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09

CHAPTER 09

Logarithmic Graphs and Exponential Models

Logs turn curves into straight lines, and exponentials describe anything that grows or decays in proportion to its size. Read the line, then read the story.

4	16	34
CONCEPTS	VARIANTS	EXAM QUESTIONS

YOU WILL MASTER

- Turn $y = ax^n$ and $y = kb^x$ into straight lines and read a, n, k, b from a graph
- Use initial values and two observations to find the constants of a model
- Find when a model reaches a target value, and its long-term value
- Judge and explain whether a model is realistic

CHAPTER 09 AT A GLANCE

Three note pages, sixteen ways to be asked

Your notes

CHAPTER 09 MAP

Log Graphs and Models

01 9.1 Logarithmic graphs

- log *y* against log *x*: $y = ax^n$
- log *y* against *x*: $y = kb^x$
- meaning of *a* and *k*
- use the line

02 9.2 Growth and decay

- initial value
- two observations
- when does it reach … ?
- predict at a time

03 9.3 Interpreting models

- long-term value
- can it reach … ?
- percentage multiplier
- compare models
- complete months

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9.3 Interpreting Exponential Models 10

Logarithmic Graphs NOTE 9.1

Your notes

Logarithmic Graphs

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Why take logs of data?

- Curves are hard to read; **straight lines** are easy: $Y = mX + c$ gives two numbers, the gradient and the intercept.
- Taking logs turns the two important curved models into straight lines:

Power model $y = ax^n$	**Exponential model** $y = kb^x$
$\log y = \log a + n \log x$	$\log y = \log k + x \log b$
↓	↓
plot $\log y$ against $\log x$ gradient = n, intercept = $\log a$	plot $\log y$ against x gradient = $\log b$, intercept = $\log k$

What is on the axes tells you which model it is.

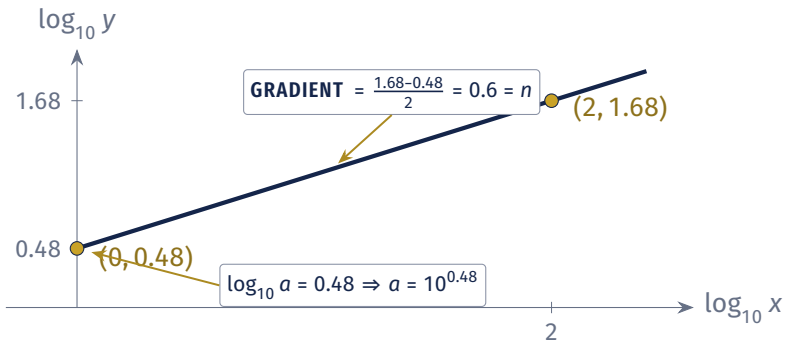
How do I find the constants from the line?

STEP 1 Take logs of the model and use the log laws to write it as $\log y$ = (gradient)×(…) + (intercept).

STEP 2 Find the gradient and intercept of the line from the two points given.

STEP 3 Match: the gradient and the intercept are **logs** of the constants (except n).

STEP 4 Undo the log: if the log is $\log_{10}$, then $a = 10^{\text{intercept}}$; if it is ln, then $a = e^{\text{intercept}}$.



$\log_{10} y$ against $\log_{10} x$: the gradient is the power n; the intercept is $\log_{10} a$, not a.

e.g. **A LINE OF** $\ln y$ **AGAINST** x **HAS GRADIENT** 0.3 **AND INTERCEPT** 1.2. **FIND** y **IN TERMS OF** x

$\ln y = 1.2 + 0.3x$ ← **WRITE THE LINE**

$y = e^{1.2+0.3x} = e^{1.2}e^{0.3x}$ ← **APPLY** $e^{(\,)}$**; SPLIT THE POWER**

ANSWER $y = 3.32e^{0.3x}$

Examiner Tips and Tricks

- The most common error: saying the intercept **is** a (or k). It is $\log a$. Undo it with the **same base** as the axes.
- In $y = kb^x$, the gradient is $\log b$, not b: $b = 10^{\text{gradient}}$.
- In $y = ax^n$, a is the value of y when $x = 1$ (not when $x = 0$). Use this when asked to interpret a.

Your notes

VISUAL NOTES · P3.09

Your notes

WHEN YOU SEE → YOU DO

axes labelled $\log_{10} y$ **and** $\log_{10} x$	→ power model $y = ax^n$: gradient n, intercept $\log a$
axes labelled $\log_{10} A$ **and** t	→ exponential model $A = pq^t$: gradient $\log q$, intercept $\log p$
"Find, to 3 decimal places, the value of p and the value of q"	→ $p = 10^{\text{intercept}}$, $q = 10^{\text{gradient}}$
"Interpret the value of a in the context of the model"	→ a is y when $x = 1$ (one unit of the input)

Worked Example

? The surface area A m^2 covered by a plant t days after the start of a study is modelled by $A = pq^t$. A graph of $\log_{10} A$ against t is a straight line through $(0, 0.45)$ and $(10, 0.95)$. Find p and q to 3 decimal places.

$\log_{10} A = \log_{10} p + t \log_{10} q$ ← TAKE LOGS; PRODUCT AND POWER LAWS

$\log_{10} p = 0.45 \Rightarrow p = 10^{0.45}$ ← INTERCEPT

$\log_{10} q = \frac{0.95-0.45}{10} = 0.05 \Rightarrow q = 10^{0.05}$ ← GRADIENT

ANSWER $p = 2.818$, $q = 1.122$

Your notes

Worked Example

? A graph of $\log_{10} y$ against $\log_{10} x$ is a straight line through $(0, 0.48)$ and $(2, 1.68)$. Find the equation connecting y and x, and interpret the constant a.

$\log_{10} y = \log_{10} a + n \log_{10} x$ ← **POWER MODEL** $y = ax^n$

$n = \frac{1.68-0.48}{2} = 0.6, \qquad a = 10^{0.48} = 3.02$ ← **GRADIENT AND INTERCEPT**

$x = 1 : \ y = a$ ← **INTERPRETATION**

ANSWER $y = 3.02x^{0.6}$; a is the value of y when $x = 1$

HOW THIS IS EXAMINED · past WMA13 papers

$\log y$ vs $\log x$ SEEN 6× IN EXAMS $\log y$ vs x SEEN 13× IN EXAMS
use a log model SEEN 1× IN EXAMS interpret a SEEN 2× IN EXAMS

Exponential Growth and Decay NOTE 9.2

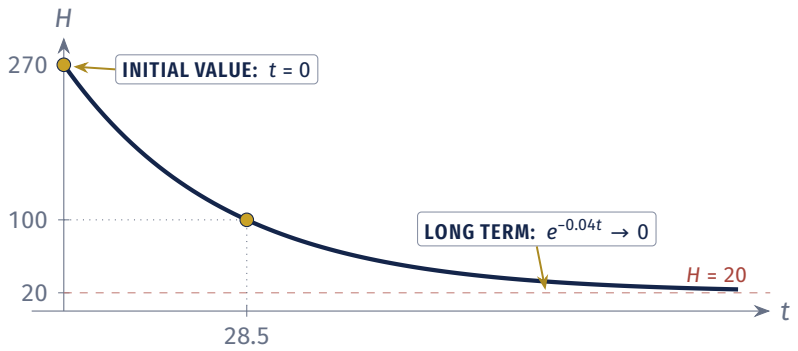
Your notes

Exponential Growth and Decay

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What do exponential models look like?

- **Growth**: $N = Ae^{kt}$ with $k > 0$. **Decay**: $N = Ae^{-kt}$ with $k > 0$.
- With an added constant, $H = Ae^{-kt} + c$, the quantity decays **towards** c, not towards 0 (e.g. a cooling object approaches room temperature).
- The rate of change is proportional to the amount (or to the distance from c): $\frac{dH}{dt} = -k(H - c)$.



$H = 250e^{-0.04t} + 20$: starts at 270, falls towards 20.

How do I find the constants?

- **Initial value**: put $t = 0$. Since $e^0 = 1$, $N(0) = A$ (or $A + c$).
- **Two observations**: substitute both, then **divide** one equation by the other to eliminate A and leave one exponential.

How do I find when a model reaches a value?

STEP 1 Set the model equal to the target value.

STEP 2 Isolate the exponential.

STEP 3 Take ln and solve for t.

STEP 4 Convert to the units asked (minutes, years, the calendar year).

Your notes

Examiner Tips and Tricks

- "Initial" always means $t = 0$, whatever the units of t.
- Keep unrounded values of constants in your calculator; round only the final answer.
- "Show by differentiation that $\frac{dH}{dt} = a + bH$": differentiate, then replace $250e^{-0.04t}$ by $H - 20$.

WHEN YOU SEE → YOU DO

"Find the initial temperature"	→ put $t = 0$
"Find the value of t for which $H = 144$"	→ set equal, isolate $e^{(\dots)}$, take ln
"Given that … 30 m^2 at the start … 50 m^2 after T days"	→ first fact gives a constant; second gives T
"Show by differentiation that $\frac{dH}{dt} = a + bH$"	→ differentiate, then eliminate the exponential using H

Worked Example

? A hot object cools so that its temperature after t minutes is $H = 250e^{-0.04t} + 20$. (a) Find the initial temperature. (b) Find t when $H = 100$. (c) Show that $\frac{dH}{dt} = a + bH$, stating a and b.

$H(0) = 250 + 20 = 270$ ← **(A)** $e^0 = 1$

$250e^{-0.04t} = 80 \Rightarrow -0.04t = \ln 0.32$ ← **(B) ISOLATE, TAKE** ln

$\frac{dH}{dt} = -10e^{-0.04t} = -0.04(H - 20)$ ← **(C)** $250e^{-0.04t} = H - 20$

ANSWER (a) 270°C (b) $t = 28.49$ (c) $\frac{dH}{dt} = 0.8 - 0.04H$: $a = 0.8$, $b = -0.04$

Your notes

Worked Example

? The value of a machine is modelled by $V = Ae^{kt}$. After 2 years it is worth \$3200 and after 5 years \$1638.40. Find A and k, and the yearly percentage loss in value.

$Ae^{2k} = 3200, \quad Ae^{5k} = 1638.4$ ← SUBSTITUTE BOTH OBSERVATIONS

$e^{3k} = 0.512 \Rightarrow k = \frac{1}{3}\ln 0.512$ ← DIVIDE TO ELIMINATE A

$A = \frac{3200}{e^{2k}} = \frac{3200}{0.64}$ ← SUBSTITUTE BACK: $e^k = 0.8$

ANSWER $A = 5000$, $k = \ln 0.8 = -0.223$; it keeps 80% each year, so loses 20% per year

HOW THIS IS EXAMINED · past WMA13 papers

initial value SEEN 15× IN EXAMS two observations SEEN 6× IN EXAMS
time to reach a target SEEN 4× IN EXAMS predict at a time SEEN 4× IN EXAMS

Interpreting Exponential Models NOTE 9.3

Your notes

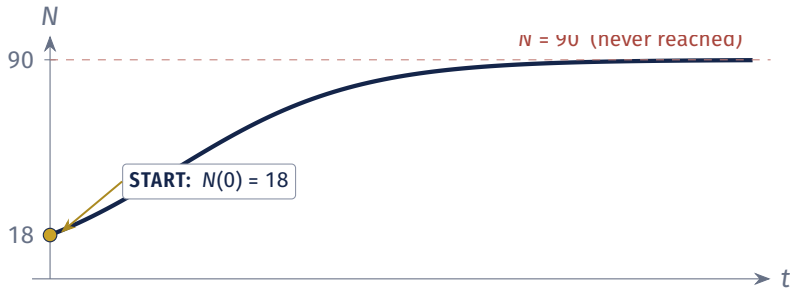
VISUAL NOTES · P3.09

Interpreting Exponential Models

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How do I find the long-term value?

- As $t \to \infty$: $e^{-kt} \to 0$ and $e^{kt} \to \infty$.
- $H = Ae^{-kt} + c \to c$: the long-term value is the **added constant**.
- For a fraction such as $N = \dfrac{90pe^{0.2t}}{3 + pe^{0.2t}}$: divide top and bottom by $e^{0.2t}$ to get $\dfrac{90p}{3e^{-0.2t} + p} \to 90$.



A limited-growth model: fast at first, then levelling off below its limit.

How do I judge whether a model is realistic?

- Find the **range** of values the model can produce for the allowed t.
- Compare with the context: can the value really be that large, that small, or negative?
- Give a **specific** reason using numbers from the model, e.g. "the model gives $N < 90$ for all t, so it can never reach 100".
- When comparing two models, refer to what each predicts at the start, in the long term, or for the data given.

Your notes

What does a multiplier mean?

- $V = Ab^t$ (or Ae^{kt} with $b = e^k$): each unit of time multiplies V by b.
- $b = 0.8$: keeps 80%, so **loses** 20% per unit of time. $b = 1.05$: **gains** 5%.

What about "complete months" or "the first year"?

- Solve for t exactly, then translate: if $t = 13.86$ months, the target is first passed during the 14th month, so the answer is 14 complete months.
- Read the inequality: "more than" and "at least" can change which whole number is correct. Check the integer values either side.

Examiner Tips and Tricks

- "Explain why … can never reach 500": show the limit (or maximum) from the model and compare: "$N \to 450$ as $t \to \infty$ and $N < 450$, so N never reaches 500".
- A negative output in a profit model means a **loss**: "a loss of about £830 000". Mention the units.
- Never say only "the model is unrealistic". Say **why**, with a number.

WHEN YOU SEE → YOU DO

"Find, according to the model, the maximum possible surface area"	→ let $t \to \infty$; divide by the largest exponential
"Explain why, according to this model, the number … can never reach 500"	→ state the limit/bound and compare
"State, giving a reason, whether the model is suitable for …"	→ evaluate the model at that value; compare with reality
"Interpret the value 0.8 in the context of the model"	→ multiplier per unit time: 20% lost each year

Your notes

Worked Example

? $N = \dfrac{90pe^{0.2t}}{3 + pe^{0.2t}}$, where N is the number of plants after t weeks. Initially there are 18 plants. (a) Show that $p = 0.75$. (b) Find the maximum possible number of plants according to the model. (c) Explain why the model predicts the number of plants will never reach 100.

$t = 0: \quad \dfrac{90p}{3 + p} = 18$ ← (A) INITIAL VALUE

$90p = 54 + 18p \Rightarrow 72p = 54$ ← SOLVE

$N = \dfrac{90p}{3e^{-0.2t} + p} \to \dfrac{90p}{p} = 90$ ← (B) DIVIDE BY $e^{0.2t}$; $e^{-0.2t} \to 0$

ANSWER (a) $p = 0.75$ (b) 90 (c) $N < 90$ for all t, so N can never reach 100

HOW THIS IS EXAMINED · past WMA13 papers

long-term value SEEN 4× IN EXAMS range feasibility SEEN 4× IN EXAMS
compare models NOT SEEN YET • BE READY multiplier as a percentage SEEN 2× IN EXAMS
mixed log model SEEN 1× IN EXAMS complete time periods SEEN 1× IN EXAMS

MISTAKE GALLERY

Where marks are lost

Your notes

× WRONG

Intercept 0.45 on a $\log_{10} A$ graph, so $p = 0.45$.

✓ RIGHT

$\log_{10} p = 0.45$, so $p = 10^{0.45} = 2.818$.

The axes are logs; undo them.

× WRONG

In $y = kb^x$, gradient = b.

✓ RIGHT

Gradient = $\log b$.

Take logs of the model before matching.

× WRONG

Long-term value of $H = 250e^{-0.04t} + 20$ is 0.

✓ RIGHT

It is 20: $e^{-0.04t} \to 0$ leaves +20.

Only the exponential part disappears.

× WRONG

$b = 0.8$ means “decreases by 80%”.

✓ RIGHT

It keeps 80%: decreases by 20%.

Multiplier = fraction retained.

× WRONG

Rounding k to −0.22 before the next part.

✓ RIGHT

Keep the exact $k = \ln 0.8$ (or full calculator value).

Early rounding changes later answers.

THE CHAPTER ON ONE PAGE

Log Graphs and Exponential Models

Your notes

9.1

Straight lines

$y = ax^n$: $\log y$ vs $\log x$, gradient n, intercept $\log a$.
$y = kb^x$: $\log y$ vs x, gradient $\log b$, intercept $\log k$.
Undo with the same base.

9.2

Constants

Initial: $t = 0$.
Two observations: divide to eliminate A.
Reaching a value: isolate, take ln.

9.3

Long term

$e^{-kt} \to 0$; the added constant is the limit.
Fraction: divide by the largest exponential.

9.3

Interpret

Range vs reality, with a number.
Multiplier b: keeps $100b\%$.
Complete periods: check the integers.

BOTTOM LINE Logs straighten curves, $e^0 = 1$ gives the start, $e^{-\infty} = 0$ gives the end, and every judgement about a model needs a number from the model.