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11

CHAPTER 11

Applications of Differentiation

The derivative is a gradient: use it for tangents and normals, set it to zero for turning points, read its sign for increasing and decreasing, and read it as a rate in a model.

4 CONCEPTS

16 VARIANTS

54 EXAM QUESTIONS

YOU WILL MASTER

- Write the tangent or normal at a point, including tangents through an outside point and vertical tangents
- Find and classify stationary points, including on trig and exponential curves
- Use turning points to count solutions and restrict parameters
- Show a function is increasing or decreasing, and find and interpret rates in models

CHAPTER 11 AT A GLANCE

Four note pages, sixteen ways to be asked

Your notes

CHAPTER 11 MAP

Applications of Differentiation

01 **11.1 Tangents and normals**

- at a point
- through an outside point
- axis intercepts and areas
- vertical tangents

02 **11.2 Stationary points**

- find coordinates
- classify
- periodic curves
- count solutions
- parameter conditions

03 **11.3 Increasing and decreasing**

- sign of $f'(x)$
- "show f is decreasing"

04 **11.4 Rates of change**

- rate at a time
- time for a given rate
- rate in terms of the quantity

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11.2 Stationary Points 7
11.3 Increasing and Decreasing Functions 10
11.4 Rates of Change in Models 12

Tangents and Normals NOTE 11.1

Your notes

Tangents and Normals

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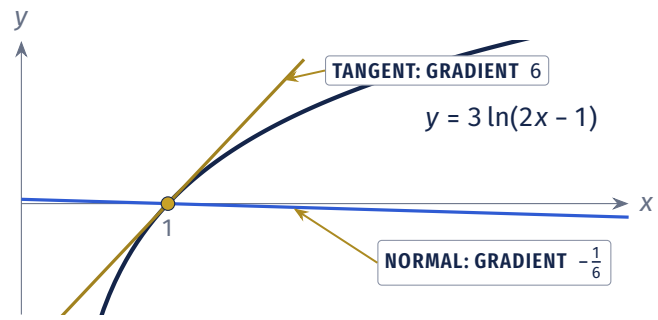
How do I find the equation of a tangent or normal?

STEP 1 Find the coordinates of the point (x_1, y_1): substitute the given coordinate into the curve.

STEP 2 Differentiate and substitute x_1 to get the gradient of the **tangent**, m.

STEP 3 For the **normal** use the negative reciprocal: $-\frac{1}{m}$.

STEP 4 Use $y - y_1 = m(x - x_1)$ and rearrange to the form asked.



The tangent touches the curve; the normal is perpendicular to it at the same point.

What if the tangent must pass through a given point?

- Write the tangent at a **general** point $x = a$: $y - f(a) = f'(a)(x - a)$.
- Substitute the coordinates of the outside point (often the origin) and solve for a.

What about curves given as $x = f(y)$?

- The gradient is $\frac{dy}{dx} = 1 \div \frac{dx}{dy}$ (see 10.4).
- A **vertical tangent** (x = constant) occurs where $\frac{dx}{dy} = 0$. Find y there, then x from the curve; the tangent is $x = \ldots$

Your notes

Examiner Tips and Tricks

- Keep exact values: "$y = mx + c$, where m and c are constants to be found" usually wants surds, π or ln.
- For areas of triangles made with axes, find each line's intercept, then sketch the triangle before computing "$\frac{1}{2}\times$ base × height".
- Normal gradient: $-\frac{1}{m}$, not $\frac{1}{m}$ and not $-m$.

WHEN YOU SEE → YOU DO

"Find the equation of the tangent to C at P. Write your answer in the form $y = mx + c$"	→ coordinates of P, f' at P, $y - y_1 = m(x - x_1)$
"the tangent to C at P passes through the origin"	→ tangent at $x = \alpha$, then put $(0, 0)$ in
"The normal to the curve at P cuts the x-axis at N. Find the exact area of triangle OPN"	→ normal, its x-intercept, then $\frac{1}{2}\times$ base × height
"The tangents at P and Q are parallel to the y-axis"	→ solve $\frac{dx}{dy} = 0$

Worked Example

? The curve C has equation $y = 3\ln(2x - 1)$. Find the equations of the tangent and the normal to C at the point where $x = 1$.

$y = 3\ln 1 = 0 \Rightarrow P(1, 0)$ ← THE POINT

$\frac{dy}{dx} = \frac{6}{2x - 1} = 6$ at $x = 1$ ← CHAIN RULE

normal gradient $= -\frac{1}{6}$ ← NEGATIVE RECIPROCAL

ANSWER tangent $y = 6x - 6$; normal $y = -\frac{1}{6}x + \frac{1}{6}$

Your notes

Worked Example

? The tangent to $y = e^{2x}$ at the point P passes through the origin. Find the exact coordinates of P.

$y - e^{2a} = 2e^{2a}(x - a)$ ← **TANGENT AT A GENERAL POINT** $x = a$

$0 - e^{2a} = 2e^{2a}(0 - a) \Rightarrow 1 = 2a$ ← **PUT** $(0, 0)$ **IN; DIVIDE BY** $e^{2a} > 0$

ANSWER $P\left(\frac{1}{2}, e\right)$

Worked Example

? $P(e, 4)$ lies on $y = 4 \ln x$. The normal to the curve at P meets the x-axis at N. Find the exact area of triangle OPN.

$\frac{dy}{dx} = \frac{4}{x} = \frac{4}{e} \Rightarrow$ normal gradient $-\frac{e}{4}$ ← **GRADIENT AT** P

$0 - 4 = -\frac{e}{4}(x - e) \Rightarrow x = e + \frac{16}{e}$ ← **NORMAL MEETS** $y = 0$

$\frac{1}{2} \times \left(e + \frac{16}{e}\right) \times 4$ ← **BASE** ON, **HEIGHT** 4

ANSWER Area $= 2e + \frac{32}{e}$

Your notes

Worked Example

? The curve C has equation $x = ye^{2y}$. (a) Find the equation of the vertical tangent to C. (b) The line $x = k$ meets C at exactly two points. Find the range of values of k.

$\frac{dx}{dy} = e^{2y}(1 + 2y) = 0 \Rightarrow y = -\frac{1}{2}$ ← PRODUCT RULE; $e^{2y} \neq 0$

$x = -\frac{1}{2}e^{-1}$ ← SUBSTITUTE INTO THE CURVE

for $y < 0$: $-\frac{1}{2e} \leq x < 0$ ← (B) x FALLS TO $-\frac{1}{2e}$ THEN RISES BACK TOWARDS 0

ANSWER (a) $x = -\frac{1}{2e}$ (b) $-\frac{1}{2e} < k < 0$

HOW THIS IS EXAMINED · past WMA13 papers

tangent or normal at a point SEEN 7× IN EXAMS intercepts and areas SEEN 2× IN EXAMS
through an outside point SEEN 1× IN EXAMS vertical tangent SEEN 2× IN EXAMS
signed gradients SEEN 1× IN EXAMS

Stationary Points NOTE 11.2

Your notes

Stationary Points

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How do I find stationary points?

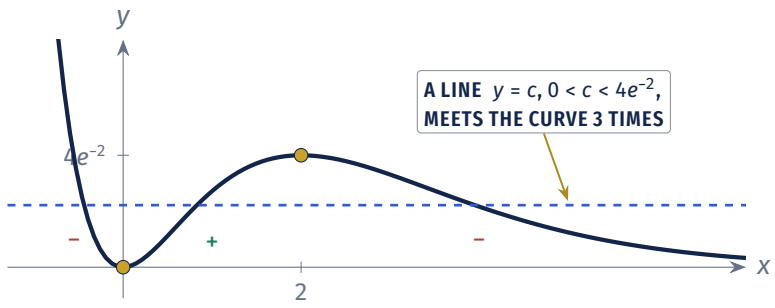
STEP 1 Differentiate, simplify and **factorise** $\frac{dy}{dx}$.

STEP 2 Solve $\frac{dy}{dx} = 0$. Factors such as e^{-x} are never zero.

STEP 3 Substitute each x into the **original** equation for y.

How do I decide if it is a maximum or a minimum?

- **Second derivative**: $\frac{d^2y}{dx^2} < 0$ maximum; > 0 minimum; $= 0$ use another method.
- **Sign of** $\frac{dy}{dx}$ either side: + → − maximum, − → + minimum.



$y = x^2e^{-x}$: a minimum at $(0, 0)$, a maximum at $(2, 4e^{-2})$, and the sign of $\frac{dy}{dx}$ between them.

How do I use turning points to count solutions?

- The number of solutions of $f(x) = c$ is the number of times the line $y = c$ meets the graph.
- Compare c with the **turning-point heights**, the values at the ends of the domain, and any asymptotes.
- To **force** a number of stationary points, look at the equation $f'(x) = 0$: e.g. a

quadratic factor has two distinct roots when its discriminant is positive.

Your notes

Examiner Tips and Tricks

- On a trig curve, list **all** stationary angles in the interval, then decide which are maxima (the question may want only those).
- "Find the exact coordinates": keep e^{-2} and ln, do not use decimals.
- A factor e^{kx} in $f'(x)$ can be divided out because it is always positive; it never gives a stationary point.

WHEN YOU SEE → YOU DO

"Find the exact coordinates of the stationary points"	→ factorise f', set each non-exponential factor to zero, use the original for y
"the curve has two turning points, at P and Q ... show that, at P and Q, ..."	→ $f'(x) = 0$ rearranged to the given form
"Given that the equation $f(x) = c$ has exactly three solutions"	→ compare c with turning heights and limits
"Find the range of values of k for which the curve has two stationary points"	→ discriminant of the factor in $f'(x)$

Worked Example

? Find the exact coordinates of the stationary points of $y = x^2e^{-x}$ and determine their nature.

$\frac{dy}{dx} = 2xe^{-x} - x^2e^{-x} = xe^{-x}(2 - x)$ ← PRODUCT RULE, THEN FACTORISE

$x = 0$ or $x = 2$ $\quad (e^{-x} \neq 0)$ ← SOLVE $\frac{dy}{dx} = 0$

signs: −, +, − ← LEFT OF 0, BETWEEN, RIGHT OF 2

ANSWER minimum $(0, 0)$; maximum $(2, 4e^{-2})$

Your notes

Worked Example

? $f(x) = e^x \sin x$, $0 \le x \le 2\pi$. Find the x-coordinates of the stationary points and identify the maximum.

$f'(x) = e^x(\sin x + \cos x) = 0$ ← PRODUCT RULE; $e^x \neq 0$

$\tan x = -1 \Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}$ ← SECOND AND FOURTH QUADRANTS

$f''(x) = 2e^x \cos x < 0$ at $\frac{3\pi}{4}$ ← SECOND DERIVATIVE TEST

ANSWER maximum at $x = \frac{3\pi}{4}$; minimum at $x = \frac{7\pi}{4}$

Worked Example

? The curve $y = (x^2 + k)e^x$ has two distinct stationary points. Find the range of possible values of k.

$\frac{dy}{dx} = e^x(x^2 + 2x + k)$ ← PRODUCT RULE AND FACTORISE

$x^2 + 2x + k = 0$ needs $b^2 - 4ac > 0$ ← TWO DISTINCT ROOTS

$4 - 4k > 0$ ← DISCRIMINANT

ANSWER $k < 1$

HOW THIS IS EXAMINED · past WMA13 papers

stationary coordinates SEEN 8× IN EXAMS periodic curves SEEN 2× IN EXAMS
count intersections SEEN 3× IN EXAMS model extremum SEEN 2× IN EXAMS
optimise a gradient SEEN 1× IN EXAMS
parameter by number of stationary points SEEN 2× IN EXAMS
prescribed gradient SEEN 2× IN EXAMS

Increasing and Decreasing Functions NOTE 11.3

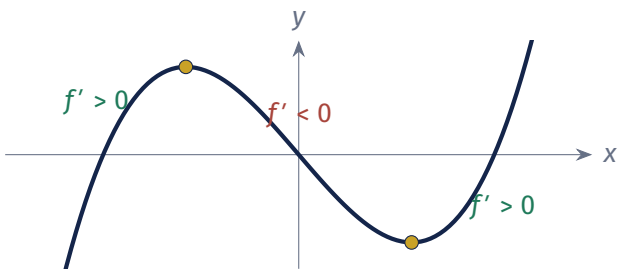
Your notes

Increasing and Decreasing Functions

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How does the derivative tell me if a function is increasing?

- $f'(x) > 0$ on an interval: f is **increasing** there.
- $f'(x) < 0$ on an interval: f is **decreasing** there.
- To **show** a function is decreasing for its whole domain, show $f'(x)$ is negative for **every** x in the domain: squares, $e^{(\dots)}$ and positive constants help.



The sign of the gradient matches the direction of the curve.

Examiner Tips and Tricks

- "You must make your reasoning clear": write a sentence, e.g. "$(x-2)^2 > 0$ so $f'(x) = -\frac{6}{(x-2)^2} < 0$ for all $x > 2$, hence f is decreasing."
- Increasing or decreasing functions are one-one, so they have inverses (C02).

Your notes

WHEN YOU SEE → YOU DO

"Show, using calculus, that f is a decreasing function. You must make your reasoning clear"	→ $f'(x)$, then argue its sign for every x
"state whether f is an increasing or a decreasing function, giving a reason"	→ sign of $f'(x)$ with a reason
"Find the range of values of x for which f is increasing"	→ solve $f'(x) > 0$

Worked Example

? $f(x) = \dfrac{2x+2}{x-2}$, $x > 2$. Show, using calculus, that f is a decreasing function.

$$f'(x) = \frac{2(x-2)-(2x+2)(1)}{(x-2)^2}$$ ← QUOTIENT RULE

$$= \frac{-6}{(x-2)^2}$$ ← SIMPLIFY THE NUMERATOR

$(x-2)^2 > 0$ for $x > 2$ ← REASONING

ANSWER $f'(x) < 0$ for all $x > 2$, so f is decreasing

HOW THIS IS EXAMINED · past WMA13 papers

sign of the first derivative SEEN 11× IN EXAMS

Rates of Change in Models NOTE 11.4

Your notes

Rates of Change in Models

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What does a derivative mean in a model?

- $\frac{dV}{dt}$ is the **rate of change** of V with respect to time: how fast V is changing at that instant.
- Its units are the units of V per unit of t, e.g. \$ per year, °C per minute.
- A **negative** rate means the quantity is decreasing. Say "decreasing at 571 dollars per year" rather than "−571".

What are the three common questions?

- **Rate at a time**: differentiate first, then substitute t. Never substitute first.
- **Time for a given rate**: differentiate, set $\frac{dV}{dt}$ equal to the rate (with its sign), solve.
- **Rate in terms of the quantity**: differentiate, then replace the exponential using the model, e.g. $\frac{dH}{dt} = -0.04(H - 20)$.

Examiner Tips and Tricks

- "The rate of increase … exactly 6 days after the start": $t = 6$, not day 7.
- "Cooling at 5°C per minute" means $\frac{dH}{dt} = -5$.
- Give units in your final answer.

WHEN YOU SEE → YOU DO

"find the rate of increase of ... exactly 6 days after the start"	→ differentiate, then $t = 6$; units
"find the time at which the temperature is falling at 5°C per minute"	→ $\frac{dH}{dt} = -5$
"Show by differentiation that $\frac{dH}{dt} = a + bH$"	→ replace the exponential using H

Your notes

Worked Example

? The value of a car is $V = 5000(0.8)^t$ dollars after t years. Find the rate of change of V when $t = 3$, and interpret it.

$\frac{dV}{dt} = 5000(0.8)^t \ln 0.8$ ← a^t RULE

$t = 3$: $5000 \times 0.512 \times \ln 0.8 = -571.25$ ← SUBSTITUTE AFTER DIFFERENTIATING

ANSWER −571 (3 s.f.): the value is falling by about 571 dollars per year at $t = 3$

Worked Example

? $H = 250e^{-0.04t} + 20$. Find the time at which the temperature is decreasing at 5°C per minute.

$\frac{dH}{dt} = -10e^{-0.04t}$ ← DIFFERENTIATE

$-10e^{-0.04t} = -5 \Rightarrow e^{-0.04t} = 0.5$ ← RATE WITH ITS SIGN

ANSWER $t = 25 \ln 2 = 17.33$ minutes

HOW THIS IS EXAMINED · past WMA13 papers

rate at a time SEEN 10× IN EXAMS time for a given rate SEEN 4× IN EXAMS
rate in terms of the quantity SEEN 1× IN EXAMS

Your notes

MISTAKE GALLERY

Where marks are lost

Your notes

× WRONG

Normal gradient = $\frac{1}{m}$.

✓ RIGHT

$-\frac{1}{m}$.

Perpendicular gradients multiply to −1.

× WRONG

Substituting $t = 3$ into V and then "differentiating" a number.

✓ RIGHT

Differentiate V first, then substitute.

A number has derivative 0.

× WRONG

$xe^{-x}(2 - x) = 0 \Rightarrow x = 0, 2$ and $e^{-x} = 0$.

✓ RIGHT

Only $x = 0$, $x = 2$: $e^{-x} > 0$.

Exponentials are never zero.

× WRONG

Using $\frac{dx}{dy} = 0$ for a horizontal tangent.

✓ RIGHT

$\frac{dx}{dy} = 0$ gives a **vertical** tangent.

Horizontal means $\frac{dy}{dx} = 0$.

× WRONG

"f is decreasing because $f'(3) < 0$".

✓ RIGHT

Show $f'(x) < 0$ for **all** x in the domain.

One value proves nothing about the whole domain.

THE CHAPTER ON ONE PAGE

Applications of Differentiation

Your notes

11.1

Lines

Tangent: $y - y_1 = m(x - x_1)$; normal: $-\frac{1}{m}$.
Through a point: tangent at $x = a$, substitute.
Vertical tangent: $\frac{dx}{dy} = 0$.

11.2

Turning points

Factorise f', solve = 0, y from the original.
$f'' < 0$ max, > 0 min.
Count solutions with turning heights.

11.3

Direction

$f' > 0$ increasing, $f' < 0$ decreasing.
"Show": sign for every x, with a reason.

11.4

Rates

Differentiate, then substitute.
Sign and units in words.
Eliminate the exponential using the model.

BOTTOM LINE The derivative is the gradient: a line's slope, zero at a turn, positive going up, and a rate in a model. Always differentiate before you substitute.