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04

CHAPTER 04

Modulus

Every modulus question is a graph question in disguise: find the corner, split into two straight branches, and keep only what is allowed.

3	12	20
CONCEPTS	VARIANTS	EXAM QUESTIONS

YOU WILL MASTER

- Sketch $|f(x)|$, $f(|x|)$ and every $a|x+p|+q$ graph with exact corners and intercepts
- Solve modulus equations by branches, and reject the answers that are not allowed
- Count solutions and find the parameter range for “exactly two roots”
- Solve modulus inequalities and write the final interval correctly

CHAPTER 04 AT A GLANCE

Three note pages, twelve ways to be asked

Your notes

CHAPTER 04 MAP

Modulus

01 4.1 Sketching modulus graphs

- $y = |f(x)|$
- $y = f(|x|)$
- $y = a|x + p| + q$
- find unknown constants

02 4.2 Solving modulus equations

- $|f(x)| = g(x)$
- $|f(x)| = |g(x)|$
- exactly n roots
- modulus = a curve
- $f(|x|) = c$

03 4.3 Solving modulus inequalities

- $|f(x)| > g(x)$
- < or ≤: the ends
- ln or quadratic inside

Contents

4.1 Sketching Graphs of Modulus Functions 3
4.2 Solving Modulus Equations 8
4.3 Solving Modulus Inequalities 14

Sketching Graphs of Modulus Functions NOTE 4.1

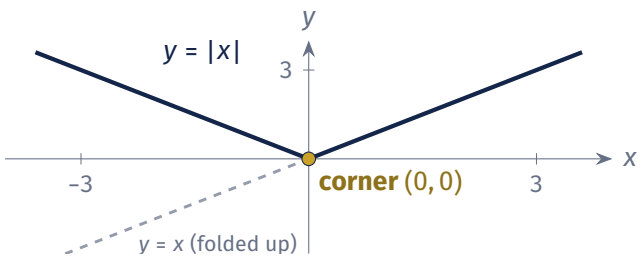
Your notes

Sketching Graphs of Modulus Functions

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What is the modulus function?

- The **modulus** of a number is its **size**: how far it is from 0, ignoring the sign.
  - $|5| = 5$ and $|-5| = 5$
  - The modulus is **never negative**: $|x| \geq 0$ for every x.
- The modulus function $y = |x|$ is defined in two **branches**:
  - $|x| = x$ when $x \geq 0$ (leave it alone)
  - $|x| = -x$ when $x < 0$ (change its sign)
- Its graph is a **V-shape** with its **corner** (vertex) at the origin.



The graph of $y = |x|$: the part of $y = x$ below the axis is folded up.

How do I sketch $y = a|x + p| + q$?

- Every graph of this type is a **V-shape**: two straight arms that meet at a **corner**.
- The corner is where the inside of the modulus equals **zero**.

STEP 1 Find the corner: solve $x + p = 0$, so $x = -p$. Its height is $y = q$.

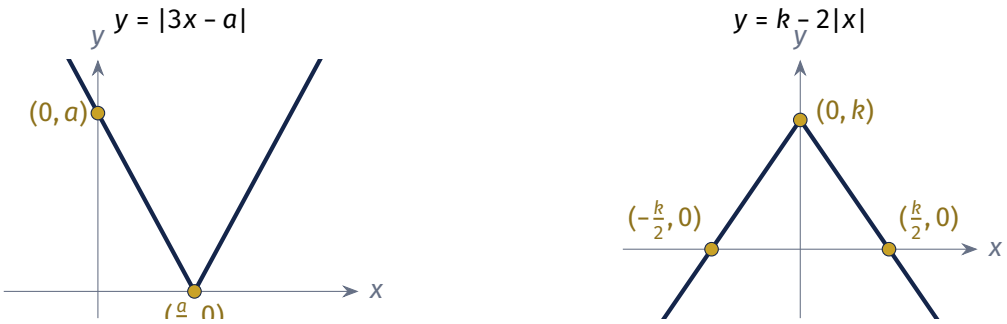
STEP 2 Decide the direction: $a > 0$ opens **upwards** (V); $a < 0$ opens **downwards** (Λ).

STEP 3 Find the y-intercept: put $x = 0$ **inside** the modulus first, then take the size.

STEP 4 Find any x-intercepts: solve $a|x + p| + q = 0$ by branches (see 4.2).

STEP 5 Draw two straight arms through these points and label every coordinate.

Your notes



Two of the most common exam graphs. Label the corner and every intercept.

e.g. **SKETCH $y = |2x - 6|$, SHOWING WHERE THE GRAPH MEETS THE AXES**

$2x - 6 = 0 \Rightarrow x = 3$ ← **CORNER ON THE x-AXIS**

$x = 0 : \ y = |-6| = 6$ ← **y-INTERCEPT: INSIDE FIRST**

ANSWER V-shape, corner $(3, 0)$, y-intercept $(0, 6)$

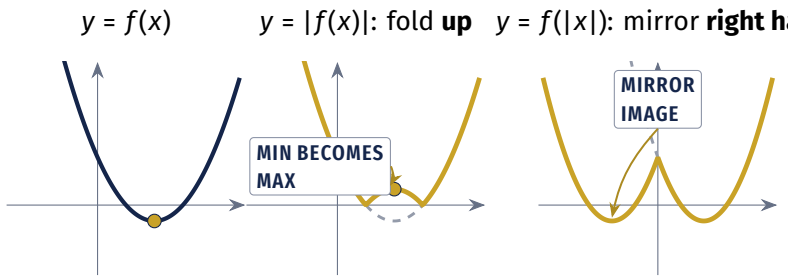
How do I sketch $y = |f(x)|$?

- The modulus acts on the **output** (y-values), so no part of the graph can be below the x-axis.
- Keep every part of $y = f(x)$ that is **on or above** the x-axis.
- **Reflect** every part that is **below** the x-axis in the x-axis: $(x, y) \to (x, -y)$.
  - x-intercepts do not move.
  - A minimum below the axis becomes a **maximum** above it.

How do I sketch $y = f(|x|)$?

- The modulus acts on the **input** (x-values), so $f(|x|)$ gives the same output for x and $-x$.
- **Delete** the part of the graph where $x < 0$.
- **Reflect** the part where $x \geq 0$ in the y-axis. The result is symmetric about the y-axis.
  - Negative y-values **stay negative**.
  - A turning point (p, q) with $p > 0$ gives a second one at $(-p, q)$.

Your notes



The same f, two different modulus graphs. Dashed: the part that moved.

What is the difference between $y = |f(x)|$ and $y = f(|x|)$?

$y = |f(x)|$ (modulus outside)

- changes the y-values
- nothing below the x-axis
- fold **up** in the x-axis

$y = f(|x|)$ (modulus inside)

- changes the x-values
- symmetric in the y-axis
- mirror the **right half** to the left

Examiner Tips and Tricks

- "On separate diagrams" means one clean sketch for each part. Do not draw them on top of each other.
- "Showing the coordinates of the points where the graph meets the axes": every intercept needs both coordinates, e.g. $(0, a)$, not just a.
- The y-intercept of $y = |3x - a|$ is $(0, a)$, never $(0, -a)$.
- Your sketch in part (a) is the tool for parts (b) and (c). Draw it carefully.

Your notes

WHEN YOU SEE → YOU DO

"sketch the graph of $y = \lvert f(x)\rvert$**"**	→ keep above the axis, **reflect** the parts below
"sketch the graph of $y = f(\lvert x\rvert)$**"**	→ keep $x \geq 0$, **copy** it into $x < 0$ by reflecting in the y-axis
"$y = \lvert 3x - a\rvert$ **... where** a **is a positive constant"**	→ corner $(\frac{a}{3}, 0)$; y-intercept $(0, a)$: answers in terms of a
"The graph has a maximum point at $(3, 7)$**" with unknowns**	→ corner ⇒ inside = 0; height ⇒ outside constant

Worked Example

? Sketch the graph of $y = 5 - \lvert 3x - 22\rvert$, showing the coordinates of the maximum point and of any points where the graph meets the coordinate axes.

$3x - 22 = 0 \Rightarrow x = \frac{22}{3}$ ← STEP 1: THE CORNER IS WHERE THE INSIDE EQUALS ZERO

$y = 5 - 0 = 5 \Rightarrow \left(\frac{22}{3}, 5\right)$ ← AT THE CORNER THE MODULUS IS 0; MINUS SIGN MEANS THE CORNER IS A MAXIMUM

$\lvert 3x - 22\rvert = 5 \Rightarrow 3x - 22 = 5$ or $3x - 22 = -5$ ← x-INTERCEPTS: SET $y = 0$ AND SPLIT INTO TWO BRANCHES

$x = 9$ or $x = \frac{17}{3}$ ← SOLVE EACH BRANCH

$x = 0 : \ y = 5 - \lvert -22\rvert = -17$ ← y-INTERCEPT: INSIDE FIRST, THEN TAKE THE SIZE

ANSWER Λ-shape, maximum $(\frac{22}{3}, 5)$, crosses the x-axis at $(\frac{17}{3}, 0)$ and $(9, 0)$, the y-axis at $(0, -17)$

Your notes

Worked Example

? The graph of $y = a - |2x - b|$, where a and b are constants, has a maximum point at $(3, 7)$. Find a and b, and the coordinates of the point where the graph crosses the y-axis.

$2(3) - b = 0 \Rightarrow b = 6$ ← THE CORNER IS AT $x = 3$, SO THE INSIDE IS ZERO THERE

$7 = a - 0 \Rightarrow a = 7$ ← AT THE CORNER THE MODULUS IS 0

$y = 7 - |0 - 6| = 1$ ← PUT $x = 0$

ANSWER $a = 7$, $b = 6$, crosses the y-axis at $(0, 1)$

Worked Example

? $f(x) = x^2 - 4x + 3$. On separate diagrams sketch (a) $y = |f(x)|$ (b) $y = f(|x|)$, giving the coordinates of any turning points.

$f(x) = (x - 1)(x - 3)$, min $(2, -1)$, $(0, 3)$ ← KEY POINTS OF $y = f(x)$ FIRST

$(2, -1) \rightarrow (2, 1)$ ← (A) THE PART BETWEEN $x = 1$ AND $x = 3$ IS FOLDED UP: THE MINIMUM BECOMES A MAXIMUM

$(2, -1) \rightarrow (-2, -1)$ ← (B) THE RIGHT HALF IS MIRRORED: A SECOND MINIMUM APPEARS; y STAYS NEGATIVE

ANSWER (a) maximum $(2, 1)$ (b) minima $(2, -1)$ and $(-2, -1)$, maximum $(0, 3)$

HOW THIS IS EXAMINED · past WMA13 papers

$|f(x)|$ fold SEEN 3× IN EXAMS $f(|x|)$ mirror NOT SEEN YET • BE READY

$a|x + p| + q$ V-shape SEEN 16× IN EXAMS find constants SEEN 3× IN EXAMS

Solving Modulus Equations NOTE 4.2

Your notes

Solving Modulus Equations

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How do I solve $|f(x)| = g(x)$?

- A modulus has two **branches**, so the equation splits into two ordinary equations:
  - $f(x) = g(x)$ (where $f(x) \geq 0$)
  - $-f(x) = g(x)$ (where $f(x) < 0$)
- Solving both always gives two candidate answers, but **not all of them are real solutions**.
- **Sketch both graphs** first: the number of crossings tells you how many answers to keep.

STEP 1 Sketch $y = |f(x)|$ and $y = g(x)$ on the same axes and count the intersections.

STEP 2 Find the corner and write the two branch equations.

STEP 3 Solve each one.

STEP 4 **Reject** any answer that is not on its own branch, or that makes $g(x) < 0$.

e.g. **SOLVE** $|x + 1| = 3$

$x + 1 = 3$ or $x + 1 = -3$

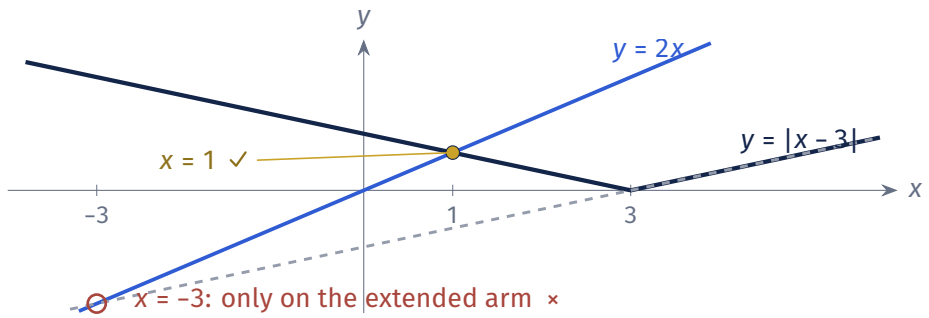
← RIGHT SIDE IS A POSITIVE CONSTANT: BOTH BRANCHES ARE FINE

ANSWER $x = 2$ or $x = -4$

Your notes

Why must I check my answers?

- Each branch equation is only true on its own side of the corner.
- An answer from the branch $x \geq 3$ that turns out to be $x = -3$ is **not** a solution: the line never meets that arm.
- A modulus cannot equal a negative number, so any answer that makes the right-hand side negative must go.



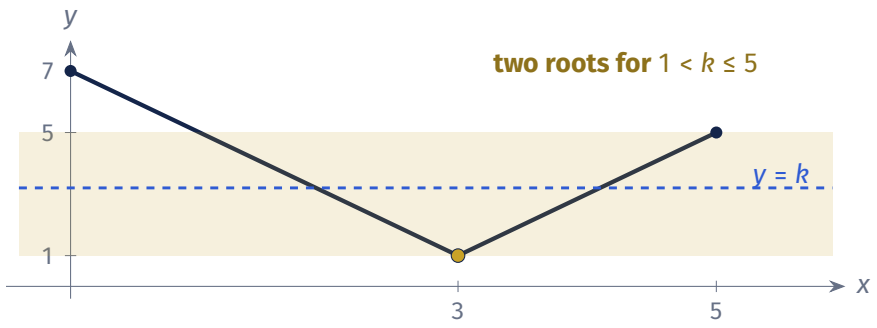
$|x - 3| = 2x$ has one solution. The algebra also produces $x = -3$, which must be rejected.

How do I solve $|f(x)| = |g(x)|$?

- Two numbers with the same size are either **equal** or **negatives** of each other:
  - $f(x) = g(x)$ or $f(x) = -g(x)$
- Alternatively **square both sides**: $(f(x))^2 = (g(x))^2$. This is safe because both sides are ≥ 0.
- Both answers are usually valid here, but substitute to check.

How do I find the values of k for "exactly two roots"?

- The roots of $f(x) = k$ are where the horizontal line $y = k$ meets $y = f(x)$.
- **Slide the line** up and down the sketch and count the crossings in each band of heights.
- Check the boundary heights one by one:
  - at the **corner** height the line touches once (one root)
  - at a **domain end** the point counts only if that end is included (≤)



$f(x) = |2x - 6| + 1$, $0 \le x \le 5$: two roots only for $1 < k \le 5$.

Your notes

How do I solve $f(|x|) = c$?

- Substitute $u = |x|$. The equation becomes $f(u) = c$, with the rule $u \ge 0$.
- Solve for u and **reject** any negative value.
- Each positive u gives **two** answers, $x = u$ and $x = -u$; $u = 0$ gives only $x = 0$.

SOLVING ANY MODULUS EQUATION

STEP 1	Sketch both sides and count the crossings
STEP 2	Find the corner: inside = 0
STEP 3	Write the two branch equations
STEP 4	Solve each branch
STEP 5	Reject answers off their branch; check the rest

Examiner Tips and Tricks

- If the question says "Solve", write the final answers clearly and **cross out or state** the rejected one with a reason: "$x = -3$ is not valid because $x \ge 3$ on this branch".
- With a parameter, e.g. $|3x - a| = |x + a|$ with $a > 0$, keep a in every answer and use $a > 0$ when you check.
- A root exactly at the corner comes out of **both** branches. List it once.

Your notes

WHEN YOU SEE → YOU DO

"Solve the equation $\|ax + b\| = \ldots$"	→ two branches, then **check** each answer against its branch
"$\|\ldots\| = \|\ldots\|$"	→ inside = inside **or** inside = –inside (or square both sides)
"Given that the equation $f(x) = k$ has exactly two roots, find the range of values of k"	→ sketch, slide $y = k$, check each boundary
"$f(\|x\|) = 48$"	→ put $u = \|x\| \geq 0$, solve, then $x = \pm u$

Worked Example

? Solve the equation $|x - 3| = 2x$.

$x \geq 3$: $x - 3 = 2x \Rightarrow x = -3$ ← RIGHT BRANCH; BUT –3 IS NOT ≥ 3, SO REJECT

$x < 3$: $-(x - 3) = 2x \Rightarrow 3 = 3x \Rightarrow x = 1$ ← LEFT BRANCH; 1 < 3 SO IT IS VALID

$|1 - 3| = 2 = 2(1)$ ← CHECK IN THE ORIGINAL EQUATION

ANSWER $x = 1$

Your notes

Worked Example

? Solve $|2x - 1| = |x + 4|$.

$2x - 1 = x + 4 \Rightarrow x = 5$ ← SAME SIGN

$2x - 1 = -(x + 4) \Rightarrow 3x = -3 \Rightarrow x = -1$ ← OPPOSITE SIGNS

$|9| = |9|, \quad |-3| = |3|$ ← BOTH CHECK

ANSWER $x = 5$ or $x = -1$

Worked Example

? $f(x) = |2x - 6| + 1$, $0 \le x \le 5$. Given that the equation $f(x) = k$ has exactly two roots, find the range of values of k.

corner $(3, 1)$, $f(0) = 7$, $f(5) = 5$ ← THE KEY HEIGHTS: CORNER AND BOTH ENDS OF THE DOMAIN

$k = 1$: one root ← THE LINE JUST TOUCHES THE CORNER

$1 < k \le 5$: two roots ← THE LINE CROSSES BOTH ARMS; $x = 5$ IS INCLUDED

$5 < k \le 7$: one root ← THE RIGHT ARM STOPS AT HEIGHT 5

ANSWER $1 < k \le 5$

Your notes

Worked Example

? Solve $x^2 - 4 = |x - 2|$.

$x \geq 2$: $x^2 - 4 = x - 2 \Rightarrow (x - 2)(x + 1) = 0$ ← $x = 2$ **VALID;** $x = -1$ **REJECTED (NOT** ≥ 2**)**

$x < 2$: $x^2 - 4 = 2 - x \Rightarrow (x + 3)(x - 2) = 0$ ← $x = -3$ **VALID;** $x = 2$ **ALREADY FOUND**

ANSWER $x = 2$ or $x = -3$

Worked Example

? $f(x) = x^2 + 2x - 8$. Solve $f(|x|) = 0$.

$u^2 + 2u - 8 = 0 \Rightarrow (u + 4)(u - 2) = 0$ ← **LET** $u = |x|$

$u = -4$ (reject), $u = 2$ ← $|x|$ **CAN NEVER BE NEGATIVE**

ANSWER $x = 2$ or $x = -2$

HOW THIS IS EXAMINED · past WMA13 papers

$|ax + b|$ = a line **SEEN 5× IN EXAMS** $|f| = |g|$ **SEEN 3× IN EXAMS**
exactly n roots **SEEN 5× IN EXAMS** modulus = a curve **SEEN 2× IN EXAMS**
$f(|x|) = c$ **SEEN 2× IN EXAMS**

Solving Modulus Inequalities NOTE 4.3

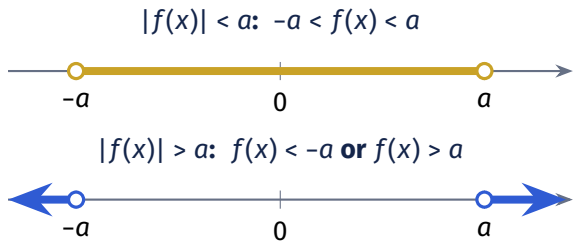
Your notes

Solving Modulus Inequalities

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How do I solve $|f(x)| < a$ and $|f(x)| > a$?

- Think of the modulus as a **distance from 0**.
- $|f(x)| < a$ means $f(x)$ is within a of 0: one **sandwich** interval
  - $-a < f(x) < a$
- $|f(x)| > a$ means $f(x)$ is further than a from 0: **two separate pieces**
  - $f(x) < -a$ or $f(x) > a$
- With ≤ or ≥ the end values are **included**.



"Less than" stays inside; "greater than" goes outside.

e.g. **SOLVE** $|x - 2| < 5$

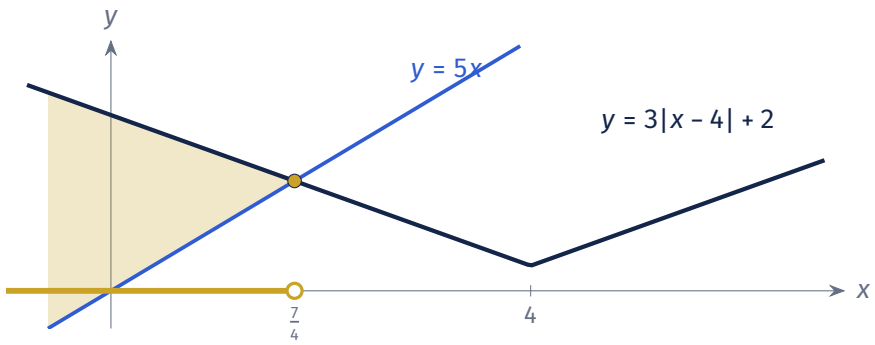
$-5 < x - 2 < 5$ ← LESS THAN: ONE SANDWICH

ANSWER $-3 < x < 7$

Your notes

How do I solve $|f(x)| > g(x)$ when $g(x)$ is not a constant?

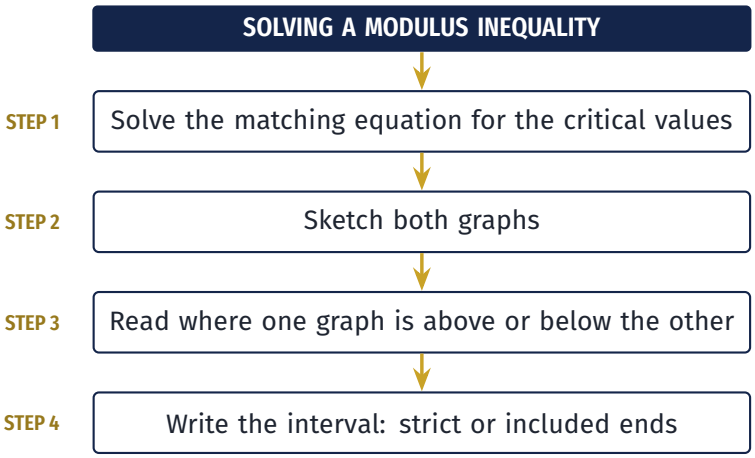
- Solve the matching **equation** $|f(x)| = g(x)$ first (see 4.2). Its solutions are the **critical values**.
- Sketch both graphs and read off where the modulus graph is **above** (>) or **below** (<) the other graph.
- Write the interval using the critical values: strict signs exclude them, ≤ and ≥ include them.



$3|x - 4| + 2 > 5x$ holds where the V is above the line: $x < \frac{7}{4}$.

What if the inside is a logarithm or a quadratic?

- Write down the **domain** first, e.g. $\ln x$ needs $x > 0$; $\ln(x + 1)$ needs $x > -1$.
- Remove the modulus with a sandwich (<) or two pieces (>), then solve each part.
- Dividing or multiplying by a **negative** number **reverses** every inequality sign.
- Finally intersect your answer with the domain.



Your notes

Examiner Tips and Tricks

- Two separate pieces must be joined by "or": write $x \le -2$ or $x \ge 4$, never $-2 \ge x \ge 4$.
- "Find, in terms of k, the range of values of x": the critical values contain k, but the sketch still decides which side to take.
- Always sketch, even roughly. Choosing the side from algebra alone is the most common lost mark.

WHEN YOU SEE → YOU DO

"Solve the inequality $\lvert \ldots \rvert > \ldots$**"**	→ solve the **equation** first, then use the sketch to choose the side
"Find, in terms of k**, the range of values of** x **that satisfy …"**	→ critical values in k; the sketch decides the side
$\lvert g(x) \rvert < c$ **(**$c > 0$**)**	→ one **sandwich**: $-c < g(x) < c$
$\lvert g(x) \rvert > c$	→ **two pieces**: $g(x) < -c$ or $g(x) > c$

Worked Example

? Solve the inequality $3|x - 4| + 2 > 5x$.

$x \ge 4$: $3x - 12 + 2 = 5x \Rightarrow x = -5$ ← EQUATION FIRST; -5 IS NOT ≥ 4, REJECT

$x < 4$: $12 - 3x + 2 = 5x \Rightarrow x = \frac{7}{4}$ ← THE ONLY CRITICAL VALUE

V above the line for $x < \frac{7}{4}$ ← READ THE SIDE FROM THE SKETCH

ANSWER $x < \frac{7}{4}$

Your notes

Worked Example

? Solve (a) $|2x + 1| \le 5$ (b) $|x - 1| \ge 3$.

$-5 \le 2x + 1 \le 5 \Rightarrow -6 \le 2x \le 4$ ← (A) ``LESS THAN'': ONE SANDWICH; SUBTRACT 1 FROM ALL THREE PARTS

$x - 1 \le -3$ or $x - 1 \ge 3$ ← (B) ``GREATER THAN'': TWO PIECES

ANSWER (a) $-3 \le x \le 2$ (b) $x \le -2$ or $x \ge 4$

Worked Example

? Solve $|3 - 2\ln x| < 1$, $x > 0$.

$-1 < 3 - 2\ln x < 1$ ← SANDWICH

$-4 < -2\ln x < -2$ ← SUBTRACT 3 FROM ALL THREE PARTS

$2 > \ln x > 1 \Rightarrow 1 < \ln x < 2$ ← DIVIDE BY −2: BOTH SIGNS REVERSE

$e < x < e^2$ ← APPLY $e^{(\,)}$; THIS IS INSIDE $x > 0$

ANSWER $e < x < e^2$

HOW THIS IS EXAMINED · past WMA13 papers

$|ax + b|$ vs a line SEEN 6× IN EXAMS < or ≤ ends NOT SEEN YET • BE READY

ln or quadratic inside SEEN 2× IN EXAMS

MISTAKE GALLERY

Where marks are lost

Your notes

× WRONG

For $y = |f(x)|$, reflecting the left half of the graph.

✓ RIGHT

Only the parts **below** the x-axis are folded up.

$|f(x)|$ changes y-values, not x-values.

× WRONG

$|x - 3| = 2x \Rightarrow x = -3$ or $x = 1$.

✓ RIGHT

$x = -3$ came from the branch $x \geq 3$, so it is rejected: $x = 1$ only.

Every branch answer must lie on its own branch.

× WRONG

y-intercept of $y = |3x - a|$ is $(0, -a)$.

✓ RIGHT

It is $(0, a)$: a modulus is never negative.

Put $x = 0$ *inside* the modulus, then take the size.

× WRONG

$|x - 1| \geq 3 \Rightarrow -2 \geq x \geq 4$.

✓ RIGHT

$x \leq -2$ or $x \geq 4$.

"Greater than" always splits into two pieces.

× WRONG

$|a + b| = |a| + |b|$.

✓ RIGHT

Only true when a and b have the same sign.

Use branches instead.

THE CHAPTER ON ONE PAGE

Modulus

Your notes

4.1

Sketch

$|f(x)|$: fold below-axis parts up.
$f(|x|)$: mirror the right half.
$a|x+p|+q$: corner $(-p, q)$; y-intercept: inside first.
Label every corner and intercept.

4.2

Equations

Split at the corner. Solve each branch. **Reject** answers off their branch or giving a negative modulus.
$|A| = |B| \Rightarrow A = \pm B$.
$f(|x|) = c$: $u = |x| \geq 0$, then $x = \pm u$.

4.2

Exactly n roots

Sketch on the domain, slide $y = k$.
Corner height: 1 root. Check each end point.

4.3

Inequalities

Solve the equation first, then read the side from the graph.
$|g| < c$: $-c < g < c$. $|g| > c$: $g < -c$ or $g > c$.
Strict sign: ends not included.

BOTTOM LINE Find the corner, split into two straight branches, and keep only the answers the graph allows.