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07

CHAPTER 07

Trigonometric Equations and R-Form

Turn the equation into one trig function of one angle, transform the interval to match, and collect every solution before shifting back.

3	11	35
CONCEPTS	VARIANTS	EXAM QUESTIONS

YOU WILL MASTER

- Write $a\sin x + b\cos x$ as $R\sin(x \pm \alpha)$ or $R\cos(x \pm \alpha)$
- Read maximum and minimum values, including reciprocals and squares, from the R-form
- Solve equations with identities, factorising and the R-form, in degrees or radians
- Transform the interval for $2x$, $x + \alpha$ and find every solution

CHAPTER 07 AT A GLANCE

Three note pages, eleven ways to be asked

Your notes

CHAPTER 07 MAP

Trigonometric Equations

01 7.1 Harmonic form

$R\sin(x \pm \alpha)$ · $R\cos(x \pm \alpha)$ · exact R and α

02 7.2 Maximum and minimum

max and min values · where they occur · $\dfrac{k}{c + f(x)}$

$c - (f(x))^2$

03 7.3 Solving trig equations

identity reduction · R-form: $a\sin x + b\cos x = c$

zero products · even powers · axis intercepts

Contents

7.1 Harmonic Form (R-Form) 3
7.2 Maximum and Minimum Values 7
7.3 Solving Trigonometric Equations 10

Harmonic Form (R-Form) NOTE 7.1

Your notes

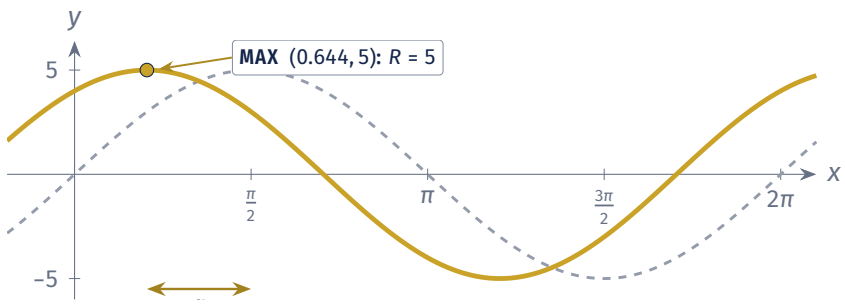
VISUAL NOTES · P3.07

Harmonic Form (R-Form)

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What is the harmonic form?

- Any expression $a\sin x + b\cos x$ is a single **shifted sine wave**. It can be written as
  - $R\sin(x \pm \alpha)$ or $R\cos(x \pm \alpha)$, with $R > 0$
- R is the **amplitude**: $R = \sqrt{a^2 + b^2}$.
- α is the **phase shift**, an acute angle (unless the question says otherwise).



$y = 3\sin x + 4\cos x$ is the wave $5\sin x$ shifted left by $\alpha \approx 0.927$.

How do I find R and α?

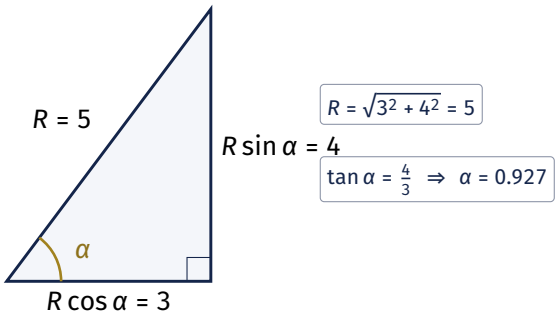
STEP 1 **Expand** the form the question asks for, e.g. $R\sin(x + \alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$.

STEP 2 **Compare coefficients** with $a\sin x + b\cos x$: $R\cos\alpha = a$, $R\sin\alpha = b$.

STEP 3 $R = \sqrt{a^2 + b^2}$ (square and add).

STEP 4 $\tan\alpha = \dfrac{R\sin\alpha}{R\cos\alpha} = \dfrac{b}{a}$ (divide). Use the **acute** angle with the correct signs.

Your notes



The coefficients form a right-angled triangle with hypotenuse R.

FORM ASKED FOR	EXPANSION TO COMPARE
$R\sin(x+\alpha)$	$R\cos\alpha\ \sin x + R\sin\alpha\ \cos x$
$R\sin(x-\alpha)$	$R\cos\alpha\ \sin x - R\sin\alpha\ \cos x$
$R\cos(x+\alpha)$	$R\cos\alpha\ \cos x - R\sin\alpha\ \sin x$
$R\cos(x-\alpha)$	$R\cos\alpha\ \cos x + R\sin\alpha\ \sin x$

Examiner Tips and Tricks

- Give R exactly ($\sqrt{29}$, not 5.39) and α to the accuracy asked, usually 3 or 4 significant figures, in the units asked (radians unless degrees are stated).
- Choose the form so that $R\cos\alpha$ and $R\sin\alpha$ are both **positive**: then α is acute and $\tan\alpha = \frac{b}{a}$ with positive numbers.
- If the angle is $2x$, keep it: $\sqrt{3}\sin 2x - \cos 2x = 2\sin\left(2x - \frac{\pi}{6}\right)$.

WHEN YOU SEE → YOU DO

"Express $8\sin x - 15\cos x$ **in the form** $R\sin(x-\alpha)$**, where** $R > 0$ **and** $0 < \alpha < \frac{\pi}{2}$**"**	→ expand, compare, $R = \sqrt{a^2+b^2}$, $\tan\alpha = \ldots$
"Give the exact value of R**"**	→ leave the surd
"Give the value of α**, in radians, to 3 decimal places"**	→ calculator in radian mode

Your notes

Worked Example

? Express $3\sin x + 4\cos x$ in the form $R\sin(x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give α to 3 decimal places.

$R\sin x\cos\alpha + R\cos x\sin\alpha$ ← EXPAND THE FORM ASKED FOR

$R\cos\alpha = 3, \quad R\sin\alpha = 4$ ← COMPARE COEFFICIENTS OF $\sin x$ AND $\cos x$

$R = \sqrt{9 + 16} = 5, \quad \tan\alpha = \frac{4}{3}$ ← SQUARE AND ADD; DIVIDE

ANSWER $3\sin x + 4\cos x = 5\sin(x + 0.927)$

Worked Example

? Express $2\cos\theta - 5\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

$R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$ ← EXPAND $R\cos(\theta + \alpha)$

$R\cos\alpha = 2, \quad R\sin\alpha = 5$ ← COMPARE: THE MINUS SIGNS MATCH

$R = \sqrt{29}, \quad \tan\alpha = \frac{5}{2}$ ← EXACT R

ANSWER $2\cos\theta - 5\sin\theta = \sqrt{29}\cos(\theta + 1.190)$

Your notes

Worked Example

? $f(x) = \sqrt{3}\sin 2x - \cos 2x$. Express $f(x)$ in the form $R\sin(2x - \alpha)$, $R > 0$, $0 < \alpha < \frac{\pi}{2}$, giving the exact values of R and α.

$R\cos\alpha = \sqrt{3}$, $\quad R\sin\alpha = 1$ ← **COMPARE WITH** $R\sin 2x\cos\alpha - R\cos 2x\sin\alpha$

$R = \sqrt{3+1} = 2$, $\quad \tan\alpha = \frac{1}{\sqrt{3}}$ ← **EXACT VALUES**

ANSWER $f(x) = 2\sin\left(2x - \frac{\pi}{6}\right)$

HOW THIS IS EXAMINED · past WMA13 papers

sine form **SEEN 5× IN EXAMS** cosine form **SEEN 4× IN EXAMS**

Maximum and Minimum Values NOTE 7.2

Your notes

Maximum and Minimum Values

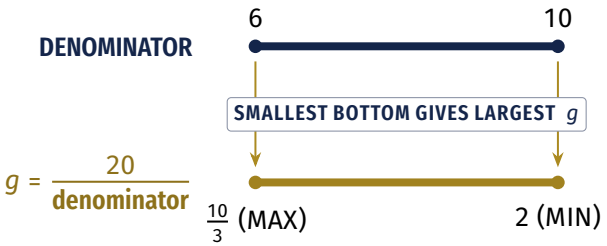
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How do I find the maximum and minimum of $a\sin x + b\cos x$?

- Write it as $R\sin(x+\alpha)$. Since $-1 \le \sin(\dots) \le 1$:
  - maximum value = R, when $\sin(x+\alpha) = 1$
  - minimum value = $-R$, when $\sin(x+\alpha) = -1$
- **Where** it happens: solve $x+\alpha = \frac{\pi}{2}$ (for the max) or $x+\alpha = \frac{3\pi}{2}$ (for the min), and keep the answer that the question asks for ("smallest positive value of x").
- For $R\cos(x+\alpha)$: maximum when $x+\alpha = 0$ (or 2π), minimum when $x+\alpha = \pi$.

What about expressions built from it?

- **Outside changes** (C03): $c + kf(x)$ has max $c + kR$ (if $k > 0$).
- **Reciprocals** $\frac{k}{c+f(x)}$ (with $c > R$ so the bottom is always positive): the fraction is **largest** when the denominator is **smallest**.
- **Squares** $c - (f(x))^2$: $(f(x))^2$ runs from 0 to R^2, because $f(x)$ passes through 0.



For $g(x) = \frac{20}{8+2\sin(2x-\frac{\pi}{6})}$: the denominator runs from 6 to 10, so g runs from 2 to $\frac{10}{3}$, the other way round.

Your notes

Examiner Tips and Tricks

- "Write down the exact maximum value": one line from R, no calculus needed.
- "Find the smallest positive value of x at which this occurs": if the solution of $x + \alpha = \frac{\pi}{2}$ is negative, add 2π (or π for a $2x$ angle).
- For a reciprocal, check that the denominator never reaches 0; for a square, check that the inside can reach 0.

WHEN YOU SEE → YOU DO

"write down the exact maximum value of $g(x)$"	→ use $\pm R$ in place of the trig part
"the smallest positive value of x for which this maximum value occurs"	→ solve $\sin(\ldots) = 1$ (or -1), shift back
"$g(x) = \frac{18}{34 - 3f(x)}$ **... minimum value of g**"	→ smallest g when the denominator is largest
"$h(\beta) = 10 - (12\sin\beta - 5\cos\beta)^2$ **... Find the range of h**"	→ square runs from 0 to R^2

Worked Example

? $f(x) = 3\sin x + 4\cos x = 5\sin(x + 0.927)$. (a) Find the maximum value of $g(x) = 3 + f(x)$ and the smallest positive x at which it occurs. (b) Find the range of $h(x) = 10 - (f(x))^2$.

$\max g = 3 + 5 = 8$ ← **(A)** $\sin(\ldots) = 1$

$x + 0.927 = \frac{\pi}{2} \Rightarrow x = 0.644$ ← **POSITIVE, SO THIS IS THE SMALLEST**

$0 \le (f(x))^2 \le 25$ ← **(B)** f **RUNS FROM** -5 **TO** 5 **AND PASSES THROUGH** 0

ANSWER (a) max 8 at $x = 0.644$ (b) $-15 \le h(x) \le 10$

Your notes

Worked Example

? $g(x) = \dfrac{20}{8 + \sqrt{3}\sin 2x - \cos 2x}$, $0 < x < \pi$. Using $\sqrt{3}\sin 2x - \cos 2x = 2\sin\left(2x - \frac{\pi}{6}\right)$, find the maximum and minimum values of g, and the value of x at which the minimum occurs.

$6 \le 8 + 2\sin\left(2x - \frac{\pi}{6}\right) \le 10$ ← RANGE OF THE DENOMINATOR: ALWAYS POSITIVE

$\max g = \frac{20}{6} = \frac{10}{3}, \quad \min g = \frac{20}{10} = 2$ ← LARGEST g FROM THE SMALLEST DENOMINATOR

$2x - \frac{\pi}{6} = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{3}$ ← MINIMUM OF g: MAXIMUM OF THE DENOMINATOR

ANSWER $\max \frac{10}{3}$, $\min 2$ at $x = \frac{\pi}{3}$

HOW THIS IS EXAMINED · past WMA13 papers

max or min SEEN 8× IN EXAMS squared expression SEEN 1× IN EXAMS
reciprocal SEEN 1× IN EXAMS angle of the extremum SEEN 3× IN EXAMS

Solving Trigonometric Equations NOTE 7.3

Your notes

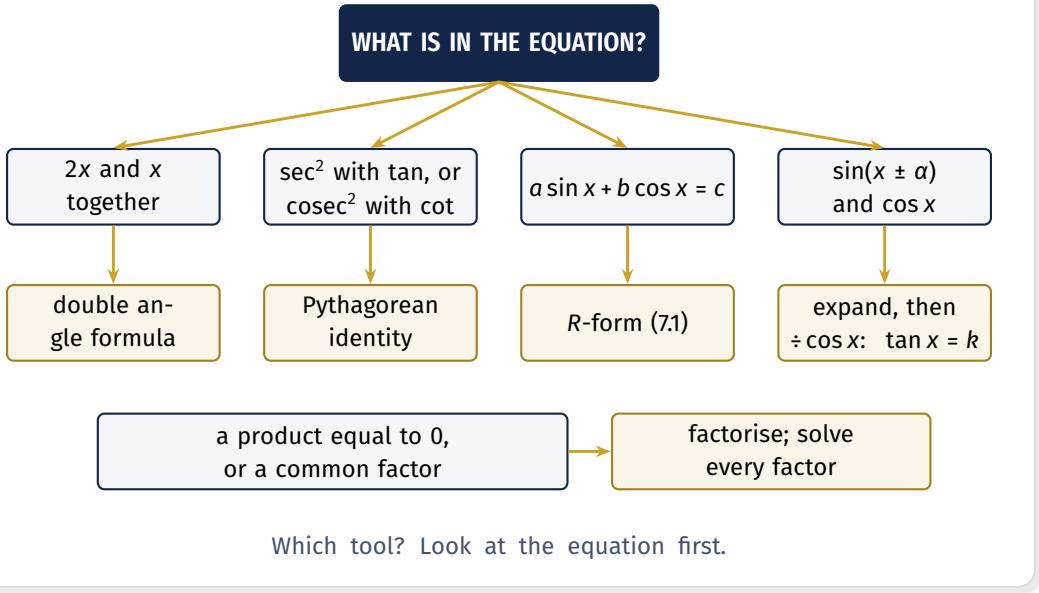
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Solving Trigonometric Equations

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What is the strategy for any trig equation?

- Aim for **one trig function** of **one angle**: sin(angle) = k, tan(angle) = k, ...
- Look at what is in the equation and choose the tool:



Which tool? Look at the equation first.

How do I find every solution in the interval?

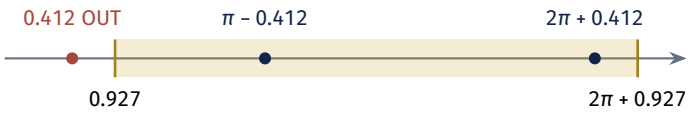
STEP 1 If the angle is $2x$, $x+\alpha$, $\frac{x}{2}$, ..., **transform the interval** first: $0 \le x < 360^\circ \Rightarrow 0 \le 2x < 720^\circ$.

STEP 2 Find the principal value with the calculator.

STEP 3 Find the **second** value in one period: sin: 180° – PV; cos: 360° – PV; tan: PV +180°.

STEP 4 Add or subtract whole periods until you leave the transformed interval.

STEP 5 **Shift back** to x (halve, subtract α, ...) and only then round.



VALUES OF $x + 0.927$ THAT LIE IN THE SHADED INTERVAL

Transforming the interval for $\sin(x + 0.927) = 0.4$, $0 \le x < 2\pi$.

Your notes

Examiner Tips and Tricks

- "Show all stages of your working. Solutions relying entirely on calculator technology are not acceptable": write the identity you use, the factorised form and the principal value.
- Never divide by $\sin x$ or $\cos x$ when it could be 0: **factorise** instead.
- For $\sin^2 x = k$ or $\tan^2 x = k$, take **both** square roots: $\tan x = \pm\sqrt{k}$.
- Check the interval: strict ($0 < x < \pi$) or inclusive ($0 \le x \le 2\pi$), degrees or radians, and the accuracy asked.
- A labelled point on a graph ("the curve meets the x-axis at R") means: set $y = 0$, solve, and pick the root that matches the picture.

WHEN YOU SEE → YOU DO

"Show that the equation $8\cos\theta = 3\operatorname{cosec}\theta$ **can be written in the form** $\sin 2\theta = k$**"**	→ $\operatorname{cosec} = \frac{1}{\sin}$, multiply up, $2\sin\theta\cos\theta = \sin 2\theta$
"Given $2\sin(x + 45^\circ) = \cos(x - 60^\circ)$**, show that** $\tan x = \ldots$**"**	→ expand both, collect $\sin x$ and $\cos x$, divide by $\cos x$
"Hence solve ... $2\sin(2\theta) = \cos(2\theta - 105^\circ)$**"**	→ same equation with a new angle: transform the interval
"Hence solve $3\sin x + 4\cos x = 2$**"**	→ R-form, then $\sin(x + \alpha) = \frac{2}{R}$ on the shifted interval
$(2x - 1)(\sec x + 2) = 0$	→ each factor separately; keep the algebraic root

Your notes

Worked Example

? Solve, for $0 \le \theta < 360^\circ$, $3\cos 2\theta + 5\sin\theta = 4$.

$3(1 - 2\sin^2\theta) + 5\sin\theta - 4 = 0$ ← $\cos 2\theta = 1 - 2\sin^2\theta$: **ONLY** sin **LEFT**

$6\sin^2\theta - 5\sin\theta + 1 = 0 \Rightarrow$
$(3\sin\theta - 1)(2\sin\theta - 1) = 0$ ← **QUADRATIC IN** $\sin\theta$

$\sin\theta = \frac{1}{3}$: $19.5^\circ, 160.5^\circ$ ← **PV AND** 180° **– PV**

$\sin\theta = \frac{1}{2}$: $30^\circ, 150^\circ$ ← **EXACT VALUES**

ANSWER $\theta = 19.5^\circ, 30^\circ, 150^\circ, 160.5^\circ$

Worked Example

? (a) Show that $2\sin(\theta+30^\circ) = 3\cos\theta$ can be written as $\tan\theta = \frac{2}{\sqrt{3}}$.

(b) Hence solve, for $0 \le x < 180^\circ$, $2\sin(2x + 30^\circ) = 3\cos 2x$.

$2\left(\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta\right) = 3\cos\theta$ ← **(A) EXPAND** $\sin(\theta + 30^\circ)$

$\sqrt{3}\sin\theta = 2\cos\theta \Rightarrow \tan\theta = \frac{2}{\sqrt{3}}$ ← **COLLECT, DIVIDE BY** $\cos\theta$

$\theta = 2x$, $0 \le 2x < 360^\circ$: $2x = 49.1^\circ, 229.1^\circ$ ← **(B) TRANSFORM THE INTERVAL;** tan: **ADD** 180°

ANSWER $x = 24.6^\circ, 114.6^\circ$

Your notes

Worked Example

? Using $3\sin x + 4\cos x = 5\sin(x + 0.927)$, solve $3\sin x + 4\cos x = 2$ for $0 \le x < 2\pi$, to 3 decimal places.

$\sin(x + 0.927) = 0.4$ ← DIVIDE BY R

$0.927 \le x + 0.927 < 7.210$ ← TRANSFORM THE INTERVAL

$x + 0.927 = \pi - 0.412, \quad 2\pi + 0.412$ ← 0.412 ITSELF IS BELOW THE INTERVAL

ANSWER $x = 1.803,\ 5.767$

Worked Example

? (i) Solve, for $0 < x < \pi$, $(2x - 1)(\sec x + 2) = 0$. (ii) Solve, for $0 < \theta < 180^\circ$, $8\cos\theta = 3\operatorname{cosec}\theta$.

$2x - 1 = 0 \Rightarrow x = \frac{1}{2}$ ← (I) ALGEBRAIC FACTOR: IN THE INTERVAL, KEEP IT

$\sec x = -2 \Rightarrow \cos x = -\frac{1}{2} \Rightarrow x = \frac{2\pi}{3}$ ← TRIG FACTOR

$8\sin\theta\cos\theta = 3 \Rightarrow 4\sin 2\theta = 3$ ← (II) MULTIPLY BY $\sin\theta$; DOUBLE ANGLE

$0 < 2\theta < 360^\circ : \ 2\theta = 48.6^\circ,\ 131.4^\circ$ ← TRANSFORM THE INTERVAL

ANSWER (i) $x = \frac{1}{2}, \frac{2\pi}{3}$ (ii) $\theta = 24.3^\circ,\ 65.7^\circ$

HOW THIS IS EXAMINED · past WMA13 papers

phase shift SEEN 7× IN EXAMS identity reduction SEEN 18× IN EXAMS
even power SEEN 1× IN EXAMS zero product SEEN 1× IN EXAMS
axis intercept SEEN 2× IN EXAMS

MISTAKE GALLERY

Where marks are lost

Your notes

× WRONG

$R = a + b$, or $R = 5.39$ when "exact" is asked.

✓ RIGHT

$R = \sqrt{a^2 + b^2}$, left as a surd.

Square and add the coefficients.

× WRONG

$\tan\alpha = \frac{a}{b}$ for $R\sin(x + \alpha)$ with $a\sin x + b\cos x$.

✓ RIGHT

$\tan\alpha = \frac{R\sin\alpha}{R\cos\alpha} = \frac{b}{a}$.

Compare coefficients carefully.

× WRONG

Solving $\sin 2x = k$ only on $0 \le 2x < 360^\circ$ when $0 \le x < 360^\circ$.

✓ RIGHT

Use $0 \le 2x < 720^\circ$: twice as many solutions.

Transform the interval first.

× WRONG

$\sin 2x = \sin x \Rightarrow 2\cos x = 1$ only.

✓ RIGHT

$\sin x(2\cos x - 1) = 0$: also $\sin x = 0$.

Dividing by $\sin x$ loses solutions.

× WRONG

$\tan^2 x = 3 \Rightarrow \tan x = \sqrt{3}$ only.

✓ RIGHT

$\tan x = \pm\sqrt{3}$.

Both square roots.

THE CHAPTER ON ONE PAGE

Trig Equations and R-Form

Your notes

7.1

R-form

Expand the form asked, compare coefficients.
$R = \sqrt{a^2 + b^2}$; $\tan\alpha = \frac{R\sin\alpha}{R\cos\alpha}$.
Exact R; α in the units asked.

7.2

Max and min

Trig part = ±1 gives ±R.
Reciprocal: order reverses. Square: from 0 to R^2.
Where: solve sin(…) = ±1.

7.3

Strategy

One function, one angle: double angle, Pythagoras, R-form, expand-and-divide, or factorise.

7.3

Every solution

Transform the interval, PV, second value, add periods, shift back, then round.

BOTTOM LINE One trig function, one angle, the right interval: get those three and every trig equation becomes routine.