

DR ESLAM AHMED
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05

CHAPTER 05

Trigonometric Functions and Graphs

Three new functions are just reciprocals, and three more are inverses on a restricted piece of the curve. Know their graphs and you know their values.

2	6	1
CONCEPTS	VARIANTS	EXAM QUESTIONS

YOU WILL MASTER

- Define and evaluate sec, cosec and cot exactly, in degrees and radians
- Sketch their graphs with asymptotes, turning points and range
- Use arcsin, arccos and arctan with their restricted domains and ranges
- Find exact values such as $\tan(\arcsin\frac{3}{5})$ using a right-angled triangle

CHAPTER 05 AT A GLANCE

Two note pages, six ways to be asked

Your notes

CHAPTER 05 MAP

Trigonometric Functions

01 5.1 Reciprocal trig functions

$\sec x = \frac{1}{\cos x}$ $\operatorname{cosec} x = \frac{1}{\sin x}$ $\cot x = \frac{1}{\tan x}$

graphs and asymptotes

02 5.2 Inverse trig functions

arcsin x arccos x arctan x exact values via triangles

Contents

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5.2 Inverse Trigonometric Functions 7

Reciprocal Trigonometric Functions NOTE 5.1

Your notes

VISUAL NOTES · P3.05

Reciprocal Trigonometric Functions

PURE 3 › TRIGONOMETRY › RECIPROCAL AND INVERSE FUNCTIONS • Exam code WMA13 • Dr Eslam Ahmed

What are sec, cosec and cot?

- They are the **reciprocals** of the three basic trig functions:

RECIPROCAL TRIG FUNCTIONS — LEARN THIS: NOT GIVEN

$$\sec x = \frac{1}{\cos x} \qquad \operatorname{cosec} x = \frac{1}{\sin x} \qquad \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

- Memory hook: look at the **third letter**. s**e**c ↔ **c**os, co**s**ec ↔ **s**in, co**t** ↔ **t**an.
- Your calculator has no sec, cosec or cot button: type 1 ÷ cos, 1 ÷ sin or 1 ÷ tan.
- Each one is **undefined** wherever its denominator is zero.
  - sec x is undefined when $\cos x = 0$: $x = 90^\circ, 270^\circ, \ldots$ $\left(\frac{\pi}{2}, \frac{3\pi}{2}, \ldots\right)$
  - cosec x and cot x are undefined when $\sin x = 0$: $x = 0^\circ, 180^\circ, 360^\circ, \ldots$

How do I find exact values?

- Find the exact value of sin, cos or tan first (with its quadrant sign), then take the **reciprocal**.
- Rationalise if needed: $\frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$.

e.g. **FIND THE EXACT VALUE OF** $\operatorname{cosec} 240^\circ$

$\sin 240^\circ = -\sin 60^\circ = -\frac{\sqrt{3}}{2}$ ← THIRD QUADRANT: SINE IS NEGATIVE

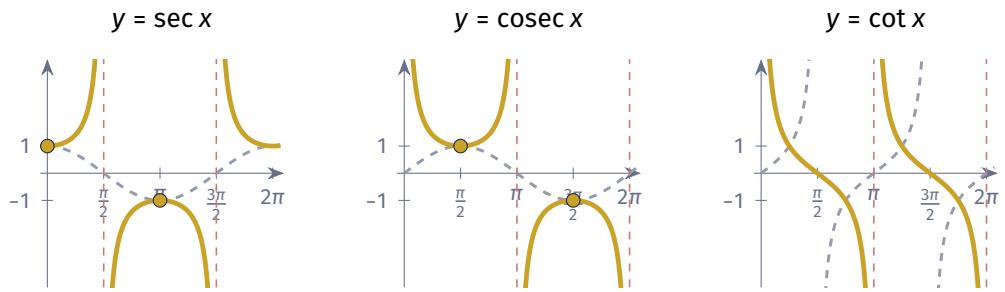
$\operatorname{cosec} 240^\circ = -\frac{2}{\sqrt{3}}$ ← TAKE THE RECIPROCAL

ANSWER $-\frac{2\sqrt{3}}{3}$

Your notes

What do their graphs look like?

- Where $\cos x = \pm 1$, $\sec x = \pm 1$: these are the **turning points** of $y = \sec x$.
- Where $\cos x = 0$, $\sec x$ has a vertical **asymptote**.
- As $\cos x$ gets small, $\sec x$ gets large: the curve **bends away** from the axis.
- Each graph has the **same period** as its basic function: 360° for sec and cosec, 180° for cot.



Dashed: the basic function. Red dashed lines: asymptotes where the basic function is zero.

GRAPH	ASYMPTOTES	PERIOD	RANGE
$\sec x$	$x = 90^\circ, 270^\circ, \ldots$	360°	$y \leq -1$ or $y \geq 1$
$\operatorname{cosec} x$	$x = 0^\circ, 180^\circ, 360^\circ, \ldots$	360°	$y \leq -1$ or $y \geq 1$
$\cot x$	$x = 0^\circ, 180^\circ, 360^\circ, \ldots$	180°	all real y

Examiner Tips and Tricks

- On a sketch, draw the asymptotes as dashed lines and label them with their equations, e.g. $x = \frac{\pi}{2}$.
- $\sec x$ and $\operatorname{cosec} x$ can never be between -1 and 1. An equation such as $\sec x = 0.5$ has **no solutions**.
- Transformations still work: $y = 3 + 2\operatorname{cosec} x$ has range $y \leq 1$ or $y \geq 5$.

Your notes

WHEN YOU SEE → YOU DO

"Find the exact value of $\sec\frac{5\pi}{6}$**"**	→ exact cos with its quadrant sign, then the reciprocal; rationalise
"Sketch $y = \operatorname{cosec} x$ **... state the equations of the asymptotes"**	→ asymptotes where $\sin x = 0$; turning points at ±1
"Solve $\operatorname{cosec} x = 3$**"**	→ turn it over: $\sin x = \frac{1}{3}$, then solve as usual
"State the range of $y = a + b\sec x$**"**	→ start from $\sec x \leq -1$ or ≥ 1, then transform

Worked Example

? Find the exact values of (a) $\sec\frac{5\pi}{6}$ (b) $\cot\frac{\pi}{3}$ (c) $\operatorname{cosec}\frac{7\pi}{4}$.

$\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} \Rightarrow \sec\frac{5\pi}{6} = -\frac{2}{\sqrt{3}}$ ← (A) SECOND QUADRANT: COS NEGATIVE

$\tan\frac{\pi}{3} = \sqrt{3} \Rightarrow \cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$ ← (B) RECIPROCAL OF TAN

$\sin\frac{7\pi}{4} = -\frac{\sqrt{2}}{2} \Rightarrow \operatorname{cosec}\frac{7\pi}{4} = -\frac{2}{\sqrt{2}}$ ← (C) FOURTH QUADRANT: SIN NEGATIVE

ANSWER (a) $-\frac{2\sqrt{3}}{3}$ (b) $\frac{\sqrt{3}}{3}$ (c) $-\sqrt{2}$

Your notes

Worked Example

? Sketch $y = 3 + 2\sec x$ for $0 \le x \le 2\pi$. State the equations of the asymptotes, the coordinates of the turning points and the range.

$\cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$ ← **ASYMPTOTES DO NOT MOVE UNDER OUTSIDE CHANGES**

$\sec 0 = 1 \to y = 5$; $\sec \pi = -1 \to y = 1$ ← **TURNING POINTS: APPLY** $3 + 2(\ )$ **TO** ± 1

$\sec x \ge 1 \Rightarrow y \ge 5$; $\sec x \le -1 \Rightarrow y \le 1$ ← **TRANSFORM THE RANGE OF** sec

ANSWER asymptotes $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$; minima $(0, 5)$, $(2\pi, 5)$; maximum $(\pi, 1)$; range $y \le 1$ or $y \ge 5$

HOW THIS IS EXAMINED · past WMA13 papers

sec NOT SEEN YET • BE READY cosec NOT SEEN YET • BE READY cot NOT SEEN YET • BE READY

Inverse Trigonometric Functions NOTE 5.2

Your notes

Inverse Trigonometric Functions

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What are arcsin, arccos and arctan?

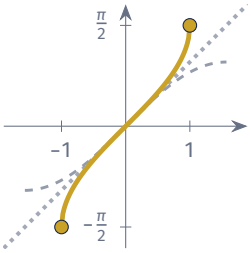
- They are the **inverse functions** of sin, cos and tan: arcsin x is the angle whose sine is x.
- sin x is **many-one**, so it has no inverse on its whole domain. It is **restricted** to a one-one piece first (see 2.3).
- The output is always in **radians** and always in the **principal range**:

FUNCTION	DOMAIN (INPUTS)	RANGE (OUTPUT ANGLES)
arcsin x	$-1 \le x \le 1$	$-\frac{\pi}{2} \le \arcsin x \le \frac{\pi}{2}$
arccos x	$-1 \le x \le 1$	$0 \le \arccos x \le \pi$
arctan x	all real x	$-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$

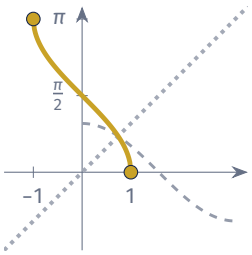
What do their graphs look like?

- Each graph is the **reflection in $y = x$** of the restricted trig graph.
- arcsin and arctan pass through the **origin** and are increasing; arccos is decreasing and passes through $\left(0, \frac{\pi}{2}\right)$.
- arctan x has horizontal asymptotes $y = \pm\frac{\pi}{2}$.

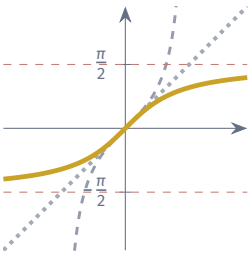
$y = \arcsin x$



$y = \arccos x$



$y = \arctan x$

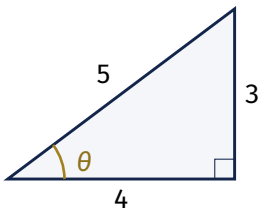


Dashed: the restricted trig function. Solid: its inverse, the reflection in $y = x$.

Your notes

How do I find exact values?

- Use the exact-value table, but choose the angle **inside the principal range**.
  - $\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$ (in $[0, \pi]$), $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$ (not $\frac{7\pi}{6}$)
- For expressions such as $\tan\left(\arcsin\frac{3}{5}\right)$: let $\theta = \arcsin\frac{3}{5}$, draw a **right-angled triangle** with $\sin\theta = \frac{3}{5}$, and read off the side you need.



$\sin\theta = \frac{3}{5}$ **(GIVEN)**

$\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$

MISSING SIDE BY PYTHAGORAS

θ = $\arcsin\frac{3}{5}$: the 3–4–5 triangle gives every other ratio of θ.

Examiner Tips and Tricks

- Your calculator must be in **radian** mode for these questions; answers are angles in radians unless stated otherwise.
- "Sketch C" for $y = \arcsin\frac{x}{2}$: mark both end points, $(-2, -\frac{\pi}{2})$ and $(2, \frac{\pi}{2})$. The graph **stops** there.
- $\arcsin x$ is often written $\sin^{-1} x$. It is **not** $\frac{1}{\sin x}$, which is cosec x.

WHEN YOU SEE → YOU DO

"$f(x) = \arcsin\frac{x}{2}$, $-2 \le x \le 2$ **... Sketch C"**	→ stretched arcsin: end points $(\pm 2, \pm\frac{\pi}{2})$, through O
"Given $x = 2\sin y$**, show that** $\frac{dy}{dx} = \ldots$**"**	→ rewrite $y = \arcsin\frac{x}{2}$ as $x = 2\sin y$ and differentiate (C10)
"Find the exact value of $\cos\left(\arctan\frac{3}{4}\right)$**"**	→ right-angled triangle; read the ratio
"Solve $\arcsin(2x - 1) = \frac{\pi}{6}$**"**	→ take sin of both sides

Your notes

Worked Example

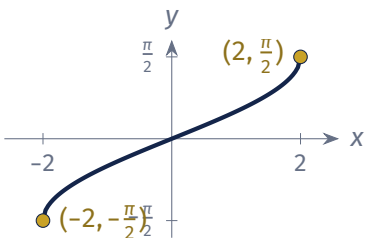
? The curve C has equation $y = \arcsin\frac{x}{2}$, $-2 \le x \le 2$. (a) Sketch C. (b) Find the exact x-coordinate of the point on C with $y = \frac{\pi}{4}$.

$x = \pm 2 \Rightarrow y = \arcsin(\pm 1) = \pm\frac{\pi}{2}$ ← (A) END POINTS OF THE GRAPH

$x = 0 \Rightarrow y = 0$ ← PASSES THROUGH THE ORIGIN, INCREASING

$\sin\frac{\pi}{4} = \frac{x}{2} \Rightarrow x = 2 \times \frac{\sqrt{2}}{2}$ ← (B) TAKE sin OF BOTH SIDES

ANSWER (a) increasing curve from $(-2, -\frac{\pi}{2})$ through O to $(2, \frac{\pi}{2})$ (b) $x = \sqrt{2}$



Worked Example

? Find the exact values of (a) $\arccos\left(-\frac{\sqrt{3}}{2}\right)$ (b) $\sin\left(2\arctan\frac{3}{4}\right)$.

$\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$ and $\frac{5\pi}{6} \in [0, \pi]$ ← (A) CHOOSE THE ANGLE IN THE PRINCIPAL RANGE

$\theta = \arctan\frac{3}{4}$: $\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$ ← (B) 3--4--5 TRIANGLE

$\sin 2\theta = 2\sin\theta\cos\theta = 2 \times \frac{3}{5} \times \frac{4}{5}$ ← DOUBLE ANGLE FORMULA (C06)

ANSWER (a) $\frac{5\pi}{6}$ (b) $\frac{24}{25}$

HOW THIS IS EXAMINED · past WMA13 papers

arcsin SEEN 1× IN EXAMS arccos NOT SEEN YET • BE READY arctan NOT SEEN YET • BE READY

Your notes

MISTAKE GALLERY

Where marks are lost

Your notes

× WRONG

$\sec x = \dfrac{1}{\sin x}$.

✓ RIGHT

$\sec x = \dfrac{1}{\cos x}$.

Third letter: **s**ec pairs with **c**os.

× WRONG

Solving $\sec x = 0.5$ and getting answers.

✓ RIGHT

No solutions: $|\sec x| \geq 1$.

$\cos x = 2$ is impossible.

× WRONG

$\arcsin\left(-\frac{1}{2}\right) = \frac{7\pi}{6}$.

✓ RIGHT

$-\frac{\pi}{6}$: the answer must be in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Inverse trig gives the principal value only.

× WRONG

$\sin^{-1} x = \dfrac{1}{\sin x}$.

✓ RIGHT

$\sin^{-1} x = \arcsin x$; $\dfrac{1}{\sin x} = \operatorname{cosec} x$.

The -1 means inverse function.

× WRONG

Sketching $y = \arcsin x$ beyond $x = \pm 1$.

✓ RIGHT

The graph stops at $(\pm 1, \pm\frac{\pi}{2})$.

The domain is $-1 \leq x \leq 1$.

THE CHAPTER ON ONE PAGE

Trigonometric Functions

Your notes

5.1

Reciprocals

$\sec = \frac{1}{\cos}$, $\operatorname{cosec} = \frac{1}{\sin}$, $\cot = \frac{1}{\tan}$.
Undefined where the denominator is 0.
Exact value: basic ratio with sign, then flip.

5.1

Graphs

Asymptotes where the basic function is 0.
Turning points at ±1.
$|\sec|, |\operatorname{cosec}| \geq 1$; cot takes every value.

5.2

Inverses

arcsin, arccos: inputs $[-1, 1]$; arctan: all inputs.
Outputs: $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $[0, \pi]$, $(-\frac{\pi}{2}, \frac{\pi}{2})$.
Graph: reflect in $y = x$.

5.2

Exact values

Pick the angle in the principal range.
trig(arc ...): draw the right-angled triangle.

BOTTOM LINE Reciprocal functions flip the values and add asymptotes; inverse functions reverse a restricted piece and always answer in the principal range.