THIS: NOT GIVEN

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c \qquad \int f'(x)\big(f(x)\big)^n\,dx = \frac{\big(f(x)\big)^{n+1}}{n+1} + c$$

- If the numerator is only a **multiple** of $f'(x)$, adjust with a constant: $\int \frac{9x}{3x^2+k}\,dx = \frac{9}{6}\int \frac{6x}{3x^2+k}\,dx = \frac{3}{2}\ln(3x^2+k)$.
- The same idea works with e: $\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$.
- $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x| = \ln|\sec x| + c$.

$\int \frac{e^{2x}}{e^{2x}+3}\,dx$ → derivative of the bottom is $2e^{2x}$, answer $\frac{1}{2}\ln(e^{2x}+3)$

$\int x(x^2+1)^4\,dx$ → derivative of the bracket is $2x$, answer $\frac{1}{10}(x^2+1)^5$

Spot the pair: the bottom (or the bracket) and its derivative.

Your notes

Examiner Tips and Tricks

- Guess the answer, differentiate it, and adjust the constant. This is quicker and safer than a substitution.
- "Hence" after a differentiation part often means the integrand is (a multiple of) that derivative.

WHEN YOU SEE → YOU DO

a fraction whose top is (a multiple of) the derivative of its bottom	→ $k \ln \lvert \text{bottom} \rvert$
a bracket to a power, multiplied by (a multiple of) the bracket's derivative	→ raise the power, divide by the new power and the constant
"Given that $\int_2^5 \frac{9x}{3x^2+k}\,dx = \ln 8$, find k"	→ integrate, apply limits, use log laws, solve

Worked Example

Find (a) $\int \frac{e^{2x}}{e^{2x}+3}\,dx$ (b) $\int x(x^2+1)^4\,dx$ (c) $\int \cos x \sin^3 x\,dx$.

$\frac{1}{2}\ln(e^{2x}+3)$ ← **(A) TOP IS HALF THE DERIVATIVE OF THE BOTTOM**

$\frac{1}{2} \cdot \frac{(x^2+1)^5}{5}$ ← **(B)** x **IS HALF OF** $2x$

$\frac{1}{4}\sin^4 x$ ← **(C)** $\cos x$ **IS THE DERIVATIVE OF** $\sin x$

ANSWER (a) $\frac{1}{2}\ln(e^{2x}+3)+c$ (b) $\frac{1}{10}(x^2+1)^5+c$ (c) $\frac{1}{4}\sin^4 x+c$

Your notes

Worked Example

? Given that $k > 0$ and $\int_1^3 \frac{2x}{x^2 + k}\,dx = \ln 2$, find k.

$\left[\ln(x^2 + k)\right]_1^3 = \ln(9 + k) - \ln(1 + k)$ ← THE TOP IS EXACTLY THE DERIVATIVE OF THE BOTTOM

$\ln \frac{9 + k}{1 + k} = \ln 2 \Rightarrow 9 + k = 2 + 2k$ ← LOG LAW, THEN COMPARE

ANSWER $k = 7$

HOW THIS IS EXAMINED · past WMA13 papers

$\frac{f'}{f}$ to ln SEEN 6× IN EXAMS $f' f^n$ SEEN 8× IN EXAMS

Integrating with Trigonometric Identities NOTE 12.3

Your notes

Integrating with Trigonometric Identities

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How do I integrate $\sin^2 x$ and $\cos^2 x$?

- There is no direct rule. Use the double angle formula to remove the square (6.3):

POWER REDUCTION LEARN THIS: NOT GIVEN

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

- So $\int \sin^2 x\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x + c$ and $\int \cos^2 x\,dx = \frac{x}{2} + \frac{1}{4}\sin 2x + c$.

What about other trig integrals?

- $\tan^2 x = \sec^2 x - 1$, so $\int \tan^2 x\,dx = \tan x - x + c$.
- $\cot^2 x = \operatorname{cosec}^2 x - 1$, and $\int \operatorname{cosec}^2 x\,dx = -\cot x + c$.
- $(\sin x + \cos x)^2 = 1 + 2\sin x \cos x = 1 + \sin 2x$.
- $\sin^3 x$: use a supplied identity such as $\sin 3x = 3\sin x - 4\sin^3 x$, rearranged to $\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$.
- $\sin x \cos x = \frac{1}{2}\sin 2x$.

INTEGRATING A TRIG EXPRESSION

STEP 1 — Is it a standard result or a $\frac{f'}{f}$ / $f'f^n$ pattern?

STEP 2 — If not: is there a square or a product?

STEP 3 — Rewrite with an identity into single sines and cosines

STEP 4 — Integrate term by term; divide by inner multipliers

Your notes

Examiner Tips and Tricks

- The question usually tells you the identity in part (a) ("Show that $(1 + 2\cos 2x)^2 \equiv p + q\cos 2x + r\cos 4x$"). Use it in part (b).
- Exact limits such as $\frac{\pi}{3}$ need exact trig values: $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$.

WHEN YOU SEE → YOU DO

$\int \sin^2 kx$ **or** $\int \cos^2 kx$	→ power reduction with $\cos 2kx$
$\int \tan^2 x$	→ $\sec^2 x - 1$
"(a) Show that $\sin 3x \equiv 3\sin x - 4\sin^3 x$ **(b) Hence find** $\int_0^{\pi/3} \sin^3 x\,dx$**"**	→ rearrange for $\sin^3 x$, integrate
$(\sin x + \cos x)^2$ **or** $(1 + 2\cos 2x)^2$	→ expand, then reduce squares

Worked Example

Find (a) $\int_0^{\pi/2} \cos^2 x\,dx$ (b) $\int \tan^2 x\,dx$ (c) $\int (\sin x + \cos x)^2\,dx$.

$\left[\frac{x}{2} + \frac{1}{4}\sin 2x\right]_0^{\pi/2} = \frac{\pi}{4}$ ← **(A) POWER REDUCTION**

$\int (\sec^2 x - 1)\,dx = \tan x - x$ ← **(B) PYTHAGOREAN IDENTITY**

$\int (1 + \sin 2x)\,dx = x - \frac{1}{2}\cos 2x$ ← **(C) EXPAND;** $\sin^2 + \cos^2 = 1$

ANSWER (a) $\frac{\pi}{4}$ (b) $\tan x - x + c$ (c) $x - \frac{1}{2}\cos 2x + c$

Your notes

Worked Example

? Using $\sin 3x \equiv 3\sin x - 4\sin^3 x$, find the exact value of $\int_0^{\pi/3} \sin^3 x \, dx$.

$\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$ ← REARRANGE THE IDENTITY

$\left[-\frac{3}{4}\cos x + \frac{1}{12}\cos 3x\right]_0^{\pi/3}$ ← INTEGRATE EACH TERM

$\left(-\frac{3}{8} - \frac{1}{12}\right) - \left(-\frac{3}{4} + \frac{1}{12}\right)$ ← $\cos\frac{\pi}{3} = \frac{1}{2}$, $\cos\pi = -1$

ANSWER $\frac{5}{24}$

HOW THIS IS EXAMINED · past WMA13 papers

$\sin^2$, $\cos^2$ SEEN 4× IN EXAMS $\tan^2$ NOT SEEN YET • BE READY
reciprocal Pythagorean NOT SEEN YET • BE READY cubic via identity SEEN 1× IN EXAMS
proved identity SEEN 1× IN EXAMS square of a sum SEEN 1× IN EXAMS

Definite Integrals and Areas NOTE 12.4

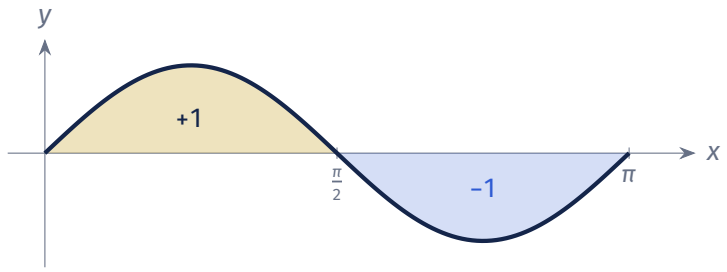
Your notes

Definite Integrals and Areas

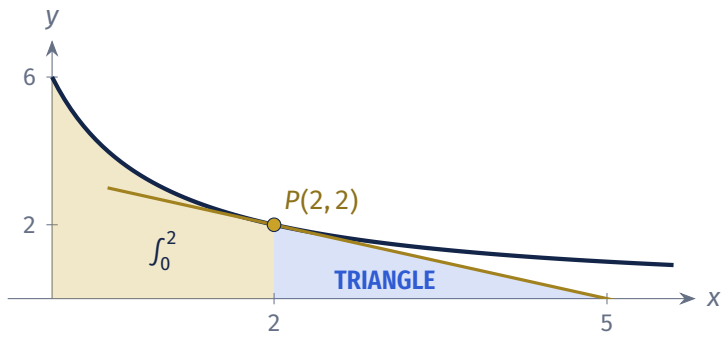
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How do I find an area under a curve?

- $\int_a^b f(x)\,dx$ gives the area between the curve and the x-axis **when the curve is above the axis**.
- Parts **below** the axis give **negative** values. Find where the curve crosses the axis, integrate each part separately, and add the sizes.
- A region bounded by a curve and a **straight line** is often "area under the curve" ± a **triangle** or trapezium. Sketch it and split it.



$\int_0^{\pi} \sin 2x\,dx = 0$, but the shaded area is $1 + 1 = 2$.



A region made of the area under a curve plus a triangle under a tangent.

Your notes

Examiner Tips and Tricks

- "Find, using algebraic integration, the exact area": show the integral with its limits and the antiderivative before substituting.
- Give the answer in the form asked, e.g. "$P + Q\ln 3$, where P and Q are rational".
- For a curve that touches the axis (a square such as $(1 - 2\cos 2x)^2$), it never goes below, so a single integral gives the area.

WHEN YOU SEE → YOU DO

"Find the exact area of the shaded region" with a curve crossing the axis	→ find the crossings, integrate each part, add sizes
"The region R is bounded by C, the line l and the x-axis"	→ curve integral ± triangle; find where l meets the axis
"touches the positive x-axis for the second time when $x = a$"	→ solve curve = 0; take the second positive root as the limit
"giving your answer in the form $a\pi + b\sqrt{3}$"	→ exact trig values at the limits

Your notes

Worked Example

? The curve $y = (1 - 2\cos 2x)^2$ touches the positive x-axis for the second time at $x = a$. (a) Show that $(1 - 2\cos 2x)^2 \equiv 3 - 4\cos 2x + 2\cos 4x$. (b) Find the exact area between the curve, the axes and $x = a$.

$1 - 4\cos 2x + 4\cos^2 2x =$
$1 - 4\cos 2x + 2(1 + \cos 4x)$ ← (A) EXPAND; POWER REDUCTION

$\cos 2x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6} \Rightarrow a = \frac{5\pi}{6}$ ← (B) TOUCHES WHERE THE BRACKET IS 0

$\left[3x - 2\sin 2x + \frac{1}{2}\sin 4x\right]_0^{5\pi/6}$ ← THE CURVE IS NEVER BELOW THE AXIS

$= \frac{5\pi}{2} - 2\left(-\frac{\sqrt{3}}{2}\right) + \frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right)$ ← EXACT VALUES AT $\frac{5\pi}{3}$ AND $\frac{10\pi}{3}$

ANSWER Area $= \frac{5\pi}{2} + \frac{3\sqrt{3}}{4}$

Worked Example

? The curve C has equation $y = \frac{6}{x+1}$, $x \geq 0$. The tangent to C at $P(2, 2)$ meets the x-axis at Q. Find the exact area of the region bounded by C, the y-axis, the x-axis and the tangent.

$\frac{dy}{dx} = -\frac{6}{(x+1)^2} = -\frac{2}{3} \Rightarrow y - 2 = -\frac{2}{3}(x - 2)$ ← TANGENT AT P

$y = 0 : \ x = 5 \Rightarrow Q(5, 0)$ ← THE TRIANGLE HAS BASE 3, HEIGHT 2

$\int_0^2 \frac{6}{x+1}\,dx = \left[6\ln(x+1)\right]_0^2 = 6\ln 3$ ← AREA UNDER THE CURVE

ANSWER Area $= 6\ln 3 + 3$

HOW THIS IS EXAMINED · past WMA13 papers

exact definite integral SEEN 1× IN EXAMS find a constant SEEN 1× IN EXAMS
area of a bounded region SEEN 2× IN EXAMS curve plus line geometry SEEN 3× IN EXAMS

Your notes

MISTAKE GALLERY

Where marks are lost

Your notes

× WRONG

$\int e^{3x}\,dx = 3e^{3x}.$

✓ RIGHT

$\frac{1}{3}e^{3x} + c.$

Integrating divides by the inner multiplier.

× WRONG

$\int \frac{1}{2x+1}\,dx = \ln|2x+1|.$

✓ RIGHT

$\frac{1}{2}\ln|2x+1| + c.$

Check by differentiating.

× WRONG

$\int \sin^2 x\,dx = \frac{\sin^3 x}{3}.$

✓ RIGHT

$\frac{x}{2} - \frac{1}{4}\sin 2x + c.$

The reverse power rule needs the derivative $\cos x$ as a factor.

× WRONG

Area = $\int_0^{\pi} \sin 2x\,dx = 0.$

✓ RIGHT

Split at $\frac{\pi}{2}$: area = 1 + 1 = 2.

Regions below the axis give negative integrals.

× WRONG

Forgetting $+ c$.

✓ RIGHT

Every indefinite integral needs $+ c$.

It is a mark on its own.

THE CHAPTER ON ONE PAGE

Integration

Your notes

12.1

Standard

$e^{kx} \to \frac{1}{k}e^{kx}$; $\frac{1}{ax+b} \to \frac{1}{a}\ln|ax+b|$.
$\cos kx \to \frac{1}{k}\sin kx$; $\sin kx \to -\frac{1}{k}\cos kx$.
$\sec^2 kx \to \frac{1}{k}\tan kx$; $+c$.

12.2

Recognition

$\frac{f'}{f} \to \ln|f|$.
$f'f^n \to \frac{f^{n+1}}{n+1}$.
Adjust constants; check by differentiating.

12.3

Identities

$\sin^2$, $\cos^2$: use $\cos 2x$.
$\tan^2 = \sec^2 - 1$.
Expand squares; use given identities.

12.4

Areas

Split where the curve crosses the axis.
Curve ± triangle for line regions.
Exact values; form asked.

BOTTOM LINE Ask "what differentiates to this?". Divide by the inner multiplier, spot f' next to f, and use an identity when there is a square.