

DR ESLAM AHMED
eliteigcse.com

08

CHAPTER 08

Exponential and Logarithmic Functions

e^x and $\ln x$ undo each other. Isolate the exponential, take logs, and always check that what you take the log of is positive.

3	8	12
CONCEPTS	VARIANTS	EXAM QUESTIONS

YOU WILL MASTER

- Sketch $y = e^{ax+b} + c$ and $y = \ln(x + a) + b$ with asymptotes and intercepts
- Solve $e^{ax+b} = p$ and $\ln(ax + b) = q$ exactly
- Solve quadratics in e^x and in $\ln x$, rejecting impossible values
- Use the laws of logarithms and check every log argument is positive

CHAPTER 08 AT A GLANCE

Two note pages, eight ways to be asked

Your notes

CHAPTER 08 MAP

Exponentials and Logarithms

01 **8.1 *e* and ln**

graph of $e^{ax+b} + c$ | graph of $\ln(x + a) + b$ | inverse functions

02 **8.2 Exponential and log equations**

$e^{ax+b} = p$ | fraction with e^{kt} | quadratic in e^x | $\ln(ax + b) = q$ | log laws | quadratic in $\ln x$

Contents

8.1 The Functions e^x and $\ln x$ 3
8.2 Solving Exponential and Logarithmic Equations 7

The Functions e^x and $\ln x$ NOTE 8.1

Your notes

The Functions e^x and $\ln x$

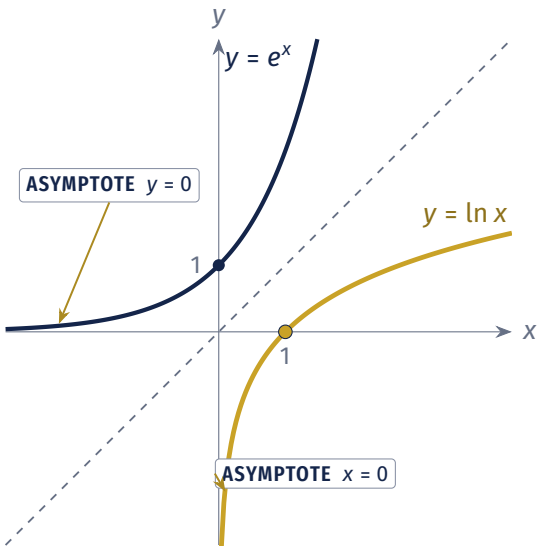
PURE 3 › EXPONENTIALS AND LOGARITHMS › E AND LN • Exam code WMA13 • Dr Eslam Ahmed

What are e^x and $\ln x$?

- $e \approx 2.718$ is the special number for which the gradient of $y = e^x$ **equals its height** at every point.
- $\ln x$ is the logarithm to base e: $\ln x = \log_e x$.
- They are **inverse functions**: $e^{\ln x} = x$ $(x > 0)$ and $\ln(e^x) = x$.
- So their graphs are reflections of each other in $y = x$.

KEY FACTS LEARN THIS: NOT GIVEN

$e^0 = 1$ $\quad \ln 1 = 0$ $\quad \ln e = 1$ $\quad y = e^x \iff x = \ln y$



$y = e^x$ and $y = \ln x$: mirror images in $y = x$.

	$y = e^x$	$y = \ln x$
domain	all real x	$x > 0$
range	$y > 0$	all real y
intercept	$(0, 1)$	$(1, 0)$
asymptote	$y = 0$	$x = 0$

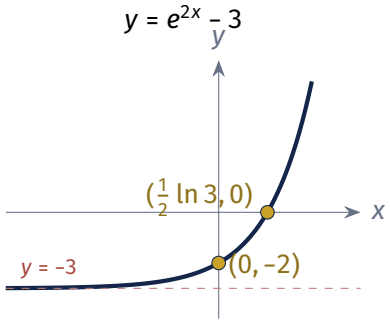
Your notes

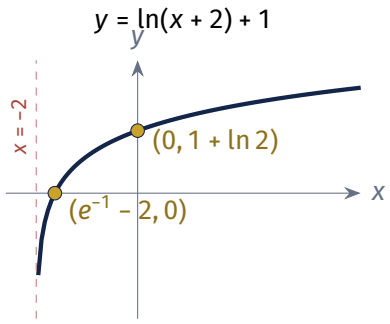
How do I sketch $y = e^{ax+b} + c$?

- $e^{(\dots)}$ is always **positive**, so the curve stays on one side of the **horizontal asymptote** $y = c$.
- $a > 0$: increasing (growth). $a < 0$: decreasing (decay), approaching $y = c$ as $x \to \infty$.
- y-intercept: put $x = 0$: $y = e^b + c$.
- x-intercept (if $c < 0$): solve $e^{ax+b} = -c$.

How do I sketch $y = \ln(x + a) + b$?

- The inside must be **positive**: $x > -a$. The **vertical asymptote** is $x = -a$.
- x-intercept: solve $\ln(x + a) = -b$, i.e. $x = e^{-b} - a$.





Exam-style sketches: label the asymptote with its equation and give every intercept exactly.

Examiner Tips and Tricks

- "On your sketch, state the equation of the asymptote": write $y = 24$ (or $x = -2$) next to a dashed line. A number alone is not an equation.
- Give intercepts **exactly**: $(\frac{1}{2}\ln 3, 0)$, not $(0.549, 0)$, unless a decimal is asked for.
- e^x can never be 0 or negative. A curve $y = e^{(\dots)} + c$ never crosses its asymptote.

Your notes

WHEN YOU SEE → YOU DO

"Sketch the graph of H against t … state the equation of the asymptote"	→ $H = Ae^{-kt} + c$: start at $A + c$, decay towards $H = c$
"Find the coordinates of the points where the curve meets the axes" for $e^{(\dots)}$ or $\ln(\dots)$	→ put $x = 0$; put $y = 0$ and solve with ln or e
"State the range of f" with $f(x) = e^x + c$	→ $f(x) > c$

Worked Example

? Sketch $y = e^{2x} - 3$, giving the equation of the asymptote and the exact coordinates of the points where the curve meets the axes.

$x \to -\infty : \ e^{2x} \to 0 \ \Rightarrow \ y \to -3$ ← **HORIZONTAL ASYMPTOTE** $y = -3$

$x = 0 : \ y = 1 - 3 = -2$ ← **y-INTERCEPT**

$e^{2x} = 3 \ \Rightarrow \ 2x = \ln 3$ ← **x-INTERCEPT: TAKE** ln

ANSWER asymptote $y = -3$; $(0, -2)$ and $\left(\frac{1}{2}\ln 3, 0\right)$

Worked Example

? Sketch $y = \ln(x + 2) + 1$, giving the asymptote and the exact intercepts.

$x + 2 > 0 \ \Rightarrow$ asymptote $x = -2$ ← **THE INSIDE OF** ln **MUST BE POSITIVE**

$x = 0 : \ y = \ln 2 + 1$ ← **y-INTERCEPT**

$\ln(x + 2) = -1 \ \Rightarrow \ x = e^{-1} - 2$ ← **x-INTERCEPT: APPLY** $e^{(\ )}$

ANSWER asymptote $x = -2$; $(0, 1 + \ln 2)$ and $(e^{-1} - 2, 0)$

HOW THIS IS EXAMINED · past WMA13 papers

exponential sketch SEEN 1× IN EXAMS logarithm sketch SEEN 1× IN EXAMS

Your notes

Solving Exponential and Logarithmic Equations NOTE 8.2

Your notes

Solving Exponential and Logarithmic Equations

PURE 3 › EXPONENTIALS AND LOGARITHMS › E AND LN • Exam code WMA13 • Dr Eslam Ahmed

How do I solve $e^{ax+b} = p$?

STEP 1 **Isolate** the exponential: get $e^{(\dots)}$ alone on one side.

STEP 2 Check the other side is **positive**. If not, there is no solution.

STEP 3 Take ln of both sides: $ax + b = \ln p$.

STEP 4 Solve, and keep the answer exact (in ln) unless a decimal is asked for.

e.g. **SOLVE** $3e^{2x-1} + 5 = 17$

$e^{2x-1} = 4$ ← **ISOLATE: SUBTRACT 5, DIVIDE BY 3**

$2x - 1 = \ln 4 = 2 \ln 2$ ← **TAKE** ln

ANSWER $x = \frac{1}{2} + \ln 2$

What if the exponential appears more than once?

- **Fraction** such as $N = \frac{600e^{0.1t}}{2 + e^{0.1t}}$: multiply up, collect the $e^{0.1t}$ terms, factorise, divide.
- **Hidden quadratic**: $e^{2x} = (e^x)^2$. Let $u = e^x$, solve the quadratic in u, **reject** $u \le 0$, then $x = \ln u$.
- $e^{-x} = \frac{1}{e^x} = \frac{1}{u}$: multiply through by u to clear it.

How do I solve log equations?

- One log: $\ln(ax + b) = q \Rightarrow ax + b = e^q$.
- Several logs: combine them into **one** log using the laws, then remove it.
- Hidden quadratic in $\ln x$: let $u = \ln x$, then $x = e^u$.

- **Check** every answer: the inside of every original log must be positive.

Your notes

LAWS OF LOGARITHMS (ANY BASE) LEARN THIS: NOT GIVEN

$$\ln a + \ln b = \ln(ab) \qquad \ln a - \ln b = \ln \frac{a}{b} \qquad k \ln a = \ln(a^k)$$

Examiner Tips and Tricks

- "Solutions based entirely on calculator technology are not acceptable": show the isolated exponential and the line where you take logs.
- $\ln(a + b) \neq \ln a + \ln b$. Only products and quotients split.
- When $u = e^x$ gives a negative root, write "$e^x = -2$ has no solution" to show you rejected it deliberately.

WHEN YOU SEE → YOU DO

"Find the value of t for which $H = 144$"	→ substitute, isolate $e^{-0.05t}$, take ln
"Find the exact value of x"	→ leave the answer with ln (or e)
e^{2x} with e^x, or e^x with e^{-x}	→ $u = e^x$: quadratic; reject $u \leq 0$
"$f(x) = \ln(\ldots)$, $x > 0$ ... Find the exact values of a such that $fg(a) = g(a)$"	→ form the equation, combine logs, then check every argument is positive

Your notes

Worked Example

? The number of animals in a reserve is modelled by $N = \dfrac{600e^{0.1t}}{2 + e^{0.1t}}$, $t \geq 0$. (a) Find the initial number. (b) Find t when $N = 400$. (c) Explain why N can never reach 600.

$t = 0 : \ N = \dfrac{600}{3} = 200$ ← (A) $e^0 = 1$

$400(2 + e^{0.1t}) = 600e^{0.1t} \Rightarrow 800 = 200e^{0.1t}$ ← (B) MULTIPLY UP AND COLLECT

$e^{0.1t} = 4 \Rightarrow t = 10 \ln 4$ ← TAKE ln

$N = \dfrac{600}{2e^{-0.1t} + 1} < 600$ ← (C) DIVIDE TOP AND BOTTOM BY $e^{0.1t}$: THE DENOMINATOR IS ALWAYS GREATER THAN 1

ANSWER (a) 200 (b) $t = 10 \ln 4 = 13.86$ (c) $N \to 600$ but $N < 600$ for every t

Worked Example

? Solve $2e^x - 7 + 3e^{-x} = 0$, giving exact answers.

$2u - 7 + \dfrac{3}{u} = 0 \Rightarrow 2u^2 - 7u + 3 = 0$ ← LET $u = e^x$; MULTIPLY BY u

$(2u - 1)(u - 3) = 0 \Rightarrow u = \frac{1}{2}, 3$ ← BOTH POSITIVE: BOTH VALID

ANSWER $x = \ln \frac{1}{2} = -\ln 2$ or $x = \ln 3$

Your notes

Worked Example

? Solve (a) $\ln(3x - 2) = 4$ (b) $\ln x + \ln(x - 2) = \ln 15$ (c) $(\ln x)^2 - 3\ln x + 2 = 0$.

$3x - 2 = e^4 \Rightarrow x = \dfrac{e^4 + 2}{3}$ ← (A) APPLY $e^{(\,)}$ TO BOTH SIDES

$\ln\big(x(x - 2)\big) = \ln 15 \Rightarrow x^2 - 2x - 15 = 0$ ← (B) PRODUCT LAW, THEN COMPARE

$x = 5$ ($x = -3$ rejected: $\ln(-3)$ undefined) ← CHECK THE ARGUMENTS

$(\ln x - 1)(\ln x - 2) = 0$ ← (C) QUADRATIC IN $\ln x$

ANSWER (a) $x = \dfrac{e^4 + 2}{3}$ (b) $x = 5$ (c) $x = e$ or $x = e^2$

HOW THIS IS EXAMINED · past WMA13 papers

$e^{ax+b} = p$ SEEN 7× IN EXAMS fraction with e^{kt} SEEN 1× IN EXAMS
quadratic in e^x SEEN 1× IN EXAMS $\ln(ax + b) = q$ SEEN 3× IN EXAMS
log laws NOT SEEN YET • BE READY quadratic in $\ln x$ SEEN 1× IN EXAMS

MISTAKE GALLERY

Where marks are lost

Your notes

× WRONG

$3e^{2x} = 12 \Rightarrow \ln(3e^{2x}) = 6x$.

✓ RIGHT

Divide first: $e^{2x} = 4$, then $2x = \ln 4$.

Isolate the exponential before taking logs.

× WRONG

$\ln(x + 3) = \ln x + \ln 3$.

✓ RIGHT

No law for the log of a sum.

Only $\ln(ab)$ and $\ln \frac{a}{b}$ split.

× WRONG

Keeping $x = -3$ from $x(x - 2) = 15$.

✓ RIGHT

Reject it: $\ln(-3)$ is undefined.

Check every original log argument.

× WRONG

$u = e^x = -2 \Rightarrow x = \ln(-2)$.

✓ RIGHT

No solution from $u = -2$.

$e^x > 0$ always.

× WRONG

Asymptote of $y = 280e^{-0.05t} + 24$ is $y = 0$.

✓ RIGHT

$H = 24$.

The added constant moves the asymptote.

THE CHAPTER ON ONE PAGE

Exponentials and Logarithms

Your notes

8.1

Graphs

e^x: through $(0, 1)$, asymptote $y = 0$, range $y > 0$.
$\ln x$: through $(1, 0)$, asymptote $x = 0$, domain $x > 0$.
Mirror images in $y = x$.

8.1

Transformed

$e^{ax+b} + c$: asymptote $y = c$.
$\ln(x + a) + b$: asymptote $x = -a$.
Exact intercepts.

8.2

Exponential equations

Isolate, check positive, take ln.
e^{2x} with e^x: $u = e^x$, reject $u \leq 0$.

8.2

Log equations

Combine to one log, then apply e.
Check every argument is positive.

BOTTOM LINE Isolate, then undo: ln removes e, e removes ln. And nothing inside a log, or equal to an exponential, may be zero or negative.