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03

CHAPTER 03

Graph Transformations

Inside the bracket moves x the "wrong" way; outside moves y the way it looks. Track a few key points and the whole graph follows.

2 CONCEPTS

4 VARIANTS

13 EXAM QUESTIONS

YOU WILL MASTER

- Map any point under $f(x) + a$, $f(x + a)$, $af(x)$, $f(ax)$, $-f(x)$ and $f(-x)$
- Combine an inside change and an outside change, e.g. $3f(2x)$ or $f(-x) - 1$
- Sketch transformed graphs with every turning point and intercept labelled
- Work backwards: find a and b from a transformed vertex or maximum

CHAPTER 03 AT A GLANCE

Two note pages, four ways to be asked

Your notes

CHAPTER 03 MAP

Graph Transformations

01 **3.1 Single transformations**

outside: $f(x) + a$, $af(x)$, $-f(x)$ | inside: $f(x + a)$, $f(ax)$, $f(-x)$

02 **3.2 Combinations of transformations**

inside and outside: $3f(2x)$ | reflection and shift: $f(-x) - 1$

find a, b from a vertex | max and min values

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Single Transformations NOTE 3.1

Your notes

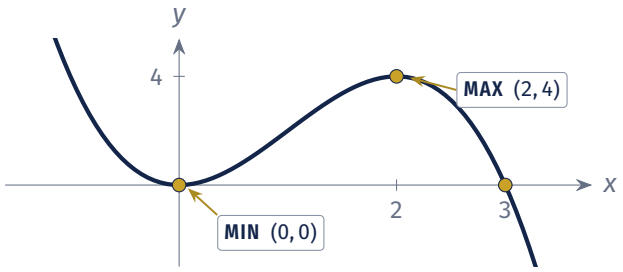
VISUAL NOTES · P3.03

Single Transformations

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What is a graph transformation?

- A **transformation** changes the equation $y = f(x)$ and so moves, stretches or reflects its graph.
- Every transformation is either **outside** the function or **inside** the bracket:
  - **outside** changes act on the y-coordinates, and do what they look like
  - **inside** changes act on the x-coordinates, and do the **opposite** of what they look like
- To sketch a transformed graph, move the **key points** (turning points, intercepts) and keep the shape.



The base curve for this page: $y = f(x)$ with a minimum at $(0, 0)$, a maximum at $(2, 4)$ and a root at $(3, 0)$.

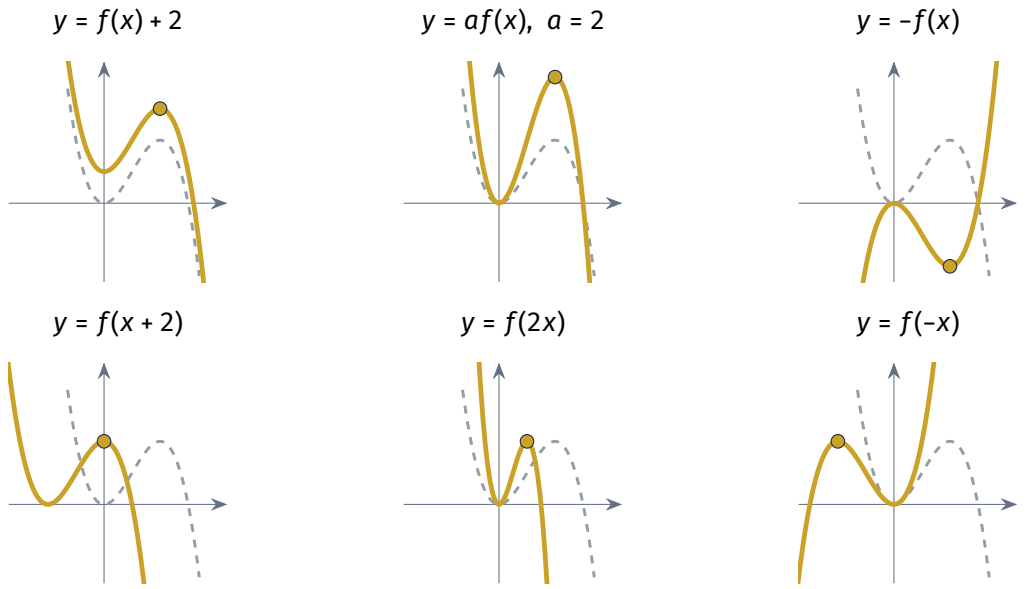
How do outside changes work?

- $y = f(x) + a$: **translation** by $\begin{pmatrix} 0 \\ a \end{pmatrix}$. $(x, y) \to (x, y + a)$
- $y = af(x)$: **vertical stretch**, scale factor a. $(x, y) \to (x, ay)$
- $y = -f(x)$: **reflection** in the x-axis. $(x, y) \to (x, -y)$
- x-coordinates never change. Roots of $af(x)$ stay where they are.

Your notes

How do inside changes work?

- $y = f(x + a)$: **translation** by $\begin{pmatrix} -a \\ 0 \end{pmatrix}$. $(x, y) \to (x - a, y)$ ($+a$ moves **left**)
- $y = f(ax)$: **horizontal stretch**, scale factor $\frac{1}{a}$. $(x, y) \to \left(\frac{x}{a}, y\right)$ ($f(2x)$ **squashes**)
- $y = f(-x)$: **reflection** in the y-axis. $(x, y) \to (-x, y)$
- y-coordinates never change. The y-intercept of $f(ax)$ stays the same.



Six single transformations of the same curve. Dashed: $y = f(x)$.

EQUATION	TRANSFORMATION	(2, 4) MAPS TO
$f(x) + 2$	translation $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$	(2, 6)
$2f(x)$	vertical stretch, scale factor 2	(2, 8)
$-f(x)$	reflection in the x-axis	(2, −4)
$f(x + 2)$	translation $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$	(0, 4)
$f(2x)$	horizontal stretch, scale factor $\frac{1}{2}$	(1, 4)
$f(-x)$	reflection in the y-axis	(−2, 4)

e.g. **WHERE DOES (2, 4) GO UNDER $y = f(x - 3)$?**

$x - 3 = 2 \Rightarrow x = 5$

← **INSIDE: SOLVE NEW INSIDE = OLD x**

ANSWER (5, 4)

Your notes

Examiner Tips and Tricks

- For an inside change, solve "new inside = old x-coordinate". For $f(2x)$ and the point $(2, 4)$: $2x = 2$, so $x = 1$. This never goes the wrong way.
- Describe a stretch with its **direction** and **scale factor**: "a stretch parallel to the x-axis, scale factor $\frac{1}{2}$".
- Asymptotes move too: $y = f(x) + 2$ moves a horizontal asymptote up by 2.

WHEN YOU SEE → YOU DO

"Find the point to which P is mapped when ... is transformed to ..."	→ inside: solve for x; outside: apply to y
"$y = f(2x)$"	→ halve the x-coordinates; y unchanged
"$y = 3f(x)$"	→ treble the y-coordinates; roots unchanged
"$y = f(-x)$" / "$y = -f(x)$"	→ inside minus: mirror in y-axis; outside minus: mirror in x-axis

Worked Example

? The point $P(6, -2)$ lies on the curve $y = f(x)$. Find the point to which P is mapped when the curve is transformed to (a) $y = f(3x)$ (b) $y = 2f(x + 1)$ (c) $y = |f(x)|$.

$3x = 6 \Rightarrow x = 2$ ← (A) INSIDE: NEW INSIDE = OLD x; y UNCHANGED

$x + 1 = 6 \Rightarrow x = 5$; $y = 2(-2) = -4$ ← (B) INSIDE SHIFT, THEN OUTSIDE STRETCH

$y = |-2| = 2$ ← (C) MODULUS OUTSIDE: NEGATIVE y IS REFLECTED UP

ANSWER (a) $(2, -2)$ (b) $(5, -4)$ (c) $(6, 2)$

HOW THIS IS EXAMINED · past WMA13 papers

vertical transformation SEEN 1× IN EXAMS horizontal transformation SEEN 1× IN EXAMS

Your notes

Combinations of Transformations NOTE 3.2

Your notes

Combinations of Transformations

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How do I apply two transformations at once?

- Separate the equation into its **inside** part and its **outside** part.
  - $y = 3f(2x)$: inside $2x$ (horizontal stretch $\frac{1}{2}$), outside ×3 (vertical stretch 3)
  - $y = f(-x) - 1$: inside $-x$ (reflect in y-axis), outside -1 (down 1)
- An inside change and an outside change **do not affect each other**: apply the inside one to x and the outside one to y.
- Two **outside** changes follow the normal order of operations: $2f(x) - 5$ means stretch first, then subtract 5.

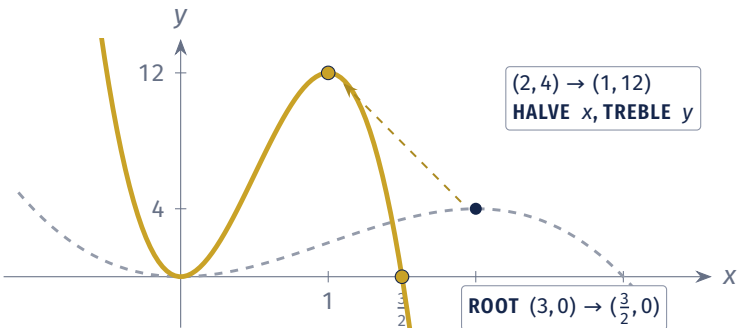
MAPPING A POINT UNDER A COMBINED TRANSFORMATION

STEP 1 Write the key point (p, q)

STEP 2 Inside: solve "new inside = p" for the new x

STEP 3 Outside: apply the outside operations to q in order

STEP 4 Plot and label the new point



$y = 3f(2x)$: squash towards the y-axis, then stretch away from the x-axis.

Your notes

How do I find a and b from a transformed point?

- Use the same rule backwards: **inside** gives an equation in the x-coordinates, **outside** gives an equation in the y-coordinates.
- For $y = af(x + b)$: new x = old $x - b$ and new $y = a\times$ old y.
- A vertex, maximum or minimum is the best point to use: it is easy to read from the question.

How do transformations change maximum and minimum values?

- Outside changes alter the **values**: if f has minimum -3, then $2f(x) - 5$ has minimum $2(-3) - 5 = -11$.
- A **negative** outside factor swaps maxima and minima: $-f$ turns a minimum of -3 into a maximum of 3.
- Inside changes alter **where** it happens: if f is minimum at $x = 1.2$, then $f(2x)$ is minimum at $x = 0.6$.

Examiner Tips and Tricks

- "On separate diagrams": one clean sketch per part, with every turning point and the y-intercept labelled with both coordinates.
- The y-intercept is found by putting $x = 0$ into the **new** equation: for $y = f(-x) - 1$ it is $f(0) - 1$.
- This topic often appears inside an R-form question (C07): "State the y coordinate of the minimum points of $y = 2f(x) - 5$". Only one line of work is needed.

WHEN YOU SEE → YOU DO

"On separate diagrams, sketch $y = 3f(2x)$ and $y = f(-x) - 1$"	→ move every key point: inside to x, outside to y
"the vertex of $y = af(x + b)$ is $(-2, 5)$"	→ equation in x gives b; equation in y gives a
"State the y coordinate of the minimum points on $y = 2f(x) - 5$"	→ apply the outside operations to the minimum value
"the smallest value of x at which a maximum occurs for $y = -f(2x)$"	→ $-$ swaps max and min; $2x$ halves the x-value

Your notes

Worked Example

? The curve $y = f(x)$ has a minimum at $(0, 0)$, a maximum at $(2, 4)$ and crosses the x-axis at $(3, 0)$. On separate diagrams sketch (i) $y = 3f(2x)$ (ii) $y = f(-x) - 1$, showing the turning points and the y-intercept.

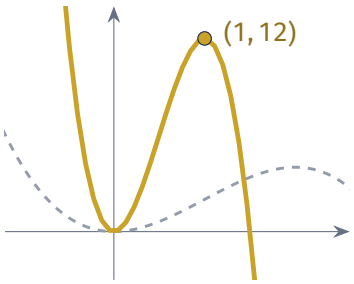
$(0, 0) \to (0, 0), \quad (2, 4) \to (1, 12)$ ← **(I) HALVE EVERY x, TREBLE EVERY y**

$(3, 0) \to \left(\frac{3}{2}, 0\right)$ ← **(I) THE ROOT MOVES TOO**

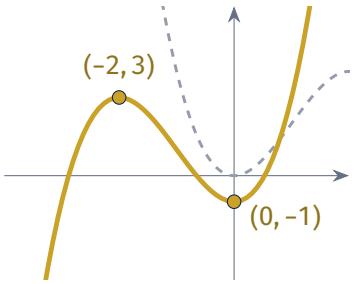
$(0, 0) \to (0, -1), \quad (2, 4) \to (-2, 3)$ ← **(II) MIRROR IN THE y-AXIS, THEN DOWN 1**

ANSWER (i) min $(0, 0)$, max $(1, 12)$ (ii) min $(0, -1)$ (also the y-intercept), max $(-2, 3)$

(i) $y = 3f(2x)$



(ii) $y = f(-x) - 1$



The two answers. Dashed: $y = f(x)$.

Your notes

Worked Example

? $f(x) = 2|x - 3| + 1$. The graph of $y = f(x)$ has its vertex at P. The graph is transformed to $y = af(x + b)$, whose vertex is $(-2, 5)$. Find P, a and b.

$x - 3 = 0 \Rightarrow P(3, 1)$ ← VERTEX: THE MODULUS IS ZERO

$x + b = 3$ when $x = -2 \Rightarrow b = 5$ ← INSIDE: NEW INSIDE = OLD x

$a \times 1 = 5 \Rightarrow a = 5$ ← OUTSIDE: NEW y = $a\times$ OLD y

ANSWER $P(3, 1)$, $a = 5$, $b = 5$

Worked Example

? $f(x)$, $x > 0$, has minimum value -3, first occurring at $x = 1.2$. (a) State the minimum value of $y = 2f(x) - 5$. (b) State the smallest x at which a maximum occurs for $y = -f(2x)$, and the maximum value.

$2(-3) - 5 = -11$ ← (A) OUTSIDE OPERATIONS ON THE MINIMUM VALUE

$2x = 1.2 \Rightarrow x = 0.6$ ← (B) THE MINIMUM OF f BECOMES A MAXIMUM OF $-f$; INSIDE HALVES x

$-(-3) = 3$ ← THE MAXIMUM VALUE

ANSWER (a) -11 (b) $x = 0.6$, maximum value 3

HOW THIS IS EXAMINED · past WMA13 papers

inside and outside SEEN 12× IN EXAMS reflection and shift SEEN 2× IN EXAMS

MISTAKE GALLERY

Where marks are lost

Your notes

× WRONG

$y = f(x + 2)$ moves the graph 2 to the right.

✓ RIGHT

It moves 2 to the **left**: $(x, y) \to (x - 2, y)$.

Inside changes do the opposite.

× WRONG

$y = f(2x)$ doubles the x-coordinates.

✓ RIGHT

It **halves** them: scale factor $\frac{1}{2}$.

Solve $2x$ = old x.

× WRONG

$y = 2f(x)$ moves the roots.

✓ RIGHT

Roots stay: $2 \times 0 = 0$.

Outside changes never touch x.

× WRONG

For $y = f(-x) - 1$, writing the y-intercept as $(0, 0)$.

✓ RIGHT

$f(0) - 1 = -1$, so $(0, -1)$.

Use the new equation at $x = 0$.

× WRONG

$y = -f(2x)$: maximum at the same place as f's maximum.

✓ RIGHT

The **minimum** of f becomes the maximum.

A negative outside factor swaps max and min.

THE CHAPTER ON ONE PAGE

Graph Transformations

Your notes

3.1

Outside: y changes

$f(x) + a$: up a.
$af(x)$: $y \times a$.
$-f(x)$: mirror in the x-axis.
Roots never move.

3.1

Inside: x changes

$f(x + a)$: left a.
$f(ax)$: $x \div a$.
$f(-x)$: mirror in the y-axis.
y-intercept never moves.

3.2

Combined

Inside to x, outside to y, independently.
Solve "new inside = old x".
Two outside steps: usual order.

3.2

Backwards and values

Vertex of $af(x + b)$: x-equation gives b, y-equation gives a.
Max and min values follow the outside operations.

BOTTOM LINE Move the key points, not the whole curve: inside changes act on x and go the opposite way, outside changes act on y and do what they say.