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10

CHAPTER 10

Differentiation Techniques

Look at the structure first: a function of a function needs the chain rule, two functions multiplied need the product rule, one divided by another needs the quotient rule.

4 CONCEPTS

13 VARIANTS

40 EXAM QUESTIONS

YOU WILL MASTER

- Differentiate e^{kx}, $\ln kx$, a^x and every trig function, including sec, cosec, cot
- Use the chain rule, including two or more layers
- Use the product and quotient rules, combined with the chain rule, and simplify
- Find $\frac{dy}{dx}$ when x is given in terms of y

CHAPTER 10 AT A GLANCE

Four note pages, thirteen ways to be asked

Your notes

CHAPTER 10 MAP

Differentiation

01 **10.1 Standard derivatives**

e^{kx}, a^x | $\ln kx$ | sin, cos, tan | sec, cosec, cot

02 **10.2 Chain rule**

one layer | two or more layers

03 **10.3 Product and quotient rules**

product | quotient | with chain | simplify first

04 **10.4 x in terms of y**

flip $\frac{dx}{dy}$ to get $\frac{dy}{dx}$

Contents

10.1 Differentiating Standard Functions . . . 3
10.2 The Chain Rule . . . 6
10.3 Product and Quotient Rules . . . 9
10.4 Differentiating x as a Function of y . . . 13

Differentiating Standard Functions NOTE 10.1

Your notes

Differentiating Standard Functions

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What are the standard derivatives?

y	$\frac{dy}{dx}$	NOTE
e^{kx}	ke^{kx}	learn
a^x	$a^x \ln a$	learn; $a > 0$
$\ln kx$	$\frac{1}{x}$	learn: the k disappears
$\sin kx$	$k \cos kx$	learn
$\cos kx$	$-k \sin kx$	learn: note the minus
$\tan kx$	$k \sec^2 kx$	formula booklet
$\sec x$	$\sec x \tan x$	formula booklet
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$	formula booklet
$\cot x$	$-\operatorname{cosec}^2 x$	formula booklet

- Angles must be in **radians** for the trig derivatives.
- $\ln kx = \ln k + \ln x$, so its derivative is just $\frac{1}{x}$.
- Differentiate a sum or difference term by term; constants multiply through.

Should I simplify first?

- Yes, if it removes a product or quotient. Use the **log laws** or split the fraction.
  - $\ln\left(x^2(x+1)\right) = 2\ln x + \ln(x+1)$
  - $\frac{x^3+2}{x} = x^2 + 2x^{-1}$

Your notes

e.g. **DIFFERENTIATE** $y = 3e^{2x} - 4\ln 5x + \sin 3x$

$6e^{2x}$ ← e^{kx}: **MULTIPLY BY** $k = 2$

$-\frac{4}{x}$ ← $\ln 5x$: **THE 5 DISAPPEARS**

ANSWER $\frac{dy}{dx} = 6e^{2x} - \frac{4}{x} + 3\cos 3x$

Examiner Tips and Tricks

- The derivatives of tan, sec, cosec, cot are in the formula booklet; the others must be learnt.
- Every derivative of a "co" function (cos, cosec, cot) starts with a minus.

WHEN YOU SEE → YOU DO

e^{kx}, $\ln kx$, $\sin kx$ **in a sum**	→	term by term; keep the inner multiplier k
a^x, **e.g.** 2^x **or** 10^y	→	$a^x \ln a$
$\ln$ **of a product, quotient or power**	→	split with the log laws first

HOW THIS IS EXAMINED · past WMA13 papers

exponential NOT SEEN YET • BE READY natural log NOT SEEN YET • BE READY
trig SEEN 1× IN EXAMS reciprocal trig NOT SEEN YET • BE READY a^x NOT SEEN YET • BE READY

Your notes

Worked Example

? Find $\frac{dy}{dx}$ for (a) $y = 2^x$ (b) $y = \sec 2x$ (c) $y = \ln\left(x^2(x+1)\right)$.

$2^x \ln 2$ ← (A) a^x RULE

$2 \sec 2x \tan 2x$ ← (B) BOOKLET RESULT WITH THE INNER MULTIPLIER 2

$y = 2\ln x + \ln(x+1) \Rightarrow \frac{2}{x} + \frac{1}{x+1}$ ← (C) LOG LAWS FIRST

ANSWER (a) $2^x \ln 2$ (b) $2 \sec 2x \tan 2x$ (c) $\frac{3x+2}{x(x+1)}$

The Chain Rule NOTE 10.2

Your notes

The Chain Rule

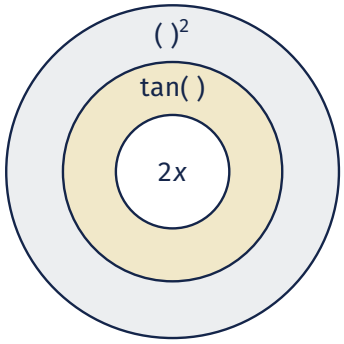
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What is the chain rule?

- It differentiates a **function of a function**, such as $(3x^2 - 5)^4$, $e^{\sin 3x}$ or $\ln(\sin 2x)$.
- Differentiate the **outside**, keep the inside the same, then multiply by the derivative of the **inside**.

THE CHAIN RULE LEARN THIS: NOT GIVEN

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$



OUTER: $2\tan 2x$

MIDDLE: $\sec^2 2x$

INNER: 2

$\frac{d}{dx}\tan^2 2x = 4\tan 2x \sec^2 2x$

Peel the layers from the outside in, and multiply what you get.

What are the common chain rule patterns?

- $\big(f(x)\big)^n \rightarrow n\big(f(x)\big)^{n-1} f'(x)$
- $e^{f(x)} \rightarrow f'(x)e^{f(x)}$
- $\ln f(x) \rightarrow \dfrac{f'(x)}{f(x)}$
- $\sin f(x) \rightarrow f'(x)\cos f(x)$, and similarly for the other trig functions

Your notes

e.g. **DIFFERENTIATE** $y = \sqrt{4x + 1}$

$y = (4x + 1)^{\frac{1}{2}} \Rightarrow \frac{1}{2}(4x + 1)^{-\frac{1}{2}} \times 4$ ← **POWER OUTSIDE,** $4x + 1$ **INSIDE**

ANSWER $\frac{dy}{dx} = \frac{2}{\sqrt{4x + 1}}$

Examiner Tips and Tricks

- Write the inside function as u if it helps, but give the final answer in x.
- $\sin^2 x$ means $(\sin x)^2$: the power is the outer layer.
- "Writing your answer in simplest form": $\frac{2\cos 2x}{\sin 2x} = 2\cot 2x$.

WHEN YOU SEE → YOU DO

a bracket raised to a power	→ power outside, bracket inside
e, ln, sin, **... of something more than** kx	→ chain rule: multiply by the inner derivative
$\tan^2(2x)$, $e^{\sin 3x}$, $\ln(\cos^2 x)$	→ two or more layers: multiply all the layer derivatives

Worked Example

? Differentiate (a) $(3x^2 - 5)^4$ (b) $\ln(\sin 2x)$ (c) $e^{\sin 3x}$ (d) $\tan^2 2x$.

$4(3x^2 - 5)^3 \times 6x$ ← **(A) POWER, THEN INNER $6x$**

$\dfrac{2\cos 2x}{\sin 2x}$ ← **(B) $\dfrac{f'}{f}$**

$3\cos 3x\, e^{\sin 3x}$ ← **(C) INNER DERIVATIVE TIMES THE SAME EXPONENTIAL**

$2\tan 2x \times \sec^2 2x \times 2$ ← **(D) THREE LAYERS**

ANSWER (a) $24x(3x^2 - 5)^3$ (b) $2\cot 2x$ (c) $3\cos 3x\, e^{\sin 3x}$ (d) $4\tan 2x \sec^2 2x$

HOW THIS IS EXAMINED · past WMA13 papers

chain **SEEN 5× IN EXAMS** nested chain **SEEN 4× IN EXAMS**

Product and Quotient Rules NOTE 10.3

Your notes

Product and Quotient Rules

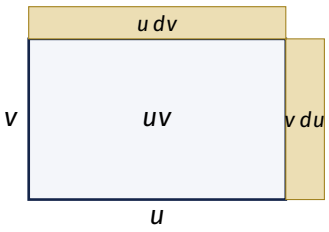
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When do I use the product rule?

- When y is two functions of x **multiplied**: $y = uv$, e.g. x^2e^{3x}, $x^2 \ln 2x$, $e^x \sin x$.

PRODUCT RULE LEARN THIS: NOT GIVEN

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$



CHANGE IN AREA $\approx u\,dv + v\,du$

Why it works: when a rectangle $u \times v$ grows, the new area is the two strips $u\,dv$ and $v\,du$.

When do I use the quotient rule?

- When y is one function **divided** by another: $y = \frac{u}{v}$.

QUOTIENT RULE IN THE FORMULA BOOKLET

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

- Order matters in the numerator: "bottom times derivative of top, **minus** top times derivative of bottom".
- Do not expand v^2 in the denominator; it often cancels with a factor in the numerator.

Your notes

How do I simplify the answer?

- Take out **common factors**: e^{3x}, powers of x, powers of a bracket.
- In a quotient with $v = (4x + 3)^2$, the numerator has a factor $(4x + 3)$: cancel one power.
- "Show that $f'(x) = \dfrac{2e^{2x}(ax + b)}{(4x + 3)^3}$" tells you the factorised shape to aim for.

Examiner Tips and Tricks

- Write u, v, u', v' in a small list before you build the formula: it earns the method mark.
- A product can always be differentiated instead of using the quotient rule: $\dfrac{u}{v} = uv^{-1}$. Use whichever is cleaner.
- The quotient rule is in the booklet; the product rule is not.

WHEN YOU SEE → YOU DO

"$y = x^2 \ln 2x$ **... find, in simplest form,** $\dfrac{dy}{dx}$"	→ product rule; factorise
"$f(x) = \dfrac{e^{2x}}{(4x + 3)^2}$ **... show that** $f'(x) = \ldots$"	→ quotient rule, chain on the bottom, cancel $(4x + 3)$
"Show by calculus that f is a decreasing function"	→ find $f'(x)$ and show it is negative for every x in the domain

Your notes

Worked Example

? Differentiate (a) $y = x^2e^{3x}$ (b) $y = x^2 \ln 2x$, simplifying fully.

$u = x^2,\ v = e^{3x};\quad u' = 2x,\ v' = 3e^{3x}$ ← **(A) LIST THE PARTS**

$x^2(3e^{3x}) + e^{3x}(2x)$ ← $uv' + vu'$

$x^2 \cdot \frac{1}{x} + 2x \ln 2x$ ← **(B)** $\frac{d}{dx} \ln 2x = \frac{1}{x}$

ANSWER (a) $xe^{3x}(3x + 2)$ (b) $x(1 + 2 \ln 2x)$

Worked Example

? $f(x) = \dfrac{e^{2x}}{(4x + 3)^2}$. Show that $f'(x) = \dfrac{2e^{2x}(ax + b)}{(4x + 3)^3}$, stating a and b.

$u = e^{2x},\ u' = 2e^{2x};\quad v = (4x+3)^2,\ v' = 8(4x+3)$ ← **CHAIN RULE FOR v'**

$f'(x) = \dfrac{(4x + 3)^2\, 2e^{2x} - e^{2x}\, 8(4x + 3)}{(4x + 3)^4}$ ← **QUOTIENT RULE**

$= \dfrac{2e^{2x}[(4x + 3) - 4]}{(4x + 3)^3}$ ← **TAKE OUT $2e^{2x}(4x + 3)$ AND CANCEL**

ANSWER $f'(x) = \dfrac{2e^{2x}(4x - 1)}{(4x + 3)^3}$: $a = 4,\ b = -1$

Your notes

Worked Example

? Show that $\frac{d}{dx}\left(\frac{\sin x}{1+\cos x}\right) = \frac{1}{1+\cos x}$.

$$\frac{(1+\cos x)\cos x - \sin x(-\sin x)}{(1+\cos x)^2}$$ ← QUOTIENT RULE

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} = \frac{\cos x + 1}{(1+\cos x)^2}$$ ← $\sin^2 + \cos^2 = 1$

ANSWER $= \frac{1}{1+\cos x}$ as required

HOW THIS IS EXAMINED · past WMA13 papers

product SEEN 1× IN EXAMS quotient SEEN 5× IN EXAMS simplify first SEEN 1× IN EXAMS
product with chain SEEN 12× IN EXAMS quotient with chain SEEN 4× IN EXAMS

Differentiating x as a Function of y NOTE 10.4

Your notes

Differentiating x as a Function of y

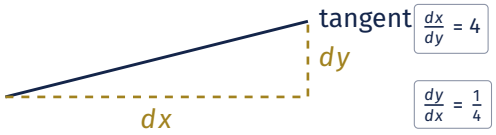
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What if x is given in terms of y?

- Equations such as $x = 4\sin^2 y$ or $x = ye^{2y}$ cannot easily be rearranged to $y = \ldots$
- Differentiate **x with respect to y**, then turn it upside down:

RECIPROCAL RULE LEARN THIS: NOT GIVEN

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \qquad \left(\frac{dx}{dy} \neq 0\right)$$



Rise and run swap: if x changes by 4 when y changes by 1, then y changes by $\frac{1}{4}$ per unit of x.

How do I give the answer in terms of x?

- The reciprocal is usually in terms of y. Use the **original equation** to replace the trig or exponential parts by x.
- Draw a right-angled triangle or use $\sin^2 + \cos^2 = 1$ to convert $\cos y$ when you know $\sin y$.
- Inverse functions use the same idea: $y = \arcsin x \Rightarrow x = \sin y \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$.

Your notes

Examiner Tips and Tricks

- This is one of the most frequent P3 questions. The steps are always: $\frac{dx}{dy}$, reciprocal, then rewrite in x using the original equation.
- Check the sign of the square root with the given range of y (e.g. $0 < y < \frac{\pi}{2}$ makes $\cos y > 0$).
- A **vertical tangent** occurs where $\frac{dx}{dy} = 0$ (see C11).

WHEN YOU SEE → YOU DO

"Given that $x = 6\sin^2 2y$ **... show that** $\frac{dy}{dx} = \frac{1}{A\sqrt{x(B-x)}}$**"**	→ $\frac{dx}{dy}$, reciprocal, then replace sin and cos using x
"$y = \log_{10}(2x+1)$ **(a) Express** x **in terms of** y **(b) Hence find** $\frac{dy}{dx}$**"**	→ $x = \frac{10^y - 1}{2}$; $\frac{dx}{dy} = \frac{10^y \ln 10}{2}$
"Given $x = 2\sin y$**, show that** $\frac{dy}{dx} = (A - x^2)^{-\frac{1}{2}}$**"**	→ $\cos y = \sqrt{1 - \sin^2 y}$, written in x

Worked Example

? Given $x = 4\sin^2 y$, $0 < y < \frac{\pi}{2}$, show that $\frac{dy}{dx} = \frac{1}{2\sqrt{4x - x^2}}$.

$\frac{dx}{dy} = 8\sin y \cos y$ ← **CHAIN RULE ON** $(\sin y)^2$

$\sin y = \frac{\sqrt{x}}{2}$, $\cos y = \sqrt{1 - \frac{x}{4}} = \frac{\sqrt{4-x}}{2}$ ← **FROM THE EQUATION;** $\cos y > 0$ **IN THE RANGE**

$\frac{dx}{dy} = 8 \cdot \frac{\sqrt{x}}{2} \cdot \frac{\sqrt{4-x}}{2} = 2\sqrt{4x - x^2}$ ← **SUBSTITUTE**

ANSWER $\frac{dy}{dx} = \frac{1}{2\sqrt{4x - x^2}}$

Your notes

Worked Example

? $y = \log_{10}(2x + 1)$. (a) Express x in terms of y. (b) Hence find $\frac{dy}{dx}$ in terms of x.

$2x + 1 = 10^y \Rightarrow x = \frac{10^y - 1}{2}$ ← (A) UNDO THE LOG

$\frac{dx}{dy} = \frac{10^y \ln 10}{2}$ ← (B) a^y RULE

$\frac{dy}{dx} = \frac{2}{10^y \ln 10} = \frac{2}{(2x + 1)\ln 10}$ ← RECIPROCAL, THEN $10^y = 2x + 1$

ANSWER $\frac{dy}{dx} = \frac{2}{(2x + 1)\ln 10}$

HOW THIS IS EXAMINED · past WMA13 papers

x in terms of y SEEN 14× IN EXAMS

MISTAKE GALLERY

Where marks are lost

Your notes

× WRONG

$\frac{d}{dx}\ln 5x = \frac{5}{x}$.

✓ RIGHT

$\frac{1}{x}$.

$\ln 5x = \ln 5 + \ln x$.

× WRONG

$\frac{d}{dx}\cos 3x = 3\sin 3x$.

✓ RIGHT

$-3\sin 3x$.

The derivative of cos is − sin.

× WRONG

$\frac{d}{dx}(x^2e^{3x}) = 2x \cdot 3e^{3x}$.

✓ RIGHT

$3x^2e^{3x} + 2xe^{3x}$.

A product needs the product rule.

× WRONG

Quotient rule with the numerator the wrong way round.

✓ RIGHT

$vu' - uv'$ on top.

The minus makes order matter.

× WRONG

$x = 4\sin^2 y \Rightarrow \frac{dy}{dx} = 8\sin y \cos y$.

✓ RIGHT

That is $\frac{dx}{dy}$; take its reciprocal.

Check which variable you differentiated.

THE CHAPTER ON ONE PAGE

Differentiation Techniques

Your notes

10.1

Standard

$e^{kx} \to ke^{kx}$; $a^x \to a^x \ln a$; $\ln kx \to \frac{1}{x}$.
$\sin kx \to k\cos kx$; $\cos kx \to -k\sin kx$.
tan, sec, cosec, cot: booklet.

10.2

Chain

Outside, keep inside, times derivative of inside.
Several layers: multiply every layer.

10.3

Product, quotient

$(uv)' = uv' + vu'$.
$\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}$.
Factorise and cancel.

10.4

$x = f(y)$

$\frac{dy}{dx} = 1 \div \frac{dx}{dy}$, then rewrite in x with the original equation.

BOTTOM LINE Read the structure: layers mean chain, a product means product rule, a fraction means quotient rule, and x in terms of y means flip $\frac{dx}{dy}$.