

DR ESLAM AHMED
eliteigcse.com

01

CHAPTER 01

Rational Algebra

Algebraic fractions behave exactly like number fractions. Factorise first, and every cancel, add and divide becomes routine.

3	6	12
CONCEPTS	VARIANTS	EXAM QUESTIONS

YOU WILL MASTER

- Factorise and cancel without losing the values x is not allowed to take
- Add, subtract, multiply and divide algebraic fractions into one simplest fraction
- Divide an improper fraction to get $Ax + B + \frac{C}{\dots}$ and check it
- Finish every “Show that” in the exact form the question asks for

CHAPTER 01 AT A GLANCE

Two note pages, six ways to be asked

Your notes

CHAPTER 01 MAP

Rational Algebra

01 **1.1 Rational expressions**

- cancel: linear denominator
- cancel: quadratic denominator
- + and −: common denominator
- × and ÷

02 **1.2 Improper algebraic fractions**

- divide by a linear divisor
- divide by a quadratic divisor
- $Ax + B + \frac{C}{(x-1)^2}$

Contents

1.1 Rational Expressions 3
1.2 Improper Algebraic Fractions 8

Rational Expressions NOTE 1.1

Your notes

Rational Expressions

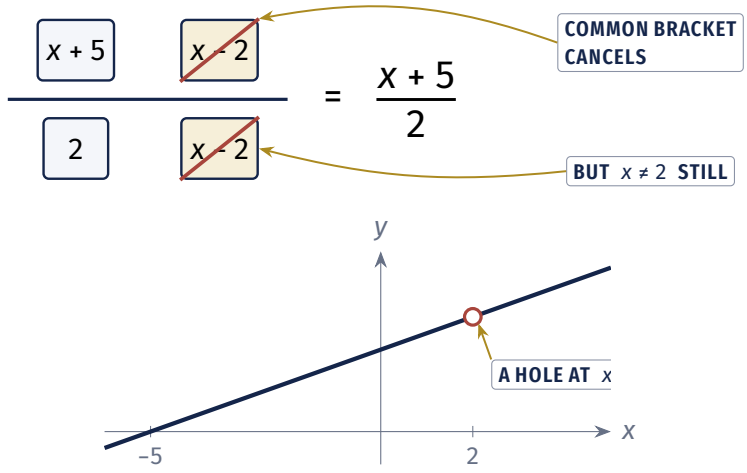
PURE 3 › ALGEBRA AND FUNCTIONS › RATIONAL EXPRESSIONS • Exam code WMA13 • Dr Eslam Ahmed

What is a rational expression?

- A **rational expression** is a fraction whose numerator and denominator are polynomials, e.g. $\frac{x^2 + 3x - 10}{2x - 4}$.
- It follows exactly the same rules as number fractions.
- It is **undefined** wherever the denominator is zero.
  - $\frac{x^2 + 3x - 10}{2x - 4}$ is undefined at $x = 2$.

How do I simplify a rational expression?

- **Factorise** the numerator and the denominator completely.
  - common factor: $2x^2 - 6x = 2x(x - 3)$
  - difference of two squares: $x^2 - 9 = (x - 3)(x + 3)$
  - quadratic: $2x^2 + 5x - 3 = (2x - 1)(x + 3)$
- **Cancel** any bracket that is a factor of the whole numerator **and** the whole denominator.
- Never cancel **terms** that are added or subtracted.
- The values that made the **original** denominator zero are still not allowed.



Only whole brackets cancel, and the forbidden value stays forbidden.

e.g. **SIMPLIFY** $\frac{x^2 - 9}{x^2 + 3x}$

$\frac{(x - 3)(x + 3)}{x(x + 3)}$ ← **FACTORISE TOP AND BOTTOM**

ANSWER $\frac{x - 3}{x}$, $x \neq 0,\ x \neq -3$

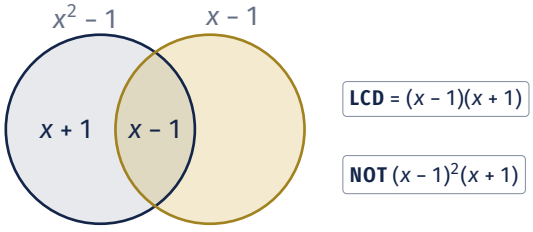
How do I add or subtract rational expressions?

STEP 1 Factorise every denominator.

STEP 2 Build the **lowest common denominator** (LCD): every different factor, once each, at its highest power.

STEP 3 Rewrite each fraction over the LCD and combine the numerators. Put each numerator in a **bracket**.

STEP 4 Expand, collect, then factorise the new numerator and cancel if you can.



Building the LCD of $\frac{\cdots}{x^2 - 1}$ and $\frac{\cdots}{x - 1}$: the shared factor is used once.

Your notes

How do I multiply or divide rational expressions?

- **Multiply**: multiply the numerators together and the denominators together, then cancel.
- **Divide**: **flip the second fraction** and multiply.
  - $\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \times \frac{D}{C}$
- Factorise **before** multiplying out, so that cancelling is easy.
- Exclude the zeros of every original denominator **and** of the numerator you divided by.

Examiner Tips and Tricks

- The most common form of this question is "Show that $f(x) = \frac{ax + b}{cx + d}$, where a, b, c and d are integers to be found". The final line must be in exactly that form.
- The answer is usually used again: to find $f^{-1}(x)$ (C02), to differentiate (C10) or to integrate (C12). A slip here costs marks later.
- Check your simplified answer by substituting a simple value such as $x = 3$ into the original and into your answer.

WHEN YOU SEE → YOU DO

"Simplify fully" / "in its simplest form"	→ factorise top and bottom completely, then cancel common **brackets**
"Show that $f(x) = \frac{ax + b}{cx + d}$, where a, b, c and d are integers"	→ combine over the LCD, factorise the new top, cancel
"Express as a single fraction"	→ one fraction, factorised, nothing left to cancel
"$x \in \mathbb{R}$, $x > 2$" written after $f(x)$	→ the domain is part of the question: keep it with your answer

Your notes

Worked Example

? Simplify fully $\dfrac{2x^2 + 5x - 3}{x^2 + x - 6}$.

$\dfrac{(2x - 1)(x + 3)}{(x + 3)(x - 2)}$ ← FACTORISE BOTH QUADRATICS

$x \neq -3, \quad x \neq 2$ ← EXCLUSIONS FROM THE ORIGINAL DENOMINATOR

$x = 0: \quad \dfrac{-3}{-6} = \dfrac{1}{2} = \dfrac{-1}{-2}$ ← QUICK CHECK WITH A SIMPLE VALUE

ANSWER $\dfrac{2x - 1}{x - 2}, \quad x \neq -3, 2$

Worked Example

? $f(x) = \dfrac{2x + 5}{x - 2} - \dfrac{3x + 9}{x^2 + x - 6}, \quad x > 2.$ Show that $f(x) = \dfrac{ax + b}{cx + d}$, where a, b, c and d are integers to be found.

$x^2 + x - 6 = (x + 3)(x - 2)$ ← FACTORISE THE DENOMINATOR: LCD IS $(x + 3)(x - 2)$

$\dfrac{(2x + 5)(x + 3) - (3x + 9)}{(x + 3)(x - 2)}$ ← ONE FRACTION; BRACKET THE SECOND NUMERATOR

$= \dfrac{2x^2 + 8x + 6}{(x + 3)(x - 2)} = \dfrac{2(x + 1)(x + 3)}{(x + 3)(x - 2)}$ ← EXPAND, COLLECT, FACTORISE

$x = 3: \quad \dfrac{11}{1} - \dfrac{18}{6} = 8 = \dfrac{2(3) + 2}{3 - 2}$ ← CHECK

ANSWER $f(x) = \dfrac{2x + 2}{x - 2}$: $a = 2, \ b = 2, \ c = 1, \ d = -2$

Your notes

Worked Example

? Simplify fully $\frac{x^2 - 9}{x^2 + 2x} \div \frac{x - 3}{x^2 - 4}$.

$\frac{(x - 3)(x + 3)}{x(x + 2)} \times \frac{(x - 2)(x + 2)}{x - 3}$ ← FLIP THE SECOND FRACTION AND FACTORISE

$\frac{(x + 3)(x - 2)}{x}$ ← CANCEL $(x - 3)$ AND $(x + 2)$

$x \neq 0, \pm 2, 3$ ← BOTH DENOMINATORS AND THE DIVISOR'S NUMERATOR

ANSWER $\frac{(x + 3)(x - 2)}{x}, \quad x \neq 0, \pm 2, 3$

HOW THIS IS EXAMINED · past WMA13 papers

cancel: linear denominator NOT SEEN YET • BE READY
cancel: quadratic denominator NOT SEEN YET • BE READY
common denominator SEEN 4× IN EXAMS product or quotient NOT SEEN YET • BE READY

Improper Algebraic Fractions NOTE 1.2

Your notes

VISUAL NOTES · P3.01

Improper Algebraic Fractions

PURE 3 › ALGEBRA AND FUNCTIONS › RATIONAL EXPRESSIONS • Exam code WMA13 • Dr Eslam Ahmed

What is an improper algebraic fraction?

- A fraction is **improper** when the degree of the numerator is **greater than or equal to** the degree of the denominator.
  - $\frac{2x^2 + 5x - 8}{x + 2}$ (degree 2 over degree 1) is improper.
  - $\frac{3}{x + 2}$ is proper.
- Like $\frac{17}{5} = 3 + \frac{2}{5}$, an improper algebraic fraction can be written as
  - **quotient** + $\frac{\text{remainder}}{\text{divisor}}$
- The **remainder** always has a lower degree than the divisor.

THE DIVISION IDENTITY LEARN THIS: NOT GIVEN

$$F(x) \equiv Q(x)\,D(x) + R(x)$$

How do I divide using long division?

STEP 1 Write both polynomials in descending powers, using 0 for any missing power.

STEP 2 Divide the leading term of what is left by the leading term of the divisor.

STEP 3 Multiply the divisor by that term and **subtract** the whole line.

STEP 4 Repeat until the degree of what is left is less than the degree of the divisor.

$$\begin{array}{r} 2x^2 + 5x - 8 \\ -2x^2 - 4x \\ \hline x - 8 \\ -x - 2 \\ \hline -10 \end{array} \quad : (x+2) = 2x + 1 + \frac{-10}{x+2}$$

Long division of $2x^2 + 5x - 8$ by $x + 2$: quotient $2x + 1$, remainder -10.

Your notes

VISUAL NOTES · P3.01

Is there a quicker way?

- Write the **identity** with unknown constants and compare.
  - $2x^2 + 5x - 8 \equiv (Ax + B)(x + 2) + C$
- Substitute the root of the divisor ($x = -2$) to find C immediately: this is the **remainder theorem**.
- Compare coefficients (highest power, constant term) to find the rest.
- This method is especially quick when the divisor is a **repeated factor** such as $(x - 1)^2$.

e.g. **FIND THE REMAINDER WHEN $x^3 - 2x + 7$ IS DIVIDED BY $x + 2$**

$F(-2) = -8 + 4 + 7$ ← **SUBSTITUTE THE ROOT OF THE DIVISOR**

ANSWER remainder = 3

DIVIDING AN IMPROPER FRACTION

- **STEP 1** Check: degree of top ≥ degree of bottom
- **STEP 2** Fill missing powers with 0
- **STEP 3** Long division or the identity
- **STEP 4** Remainder theorem check

Your notes

Examiner Tips and Tricks

- When the question gives the form, e.g. "$Ax + B + \frac{C}{x+2}$", the identity method is usually fastest and needs no long division layout.
- Divide by the **expanded** quadratic: $(x-1)^2 = x^2 - 2x + 1$.
- This division is almost always followed by "Hence" integration (C12): $\int \frac{C}{x+2}\,dx = C\ln|x+2|$.

WHEN YOU SEE → YOU DO

"Write $g(x)$ in the form $Ax + B + \frac{C}{x+2}$, where A, B and C are integers to be found"	→ long division by $(x+2)$, or the identity and comparing coefficients
"Given $h(x) = Ax + B + \frac{C}{(x-1)^2}$ ... find A, B and C"	→ identity; put $x = 1$ first
"Hence use algebraic integration" / "Hence find $\int h(x)\,dx$"	→ integrate the divided form term by term

Worked Example

? $g(x) = \frac{2x^2 + 5x - 8}{x+2}$, $x > -2$. Write $g(x)$ in the form $Ax + B + \frac{C}{x+2}$, where A, B and C are integers to be found.

$2x(x+2) = 2x^2 + 4x$; subtract: $x - 8$ ← **DIVIDE LEADING TERMS:** $2x^2 \div x = 2x$

$1(x+2) = x + 2$; subtract: -10 ← **NEXT TERM OF THE QUOTIENT:** $+1$

$2(-2)^2 + 5(-2) - 8 = -10$ ← **REMAINDER THEOREM CHECK**

ANSWER $g(x) = 2x + 1 - \frac{10}{x+2}$: $A = 2$, $B = 1$, $C = -10$

Your notes

Worked Example

? $h(x) = \dfrac{2x^3 - 7x^2 + 8x + 1}{(x-1)^2}$, $x > 1$. Given $h(x) = Ax + B + \dfrac{C}{(x-1)^2}$, find A, B and C.

$2x^3 - 7x^2 + 8x + 1 \equiv (Ax + B)(x-1)^2 + C$ ← MULTIPLY THROUGH BY $(x-1)^2$

$x = 1$: $2 - 7 + 8 + 1 = C \Rightarrow C = 4$ ← THE BRACKET VANISHES

x^3 : $A = 2$ ← COMPARE THE HIGHEST POWER

$x = 0$: $1 = B + C \Rightarrow B = -3$ ← COMPARE CONSTANTS

ANSWER $h(x) = 2x - 3 + \dfrac{4}{(x-1)^2}$

HOW THIS IS EXAMINED · past WMA13 papers

linear divisor SEEN 2× IN EXAMS quadratic divisor SEEN 6× IN EXAMS

MISTAKE GALLERY

Where marks are lost

Your notes

× WRONG

$\frac{x+6}{x} = 6$

✓ RIGHT

No cancelling: $\frac{x+6}{x} = 1 + \frac{6}{x}$.

Only a bracket that multiplies everything can be cancelled.

× WRONG

$\frac{(x+3)(2x-1)}{(x+3)(x-2)} = \frac{2x-1}{x-2}$ (nothing else).

✓ RIGHT

Add $x \neq -3$, $x \neq 2$.

Exclusions come from the original denominator.

× WRONG

$\frac{A}{B} - \frac{3x+9}{B} = \frac{A-3x+9}{B}$

✓ RIGHT

$\frac{A-(3x+9)}{B} = \frac{A-3x-9}{B}$

The minus sign applies to the whole numerator.

× WRONG

Dividing $x^3 - 2x + 1$ with no gap for x^2.

✓ RIGHT

Write $x^3 + 0x^2 - 2x + 1$ first.

Missing powers shift every column.

× WRONG

Stopping at $\frac{2(x+1)(x+3)}{(x+3)(x-2)}$.

✓ RIGHT

Cancel and match $\frac{ax+b}{cx+d}$.

"Show that" wants the exact final form.

THE CHAPTER ON ONE PAGE

Rational Algebra

Your notes

1.1

Cancel

Factorise top and bottom completely.
Cancel whole brackets only, never terms.
Keep the **original** exclusions.

1.1

Add and subtract

LCD: every factor once.
Bracket the second numerator after a minus.
Factorise the new top and cancel.

1.1

Multiply and divide

×: straight across.
÷: flip the **second** fraction.
Exclude zeros of the divisor's top too.

1.2

Divide

Top degree ≥ bottom: divide.
$F \equiv Q\,D + R$, degree R < degree D.
Put the root of D in to get R fast.

BOTTOM LINE Factorise first. Cancel only whole brackets, keep the forbidden values, and finish in the exact form the question asks for.