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02

CHAPTER 02

Functions, Composition and Inverses

A function is a machine: know what it accepts (domain), what it can produce (range), how to chain two machines, and how to run one backwards.

4	13	26
CONCEPTS	VARIANTS	EXAM QUESTIONS

YOU WILL MASTER

- State domains and ranges, including the range on an interval with a turning point
- Evaluate and form composite functions in the right order: fg means g first
- Solve equations such as $gg(x) = 78$ and $f\left(\frac{1}{a}\right) = g(a + 3)$
- Find f^{-1} with its domain, sketch it, and find where f meets f^{-1}

CHAPTER 02 AT A GLANCE

Three note pages, thirteen ways to be asked

Your notes

CHAPTER 02 MAP

Functions

01 2.1 Language of functions

- domain and range
- one-one and many-one
- range on an interval
- restricted domain

02 2.2 Composite functions

- evaluate $fg(5)$
- form $fg(x)$
- domain of fg
- solve $gg(x) = k$
- $f\left(\frac{1}{a}\right) = g(a + 3)$

03 2.3 Inverse functions

- find $f^{-1}(x)$
- restrict, then invert
- graph of f^{-1}
- $f^{-1}(k)$ without the formula
- $f(x) = f^{-1}(x)$

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Language of Functions NOTE 2.1

Your notes

Language of Functions

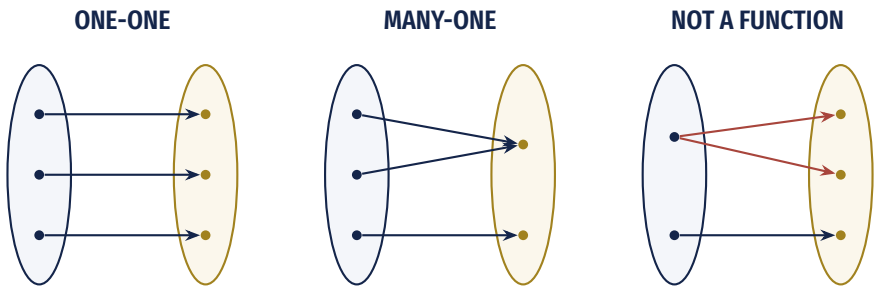
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What is a function?

- A **function** is a rule that gives **exactly one output** for every allowed input.
- The **domain** is the set of allowed inputs (x-values). It is written after the rule: $f(x) = 2x^2 - 5,\ x \geq 0$.
- The **range** is the set of outputs it actually produces (y-values). Write it using $f(x)$, e.g. $f(x) \geq -5$, **not** using x.
- Notation: $f(x) = \ldots$ or $f : x \mapsto \ldots$ mean the same thing.

What are one-one and many-one functions?

- **One-one**: every output comes from only one input. A horizontal line meets the graph at most once.
- **Many-one**: some output comes from two or more inputs, e.g. x^2 gives 4 from $x = 2$ and $x = -2$.
- **One-many** rules (e.g. $\pm\sqrt{x}$) are **not functions** at all.
- Only a **one-one** function has an inverse.



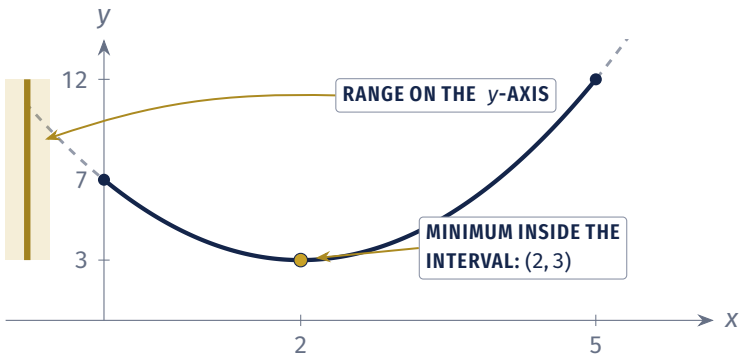
Mapping diagrams. Each input may send only one arrow.

How do I find the range on an interval?

- Sketch the graph **only over the given domain**.
- Find the value at **each end** of the interval.

- Find any **turning point inside** the interval: it may be the smallest or largest value.
- The range runs from the lowest to the highest of these values. Use ≤ for values that are reached, < for values only approached.

Your notes



$f(x) = x^2 - 4x + 7$, $0 \le x \le 5$: the minimum is **inside** the interval, so the range is $3 \le f(x) \le 12$, not $7 \le f(x) \le 12$.

e.g. **STATE THE RANGE OF** $f(x) = 2x^2 - 5$, $x \ge 0$

$f(0) = -5$ and f increases for $x \ge 0$ ← THE LOWEST POINT IS AT THE START OF THE DOMAIN

ANSWER $f(x) \ge -5$

Examiner Tips and Tricks

- The range must be written in terms of $f(x)$ (or y): "$f(x) \ge -5$", never "$x \ge -5$".
- For a rational function such as $\frac{2x+2}{x-2}$, $x > 2$, think about what happens as $x \to 2$ and as $x \to \infty$: the outputs approach (but never reach) 2, so the range is $f(x) > 2$.
- Always check whether the domain contains a turning point. Using only the end values is the most common error in this topic.

Your notes

WHEN YOU SEE → YOU DO

"State the range of f"	→	sketch over the domain; end values and turning points; answer in $f(x)$
"$0 \le x \le 5$" with a quadratic	→	is the vertex inside? then it gives the min or max
"Given that f is one-one / has an inverse, find the smallest k" with domain $x \ge k$	→	k is the x-coordinate of the turning point

Worked Example

? $f(x) = x^2 - 4x + 7,\ 0 \le x \le 5$. Find the range of f.

$f(x) = (x - 2)^2 + 3$ ← COMPLETE THE SQUARE: VERTEX $(2, 3)$

$x = 2$ is inside $[0, 5] \Rightarrow$ min $= 3$ ← TURNING POINT INSIDE THE INTERVAL

$f(0) = 7, \quad f(5) = 12$ ← BOTH ENDS: THE LARGER GIVES THE MAXIMUM

ANSWER $3 \le f(x) \le 12$

Worked Example

? $f(x) = (x - 3)^2 + 1,\ x \ge k$. Given that f is one-one, find the least value of k and state the range of f for this k.

vertex at $x = 3$ ← LEFT OF THE VERTEX f DECREASES, RIGHT OF IT f INCREASES

$k = 3$ ← START THE DOMAIN AT THE TURNING POINT SO NO OUTPUT REPEATS

ANSWER least $k = 3$; range $f(x) \ge 1$

HOW THIS IS EXAMINED · past WMA13 papers

algebraic restrictions SEEN 1× IN EXAMS restricted branch SEEN 3× IN EXAMS
range on an interval SEEN 10× IN EXAMS

Your notes

Composite Functions NOTE 2.2

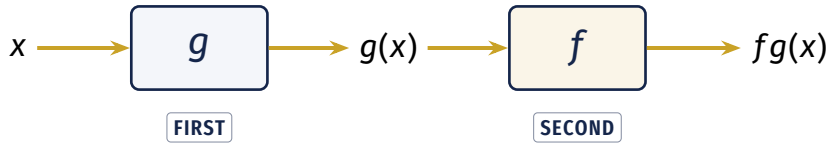
Your notes

Composite Functions

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What is a composite function?

- A **composite function** applies one function and then another to the result.
- $fg(x)$ means $f\big(g(x)\big)$: apply g **first**, then f. Read it from the **inside out** (right to left).
- In general $fg(x) \neq gf(x)$: the order matters.
- $f^2(x)$ or $ff(x)$ means apply f twice.



$fg(x)$: the input goes into g first, and its output goes into f.

How do I find $fg(x)$ or evaluate $fg(5)$?

- **A value**: work out $g(5)$ first, then put that number into f.
- **An expression**: replace every x in $f(x)$ by the whole expression $g(x)$, in brackets, then simplify.
- Keep exact values (fractions, surds, ln, e) all the way through.

What is the domain of a composite function?

- x must be allowed in g, **and** the output $g(x)$ must be allowed in f.
- So the domain of fg is the set of x in the domain of g for which $g(x)$ lies in the domain of f.

How do I solve equations such as $gg(x) = k$ or $f\left(\frac{1}{a}\right) = g(a+3)$?

- Form the expression first, then solve: $gg(x) = g\big(g(x)\big)$.

- Look for a substitution: in $(x^2 - 3)^2 - 3 = 78$ treat $x^2 - 3$ as one block.
- $f\left(\frac{1}{a}\right) = g(a + 3)$ is **not** a composite: put $\frac{1}{a}$ into f and $a + 3$ into g separately.
- Reject any answer that makes an input illegal (e.g. a zero denominator).

Your notes

e.g. $f(x) = 3x - 1$, $g(x) = x^2$. **FIND** $fg(2)$ **AND** $gf(2)$

$g(2) = 4 \Rightarrow f(4) = 11$ ← fg: g **FIRST**

$f(2) = 5 \Rightarrow g(5) = 25$ ← gf: f **FIRST**

ANSWER $fg(2) = 11$, $gf(2) = 25$: the order matters

Examiner Tips and Tricks

- Write the middle step: "$fg(5) = f(7) = \ldots$". It earns the method mark even if the arithmetic slips.
- When forming $fg(x)$ put $g(x)$ in a **bracket** before you substitute.
- "Solve $gg(x) = 126$": after expanding, a quartic in x is usually a quadratic in x^2. Remember x^2 = negative has no real solutions.

WHEN YOU SEE → YOU DO

"Find the value of $fg(5)$" / "Find $gf(2)$"	→ inside function first; write the middle value
"Find $fg(x)$, giving your answer in its simplest form"	→ substitute $g(x)$ in brackets into f, then simplify
"Solve the equation $gg(x) = 126$"	→ form $g(g(x))$, then treat the inner bracket as one block
"Solve the equation $f\left(\frac{1}{a}\right) = g(a + 3)$"	→ two separate substitutions, then solve; check denominators

Your notes

Worked Example

? $f(x) = \frac{x+7}{x-2}$, $x \neq 2$ and $g(x) = 2x - 1$. (a) Find $fg(4)$. (b) Find $fg(x)$ in its simplest form.

$g(4) = 7$ ← **(A) g FIRST**

$f(7) = \frac{14}{5}$ ← **THEN f**

$fg(x) = \frac{(2x-1)+7}{(2x-1)-2}$ ← **(B) REPLACE x IN f BY $(2x-1)$**

ANSWER (a) $fg(4) = \frac{14}{5}$ (b) $fg(x) = \frac{2x+6}{2x-3}$, $x \neq \frac{3}{2}$

Worked Example

? $g(x) = x^2 - 3$, $x \in \mathbb{R}$. Solve $gg(x) = 78$.

$(x^2-3)^2 - 3 = 78$ ← **FORM $g(g(x))$**

$(x^2-3)^2 = 81 \Rightarrow x^2 - 3 = \pm 9$ ← **TREAT $x^2 - 3$ AS ONE BLOCK**

$x^2 = 12$ or $x^2 = -6$ (no real roots) ← **REJECT THE NEGATIVE SQUARE**

ANSWER $x = \pm 2\sqrt{3}$

Your notes

Worked Example

? $f(x) = \dfrac{x+7}{x-2}$, $x \neq 2$ and $g(x) = 2x - 1$. Solve $f\left(\dfrac{1}{a}\right) = g(a+3)$.

$f\left(\frac{1}{a}\right) = \dfrac{\frac{1}{a}+7}{\frac{1}{a}-2} = \dfrac{1+7a}{1-2a}$ ← MULTIPLY TOP AND BOTTOM BY a

$g(a+3) = 2a + 5$ ← SEPARATE SUBSTITUTION

$1 + 7a = (2a + 5)(1 - 2a) \Rightarrow 4a^2 + 15a - 4 = 0$ ← CROSS-MULTIPLY AND COLLECT

$(4a - 1)(a + 4) = 0$ ← CHECK: $\frac{1}{a} \neq 2$ FOR BOTH

ANSWER $a = \frac{1}{4}$ or $a = -4$

Worked Example

? $f(x) = \ln(x - 1)$, $x > 1$ and $g(x) = 3 - x^2$, $x \in \mathbb{R}$. Find the domain of fg.

$g(x) > 1$ ← THE OUTPUT OF g MUST BE A VALID INPUT FOR f

$3 - x^2 > 1 \Rightarrow x^2 < 2$ ← SOLVE THE INEQUALITY

ANSWER $-\sqrt{2} < x < \sqrt{2}$

HOW THIS IS EXAMINED · past WMA13 papers

form $fg(x)$ SEEN 3× IN EXAMS domain of fg NOT SEEN YET • BE READY
evaluate $fg(5)$ SEEN 11× IN EXAMS solve a composite equation SEEN 6× IN EXAMS
$f(u) = g(v)$ SEEN 1× IN EXAMS

Inverse Functions NOTE 2.3

Your notes

Inverse Functions

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What is an inverse function?

- The **inverse** f^{-1} reverses f: if $f(3) = 10$ then $f^{-1}(10) = 3$.
- $ff^{-1}(x) = f^{-1}f(x) = x$.
- Only **one-one** functions have an inverse. A many-one function must first be **restricted** to a one-one part.
- $f^{-1}(x)$ is **not** $\frac{1}{f(x)}$.

f

- domain: inputs of f
- range: outputs of f

f^{-1}

- domain = **range of** f
- range = **domain of** f

How do I find $f^{-1}(x)$?

STEP 1 Write $y = f(x)$.

STEP 2 Rearrange to make x the subject.

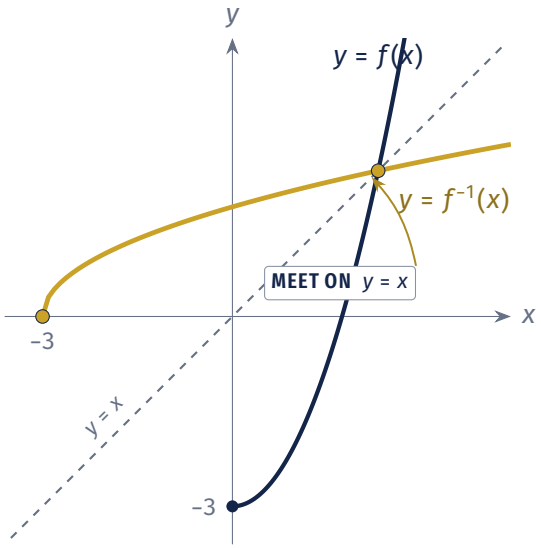
STEP 3 Swap: replace y by x, and write $f^{-1}(x) = \ldots$

STEP 4 State the domain of f^{-1}: it is the **range of** f.

Your notes

What does the graph of $y = f^{-1}(x)$ look like?

- It is the **reflection** of $y = f(x)$ in the line $y = x$.
- Every point (a, b) on f becomes (b, a) on f^{-1}; intercepts swap axes; asymptotes swap ($y = 3$ becomes $x = 3$).
- If f is **increasing**, f and f^{-1} can only meet **on** the line $y = x$, so solve $f(x) = x$.



$f(x) = x^2 - 3,\ x \geq 0$ and its inverse: mirror images in $y = x$, meeting on the line.

How do I find $f^{-1}(k)$ without the formula?

- $f^{-1}(k)$ is the input that makes f give k. Solve $f(x) = k$, keeping only the solution in the domain of f.

e.g. $f(x) = x^3 + 2x - 1$. **FIND** $f^{-1}(11)$

$x^3 + 2x - 1 = 11 \Rightarrow x = 2$ ← **SOLVE** $f(x) = 11$ **(TRY SMALL INTEGERS)**

ANSWER $f^{-1}(11) = 2$

Your notes

Examiner Tips and Tricks

- The answer "$f^{-1}(x) = \ldots$" needs its domain to score full marks: "$x > 3$", taken from the range of f.
- With a quadratic, choose the ± root that matches the domain of f: for $x \geq 3$ use $3 + \sqrt{\ldots}$.
- For a **decreasing** function, $f(x) = f^{-1}(x)$ is not guaranteed to happen only on $y = x$. Solve $f(x) = f^{-1}(x)$ directly and check the answer is in both domains.

WHEN YOU SEE → YOU DO

"Find $f^{-1}(x)$" / "Find f^{-1}"	→ $y = f(x)$, make x the subject, swap, state the domain
"State the domain of f^{-1}"	→ it is the range of f
"On Diagram 1, sketch the curve with equation $y = f^{-1}(x)$"	→ reflect in $y = x$; swap intercepts and asymptotes
"find the exact x coordinate of P" where $y = f(x)$ meets $y = f^{-1}(x)$	→ increasing f: solve $f(x) = x$; otherwise solve $f(x) = f^{-1}(x)$

Worked Example

? $f(x) = \dfrac{x+7}{x-2}$, $x > 2$. Find $f^{-1}(x)$ and state its domain.

$y(x-2) = x + 7 \Rightarrow xy - 2y = x + 7$ ← MULTIPLY OUT

$x(y-1) = 2y + 7 \Rightarrow x = \dfrac{2y+7}{y-1}$ ← COLLECT THE x TERMS, FACTORISE

$x > 2 \Rightarrow f(x) > 1$ ← RANGE OF f: AS $x \to \infty$, $f \to 1$; AS $x \to 2^+$, $f \to \infty$

ANSWER $f^{-1}(x) = \dfrac{2x+7}{x-1}$, $x > 1$

Your notes

Worked Example

? $f(x) = 4 + 3\ln(x - 1),\ x > 1$ and $g(x) = 2e^x + 3,\ x \in \mathbb{R}$. Find $f^{-1}(x)$ and $g^{-1}(x)$, each with its domain.

$\ln(x - 1) = \dfrac{y - 4}{3} \Rightarrow x = 1 + e^{\frac{y-4}{3}}$ ← **UNDO** +4, ×3, **THEN** ln

range of f is $\mathbb{R}$ ← **SO** f^{-1} **ACCEPTS EVERY** x

$e^x = \dfrac{y - 3}{2} \Rightarrow x = \ln\left(\dfrac{y - 3}{2}\right)$ ← **UNDO** +3, ×2, **THEN** e

ANSWER $f^{-1}(x) = 1 + e^{\frac{x-4}{3}},\ x \in \mathbb{R}$ $\quad g^{-1}(x) = \ln\left(\dfrac{x - 3}{2}\right),\ x > 3$

Worked Example

? $f(x) = x^2 - 6x + 5,\ x \geq 3$. Find $f^{-1}(x)$, stating its domain.

$y = (x - 3)^2 - 4$ ← **COMPLETE THE SQUARE**

$x - 3 = \pm\sqrt{y + 4} \Rightarrow x = 3 + \sqrt{y + 4}$ ← **TAKE +** **BECAUSE** $x \geq 3$

ANSWER $f^{-1}(x) = 3 + \sqrt{x + 4},\ x \geq -4$

Your notes

Worked Example

? $f(x) = x^2 - 3,\ x \geq 0$. The curves $y = f(x)$ and $y = f^{-1}(x)$ meet at P. Find the exact x-coordinate of P.

f is increasing, so P lies on $y = x$ ← THE CURVES ARE MIRROR IMAGES IN $y = x$

$x^2 - 3 = x \Rightarrow x^2 - x - 3 = 0$ ← SOLVE $f(x) = x$

$x = \dfrac{1 \pm \sqrt{13}}{2}$ ← REJECT THE NEGATIVE ROOT: $x \geq 0$

ANSWER $x = \dfrac{1 + \sqrt{13}}{2}$

Worked Example

? $f(x) = \dfrac{6}{x + 1},\ x > 0$. Find $f^{-1}(x)$ and solve $f^{-1}(x) = f(x)$.

$f^{-1}(x) = \dfrac{6}{x} - 1, \quad 0 < x < 6$ ← RANGE OF f IS $0 < f(x) < 6$

$\dfrac{6}{x + 1} = \dfrac{6}{x} - 1 \Rightarrow 6x = 6(x + 1) - x(x + 1)$ ← f IS DECREASING: SOLVE THE ACTUAL EQUATION

$x^2 + x - 6 = 0 \Rightarrow x = 2$ ($x = -3$ rejected) ← CHECK 2 IS IN BOTH DOMAINS

ANSWER $x = 2$

HOW THIS IS EXAMINED · past WMA13 papers

find $f^{-1}(x)$ SEEN 16× IN EXAMS restrict then invert NOT SEEN YET • BE READY
graph of f^{-1} SEEN 2× IN EXAMS $f^{-1}(k)$ directly SEEN 2× IN EXAMS $f = f^{-1}$ SEEN 3× IN EXAMS

MISTAKE GALLERY

Where marks are lost

Your notes

× WRONG

Range of $f(x) = 2x^2 - 5$, $x \geq 0$: "$x \geq -5$".

✓ RIGHT

$f(x) \geq -5$.

The range describes outputs.

× WRONG

Range of $x^2 - 4x + 7$ on $0 \leq x \leq 5$ from the ends only: $7 \leq f(x) \leq 12$.

✓ RIGHT

$3 \leq f(x) \leq 12$: the vertex $(2, 3)$ is inside.

Check for a turning point in the interval.

× WRONG

$fg(4) = g(f(4))$.

✓ RIGHT

$fg(4) = f(g(4))$: g first.

Read composites from the inside out.

× WRONG

$f^{-1}(x) = \dfrac{1}{f(x)}$.

✓ RIGHT

f^{-1} undoes f; it is not the reciprocal.

The -1 is notation, not a power.

× WRONG

$f^{-1}(x) = 3 \pm \sqrt{x + 4}$.

✓ RIGHT

$3 + \sqrt{x + 4}$ only, because $x \geq 3$ in f.

An inverse must be a function: one output.

THE CHAPTER ON ONE PAGE

Functions

Your notes

2.1

Domain and range

Domain: allowed x. Range: outputs, written with $f(x)$.
On an interval: both ends **and** any turning point inside.

2.2

Composites

$fg(x) = f(g(x))$: g first.
Substitute $g(x)$ in brackets.
Domain of fg: $g(x)$ must be allowed in f.

2.3

Inverses

$y = f(x)$, make x the subject, swap.
Domain of f^{-1} = range of f.
Graph: reflect in $y = x$.

2.3

f meets f^{-1}

Increasing f: solve $f(x) = x$.
Otherwise solve $f(x) = f^{-1}(x)$ and check both domains.

BOTTOM LINE Know what goes in, what comes out, which machine runs first, and how to run it backwards. Always state the domain.