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06

CHAPTER 06

Trigonometric Identities and Exact Values

Every identity in this chapter comes from two facts: $\sin^2 + \cos^2 = 1$ and the addition formulae. Learn where they come from and you can rebuild any of them in the exam.

3	11	22
CONCEPTS	VARIANTS	EXAM QUESTIONS

YOU WILL MASTER

- Use $\sec^2 x = 1 + \tan^2 x$ and $\operatorname{cosec}^2 x = 1 + \cot^2 x$ to simplify and substitute
- Expand $\sin(A \pm B)$, $\cos(A \pm B)$, $\tan(A \pm B)$ and find exact values with the right signs
- Choose the right form of $\cos 2A$ every time
- Write clear, one-directional proofs of trigonometric identities

CHAPTER 06 AT A GLANCE

Four note pages, eleven ways to be asked

Your notes

CHAPTER 06 MAP

Trigonometric Identities

01 **6.1 Further identities**

- $\sec^2 = 1 + \tan^2$
- $\operatorname{cosec}^2 = 1 + \cot^2$
- $t = \tan x$ substitution

02 **6.2 Compound angle formulae**

- $\sin(A \pm B)$, $\cos(A \pm B)$
- $\tan(A \pm B)$
- exact values with quadrant signs

03 **6.3 Double angle formulae**

- $\sin 2A$, $\tan 2A$
- three forms of $\cos 2A$
- half angles

04 **6.4 Trigonometric proof**

- compound and double angle proofs
- reciprocal identity proofs

Contents

6.1 Further Trigonometric Identities 3
6.2 Compound Angle Formulae 7
6.3 Double Angle Formulae 10
6.4 Trigonometric Proof 13

Further Trigonometric Identities NOTE 6.1

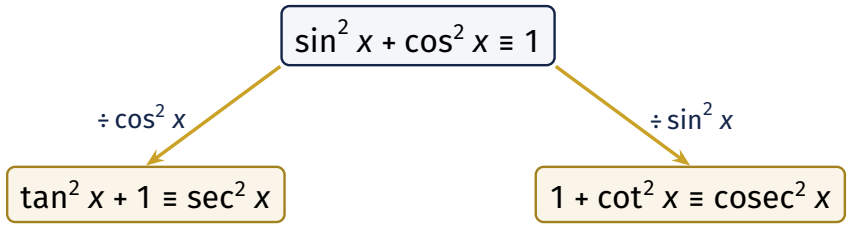
Your notes

Further Trigonometric Identities

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Where do the new Pythagorean identities come from?

- Start from the identity you already know: $\sin^2 x + \cos^2 x \equiv 1$.
- **Divide every term** by $\cos^2 x$ or by $\sin^2 x$:



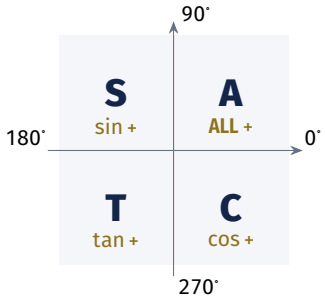
Two new identities from one old one.

PYTHAGOREAN IDENTITIES LEARN THIS: NOT GIVEN

$$\sec^2 x \equiv 1 + \tan^2 x \qquad \operatorname{cosec}^2 x \equiv 1 + \cot^2 x$$

How do I use them?

- To **simplify**: replace $\sec^2 x - 1$ by $\tan^2 x$, $\operatorname{cosec}^2 x - 1$ by $\cot^2 x$.
- To make an equation contain **one** trig function: an equation with $\sec^2 x$ and $\tan x$ becomes a quadratic in $\tan x$.
- To find one ratio from another: from $\tan x$ you get $\sec^2 x$, then $\sec x$ with the **correct sign** for the quadrant.
- The substitution $t = \tan x$ turns many mixed expressions into algebra: $\sec^2 x = 1 + t^2$, $\cot x = \frac{1}{t}$, $\tan 2x = \frac{2t}{1 - t^2}$.



Which ratios are positive in each quadrant (CAST). Use it whenever a square root appears.

Your notes

e.g. **SIMPLIFY** $\dfrac{\sec^2 x - 1}{\operatorname{cosec}^2 x - 1}$

$\dfrac{\tan^2 x}{\cot^2 x}$ ← **USE BOTH PYTHAGOREAN IDENTITIES**

$\tan^2 x \times \tan^2 x$ ← **DIVIDING BY** $\cot^2 x$ **IS MULTIPLYING BY** $\tan^2 x$

ANSWER $\tan^4 x$

Examiner Tips and Tricks

- These two identities are **not** in the formula booklet: learn them, or rebuild them by dividing $\sin^2 + \cos^2 = 1$.
- "Use the substitution $t = \tan x$ to show that ..." means: write **every** term in terms of t, then clear the fractions.
- After squaring or using $\sec^2$, always choose the sign of the square root using the quadrant.

Your notes

WHEN YOU SEE → YOU DO

$\sec^2 x$ **together with** $\tan x$ **in one expression**	→ $\sec^2 x = 1 + \tan^2 x$: everything in $\tan x$
$\operatorname{cosec}^2 x$ **together with** $\cot x$	→ $\operatorname{cosec}^2 x = 1 + \cot^2 x$: everything in $\cot x$
"Use the substitution $t = \tan x$ **to show that the equation ... can be written in the form ..."**	→ replace each ratio by its t-form, multiply out the denominators
"Given $\tan x = -\frac{3}{4}$ **and** x **is obtuse, find the exact value of** $\sec x$**"**	→ $\sec^2 = 1 + \tan^2$, then the sign from CAST

Worked Example

? Given that $\tan x = -\frac{3}{4}$ and $90^\circ < x < 180^\circ$, find the exact value of $\sec x$ and of $\operatorname{cosec} x$.

$\sec^2 x = 1 + \frac{9}{16} = \frac{25}{16}$ ← PYTHAGOREAN IDENTITY

$\sec x = -\frac{5}{4}$ ← SECOND QUADRANT: cos IS NEGATIVE, SO sec IS TOO

$\operatorname{cosec}^2 x = 1 + \cot^2 x = 1 + \frac{16}{9} = \frac{25}{9} \Rightarrow$
$\operatorname{cosec} x = \frac{5}{3}$ ← SECOND QUADRANT: sin IS POSITIVE

ANSWER $\sec x = -\frac{5}{4}$, $\operatorname{cosec} x = \frac{5}{3}$

Your notes

Worked Example

? (a) Use the substitution $t = \tan x$ to show that the equation $2\tan 2x + \cot x \sec^2 x = 0$ can be written as $t^4 - 4t^2 - 1 = 0$. (b) Hence solve the equation for $0 \leq x < 360^\circ$, to one decimal place.

$$\frac{4t}{1-t^2} + \frac{1+t^2}{t} = 0$$

← **(A)** $\tan 2x = \frac{2t}{1-t^2}$, $\cot x = \frac{1}{t}$, $\sec^2 x = 1 + t^2$

$4t^2 + (1+t^2)(1-t^2) = 0 \Rightarrow t^4 - 4t^2 - 1 = 0$

← **MULTIPLY BY** $t(1-t^2)$ **AND REARRANGE**

$t^2 = 2 + \sqrt{5}$ $(t^2 = 2 - \sqrt{5} < 0$ rejected$)$

← **(B) QUADRATIC IN** t^2

$\tan x = \pm 2.058\ldots$

← **BOTH SIGNS: FOUR ANGLES IN** 0° **TO** 360°

ANSWER $x = 64.1^\circ, 115.9^\circ, 244.1^\circ, 295.9^\circ$

HOW THIS IS EXAMINED · past WMA13 papers

reciprocal Pythagorean **SEEN 1× IN EXAMS**
mixed reciprocal and double angle **SEEN 1× IN EXAMS**

Compound Angle Formulae NOTE 6.2

Your notes

Compound Angle Formulae

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What are the compound angle formulae?

- They expand the trig of a **sum or difference** of two angles. They are in the formula booklet:

COMPOUND ANGLE FORMULAE — IN THE FORMULA BOOKLET

$$\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

- $\sin(A + B) \neq \sin A + \sin B$. You must use the formula.
- Sign pattern: sin keeps the **same sign**; cos **changes the sign** ($\cos(A + B)$ has a minus).

sin: $\sin A \cos B \pm \cos A \sin B$ — SAME SIGN AS THE BRACKET

cos: $\cos A \cos B \mp \sin A \sin B$ — OPPOSITE SIGN TO THE BRACKET

Read the signs from the formula, not from memory.

How do I find exact values with them?

- Split the angle into two exact angles: $75^\circ = 45^\circ + 30^\circ$, $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$.
- If you are given $\sin A$ only, find $\cos A$ with a **right-angled triangle** (or $\sin^2 + \cos^2 = 1$), then fix its **sign** from the quadrant.

Your notes

e.g. **FIND THE EXACT VALUE OF** $\sin 75^\circ$

$\sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$ ← **SPLIT INTO EXACT ANGLES**

$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$ ← **EXACT VALUES**

ANSWER $\sin 75^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}$

Examiner Tips and Tricks

- In equations such as $\sin(x + 30^\circ) = 2\cos x$: expand, then divide by $\cos x$ to get an equation in $\tan x$.
- $\tan(x + 45^\circ) = \frac{\tan x + 1}{1 - \tan x}$, because $\tan 45^\circ = 1$. This appears often.
- Write the expanded formula with the actual angles **before** substituting numbers: it is the method mark.

WHEN YOU SEE → YOU DO

"Find the exact value of $\cos 15^\circ$**" / "of** $\tan \frac{7\pi}{12}$**"**	→ split into $45^\circ, 30^\circ, 60^\circ$ (or $\frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{3}$)
"Given $\sin A = \frac{3}{5}$ **... B is obtuse ... find** $\cos(A + B)$**"**	→ triangle for the missing ratios; CAST for their signs
"Using the identity for $\cos(A + B)$**, prove that ..."**	→ put $B = A$ (see 6.4)
$\sin(x + \alpha)$ **or** $\cos(x - \alpha)$ **inside an equation**	→ expand and collect $\sin x$ and $\cos x$

Your notes

Worked Example

? $\sin A = \frac{3}{5}$ where A is acute, and $\cos B = -\frac{5}{13}$ where B is obtuse. Find the exact value of $\cos(A + B)$.

$\cos A = \frac{4}{5}$ ← **3--4--5 TRIANGLE; A ACUTE SO POSITIVE**

$\sin B = \frac{12}{13}$ ← **5--12--13 TRIANGLE; B OBTUSE SO** $\sin B > 0$

$\cos(A + B) = \frac{4}{5}\left(-\frac{5}{13}\right) - \frac{3}{5} \cdot \frac{12}{13}$ ← $\cos A \cos B - \sin A \sin B$

ANSWER $\cos(A + B) = -\frac{56}{65}$

Worked Example

? Given that $\tan(x + 45^\circ) = 5$, find the exact value of $\tan x$.

$\frac{\tan x + 1}{1 - \tan x} = 5$ ← $\tan(A + B)$ **WITH** $\tan 45^\circ = 1$

$\tan x + 1 = 5 - 5\tan x \Rightarrow 6\tan x = 4$ ← **CROSS-MULTIPLY AND COLLECT**

ANSWER $\tan x = \frac{2}{3}$

HOW THIS IS EXAMINED · past WMA13 papers

compound angle identity **SEEN 2× IN EXAMS**
exact compound value **NOT SEEN YET • BE READY** standard exact angle **SEEN 1× IN EXAMS**

Double Angle Formulae NOTE 6.3

Your notes

VISUAL NOTES · P3.06

Double Angle Formulae

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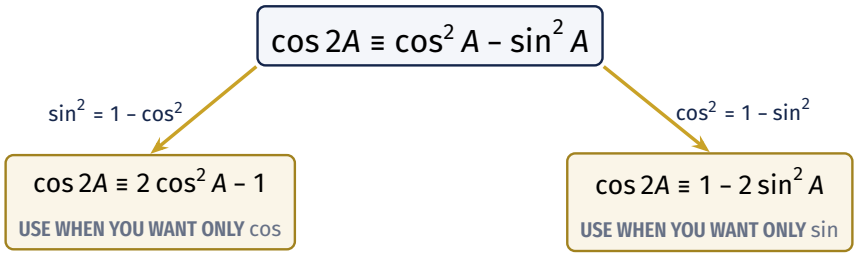
What are the double angle formulae?

- Put $B = A$ in the compound angle formulae:

DOUBLE ANGLE FORMULAE LEARN THIS: NOT GIVEN

$$\sin 2A \equiv 2\sin A\cos A \qquad \tan 2A \equiv \frac{2\tan A}{1-\tan^2 A}$$

- $\cos 2A$ has **three forms**. Choose the one that leaves only one trig function in your expression:



The three forms of $\cos 2A$ and when to use each.

How do I use double angles backwards (half angles)?

- Rearranging the $\cos 2A$ forms gives the **power-reduction** identities, used for integration (C12):
  - $\cos^2 A \equiv \frac{1+\cos 2A}{2}$ $\qquad \sin^2 A \equiv \frac{1-\cos 2A}{2}$
- With $A = \frac{\theta}{2}$ they give **half-angle** values: $\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$.
- Choose the sign of the square root from the quadrant of the **half** angle.
- Also: $2\sin A\cos A = \sin 2A$, so $\sin A\cos A = \frac{1}{2}\sin 2A$.

Your notes

e.g. **FIND THE EXACT VALUE OF** $\cos 15^\circ$ **USING A DOUBLE ANGLE FORMULA**

$\cos 30^\circ = 2\cos^2 15^\circ - 1$ ← $A = 15^\circ$

$\cos^2 15^\circ = \dfrac{1 + \frac{\sqrt{3}}{2}}{2} = \dfrac{2 + \sqrt{3}}{4}$ ← **REARRANGE**

ANSWER $\cos 15^\circ = \dfrac{\sqrt{2 + \sqrt{3}}}{2}$ (positive: 15° is acute)

Examiner Tips and Tricks

- An equation containing both $\cos 2x$ and $\sin x$: use $1 - 2\sin^2 x$ to get a quadratic in $\sin x$. With $\cos x$: use $2\cos^2 x - 1$.
- An equation containing $\sin 2x$ and $\sin x$ (or $\cos x$): write $\sin 2x = 2\sin x \cos x$ and **factorise**. Never divide by $\sin x$ or $\cos x$: you lose solutions.
- $\cos 2A \equiv 2\cos^2 A - 1$ is **not** in the formula booklet in this form; it is a common "prove" question (6.4).

WHEN YOU SEE → YOU DO

an expression with $2x$ **and** x **together**	→ double angle formula to make every angle x
"$\cos 2x$" **with** $\sin x$ **(or** $\cos x$**)**	→ choose $1 - 2\sin^2 x$ (or $2\cos^2 x - 1$)
"Given $\cos A = \frac{1}{3}$, $\frac{3\pi}{2} < A < 2\pi$**, find the exact value of** $\sin 2A$**"**	→ missing ratio with its quadrant sign, then the formula
$\int \cos^2 x\,dx$ **or** $\int \sin^2 x\,dx$	→ power-reduction identity first (C12)

Your notes

Worked Example

? Given that $\cos A = \frac{1}{3}$ and $\frac{3\pi}{2} < A < 2\pi$, find the exact values of $\sin 2A$, $\cos 2A$ and $\tan 2A$.

$\sin A = -\sqrt{1 - \frac{1}{9}} = -\frac{2\sqrt{2}}{3}$ ← **FOURTH QUADRANT:** sin **NEGATIVE**

$\sin 2A = 2\left(-\frac{2\sqrt{2}}{3}\right)\left(\frac{1}{3}\right) = -\frac{4\sqrt{2}}{9}$ ← $2 \sin A \cos A$

$\cos 2A = 2\left(\frac{1}{9}\right) - 1 = -\frac{7}{9}$ ← $2\cos^2 A - 1$: **ONLY** cos **NEEDED**

$\tan 2A = \frac{-4\sqrt{2}/9}{-7/9}$ ← $\frac{\sin 2A}{\cos 2A}$

ANSWER $\sin 2A = -\frac{4\sqrt{2}}{9}$, $\cos 2A = -\frac{7}{9}$, $\tan 2A = \frac{4\sqrt{2}}{7}$

HOW THIS IS EXAMINED · past WMA13 papers

double angle identity SEEN 5× IN EXAMS half angle NOT SEEN YET • BE READY
exact double angle value NOT SEEN YET • BE READY half angle sign NOT SEEN YET • BE READY

Trigonometric Proof NOTE 6.4

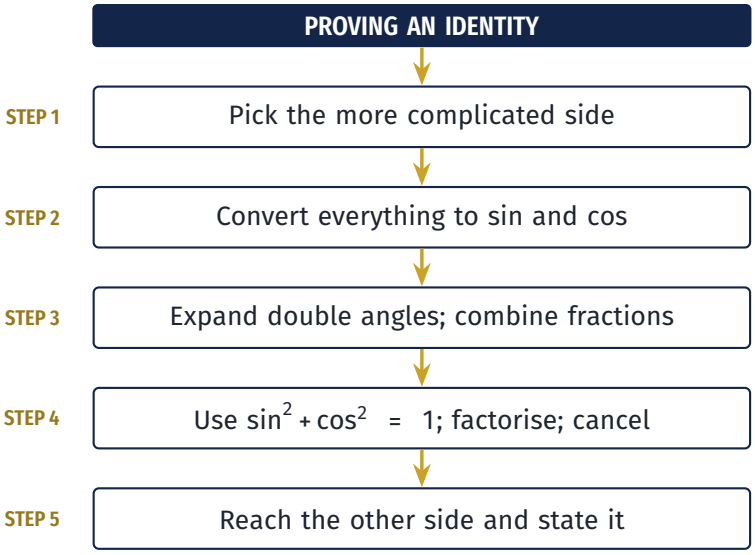
Your notes

Trigonometric Proof

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How do I prove a trigonometric identity?

- Start with **one side** (usually the more complicated one) and transform it, step by step, until it becomes the other side.
- **Never** work on both sides at once, and never start from the identity you are proving.
- Useful moves:
  - write sec, cosec, cot, tan in terms of sin and cos
  - expand double or compound angles
  - combine fractions over a common denominator
  - use $\sin^2 + \cos^2 = 1$ (and its rearrangements) to replace squares
  - factorise, especially a difference of two squares
- Finish by writing the target expression, then "≡ RHS" or "as required".



Your notes

Examiner Tips and Tricks

- Show **every** step: a proof that jumps two lines loses the final mark.
- Use the identity sign ≡ in your working, or "LHS = …".
- "Hence" after a proof: the next part is usually an equation that becomes easy using what you proved (C07).

WHEN YOU SEE → YOU DO

"Prove that …" / "Show that … ≡ …"	→ one side only; convert to sin and cos
"Using the identity for $\cos(A + B)$**, prove that** $\cos 2A \equiv 2\cos^2 A - 1$**"**	→ put $B = A$, then replace $\sin^2 A$
$1 - \cos 2x$ **or** $1 + \cos 2x$ **in a proof**	→ $2\sin^2 x$ or $2\cos^2 x$
"(a) Prove … (b) Hence solve …"	→ use (a) to replace the left side in (b)

Worked Example

? Using the identity for $\cos(A + B)$, prove that $\cos 2A \equiv 2\cos^2 A - 1$.

$\cos 2A = \cos(A + A) = \cos A \cos A - \sin A \sin A$ ← **PUT** $B = A$

$= \cos^2 A - (1 - \cos^2 A)$ ← $\sin^2 A = 1 - \cos^2 A$

ANSWER $= 2\cos^2 A - 1$ as required

Your notes

Worked Example

? Prove that $\operatorname{cosec} 2x - \cot 2x \equiv \tan x$.

$$\text{LHS} = \frac{1}{\sin 2x} - \frac{\cos 2x}{\sin 2x} = \frac{1 - \cos 2x}{\sin 2x}$$ ← **WRITE IN** sin **AND** cos; **ONE FRACTION**

$$= \frac{2\sin^2 x}{2\sin x \cos x}$$ ← $1 - \cos 2x = 2\sin^2 x$; **DOUBLE ANGLE FOR** sin

$$= \frac{\sin x}{\cos x}$$ ← **CANCEL** $2\sin x$

ANSWER $= \tan x = \text{RHS}$

Worked Example

? Prove that $\sec x - \cos x \equiv \sin x \tan x$.

$$\text{LHS} = \frac{1}{\cos x} - \cos x = \frac{1 - \cos^2 x}{\cos x}$$ ← **COMMON DENOMINATOR**

$$= \frac{\sin^2 x}{\cos x} = \sin x \cdot \frac{\sin x}{\cos x}$$ ← **PYTHAGOREAN IDENTITY**

ANSWER $= \sin x \tan x = \text{RHS}$

Worked Example

? Prove that $\cos 3x \equiv 4\cos^3 x - 3\cos x$.

$$\cos(2x + x) = \cos 2x \cos x - \sin 2x \sin x$$ ← **COMPOUND ANGLE**

$$= (2\cos^2 x - 1)\cos x - 2\sin^2 x \cos x$$ ← **DOUBLE ANGLES**

$$= 2\cos^3 x - \cos x - 2(1 - \cos^2 x)\cos x$$ ← $\sin^2 x = 1 - \cos^2 x$

ANSWER $= 4\cos^3 x - 3\cos x = \text{RHS}$

HOW THIS IS EXAMINED · past WMA13 papers

prove a compound or double angle identity **SEEN 9× IN EXAMS**
prove a reciprocal identity **SEEN 4× IN EXAMS**

Your notes

MISTAKE GALLERY

Where marks are lost

Your notes

× WRONG

$\sin(A + B) = \sin A + \sin B$.

✓ RIGHT

$\sin A \cos B + \cos A \sin B$.

Trig functions do not distribute over +.

× WRONG

$\cos(A + B) = \cos A \cos B + \sin A \sin B$.

✓ RIGHT

$\cos A \cos B - \sin A \sin B$.

cos swaps the sign.

× WRONG

$\tan x = -\frac{3}{4}$, obtuse, so $\sec x = +\frac{5}{4}$.

✓ RIGHT

$\sec x = -\frac{5}{4}$: cos is negative in the second quadrant.

Square roots need a sign from CAST.

× WRONG

Dividing $\sin 2x = \sin x$ by $\sin x$.

✓ RIGHT

Factorise: $\sin x(2\cos x - 1) = 0$.

Dividing loses the solutions $\sin x = 0$.

× WRONG

Proving by working on both sides until they “meet”.

✓ RIGHT

Transform one side into the other.

A proof must flow in one direction.

THE CHAPTER ON ONE PAGE

Trigonometric Identities

Your notes

6.1 Pythagorean

$\sin^2 + \cos^2 = 1$; $\sec^2 = 1 + \tan^2$; $\operatorname{cosec}^2 = 1 + \cot^2$.
$t = \tan x$ turns mixed ratios into algebra.

6.2 Compound

$\sin(A \pm B)$: same sign. $\cos(A \pm B)$: opposite sign.
Missing ratio: triangle, then CAST sign.

6.3 Double

$\sin 2A = 2 \sin A \cos A$.
$\cos 2A = \cos^2 - \sin^2 = 2\cos^2 - 1 = 1 - 2\sin^2$.
$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$.

6.4 Proof

One side only. Convert to sin, cos. Combine, use Pythagoras, factorise, cancel.
End with "= RHS".

BOTTOM LINE Two roots grow the whole tree: $\sin^2 + \cos^2 = 1$ and the addition formulae. Pick the form that leaves one trig function, and keep every sign honest.