

EDEXCEL INTERNATIONAL A LEVEL

# WMA12 Pure 2 Past Paper Solutions

The full WMA12 bank rebuilt in original paper order, with each question followed by its worked solution.

**190**paper-order  
questions**WMA12**

Pure 2

**Past-paper  
solutions**standalone IAL  
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**PAST PAPER**

# **WMA12/01 October 2019**

**October 2019 | 10 questions | 75 marks**

**10**

questions

**75**

marks

**Answers**

worked solution  
after each  
question

# Question 1

## Applications of Differentiation

1. A curve  $C$  has equation  $y = 2x^2(x - 5)$

(a) Find, using calculus, the  $x$  coordinates of the stationary points of  $C$ .

(4)

(b) Hence find the values of  $x$  for which  $y$  is increasing.

(2)

# Worked Solution - Question 1

Topic group

**1. Expand the curve**

$$y = 2x^2(x - 5) = 2x^3 - 10x^2.$$

**2. Differentiate**

$$\frac{dy}{dx} = 6x^2 - 20x = 2x(3x - 10).$$

**3. Find the stationary values of  $x$** 

At a stationary point,  $\frac{dy}{dx} = 0$ . Hence  $2x(3x - 10) = 0$ , so  $x = 0$  or  $x = \frac{10}{3}$ .

**4. Use the sign of the derivative**

Since  $2x(3x - 10)$  is positive outside its two roots and negative between them,  $y$  is increasing for  $x < 0$  and for  $x > \frac{10}{3}$ .

**Final answer**

$$(a) x = 0, \frac{10}{3}. \quad (b) x < 0 \text{ or } x > \frac{10}{3}.$$

## Question 2

### Modelling with Sequences & Series

2. The adult population of a town at the start of 2019 is 25 000

A model predicts that the adult population will increase by 2% each year, so that the number of adults in the population at the start of each year following 2019 will form a geometric sequence.

- (a) Find, according to the model, the adult population of the town at the start of 2032 **(3)**

It is also modelled that every member of the adult population gives £5 to local charity at the start of each year.

- (b) Find, according to these models, the total amount of money that would be given to local charity by the adult population of the town from 2019 to 2032 inclusive. Give your answer to the nearest £1 000 **(3)**

# Worked Solution - Question 2

Topic group

## 1. Identify the common ratio

An increase of 2% each year gives the geometric ratio  $r = 1.02$ .

## 2. Count the years to 2032

From the start of 2019 to the start of 2032 there are 13 yearly increases, so the population is  $25000(1.02)^{13}$ .

## 3. Calculate the 2032 population

$25000(1.02)^{13} \approx 32340.7$ , so the model gives about 32341 adults.

## 4. Set up the total number of adult payments

From 2019 to 2032 inclusive there are 14 yearly populations, so the total number of adult payments is  $25000 \frac{1.02^{14} - 1}{1.02 - 1}$ .

## 5. Multiply by the charity amount

The total money is  $5 \times 25000 \frac{1.02^{14} - 1}{0.02} \approx 1\,996\,766$ , so this is about £1,996,766.

## 6. Round as requested

To the nearest £1000, this is £1,997,000.

### Final answer

(a) 32341 adults, approximately.

(b) £1,997,000.

## Question 3

## Binomial Expansion

3. (a) Find the first 4 terms, in ascending powers of  $x$ , in the binomial expansion of

$$\left(1 + \frac{x}{4}\right)^{12}$$

giving each coefficient in its simplest form.

(3)

- (b) Find the term independent of  $x$  in the expansion of

$$\left(\frac{x^2 + 8}{x^5}\right)\left(1 + \frac{x}{4}\right)^{12}$$

(3)

## Worked Solution - Question 3

**1. Write the binomial terms**

$$\left(1 + \frac{x}{4}\right)^{12} = 1 + \binom{12}{1} \frac{x}{4} + \binom{12}{2} \left(\frac{x}{4}\right)^2 + \binom{12}{3} \left(\frac{x}{4}\right)^3 + \dots$$

**2. Simplify the first four terms**

This gives  $1 + 3x + \frac{33}{8}x^2 + \frac{55}{16}x^3$ .

**3. Rewrite the prefactor**

$$\frac{x^2 + 8}{x^5} = x^{-3} + 8x^{-5}.$$

**4. Find the constant contributions**

The  $x^{-3}$  part needs the  $x^3$  coefficient, which is  $\frac{55}{16}$ . The  $8x^{-5}$  part needs the  $x^5$  coefficient, which is  $8 \binom{12}{5} \left(\frac{1}{4}\right)^5 = 8 \cdot \frac{99}{128} = \frac{99}{16}$ .

**5. Add the constant terms**

The term independent of  $x$  is  $\frac{55}{16} + \frac{99}{16} = \frac{154}{16} = \frac{77}{8}$ .

**Final answer**

$$(a) 1 + 3x + \frac{33}{8}x^2 + \frac{55}{16}x^3. \quad (b) \frac{77}{8}.$$

## Question 4

## Polynomials

4.  $f(x) = (x - 3)(3x^2 + x + a) - 35$  where  $a$  is a constant

(a) State the remainder when  $f(x)$  is divided by  $(x - 3)$ . (1)

Given  $(3x - 2)$  is a factor of  $f(x)$ ,

(b) show that  $a = -17$  (2)

(c) Using algebra and showing each step of your working, fully factorise  $f(x)$ . (5)

## Worked Solution - Question 4

**1. Use the remainder theorem**

When  $f(x)$  is divided by  $(x - 3)$ , the remainder is  $f(3)$ . Since  $f(3) = (3 - 3)(3 \cdot 3^2 + 3 + a) - 35$ , the remainder is  $-35$ .

**2. Use the given factor**

Since  $(3x - 2)$  is a factor,  $x = \frac{2}{3}$  is a root and  $f\left(\frac{2}{3}\right) = 0$ .

**3. Solve for a**

$$0 = \left(\frac{2}{3} - 3\right) \left(3\left(\frac{2}{3}\right)^2 + \frac{2}{3} + a\right) - 35 = -\frac{7}{3}(a + 2) - 35. \text{ Hence}$$
$$-7(a + 2) = 105, \text{ so } a = -17.$$

**4. Substitute a into f(x)**

$$f(x) = (x - 3)(3x^2 + x - 17) - 35 = 3x^3 - 8x^2 - 20x + 16.$$

**5. Divide by the known factor**

Dividing  $3x^3 - 8x^2 - 20x + 16$  by  $(3x - 2)$  gives  $x^2 - 2x - 8$ .

**6. Factorise fully**

$$x^2 - 2x - 8 = (x + 2)(x - 4), \text{ so } f(x) = (3x - 2)(x + 2)(x - 4).$$

**Final answer**

$$(a) -35. \quad (b) a = -17. \quad (c) f(x) = (3x - 2)(x + 2)(x - 4).$$

## Question 5

## Integration

5. (a) Given  $0 < a < 1$ , sketch the curve with equation

$$y = a^x$$

showing the coordinates of the point at which the curve crosses the y-axis.

(2)

$x$	2	2.5	3	3.5	4
$y$	4.25	6.427	9.125	12.34	16.06

The table above shows corresponding values of  $x$  and  $y$  for  $y = x^2 + \left(\frac{1}{2}\right)^x$

The values of  $y$  are given to 4 significant figures as appropriate.

Using the trapezium rule with all the values of  $y$  in the given table,

- (b) obtain an estimate for  $\int_2^4 \left(x^2 + \left(\frac{1}{2}\right)^x\right) dx$

(3)

Using your answer to part (b) and making your method clear, estimate

- (c)  $\int_2^4 \left(x(x-3) + \left(\frac{1}{2}\right)^x\right) dx$

(2)

# Worked Solution - Question 5

## 1. Describe the sketch

For  $0 < a < 1$ ,  $y = a^x$  is a decreasing exponential curve. At  $x = 0$ ,  $y = a^0 = 1$ , so it crosses the  $y$ -axis at  $(0, 1)$ .

## 2. Find the strip width

The  $x$  values increase by  $0.5$ , so  $h = 0.5$ .

## 3. Apply the trapezium rule

$$A \approx \frac{0.5}{2} \{4.25 + 16.06 + 2(6.427 + 9.125 + 12.34)\}.$$

## 4. Calculate the estimate

$A \approx 0.25(76.094) = 19.0235$ , so the estimate is **19.0** to 3 significant figures.

## 5. Relate the two integrands

$$x(x - 3) + \left(\frac{1}{2}\right)^x = x^2 + \left(\frac{1}{2}\right)^x - 3x.$$

## 6. Subtract the exact integral of $3x$

$$\int_2^4 3x \, dx = \left[\frac{3}{2}x^2\right]_2^4 = 24 - 6 = 18.$$

## 7. Estimate the new integral

Using the estimate from part (b), the required value is  $19.0235 - 18 = 1.0235$ , so the estimate is **1.0**.

**Final answer**

(a) A decreasing exponential curve crossing the  $y$ -axis at  $(0, 1)$ .

(b) 19.0. (c) 1.0.

## Question 6

6.

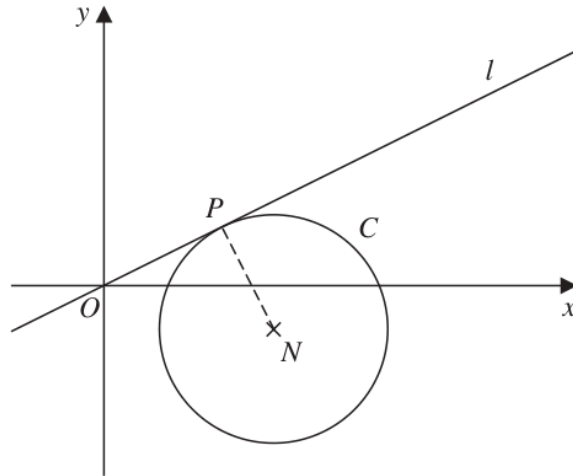


Figure 1

Figure 1 shows a sketch of a circle  $C$  with centre  $N(4, -1)$ .

The line  $l$  with equation  $y = \frac{1}{2}x$  is a tangent to  $C$  at the point  $P$ .

Find

(a) the equation of line  $PN$  in the form  $y = mx + c$ , where  $m$  and  $c$  are constants, (2)

(b) the equation of  $C$ . (5)

## Worked Solution - Question 6

**1. Use perpendicular gradients**

The tangent has gradient  $\frac{1}{2}$ , so the radius  $PN$  has gradient  $-2$ .

**2. Find the equation of  $PN$** 

Line  $PN$  passes through  $N(4, -1)$ , so  $y + 1 = -2(x - 4)$ . Therefore  $y = -2x + 7$ .

**3. Find point  $P$** 

At  $P$ , the tangent  $y = \frac{1}{2}x$  meets the radius  $y = -2x + 7$ . Thus  $\frac{1}{2}x = -2x + 7$ , so  $x = \frac{14}{5}$  and  $y = \frac{7}{5}$ .

**4. Find the radius squared**

$$r^2 = \left(\frac{14}{5} - 4\right)^2 + \left(\frac{7}{5} + 1\right)^2 = \left(-\frac{6}{5}\right)^2 + \left(\frac{12}{5}\right)^2 = \frac{36}{5}.$$

**5. Write the circle equation**

With centre  $(4, -1)$  and radius squared  $\frac{36}{5}$ , the circle is

$$(x - 4)^2 + (y + 1)^2 = \frac{36}{5}.$$

**Final answer**

$$(a) y = -2x + 7. \quad (b) (x - 4)^2 + (y + 1)^2 = \frac{36}{5}.$$

## Question 7

## Laws of Logarithms

7. Given  $\log_a b = k$ , find, in simplest form in terms of  $k$ ,

(i)  $\log_a \left( \frac{\sqrt{a}}{b} \right)$  (2)

(ii)  $\frac{\log_a a^2 b}{\log_a b^3}$  (2)

(iii)  $\sum_{n=1}^{50} (k + \log_a b^n)$  (3)

## Worked Solution - Question 7

**1. Use the given logarithm**

Since  $\log_a b = k$ , we can replace every  $\log_a b$  by  $k$ .

**2. Simplify part (i)**

$$\log_a \left( \frac{\sqrt{a}}{b} \right) = \log_a(a^{1/2}) - \log_a b = \frac{1}{2} - k.$$

**3. Simplify the numerator and denominator**

$$\log_a(a^2b) = \log_a a^2 + \log_a b = 2 + k, \text{ and } \log_a(b^3) = 3 \log_a b = 3k.$$

**4. Write part (ii)**

$$\text{Therefore } \frac{\log_a(a^2b)}{\log_a(b^3)} = \frac{k+2}{3k}.$$

**5. Simplify the summand**

$$\log_a(b^n) = n \log_a b = nk, \text{ so } k + \log_a(b^n) = k + nk = (n+1)k.$$

**6. Sum the series**

$$\sum_{n=1}^{50} (n+1)k = k \left( \sum_{n=1}^{50} n + 50 \right) = k(1275 + 50) = 1325k.$$

**Final answer**

$$(i) \frac{1}{2} - k. \quad (ii) \frac{k+2}{3k}. \quad (iii) 1325k.$$

## Question 8

8. Solutions relying on calculator technology are not acceptable in this question.

(i)

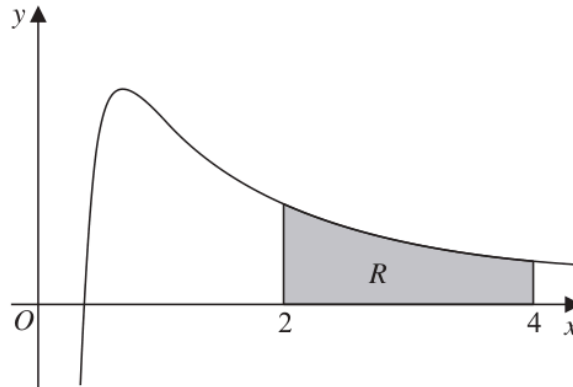


Figure 2

Figure 2 shows a sketch of part of a curve with equation

$$y = \frac{8\sqrt{x} - 5}{2x^2} \quad x > 0$$

The region  $R$ , shown shaded in Figure 2, is bounded by the curve, the line with equation  $x = 2$ , the  $x$ -axis and the line with equation  $x = 4$

Find the exact area of  $R$ .

(5)

(ii) Find the value of the constant  $k$  such that

$$\int_{-3}^6 \left( \frac{1}{2}x^2 + k \right) dx = 55$$

(4)

## Worked Solution - Question 8

**1. Rewrite the curve**

$$y = \frac{8\sqrt{x} - 5}{2x^2} = 4x^{-3/2} - \frac{5}{2}x^{-2}.$$

**2. Integrate the curve**

$$\int \left( 4x^{-3/2} - \frac{5}{2}x^{-2} \right) dx = -8x^{-1/2} + \frac{5}{2x}.$$

**3. Apply the limits for the area**

$$\left[ -8x^{-1/2} + \frac{5}{2x} \right]_2^4 = \left( -4 + \frac{5}{8} \right) - \left( -4\sqrt{2} + \frac{5}{4} \right).$$

**4. Simplify the exact area**

This gives  $4\sqrt{2} - \frac{37}{8}$ .

**5. Integrate the second expression**

$$\int \left( \frac{1}{2}x^2 + k \right) dx = \frac{x^3}{6} + kx.$$

**6. Use the limits**

$$\left[ \frac{x^3}{6} + kx \right]_{-3}^6 = (36 + 6k) - \left( -\frac{9}{2} - 3k \right) = \frac{81}{2} + 9k.$$

**7. Solve for k**

$$\frac{81}{2} + 9k = 55, \text{ so } 9k = \frac{29}{2} \text{ and } k = \frac{29}{18}.$$

**Final answer**

$$(i) 4\sqrt{2} - \frac{37}{8}. \quad (ii) k = \frac{29}{18}.$$

## Question 9

## Trigonometric Equations

9. Solutions based entirely on graphical or numerical methods are not acceptable in this question.

- (i) Solve, for  $0 \leq \theta < 180^\circ$ , the equation

$$3 \sin(2\theta - 10^\circ) = 1$$

giving your answers to one decimal place.

(4)

- (ii) The first three terms of an arithmetic sequence are

$$\sin \alpha, \frac{1}{\tan \alpha} \text{ and } 2 \sin \alpha$$

where  $\alpha$  is a constant.

- (a) Show that  $2 \cos \alpha = 3 \sin^2 \alpha$

(3)

Given that  $\pi < \alpha < 2\pi$ ,

- (b) find, showing all working, the value of  $\alpha$  to 3 decimal places.

(5)

# Worked Solution - Question 9

## 1. Solve the sine equation

Let  $u = 2\theta - 10^\circ$ . Then  $\sin u = \frac{1}{3}$ .

## 2. Find the valid angle values

Since  $0 \leq \theta < 180^\circ$ ,  $-10^\circ \leq u < 350^\circ$ . In this interval,  $u = 19.471\dots^\circ$  or  $u = 160.529\dots^\circ$ .

## 3. Return to theta

$\theta = \frac{u + 10^\circ}{2}$ , giving  $\theta = 14.7^\circ$  and  $85.3^\circ$  to one decimal place.

## 4. Use the arithmetic sequence condition

For three consecutive terms of an arithmetic sequence, second minus first equals third minus second. Hence  $\frac{1}{\tan \alpha} - \sin \alpha = 2 \sin \alpha - \frac{1}{\tan \alpha}$ .

## 5. Prove the required identity

This gives  $\frac{2}{\tan \alpha} = 3 \sin \alpha$ . Since  $\frac{1}{\tan \alpha} = \frac{\cos \alpha}{\sin \alpha}$ , we get  $\frac{2 \cos \alpha}{\sin \alpha} = 3 \sin \alpha$ , so  $2 \cos \alpha = 3 \sin^2 \alpha$ .

## 6. Form a quadratic in cos alpha

Using  $\sin^2 \alpha = 1 - \cos^2 \alpha$ ,  $2 \cos \alpha = 3(1 - \cos^2 \alpha)$ , so  $3 \cos^2 \alpha + 2 \cos \alpha - 3 = 0$ .

## 7. Solve in the given interval

$\cos \alpha = \frac{-1 + \sqrt{10}}{3}$ ; the other root is less than  $-1$ . Since  $\pi < \alpha < 2\pi$  and cosine is positive,  $\alpha$  is in the fourth quadrant.

### 8. State alpha

$$\alpha = 2\pi - \cos^{-1} \left( \frac{-1 + \sqrt{10}}{3} \right) = 5.517 \text{ radians to 3 decimal places.}$$

#### Final answer

(i)  $\theta = 14.7^\circ, 85.3^\circ$ . (ii)(a) Proven.

(ii)(b)  $\alpha = 5.517$  radians.

**Question 10****Applications of Differentiation**

10. The curve  $C$  has equation

$$y = ax^3 - 3x^2 + 3x + b$$

where  $a$  and  $b$  are constants.

Given that

- the point  $(2, 5)$  lies on  $C$
- the gradient of the curve at  $(2, 5)$  is 7

(a) find the value of  $a$  and the value of  $b$ .

**(4)**

(b) Prove that  $C$  has no turning points.

**(3)**

# Worked Solution - Question 10

Topic group

**1. Use the point on the curve**

Substitute  $(2, 5)$  into  $y = ax^3 - 3x^2 + 3x + b$ :  $5 = 8a - 12 + 6 + b$ , so  $8a + b = 11$ .

**2. Differentiate**

$$\frac{dy}{dx} = 3ax^2 - 6x + 3.$$

**3. Use the given gradient**

At  $x = 2$ , the gradient is 7, so  $12a - 12 + 3 = 7$ . Hence  $12a = 16$  and  $a = \frac{4}{3}$ .

**4. Find b**

Using  $8a + b = 11$ ,  $b = 11 - 8\left(\frac{4}{3}\right) = \frac{1}{3}$ .

**5. Check for turning points**

With  $a = \frac{4}{3}$ ,  $\frac{dy}{dx} = 4x^2 - 6x + 3$ .

**6. Use the discriminant**

For  $4x^2 - 6x + 3 = 0$ , the discriminant is  $(-6)^2 - 4(4)(3) = 36 - 48 = -12 < 0$ .

**7. Conclude**

The derivative is never zero, so the curve has no turning points.

**Final answer**

(a)  $a = \frac{4}{3}$ ,  $b = \frac{1}{3}$ . (b) Proven:  $C$  has no turning points.

**PAST PAPER**

# **WMA12/01 January 2020**

January 2020 | 10 questions | 75 marks

**10**

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**75**

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**Answers**

worked solution  
after each  
question

## Question 1

1. The table below shows corresponding values of  $x$  and  $y$  for  $y = \log_2(2x)$

The values of  $y$  are given to 2 decimal places as appropriate.

$x$	2	5	8	11	14
$y$	2	3.32	4	4.46	4.81

Using the trapezium rule with all the values of  $y$  in the given table,

- (a) obtain an estimate for  $\int_2^{14} \log_2(2x) dx$ , giving your answer to one decimal place. (3)

Using your answer to part (a) and making your method clear, estimate

(b) (i)  $\int_2^{14} \frac{\log_2(4x^2)}{5} dx$

(ii)  $\int_2^{14} \log_2\left(\frac{2}{x}\right) dx$  (4)

## Worked Solution - Question 1

**1. Use the trapezium rule**

The strip width is  $h = 3$ . Therefore

$$\int_2^{14} \log_2(2x) dx \approx \frac{3}{2} \{2 + 4.81 + 2(3.32 + 4 + 4.46)\}.$$

**2. Calculate part (a)**

The estimate is  $\frac{3}{2}(30.37) = 45.555$ , so to one decimal place the answer is **45.6**.

**3. Rewrite part (b)(i)**

$\log_2(4x^2) = \log_2\{(2x)^2\} = 2 \log_2(2x)$ , so

$$\int_2^{14} \frac{\log_2(4x^2)}{5} dx = \frac{2}{5} \int_2^{14} \log_2(2x) dx.$$

**4. Estimate part (b)(i)**

Using part (a), the estimate is  $\frac{2}{5}(45.555) = 18.222$ , so **18.2** to one decimal place.

**5. Rewrite part (b)(ii)**

$$\log_2\left(\frac{2}{x}\right) = 2 - \log_2(2x).$$

**6. Estimate part (b)(ii)**

$$\int_2^{14} \log_2\left(\frac{2}{x}\right) dx \approx \int_2^{14} 2 dx - 45.555 = 24 - 45.555 = -21.555, \text{ so } -21.6.$$

**Final answer**

(a) 45.6. (b)(i) 18.2. (b)(ii) -21.6.

## Question 2

## Binomial Expansion

2. One of the terms in the binomial expansion of  $(3 + ax)^6$ , where  $a$  is a constant, is  $540x^4$

(a) Find the possible values of  $a$ .

(4)

(b) Hence find the term independent of  $x$  in the expansion of

$$\left(\frac{1}{81} + \frac{1}{x^6}\right)(3 + ax)^6$$

(3)

## Worked Solution - Question 2

**1. Identify the  $x$  to the fourth term**

In  $(3 + ax)^6$ , the  $x^4$  term is  $\binom{6}{4}3^2(ax)^4$ .

**2. Form the equation for  $a$** 

$\binom{6}{4}3^2a^4x^4 = 15 \cdot 9a^4x^4 = 135a^4x^4$ . Since this term is  $540x^4$ ,  $135a^4 = 540$ .

**3. Solve for  $a$** 

$a^4 = 4$ , so the real possible values are  $a = \sqrt{2}$  or  $a = -\sqrt{2}$ .

**4. Find the constant contribution from 1 over 81**

The constant term from  $\frac{1}{81}(3 + ax)^6$  is  $\frac{1}{81} \cdot 3^6 = 9$ .

**5. Find the constant contribution from  $x$  to the minus six**

The  $x^6$  term in  $(3 + ax)^6$  is  $a^6x^6$ . Since  $a = \pm\sqrt{2}$ ,  $a^6 = 8$ , so  $\frac{1}{x^6} \cdot a^6x^6$  contributes 8.

**6. Add the constant terms**

The term independent of  $x$  is  $9 + 8 = 17$ .

**Final answer**

(a)  $a = \pm\sqrt{2}$ . (b) 17.

## Question 3

## Polynomials

3.

$$f(x) = 6x^3 + 17x^2 + 4x - 12$$

(a) Use the factor theorem to show that  $(2x + 3)$  is a factor of  $f(x)$ . (2)

(b) Hence, using algebra, write  $f(x)$  as a product of three linear factors. (4)

(c) Solve, for  $\frac{\pi}{2} < \theta < \pi$ , the equation

$$6 \tan^3 \theta + 17 \tan^2 \theta + 4 \tan \theta - 12 = 0$$

giving your answers to 3 significant figures. (2)

## Worked Solution - Question 3

**1. Use the factor theorem**

For  $(2x + 3)$ , use  $x = -\frac{3}{2}$ . Substituting gives

$$f\left(-\frac{3}{2}\right) = 6\left(-\frac{3}{2}\right)^3 + 17\left(-\frac{3}{2}\right)^2 + 4\left(-\frac{3}{2}\right) - 12 = 0.$$

**2. State the factor result**

Since  $f\left(-\frac{3}{2}\right) = 0$ ,  $(2x + 3)$  is a factor of  $f(x)$ .

**3. Divide by the known factor**

$$6x^3 + 17x^2 + 4x - 12 = (2x + 3)(3x^2 + 4x - 4).$$

**4. Factorise the quadratic**

$$3x^2 + 4x - 4 = (3x - 2)(x + 2), \text{ so } f(x) = (2x + 3)(3x - 2)(x + 2).$$

**5. Use the factorisation for tan theta**

Let  $t = \tan \theta$ . Then  $(2t + 3)(3t - 2)(t + 2) = 0$ , so  $t = -\frac{3}{2}$ ,  $t = \frac{2}{3}$ , or  $t = -2$ .

**6. Choose the roots in the interval**

For  $\frac{\pi}{2} < \theta < \pi$ ,  $\tan \theta$  is negative, so use  $\tan \theta = -\frac{3}{2}$  and  $\tan \theta = -2$ .

**7. Find theta**

This gives  $\theta = \pi - \tan^{-1}\left(\frac{3}{2}\right) = 2.16$  and  $\theta = \pi - \tan^{-1}(2) = 2.03$ , to 3 significant figures.

**Final answer**

(a) Proven.

(b)  $f(x) = (2x + 3)(3x - 2)(x + 2)$ . (c)  $\theta = 2.03, 2.16$  radians.

## Question 4

Integration

4.

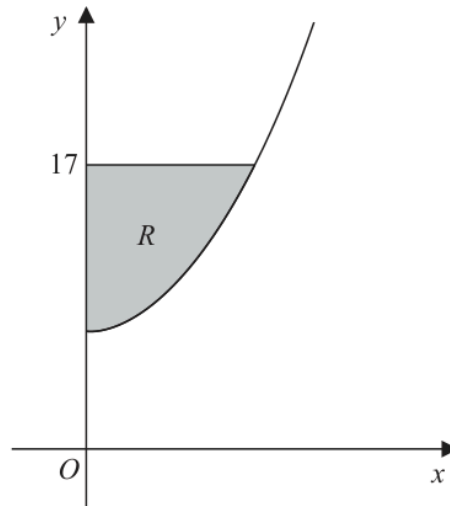


Figure 1

Figure 1 shows a sketch of the curve with equation

$$y = 2x^2 + 7 \quad x \geq 0$$

The finite region  $R$ , shown shaded in Figure 1, is bounded by the curve, the  $y$ -axis and the line with equation  $y = 17$

Find the exact area of  $R$ .

(6)

# Worked Solution - Question 4

## 1. Find the intersection with $y$ equals 17

The curve is  $y = 2x^2 + 7$ . Set  $2x^2 + 7 = 17$  to get  $x^2 = 5$ , so the right-hand limit is  $x = \sqrt{5}$ .

## 2. Set up the area

The shaded area is between  $y = 17$  and  $y = 2x^2 + 7$ , from  $x = 0$  to  $x = \sqrt{5}$ .

## 3. Integrate

$$A = \int_0^{\sqrt{5}} \{17 - (2x^2 + 7)\} dx = \int_0^{\sqrt{5}} (10 - 2x^2) dx.$$

## 4. Evaluate the exact area

$$A = \left[ 10x - \frac{2}{3}x^3 \right]_0^{\sqrt{5}} = 10\sqrt{5} - \frac{2}{3}(5\sqrt{5}) = \frac{20\sqrt{5}}{3}.$$

**Final answer**

$$\frac{20\sqrt{5}}{3}.$$

**Question 5****Modelling with Sequences & Series**

5. **In this question you must show all stages of your working.**  
**Solutions relying on calculator technology are not acceptable.**

A colony of bees is being studied.

The number of bees in the colony at the start of the study was 30 000

Three years after the start of the study, the number of bees in the colony is 34 000

A model predicts that the number of bees in the colony will increase by  $p\%$  each year, so that the number of bees in the colony at the end of each year of study forms a geometric sequence.

Assuming the model,

- (a) find the value of  $p$ , giving your answer to 2 decimal places. (3)

According to the model, at the end of  $N$  years of study the number of bees in the colony exceeds 75 000

- (b) Find, showing all steps in your working, the smallest integer value of  $N$ . (5)

# Worked Solution - Question 5

Topic group

## 1. Set up the growth ratio

Let the yearly ratio be  $r$ . After 3 years,  $30000r^3 = 34000$ .

## 2. Find $p$

$r^3 = \frac{17}{15}$ , so  $r = \left(\frac{17}{15}\right)^{1/3} \approx 1.04257$ . Therefore  $p = (r - 1) \times 100 \approx 4.26$ .

## 3. Set up the inequality

After  $N$  years the model gives  $30000r^N$ . We need  $30000r^N > 75000$ , so  $r^N > 2.5$ .

## 4. Take logarithms

$N \log r > \log(2.5)$ , so  $N > \frac{\log(2.5)}{\log r}$ .

## 5. Find the smallest integer

Using  $r \approx 1.04257$ ,  $\frac{\log(2.5)}{\log r} \approx 21.98$ . Hence the smallest integer value is  $N = 22$ .

### Final answer

(a)  $p = 4.26$ . (b)  $N = 22$ .

## Question 6

Circles

6. The circle  $C$  has equation

$$x^2 + y^2 + 6x - 4y - 14 = 0$$

(a) Find

(i) the coordinates of the centre of  $C$ ,

(ii) the exact radius of  $C$ .

(3)

The line with equation  $y = k$ , where  $k$  is a constant, is a tangent to  $C$ .

(b) Find the possible values of  $k$ .

(2)

The line with equation  $y = p$ , where  $p$  is a negative constant, is a chord of  $C$ .

Given that the length of this chord is 4 units,

(c) find the value of  $p$ .

(3)

## Worked Solution - Question 6

**1. Complete the square**

$$x^2 + 6x = (x + 3)^2 - 9 \text{ and } y^2 - 4y = (y - 2)^2 - 4.$$

**2. Find the centre and radius**

The circle becomes  $(x + 3)^2 + (y - 2)^2 = 27$ . Hence the centre is  $(-3, 2)$  and the radius is  $\sqrt{27} = 3\sqrt{3}$ .

**3. Use horizontal tangents**

A horizontal tangent has equation  $y = k$ . The possible tangent levels are one radius above and below the centre, so  $k = 2 \pm 3\sqrt{3}$ .

**4. Use half the chord length**

The chord has length 4, so half the chord is 2. If the chord is  $y = p$ , its perpendicular distance from the centre is  $|p - 2|$ .

**5. Apply Pythagoras**

$$(p - 2)^2 + 2^2 = (3\sqrt{3})^2 = 27, \text{ so } (p - 2)^2 = 23.$$

**6. Choose the negative value**

Since  $p$  is negative,  $p = 2 - \sqrt{23}$ .

**Final answer**

$$(a)(i) (-3, 2). \quad (a)(ii) 3\sqrt{3}. \quad (b) k = 2 \pm 3\sqrt{3}. \quad (c) p = 2 - \sqrt{23}.$$

## Question 7

## Trigonometric Equations

7. (a) Show that the equation

$$8 \tan \theta = 3 \cos \theta$$

may be rewritten in the form

$$3 \sin^2 \theta + 8 \sin \theta - 3 = 0$$

(3)

- (b) Hence solve, for  $0 \leq x \leq 90^\circ$ , the equation

$$8 \tan 2x = 3 \cos 2x$$

giving your answers to 2 decimal places.

(4)

# Worked Solution - Question 7

Topic group

**1. Rewrite tan theta**

$$8 \tan \theta = 3 \cos \theta \text{ becomes } 8 \frac{\sin \theta}{\cos \theta} = 3 \cos \theta.$$

**2. Clear the denominator**

$$\text{Multiplying by } \cos \theta \text{ gives } 8 \sin \theta = 3 \cos^2 \theta.$$

**3. Use the identity**

$$\text{Since } \cos^2 \theta = 1 - \sin^2 \theta, 8 \sin \theta = 3(1 - \sin^2 \theta), \text{ so } 3 \sin^2 \theta + 8 \sin \theta - 3 = 0.$$

**4. Apply the result to  $2x$** 

$$\text{For } 8 \tan 2x = 3 \cos 2x, \text{ we get } 3 \sin^2(2x) + 8 \sin(2x) - 3 = 0.$$

**5. Solve for  $\sin 2x$** 

$$(3 \sin 2x - 1)(\sin 2x + 3) = 0. \text{ Since } \sin 2x \text{ cannot be } -3, \sin 2x = \frac{1}{3}.$$

**6. Find  $x$** 

For  $0 \leq x \leq 90^\circ$ ,  $0 \leq 2x \leq 180^\circ$ . Thus  $2x = 19.471 \dots^\circ$  or  $160.529 \dots^\circ$ , so  $x = 9.74^\circ$  or  $80.26^\circ$ .

**Final answer**

(a) Proven.

(b)  $x = 9.74^\circ, 80.26^\circ$ .

## Question 8

## Sequences &amp; Series

8. (i) An arithmetic series has first term  $a$  and common difference  $d$ .

Prove that the sum to  $n$  terms of this series is

$$\frac{n}{2}\{2a + (n-1)d\} \quad (3)$$

- (ii) A sequence  $u_1, u_2, u_3, \dots$  is given by

$$u_n = 5n + 3(-1)^n$$

Find the value of

(a)  $u_5$  (1)

(b)  $\sum_{n=1}^{59} u_n$  (3)

---

## Worked Solution - Question 8

**1. Write the series forwards**

$$S_n = a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d).$$

**2. Write the series backwards**

$$S_n = (a + (n - 1)d) + (a + (n - 2)d) + \cdots + a.$$

**3. Add the two forms**

Each pair adds to  $2a + (n - 1)d$ , and there are  $n$  pairs. Hence

$$2S_n = n\{2a + (n - 1)d\}.$$

**4. Complete the proof**

Dividing by 2 gives  $S_n = \frac{n}{2}\{2a + (n - 1)d\}$ .

**5. Find  $u_5$** 

$$u_5 = 5(5) + 3(-1)^5 = 25 - 3 = 22.$$

**6. Sum the sequence**

$$\sum_{n=1}^{59} u_n = 5 \sum_{n=1}^{59} n + 3 \sum_{n=1}^{59} (-1)^n.$$

**7. Calculate the total**

$5 \sum_{n=1}^{59} n = 5 \cdot \frac{59 \cdot 60}{2} = 8850$ , and  $\sum_{n=1}^{59} (-1)^n = -1$ . Therefore the total is  $8850 - 3 = 8847$ .

**Final answer**

(i) Proven.

(ii)(a) 22. (ii)(b) 8847.

**Question 9****Laws of Logarithms**

9. (a) Sketch the curve with equation

$$y = 3 \times 4^x$$

showing the coordinates of any points of intersection with the coordinate axes.

(2)

The curve with equation  $y = 6^{1-x}$  meets the curve with equation  $y = 3 \times 4^x$  at the point  $P$ .

- (b) Show that the  $x$  coordinate of  $P$  is  $\frac{\log_{10} 2}{\log_{10} 24}$

(5)

## Worked Solution - Question 9

**1. Describe the sketch**

$y = 3 \times 4^x$  is an increasing exponential curve. At  $x = 0$ ,  $y = 3$ , so the curve crosses the  $y$ -axis at  $(0, 3)$  and has no  $x$ -axis intersection.

**2. Equate the two curves**

At  $P$ ,  $6^{1-x} = 3 \times 4^x$ .

**3. Take logarithms**

$$(1 - x) \log_{10} 6 = \log_{10} 3 + x \log_{10} 4.$$

**4. Collect the  $x$  terms**

$$\log_{10} 6 - \log_{10} 3 = x(\log_{10} 4 + \log_{10} 6).$$

**5. Simplify the logarithms**

$$\log_{10} 6 - \log_{10} 3 = \log_{10} 2 \text{ and } \log_{10} 4 + \log_{10} 6 = \log_{10} 24.$$

**6. State the  $x$  coordinate**

$$\text{Therefore } x = \frac{\log_{10} 2}{\log_{10} 24}.$$

**Final answer**

(a) Increasing exponential curve through  $(0, 3)$ , with no  $x$ -axis intersection.

$$(b) x = \frac{\log_{10} 2}{\log_{10} 24}.$$

## Question 10

## Applications of Differentiation

10. A curve  $C$  has equation

$$y = 4x^3 - 9x + \frac{k}{x} \quad x > 0$$

where  $k$  is a constant.

The point  $P$  with  $x$  coordinate  $\frac{1}{2}$  lies on  $C$ .

Given that  $P$  is a stationary point of  $C$ ,

(a) show that  $k = -\frac{3}{2}$  (4)

(b) Determine the nature of the stationary point at  $P$ , justifying your answer. (2)

The curve  $C$  has a second stationary point.

(c) Using algebra, find the  $x$  coordinate of this second stationary point. (4)

**(Total 10 marks)**

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# Worked Solution - Question 10

Topic group

## 1. Differentiate the curve

Write  $y = 4x^3 - 9x + kx^{-1}$ . Then  $\frac{dy}{dx} = 12x^2 - 9 - kx^{-2}$ .

## 2. Use the stationary point

At  $x = \frac{1}{2}$ ,  $\frac{dy}{dx} = 0$ , so  $0 = 12\left(\frac{1}{2}\right)^2 - 9 - k\left(\frac{1}{2}\right)^{-2} = 3 - 9 - 4k$ .

## 3. Find $k$

$-6 - 4k = 0$ , hence  $k = -\frac{3}{2}$ .

## 4. Use the second derivative

With  $k = -\frac{3}{2}$ ,  $\frac{dy}{dx} = 12x^2 - 9 + \frac{3}{2x^2}$  and  $\frac{d^2y}{dx^2} = 24x - 3x^{-3}$ .

## 5. Determine the nature of $P$

At  $x = \frac{1}{2}$ ,  $\frac{d^2y}{dx^2} = 12 - 24 = -12 < 0$ , so  $P$  is a local maximum.

## 6. Find the other stationary point

Set  $12x^2 - 9 + \frac{3}{2x^2} = 0$ . Multiplying by  $2x^2$  gives  $24x^4 - 18x^2 + 3 = 0$ , or  $8x^4 - 6x^2 + 1 = 0$ .

## 7. Solve the quadratic in $x$ squared

$(4x^2 - 1)(2x^2 - 1) = 0$ . The given point has  $x = \frac{1}{2}$ , so the second stationary point has  $x^2 = \frac{1}{2}$ .

### 8. State the positive $x$ coordinate

Since  $x > 0$ , the second stationary point has  $x = \frac{1}{\sqrt{2}}$ .

#### Final answer

(a)  $k = -\frac{3}{2}$ . (b) local maximum.

(c)  $x = \frac{1}{\sqrt{2}}$ .

**PAST PAPER**

# **WMA12/01 January 2021**

January 2021 | 10 questions | 75 marks

**10**

questions

**75**

marks

**Answers**

worked solution  
after each  
question

# Question 1

## Polynomials

1. 
$$f(x) = x^4 + ax^3 - 3x^2 + bx + 5$$

where  $a$  and  $b$  are constants.

When  $f(x)$  is divided by  $(x + 1)$ , the remainder is 4

(a) Show that  $a + b = -1$

(2)

When  $f(x)$  is divided by  $(x - 2)$ , the remainder is  $-23$

(b) Find the value of  $a$  and the value of  $b$ .

(4)

## Worked Solution - Question 1

**1. Use the first remainder**

Dividing by  $(x + 1)$  means use  $x = -1$ . Hence

$$f(-1) = 1 - a - 3 - b + 5 = 3 - a - b.$$

**2. Show the required relation**

The remainder is 4, so  $3 - a - b = 4$ . Therefore  $a + b = -1$ .

**3. Use the second remainder**

Dividing by  $(x - 2)$  means use  $x = 2$ . Hence

$$f(2) = 16 + 8a - 12 + 2b + 5 = 9 + 8a + 2b.$$

**4. Form the second equation**

The remainder is  $-23$ , so  $9 + 8a + 2b = -23$ , which gives  $4a + b = -16$ .

**5. Solve simultaneously**

Subtract  $a + b = -1$  from  $4a + b = -16$  to get  $3a = -15$ , so  $a = -5$ .

**6. Find b**

Using  $a + b = -1$ ,  $-5 + b = -1$ , hence  $b = 4$ .

**Final answer**

(a) Proven:  $a + b = -1$ .

(b)  $a = -5$ ,  $b = 4$ .

**Question 2****Applications of Differentiation**

2. A curve has equation

$$y = x^3 - x^2 - 16x + 2$$

- (a) Using calculus, find the  $x$  coordinates of the stationary points of the curve. **(4)**
- (b) Justify, by further calculus, the nature of all of the stationary points of the curve. **(3)**

# Worked Solution - Question 2

Topic group

**1. Differentiate the curve**

For  $y = x^3 - x^2 - 16x + 2$ ,  $\frac{dy}{dx} = 3x^2 - 2x - 16$ .

**2. Set the gradient to zero**

Stationary points occur when  $3x^2 - 2x - 16 = 0$ .

**3. Factorise**

$3x^2 - 2x - 16 = (3x - 8)(x + 2)$ , so  $x = \frac{8}{3}$  or  $x = -2$ .

**4. Differentiate again**

$$\frac{d^2y}{dx^2} = 6x - 2.$$

**5. Classify  $x$  equals negative two**

At  $x = -2$ ,  $6x - 2 = -14 < 0$ , so the stationary point is a local maximum.

**6. Classify  $x$  equals eight over three**

At  $x = \frac{8}{3}$ ,  $6x - 2 = 14 > 0$ , so the stationary point is a local minimum.

**Final answer**

(a)  $x = -2$ ,  $\frac{8}{3}$ . (b) local maximum at  $x = -2$  and local minimum at  $x = \frac{8}{3}$ .

**Question 3****Laws of Logarithms**

3. (i) Solve

$$7^{x+2} = 3$$

giving your answer in the form  $x = \log_7 a$  where  $a$  is a rational number in its simplest form.

(3)

- (ii) Using the laws of logarithms, solve

$$1 + \log_2 y + \log_2 (y + 4) = \log_2 (5 - y)$$

(5)

## Worked Solution - Question 3

**1. Rewrite the exponential equation**

From  $7^{x+2} = 3$ , divide by  $7^2$  to get  $7^x = \frac{3}{49}$ .

**2. Write  $x$  as a logarithm**

Therefore  $x = \log_7 \left( \frac{3}{49} \right)$ .

**3. Convert one to a logarithm**

Since  $1 = \log_2 2$ , the equation becomes  
 $\log_2 2 + \log_2 y + \log_2(y + 4) = \log_2(5 - y)$ .

**4. Combine the logarithms**

$\log_2\{2y(y + 4)\} = \log_2(5 - y)$ .

**5. Solve the quadratic**

Thus  $2y(y + 4) = 5 - y$ , so  $2y^2 + 9y - 5 = 0 = (2y - 1)(y + 5)$ .

**6. Check the domain**

The original logarithms require  $y > 0$  and  $y < 5$ , so  $y = -5$  is invalid and  $y = \frac{1}{2}$ .

**Final answer**

(i)  $x = \log_7 \left( \frac{3}{49} \right)$ .    (ii)  $y = \frac{1}{2}$ .

## Question 4

## Binomial Expansion

4. (a) Find the first three terms, in ascending powers of  $x$ , of the binomial expansion of

$$(2 + px)^6$$

where  $p$  is a constant. Give each term in simplest form.

(4)

Given that in the expansion of

$$\left(3 - \frac{1}{2}x\right)(2 + px)^6$$

the coefficient of  $x^2$  is  $-\frac{3}{4}$

- (b) find the possible values of  $p$ .

(4)

## Worked Solution - Question 4

**1. Write the first binomial terms**

$$(2 + px)^6 = 2^6 + \binom{6}{1}2^5(px) + \binom{6}{2}2^4(px)^2 + \dots$$

**2. Simplify the expansion**

The first three terms are  $64 + 192px + 240p^2x^2$ .

**3. Find the coefficient of x squared**

In  $(3 - \frac{1}{2}x)(2 + px)^6$ , the  $x^2$  coefficient is  $3(240p^2) - \frac{1}{2}(192p)$ .

**4. Form the equation**

$$\text{So } 720p^2 - 96p = -\frac{3}{4}.$$

**5. Solve the quadratic**

Multiplying by 4 gives  $2880p^2 - 384p + 3 = 0$ , or  $960p^2 - 128p + 1 = 0$ .

**6. State the values of p**

$$p = \frac{128 \pm 112}{1920}, \text{ so } p = \frac{1}{8} \text{ or } p = \frac{1}{120}.$$

**Final answer**

$$(a) 64 + 192px + 240p^2x^2. \quad (b) p = \frac{1}{8} \text{ or } p = \frac{1}{120}.$$

**Question 5****Proof**

5. (i) Use algebra to prove that for all  $x \geq 0$

$$3x + 1 \geq 2\sqrt{3x} \quad (3)$$

- (ii) Show that the following statement is not true.

“The sum of three consecutive prime numbers is always a multiple of 5”  
(1)

# Worked Solution - Question 5

Topic group

## 1. Start from a square

For  $x \geq 0$ ,  $\sqrt{3x}$  is real and  $(\sqrt{3x} - 1)^2 \geq 0$ .

## 2. Expand the square

$$(\sqrt{3x} - 1)^2 = 3x - 2\sqrt{3x} + 1 \geq 0.$$

## 3. Rearrange the inequality

Therefore  $3x + 1 \geq 2\sqrt{3x}$ , as required.

## 4. Give a counterexample

The consecutive prime numbers **5**, **7**, and **11** have sum  $5 + 7 + 11 = 23$ .

## 5. Complete the disproof

**23** is not a multiple of **5**, so the statement is not true.

### Final answer

(i) Proven.

(ii) For example,  $5 + 7 + 11 = 23$ , which is not a multiple of **5**.

## Question 6

## Trigonometric Equations

6. (a) Show that the equation

$$\frac{3\sin\theta\cos\theta}{2\sin\theta-1} = 5\tan\theta \quad \sin\theta \neq \frac{1}{2}$$

can be written in the form

$$3\sin^3\theta + 10\sin^2\theta - 8\sin\theta = 0 \quad (4)$$

- (b) Hence solve, for  $-\frac{\pi}{4} < x < \frac{\pi}{4}$

$$\frac{3\sin 2x\cos 2x}{2\sin 2x-1} = 5\tan 2x$$

giving your answers to 3 decimal places where appropriate.

(4)

# Worked Solution - Question 6

Topic group

**1. Clear the fraction**

Starting with  $\frac{3 \sin \theta \cos \theta}{2 \sin \theta - 1} = 5 \tan \theta$ , multiply by  $(2 \sin \theta - 1) \cos \theta$ .

**2. Use tan theta**

This gives  $3 \sin \theta \cos^2 \theta = 5 \sin \theta (2 \sin \theta - 1)$ .

**3. Use cos squared**

Substitute  $\cos^2 \theta = 1 - \sin^2 \theta$ :  $3 \sin \theta (1 - \sin^2 \theta) = 10 \sin^2 \theta - 5 \sin \theta$ .

**4. Collect terms**

Rearranging gives  $3 \sin^3 \theta + 10 \sin^2 \theta - 8 \sin \theta = 0$ .

**5. Factor the sine equation**

$3s^3 + 10s^2 - 8s = 0$  gives  $s(3s - 2)(s + 4) = 0$ , where  $s = \sin 2x$ .

**6. Use the valid sine values**

Since  $-\frac{\pi}{2} < 2x < \frac{\pi}{2}$ , the valid values are  $\sin 2x = 0$  and  $\sin 2x = \frac{2}{3}$ .

**7. Find x**

$2x = 0$  gives  $x = 0$ . Also  $2x = \sin^{-1} \left( \frac{2}{3} \right)$ , so  $x = 0.364863 \dots = 0.365$  to 3 decimal places.

**Final answer**

(a) Proven.

(b)  $x = 0, 0.365$ .

## Question 7

7.

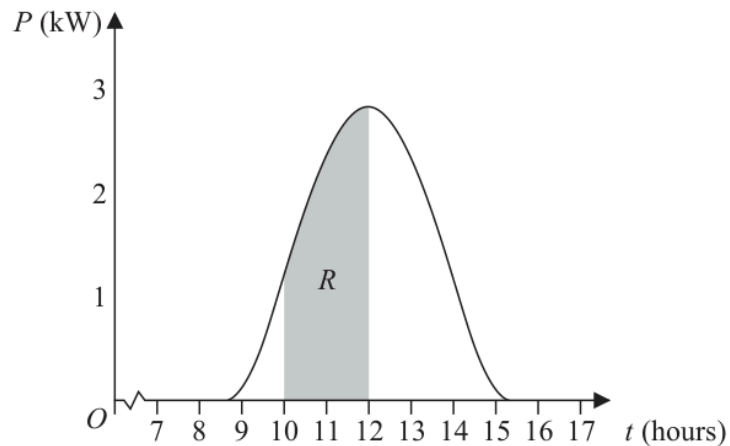


Figure 1

Solar panels are installed on the roof of a building.

The power,  $P$ , produced on a particular day, in kW, can be modelled by the equation

$$P = 0.95 + 2^{t-12} + 2^{12-t} - (t-12)^2 \quad 8.5 \leq t \leq 15.2$$

where  $t$  is the time in hours after midnight. The graph of  $P$  against  $t$  is shown in Figure 1.

A table of values of  $t$  and  $P$  is shown below, with the values of  $P$  given to 4 significant figures where appropriate.

Time, $t$ (hours)	10	10.5	11	11.5	12
Power, $P$ (kW)		1.882	2.45		2.95

(a) Use the given equation to complete the table, giving the values of  $P$  to 4 significant figures where appropriate.

(2)

The amount of energy, in kWh, produced between 10:00 and 12:00 can be found by calculating the area of region  $R$ , shown shaded in Figure 1.

(b) Use the trapezium rule, with all the values of  $P$  in the completed table, to find an estimate for the amount of energy produced between 10:00 and 12:00. Give your answer to 2 decimal places.

(4)

# Worked Solution - Question 7

**1. Use the model at t equals ten**

$$P(10) = 0.95 + 2^{-2} + 2^2 - (10 - 12)^2 = 0.95 + 0.25 + 4 - 4 = 1.20.$$

**2. Use the model at t equals eleven point five**

$$P(11.5) = 0.95 + 2^{-0.5} + 2^{0.5} - (11.5 - 12)^2 = 2.821 \dots$$

**3. Identify the trapezium width**

The time values increase by 0.5, so  $h = 0.5$ .

**4. Apply the trapezium rule**

The estimate is  $\frac{0.5}{2} \{1.20 + 2.95 + 2(1.882 + 2.45 + 2.821 \dots)\}$ .

**5. Calculate the energy**

This gives 4.614..., so the estimated energy is **4.61 kWh** to 2 decimal places.

**Final answer**

(a)  $P(10) = 1.20$ ,  $P(11.5) = 2.821$ . (b) 4.61 kWh.

## Question 8

## Sequences &amp; Series

8. A sequence  $a_1, a_2, a_3, \dots$  is defined by

$$a_{n+1} = 2(a_n + 3)^2 - 7$$
$$a_1 = p - 3$$

where  $p$  is a constant.

(a) Find an expression for  $a_2$  in terms of  $p$ , giving your answer in simplest form.

(1)

Given that  $\sum_{n=1}^3 a_n = p + 15$

(b) find the possible values of  $a_2$

(6)

## Worked Solution - Question 8

Topic group

**1. Find a two**

Since  $a_1 = p - 3$ , we have  $a_1 + 3 = p$ . Therefore  
 $a_2 = 2(a_1 + 3)^2 - 7 = 2p^2 - 7$ .

**2. Find a three**

$a_2 + 3 = 2p^2 - 4$ , so  $a_3 = 2(2p^2 - 4)^2 - 7 = 8(p^2 - 2)^2 - 7$ .

**3. Use the given sum**

$a_1 + a_2 + a_3 = p + 15$ .

**4. Substitute the expressions**

$(p - 3) + (2p^2 - 7) + 8(p^2 - 2)^2 - 7 = p + 15$ .

**5. Simplify**

Cancelling  $p$  and expanding gives  $8p^4 - 30p^2 = 0$ , so  $2p^2(4p^2 - 15) = 0$ .

**6. Find the possible a two values**

If  $p = 0$ , then  $a_2 = -7$ . If  $p^2 = \frac{15}{4}$ , then  $a_2 = 2 \cdot \frac{15}{4} - 7 = \frac{1}{2}$ .

**Final answer**

(a)  $a_2 = 2p^2 - 7$ . (b)  $a_2 = -7$  or  $\frac{1}{2}$ .

## Question 9

9. A circle  $C$  has equation

$$(x - k)^2 + (y - 2k)^2 = k + 7$$

where  $k$  is a positive constant.

- (a) Write down, in terms of  $k$ ,

- (i) the coordinates of the centre of  $C$ ,
- (ii) the radius of  $C$ .

(2)

Given that the point  $P(2, 3)$  lies on  $C$

- (b) (i) show that  $5k^2 - 17k + 6 = 0$

- (ii) hence find the possible values of  $k$ .

(3)

The tangent to the circle at  $P$  intersects the  $x$ -axis at point  $T$ .

Given that  $k < 2$

- (c) calculate the exact area of triangle  $OPT$ .

(5)

## Worked Solution - Question 9

**1. Read the centre and radius**

From  $(x - k)^2 + (y - 2k)^2 = k + 7$ , the centre is  $(k, 2k)$  and the radius is  $\sqrt{k + 7}$ .

**2. Substitute point P**

Since  $P(2, 3)$  lies on the circle,  $(2 - k)^2 + (3 - 2k)^2 = k + 7$ .

**3. Show the quadratic**

Expanding gives  $k^2 - 4k + 4 + 4k^2 - 12k + 9 = k + 7$ , hence  $5k^2 - 17k + 6 = 0$ .

**4. Solve for k**

$5k^2 - 17k + 6 = (5k - 2)(k - 3)$ , so  $k = \frac{2}{5}$  or  $k = 3$ .

**5. Use the condition k less than two**

For the tangent calculation,  $k < 2$ , so the centre is  $\left(\frac{2}{5}, \frac{4}{5}\right)$ .

**6. Find the tangent gradient**

The radius from the centre to  $P$  has gradient  $\frac{3 - 4/5}{2 - 2/5} = \frac{11}{8}$ , so the tangent gradient is  $-\frac{8}{11}$ .

**7. Find the x intercept**

The tangent is  $y - 3 = -\frac{8}{11}(x - 2)$ . Setting  $y = 0$  gives  $x = \frac{49}{8}$ , so  $T = \left(\frac{49}{8}, 0\right)$ .

### 8. Calculate the area

Triangle  $OPT$  has base  $\frac{49}{8}$  on the  $x$ -axis and height  $3$ , so its area is

$$\frac{1}{2} \cdot \frac{49}{8} \cdot 3 = \frac{147}{16}.$$

#### Final answer

(a)(i)  $(k, 2k)$ . (a)(ii)  $\sqrt{k+7}$ . (b)(ii)  $k = 3$  or  $\frac{2}{5}$ . (c)  $\frac{147}{16}$ .

**Question 10****Modelling with Sequences & Series****10. In this question you must show detailed reasoning.**

Owen wants to train for 12 weeks in preparation for running a marathon.

During the 12-week period he will run every Sunday and every Wednesday.

- On Sunday in week 1 he will run 15 km
- On Sunday in week 12 he will run 37 km

He considers two different 12-week training plans.

In training plan *A*, he will increase the distance he runs each Sunday by the same amount.

- (a) Calculate the distance he will run on Sunday in week 5 under training plan *A*. (3)

In training plan *B*, he will increase the distance he runs each Sunday by the same percentage.

- (b) Calculate the distance he will run on Sunday in week 5 under training plan *B*.  
Give your answer in km to one decimal place. (3)

Owen will also run a fixed distance,  $x$  km, each Wednesday over the 12-week period.

Given that

- $x$  is an integer
- the total distance that Owen will run on Sundays and Wednesdays over the 12 weeks will not exceed 360 km

- (c) (i) find the maximum value of  $x$ , if he uses training plan *A*,  
(ii) find the maximum value of  $x$ , if he uses training plan *B*. (5)

**(Total 11 marks)**

---

# Worked Solution - Question 10

Topic group

## 1. Find the arithmetic difference

Plan A has 12 Sunday runs from 15 km to 37 km, so there are 11 equal increases.

The common difference is  $\frac{37 - 15}{11} = 2$ .

## 2. Find week five for plan A

Week 5 is the fifth term, so the distance is  $15 + 4(2) = 23$  km.

## 3. Find the geometric ratio

For plan B,  $15r^{11} = 37$ , so  $r = \left(\frac{37}{15}\right)^{1/11}$ .

## 4. Find week five for plan B

The fifth Sunday distance is  $15r^4 = 15\left(\frac{37}{15}\right)^{4/11} = 20.829\dots$ , so it is 20.8 km.

## 5. Sum the Sunday runs for plan A

The arithmetic total is  $\frac{12}{2}(15 + 37) = 312$  km.

## 6. Find the Wednesday limit for plan A

The total distance condition gives  $312 + 12x \leq 360$ , so  $x \leq 4$ . The maximum integer is 4.

## 7. Sum the Sunday runs for plan B

The geometric total is  $15\frac{r^{12} - 1}{r - 1} = 294.185\dots$  km.

## 8. Find the Wednesday limit for plan B

Now  $294.185\dots + 12x \leq 360$ , so  $x \leq 5.484\dots$ . The maximum integer is 5.

**Final answer**

(a) 23 km. (b) 20.8 km. (c)(i)  $x = 4$ . (c)(ii)  $x = 5$ .

**PAST PAPER**

# **WMA12/01 May/June 2021**

**May/June 2021 | 9 questions | 75 marks**

**9**

questions

**75**

marks

**Answers**

worked solution  
after each  
question

# Question 1

## Modelling with Sequences & Series

1. Adina is saving money to buy a new computer. She saves £5 in week 1, £5.25 in week 2, £5.50 in week 3 and so on until she has enough money, in total, to buy the computer.

She decides to model her savings using either an arithmetic series or a geometric series.

Using the information given,

- (a) (i) state with a reason whether an arithmetic series or a geometric series should be used,
- (ii) write down an expression, in terms of  $n$ , for the amount, in pounds (£), saved in week  $n$ .

(3)

Given that the computer Adina wants to buy costs £350

- (b) find the number of weeks it will take for Adina to save enough money to buy the computer.

(4)

# Worked Solution - Question 1

Topic group

## 1. Choose the model

The weekly amounts are **5**, **5.25**, **5.50**, and so on. The increase is a constant **0.25**, so an arithmetic series should be used.

## 2. Write the $n$ th amount

The first amount is **5** and the common difference is **0.25**, so the amount saved in week  $n$  is  $5 + 0.25(n - 1)$ .

## 3. Set up the total

After  $n$  weeks,  $S_n = \frac{n}{2} \{2(5) + (n - 1)(0.25)\}$ .

## 4. Form the inequality

Adina needs  $S_n \geq 350$ , so  $\frac{n}{2}(9.75 + 0.25n) \geq 350$ .

## 5. Solve for $n$

This is  $n^2 + 39n - 2800 \geq 0$ . The positive root is **36.89...**, so the first whole number that works is  $n = 37$ .

## 6. Check the week

After **36** weeks the total is below **350**, and after **37** weeks it is **351.50**, so it takes **37** weeks.

**Final answer**

(a)(i) Arithmetic series.

(a)(ii)  $5 + 0.25(n - 1)$ .

(b) **37** weeks.

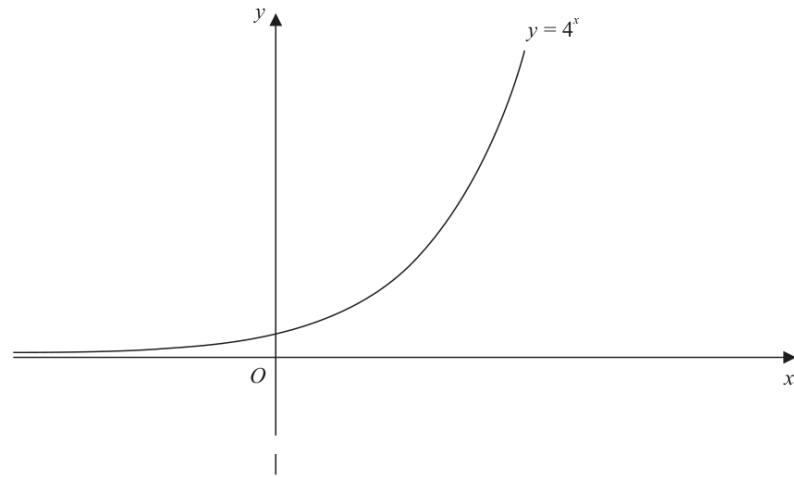
WMA12/01 MAY/JUNE 2021

8 marks

## Question 2

Laws of Logarithms

2.



**Figure 1**

Figure 1 shows a sketch of the curve with equation  $y = 4^x$

A copy of Figure 1, labelled Diagram 1, is shown on the next page.

(a) On Diagram 1, sketch the curve with equation

(i)  $y = 2^x$

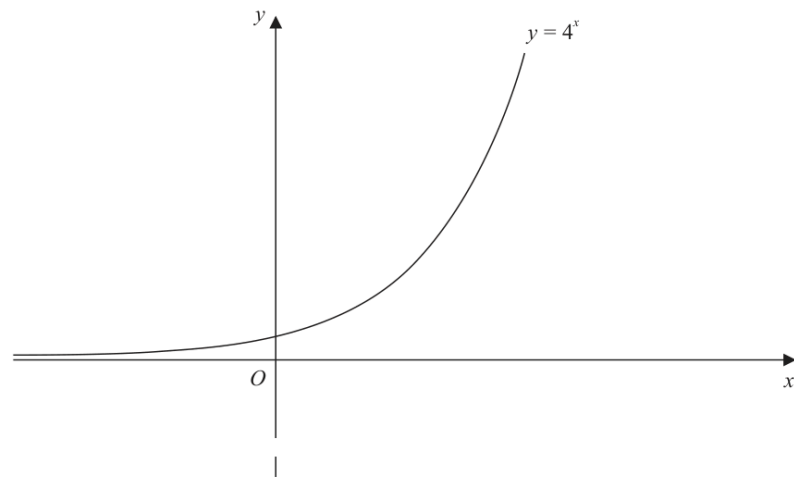
(ii)  $y = 4^x - 6$

Label clearly the coordinates of any points of intersection with the coordinate axes. **(4)**

The curve with equation  $y = 2^x$  meets the curve with equation  $y = 4^x - 6$  at the point  $P$ .

(b) Using algebra, find the exact coordinates of  $P$ . **(4)**

**Question 2 continued**



**Diagram 1**

## Worked Solution - Question 2

### 1. Describe $y$ equals two to the $x$

$y = 2^x$  is an increasing exponential curve. It crosses the  $y$ -axis at  $(0, 1)$  and has no  $x$ -axis intercept.

### 2. Describe $y$ equals four to the $x$ minus six

$y = 4^x - 6$  is the graph of  $y = 4^x$  translated 6 units down. It crosses the  $y$ -axis at  $(0, -5)$ .

### 3. Find the $x$ -axis intercept

For  $y = 4^x - 6$ , set  $4^x - 6 = 0$ . Hence  $4^x = 6$  and  $x = \log_4 6$ .

### 4. Set up the intersection equation

At  $P$ ,  $2^x = 4^x - 6$ .

### 5. Use a substitution

Let  $u = 2^x$ . Then  $4^x = (2^x)^2 = u^2$ , so  $u = u^2 - 6$ .

### 6. Solve for $u$

$u^2 - u - 6 = 0$ , so  $(u - 3)(u + 2) = 0$ . Since  $u = 2^x > 0$ ,  $u = 3$ .

### 7. State $P$

$2^x = 3$ , so  $x = \log_2 3$ , and  $y = 3$ . Therefore  $P = (\log_2 3, 3)$ .

**Final answer**

(a)(i) Sketch  $y = 2^x$  through  $(0, 1)$  with no  $x$ -axis intercept.

(a)(ii) Sketch  $y = 4^x - 6$  through  $(0, -5)$  and  $(\log_4 6, 0)$ .

(b)  $P = (\log_2 3, 3)$ .

## Question 3

Proof

3. (i) Prove that for all single digit prime numbers,  $p$ ,

$$p^3 + p \text{ is a multiple of } 10 \quad (2)$$

- (ii) Show, using algebra, that for  $n \in \mathbb{N}$

$$(n + 1)^3 - n^3 \text{ is not a multiple of } 3 \quad (3)$$

## Worked Solution - Question 3

Topic group

### 1. List the cases

The single digit prime numbers are **2, 3, 5, and 7**.

### 2. Check each prime

$$2^3 + 2 = 10, 3^3 + 3 = 30, 5^3 + 5 = 130, \text{ and } 7^3 + 7 = 350.$$

### 3. Conclude part one

Each result is a multiple of **10**, so  $p^3 + p$  is a multiple of **10** for all single digit primes.

### 4. Expand the difference of cubes

$$(n + 1)^3 - n^3 = (n^3 + 3n^2 + 3n + 1) - n^3 = 3n^2 + 3n + 1.$$

### 5. Factor the multiple of three

$$3n^2 + 3n + 1 = 3n(n + 1) + 1.$$

### 6. Use the remainder

$3n(n + 1)$  is a multiple of **3**, so  $3n(n + 1) + 1$  is one more than a multiple of **3**. Therefore it is not a multiple of **3**.

#### Final answer

(i) Proven by checking  $p = 2, 3, 5, 7$ .

(ii) Proven:  $(n + 1)^3 - n^3$  leaves remainder **1** on division by **3**.

## Question 4

## Binomial Expansion

4. (a) Find, in ascending powers of  $x$ , up to and including the term in  $x^3$ , the binomial expansion of

$$\left(2 + \frac{x}{8}\right)^{13}$$

fully simplifying each coefficient.

(4)

- (b) Use the answer to part (a) to find an approximation for  $2.0125^{13}$

Give your answer to 3 decimal places.

(3)

Without calculating  $2.0125^{13}$

- (c) state, with a reason, whether the answer to part (b) is an overestimate or an underestimate.

(1)

## Worked Solution - Question 4

**1. Write the binomial terms**

$$\left(2 + \frac{x}{8}\right)^{13} = 2^{13} + \binom{13}{1}2^{12}\frac{x}{8} + \binom{13}{2}2^{11}\left(\frac{x}{8}\right)^2 + \binom{13}{3}2^{10}\left(\frac{x}{8}\right)^3 + \dots$$

**2. Simplify the expansion**

This gives  $8192 + 6656x + 2496x^2 + 572x^3 + \dots$ .

**3. Match the value**

Since  $2.0125 = 2 + 0.0125 = 2 + \frac{0.1}{8}$ , use  $x = 0.1$  in the expansion.

**4. Calculate the approximation**

$$8192 + 6656(0.1) + 2496(0.1)^2 + 572(0.1)^3 = 8883.132.$$

**5. Decide the direction**

The next terms in the expansion are positive because  $x = 0.1 > 0$ , so stopping after the  $x^3$  term gives an underestimate.

**Final answer**

(a)  $8192 + 6656x + 2496x^2 + 572x^3$ . (b) 8883.132. (c)

Underestimate.

## Question 5

5.

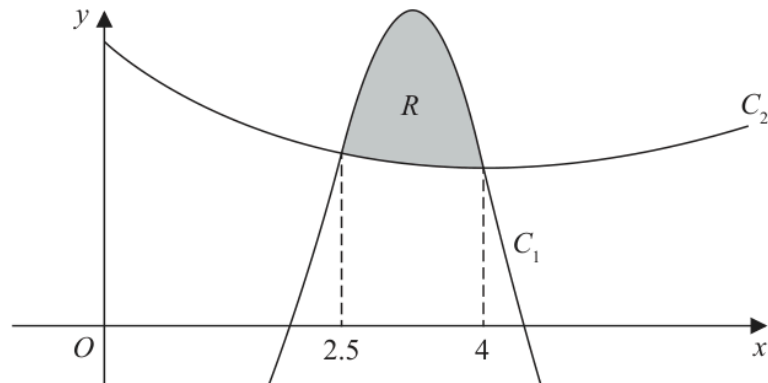


Figure 2

Figure 2 shows a sketch of part of the graph of the curves  $C_1$  and  $C_2$

The curves intersect when  $x = 2.5$  and when  $x = 4$

A table of values for some points on the curve  $C_1$  is shown below, with  $y$  values given to 3 decimal places as appropriate.

$x$	2.5	2.75	3	3.25	3.5	3.75	4
$y$	5.453	7.764	9.375	9.964	9.367	7.626	5

Using the trapezium rule with all the values of  $y$  in the table,

- (a) find, to 2 decimal places, an estimate for the area bounded by the curve  $C_1$ , the line with equation  $x = 2.5$ , the  $x$ -axis and the line with equation  $x = 4$
- (4)

The curve  $C_2$  has equation

$$y = x^{\frac{3}{2}} - 3x + 9 \quad x > 0$$

- (b) Find  $\int \left( x^{\frac{3}{2}} - 3x + 9 \right) dx$
- (3)

The region  $R$ , shown shaded in Figure 2, is bounded by the curves  $C_1$  and  $C_2$

- (c) Use the answers to part (a) and part (b) to find, to one decimal place, an estimate for the area of the region  $R$ .
- (3)

# Worked Solution - Question 5

## 1. Identify the strip width

The  $x$  values increase by **0.25**, so  $h = 0.25$ .

## 2. Apply the trapezium rule

The area under  $C_1$  is approximately

$$\frac{0.25}{2} \{5.453 + 5 + 2(7.764 + 9.375 + 9.964 + 9.367 + 7.626)\}.$$

## 3. Calculate the estimate

This gives **12.330625**, so the estimate is **12.33** to 2 decimal places.

## 4. Integrate C two

$$\int (x^{3/2} - 3x + 9) dx = \frac{2}{5}x^{5/2} - \frac{3}{2}x^2 + 9x + c.$$

## 5. Find the area under C two

Between  $x = 2.5$  and  $x = 4$ , the exact integral of  $C_2$  is

$$\left[ \frac{2}{5}x^{5/2} - \frac{3}{2}x^2 + 9x \right]_{2.5}^4 = 7.722 \dots$$

## 6. Find the shaded area

The shaded region is between  $C_1$  and  $C_2$ , so its area is approximately

$$12.330625 - 7.722 \dots = 4.608 \dots$$

## 7. Round the result

To one decimal place, the area of region  $R$  is **4.6**.

**Final answer**

(a) 12.33. (b)  $\frac{2}{5}x^{5/2} - \frac{3}{2}x^2 + 9x + c$ . (c) 4.6.

## Question 6

6. A circle has equation

$$x^2 - 6x + y^2 + 8y + k = 0$$

where  $k$  is a positive constant.

Given that the  $x$ -axis is a tangent to this circle,

- (a) find the value of  $k$ .

(3)

The circle meets the coordinate axes at the points  $R$ ,  $S$  and  $T$ .

- (b) Find the exact area of the triangle  $RST$ .

(4)

## Worked Solution - Question 6

**1. Complete the square**

$$x^2 - 6x = (x - 3)^2 - 9 \text{ and } y^2 + 8y = (y + 4)^2 - 16.$$

**2. Write centre-radius form**

The circle becomes  $(x - 3)^2 + (y + 4)^2 = 25 - k$ .

**3. Use the tangent condition**

The centre is  $(3, -4)$ . Since the  $x$ -axis is tangent, the radius is the distance from the centre to the  $x$ -axis, which is 4.

**4. Find  $k$** 

Thus  $25 - k = 4^2 = 16$ , so  $k = 9$ .

**5. Find the intercepts**

With  $k = 9$ , the circle is  $(x - 3)^2 + (y + 4)^2 = 16$ . On the  $x$ -axis,  $y = 0$ , giving  $x = 3$ .

**6. Find the  $y$ -axis points**

On the  $y$ -axis,  $x = 0$ , so  $9 + (y + 4)^2 = 16$ . Hence  $y = -4 \pm \sqrt{7}$ .

**7. Calculate the triangle area**

The vertical base on the  $y$ -axis has length  $2\sqrt{7}$ , and its perpendicular distance to  $(3, 0)$  is 3. The area is  $\frac{1}{2}(2\sqrt{7})(3) = 3\sqrt{7}$ .

**Final answer**

(a)  $k = 9$ . (b)  $3\sqrt{7}$ .

**Question 7****Laws of Logarithms**

7. (a) Given that

$$3 \log_3(2x - 1) = 2 + \log_3(14x - 25)$$

show that

$$2x^3 - 3x^2 - 30x + 56 = 0 \quad (4)$$

- (b) Show that  $-4$  is a root of this cubic equation. (2)

- (c) Hence, using algebra and showing each step of your working, solve

$$3 \log_3(2x - 1) = 2 + \log_3(14x - 25) \quad (4)$$

# Worked Solution - Question 7

**1. Use the power law**

$$3 \log_3(2x - 1) = \log_3(2x - 1)^3.$$

**2. Rewrite two as a logarithm**

Since  $2 = \log_3 9$ , the right side is

$$\log_3 9 + \log_3(14x - 25) = \log_3\{9(14x - 25)\}.$$

**3. Equate the arguments**

$$\text{So } (2x - 1)^3 = 9(14x - 25).$$

**4. Expand and simplify**

$$8x^3 - 12x^2 + 6x - 1 = 126x - 225, \text{ which simplifies to}$$
$$2x^3 - 3x^2 - 30x + 56 = 0.$$

**5. Check the given root**

Substituting  $x = -4$  gives  $2(-64) - 3(16) - 30(-4) + 56 = 0$ , so  $-4$  is a root.

**6. Factorise the cubic**

Dividing by  $(x + 4)$  gives  $2x^2 - 11x + 14 = (2x - 7)(x - 2)$ .

**7. Apply the logarithm domain**

The possible roots are  $-4$ ,  $2$ , and  $\frac{7}{2}$ . The original logarithms require  $2x - 1 > 0$  and  $14x - 25 > 0$ , so  $x > \frac{25}{14}$ .

**8. State the valid solutions**

Therefore the solutions are  $x = 2$  and  $x = \frac{7}{2}$ .

**Final answer**

(a) Proven.

(b)  $-4$  is a root of the cubic.

(c)  $x = 2, \frac{7}{2}$ .

## Question 8

## Trigonometric Equations

8. In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

(i) Solve, for  $0 < \theta < 360^\circ$ , the equation

$$3 \sin(\theta + 30^\circ) = 7 \cos(\theta + 30^\circ)$$

giving your answers to one decimal place.

(4)

(ii) (a) Show that the equation

$$3 \sin^3 x = 5 \sin x - 7 \sin x \cos x$$

can be written in the form

$$\sin x (a \cos^2 x + b \cos x + c) = 0$$

where  $a$ ,  $b$  and  $c$  are constants to be found.

(b) Hence solve for  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$  the equation

$$3 \sin^3 x = 5 \sin x - 7 \sin x \cos x$$

(6)

# Worked Solution - Question 8

Topic group

## 1. Solve for theta plus thirty

Let  $u = \theta + 30^\circ$ . Then  $3 \sin u = 7 \cos u$ , so  $\tan u = \frac{7}{3}$ .

## 2. Find u in the interval

Since  $0 < \theta < 360^\circ$ ,  $30^\circ < u < 390^\circ$ . Thus  $u = 66.801\dots^\circ$  or  $246.801\dots^\circ$ .

## 3. Return to theta

$\theta = u - 30^\circ$ , giving  $\theta = 36.8^\circ$  and  $216.8^\circ$  to one decimal place.

## 4. Rearrange the second equation

$3 \sin^3 x = 5 \sin x - 7 \sin x \cos x$  gives  $3 \sin^3 x - 5 \sin x + 7 \sin x \cos x = 0$ .

## 5. Factor out sine

$\sin x(3 \sin^2 x - 5 + 7 \cos x) = 0$ .

## 6. Use sin squared

Since  $\sin^2 x = 1 - \cos^2 x$ , this becomes  $\sin x(-3 \cos^2 x + 7 \cos x - 2) = 0$ , or  $\sin x(3 \cos^2 x - 7 \cos x + 2) = 0$ .

## 7. Solve in the interval

For  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ ,  $\sin x = 0$  gives  $x = 0$ . Also  $3 \cos^2 x - 7 \cos x + 2 = 0$  gives  $(3 \cos x - 1)(\cos x - 2) = 0$ .

## 8. State the solutions

$\cos x = 2$  is impossible, so  $\cos x = \frac{1}{3}$ . The solutions are  $x = -\cos^{-1}\left(\frac{1}{3}\right)$ ,  $0$ , and  $\cos^{-1}\left(\frac{1}{3}\right)$ .

**Final answer**

(i)  $\theta = 36.8^\circ, 216.8^\circ$ . (ii)(a)  $a = 3, b = -7, c = 2$ . (ii)(b)  $x = -\cos^{-1}\left(\frac{1}{3}\right), 0, \cos^{-1}\left(\frac{1}{3}\right)$

## Question 9

## Applications of Differentiation

9.

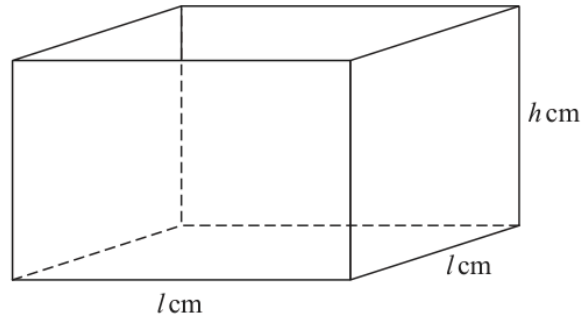


Figure 3

Figure 3 shows a sketch of a square based, open top box.

The height of the box is  $h$  cm, and the base edges each have length  $l$  cm.

Given that the volume of the box is  $250\,000$  cm<sup>3</sup>

(a) show that the external surface area,  $S$  cm<sup>2</sup>, of the box is given by

$$S = \frac{250\,000}{h} + 2000\sqrt{h} \quad (3)$$

(b) Use algebraic differentiation to show that  $S$  has a stationary point when  $h = 250^k$  where  $k$  is a rational constant to be found. (5)

(c) Justify by further differentiation that this value of  $h$  gives the minimum external surface area of the box. (2)

(Total 10 marks)

# Worked Solution - Question 9

Topic group

**1. Use the volume**

The volume is  $l^2h = 250000$ , so  $l^2 = \frac{250000}{h}$ .

**2. Write the surface area**

The open box has one square base and four side faces, so  $S = l^2 + 4lh$ .

**3. Substitute l**

Since  $l = \sqrt{\frac{250000}{h}} = \frac{500}{\sqrt{h}}$ ,  $4lh = 4 \cdot \frac{500}{\sqrt{h}} \cdot h = 2000\sqrt{h}$ .

**4. Show S**

Therefore  $S = \frac{250000}{h} + 2000\sqrt{h}$ .

**5. Differentiate S**

$S = 250000h^{-1} + 2000h^{1/2}$ , so  $\frac{dS}{dh} = -250000h^{-2} + 1000h^{-1/2}$ .

**6. Set the derivative to zero**

$-250000h^{-2} + 1000h^{-1/2} = 0$ , so  $1000h^{-1/2} = 250000h^{-2}$ .

**7. Solve for h**

Multiplying by  $h^2$  gives  $1000h^{3/2} = 250000$ , so  $h^{3/2} = 250$  and  $h = 250^{2/3}$ .

**8. Find k**

Since  $h = 250^k$ , we have  $k = \frac{2}{3}$ .

### 9. Use the second derivative

$$\frac{d^2 S}{dh^2} = 500000h^{-3} - 500h^{-3/2}.$$

### 10. Check the sign

At  $h = 250^{2/3}$ ,  $h^{3/2} = 250$ , so the second derivative is

$$\frac{500000}{250^2} - \frac{500}{250} = 8 - 2 = 6 > 0.$$

### 11. Conclude

The stationary point is therefore a minimum.

#### Final answer

(a) Proven:  $S = \frac{250000}{h} + 2000\sqrt{h}$ .

(b)  $h = 250^{2/3}$ , so  $k = \frac{2}{3}$ .

(c) Minimum.

**PAST PAPER**

# **WMA12/01 October 2021**

**October 2021 | 10 questions | 75 marks**

**10**

questions

**75**

marks

**Answers**

worked solution  
after each  
question

## Question 1

## Binomial Expansion

1. The first three terms, in ascending powers of  $x$ , of the binomial expansion of  $(1 + kx)^{16}$  are

$$1, -4x \text{ and } px^2$$

where  $k$  and  $p$  are constants.

- (a) Find, in simplest form,

(i) the value of  $k$

(ii) the value of  $p$

(3)

$$g(x) = \left(2 + \frac{16}{x}\right)(1 + kx)^{16}$$

Using the value of  $k$  found in part (a),

- (b) find the term in  $x^2$  in the expansion of  $g(x)$ .

(3)

## Worked Solution - Question 1

**1. Compare the linear term**

In  $(1 + kx)^{16}$ , the coefficient of  $x$  is  $16k$ .

**2. Find  $k$** 

Since the coefficient of  $x$  is  $-4$ ,  $16k = -4$ , so  $k = -\frac{1}{4}$ .

**3. Find  $p$** 

The coefficient of  $x^2$  is  $\binom{16}{2}k^2 = 120 \left(\frac{1}{16}\right) = \frac{15}{2}$ .

**4. Find the needed binomial terms**

For  $k = -\frac{1}{4}$ , the  $x^2$  coefficient in  $(1 + kx)^{16}$  is  $\frac{15}{2}$  and the  $x^3$  coefficient is  $\binom{16}{3} \left(-\frac{1}{4}\right)^3 = -\frac{35}{4}$ .

**5. Build the  $x$  squared term in  $g$** 

In  $g(x) = \left(2 + \frac{16}{x}\right)(1 + kx)^{16}$ , the  $x^2$  term comes from  $2 \left(\frac{15}{2}x^2\right)$  and  $\frac{16}{x} \left(-\frac{35}{4}x^3\right)$ .

**6. Simplify**

The required term is  $15x^2 - 140x^2 = -125x^2$ .

**Final answer**

$$(a)(i) k = -\frac{1}{4}. \quad (a)(ii) p = \frac{15}{2}. \quad (b) -125x^2.$$

## Question 2

## Sequences &amp; Series

2. A sequence is defined by

$$\begin{aligned}u_1 &= 6 \\ u_{n+1} &= ku_n + 3\end{aligned}$$

where  $k$  is a positive constant.

(a) Find, in terms of  $k$ , an expression for  $u_3$

(2)

Given that  $\sum_{n=1}^3 u_n = 117$

(b) find the value of  $k$ .

(3)

## Worked Solution - Question 2

Topic group

**1. Find  $u$  two**

$$u_2 = ku_1 + 3 = 6k + 3.$$

**2. Find  $u$  three**

$$u_3 = ku_2 + 3 = k(6k + 3) + 3 = 6k^2 + 3k + 3.$$

**3. Use the given sum**

$$u_1 + u_2 + u_3 = 117, \text{ so } 6 + (6k + 3) + (6k^2 + 3k + 3) = 117.$$

**4. Simplify the equation**

$$6k^2 + 9k + 12 = 117, \text{ hence } 2k^2 + 3k - 35 = 0.$$

**5. Solve for  $k$** 

$$(2k - 7)(k + 5) = 0. \text{ Since } k \text{ is positive, } k = \frac{7}{2}.$$

**Final answer**

$$(a) u_3 = 6k^2 + 3k + 3. \quad (b) k = \frac{7}{2}.$$

## Question 3

3.

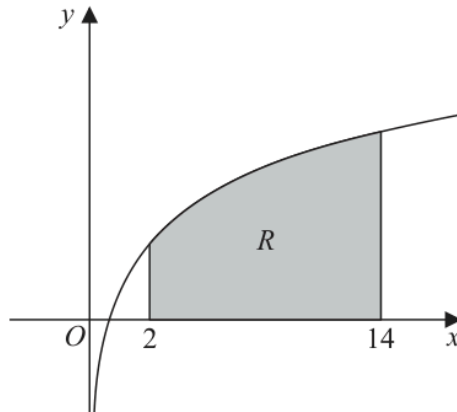


Figure 1

Figure 1 shows a sketch of part of the curve with equation  $y = \log_{10} x$

The region  $R$ , shown shaded in Figure 1, is bounded by the curve, the line with equation  $x = 2$ , the  $x$ -axis and the line with equation  $x = 14$

Using the trapezium rule with four strips of equal width,

(a) show that the area of  $R$  is approximately 10.10 (3)

(b) Explain how the trapezium rule could be used to obtain a more accurate estimate for the area of  $R$ . (1)

(c) Using the answer to part (a) and making your method clear, estimate the value of

(i)  $\int_2^{14} \log_{10} \sqrt{x} \, dx$

(ii)  $\int_2^{14} \log_{10} 100x^3 \, dx$

(4)

# Worked Solution - Question 3

**1. Find the strip width**

Four equal strips from  $x = 2$  to  $x = 14$  have width  $h = \frac{14 - 2}{4} = 3$ .

**2. List the ordinates**

Use  $x = 2, 5, 8, 11, 14$  and  $y = \log_{10} x$ .

**3. Apply the trapezium rule**

$$A \approx \frac{3}{2} \{ \log_{10} 2 + \log_{10} 14 + 2(\log_{10} 5 + \log_{10} 8 + \log_{10} 11) \} = 10.10 \dots$$

**4. Improve the estimate**

Use more strips of smaller width to make the trapezium rule estimate more accurate.

**5. Estimate the square-root integral**

$$\log_{10} \sqrt{x} = \frac{1}{2} \log_{10} x, \text{ so } \int_2^{14} \log_{10} \sqrt{x} dx \approx \frac{1}{2} (10.10) = 5.05.$$

**6. Rewrite the second logarithm**

$$\log_{10}(100x^3) = \log_{10} 100 + \log_{10} x^3 = 2 + 3 \log_{10} x.$$

**7. Estimate the second integral**

$$\int_2^{14} (2 + 3 \log_{10} x) dx \approx 2(12) + 3(10.10) = 54.30.$$

**Final answer**

(a) 10.10. (b) Use more strips.

(c)(i) 5.05. (c)(ii) 54.30.

## Question 4

## Polynomials

4.

$$f(x) = (x^2 - 2)(2x - 3) - 21$$

- (a) State the value of the remainder when  $f(x)$  is divided by  $(2x - 3)$  (1)
- (b) Use the factor theorem to show that  $(x - 3)$  is a factor of  $f(x)$  (2)
- (c) Hence,
- (i) factorise  $f(x)$
- (ii) show that the equation  $f(x) = 0$  has only one real root. (5)

## Worked Solution - Question 4

**1. Use the remainder theorem**

Dividing by  $(2x - 3)$  means use  $x = \frac{3}{2}$ .

**2. Find the remainder**

$$f\left(\frac{3}{2}\right) = \left(\frac{9}{4} - 2\right)(0) - 21 = -21.$$

**3. Check  $x$  equals three**

$$f(3) = (9 - 2)(6 - 3) - 21 = 7 \cdot 3 - 21 = 0.$$

**4. Use the factor theorem**

Since  $f(3) = 0$ ,  $(x - 3)$  is a factor of  $f(x)$ .

**5. Expand  $f$** 

$$f(x) = (x^2 - 2)(2x - 3) - 21 = 2x^3 - 3x^2 - 4x - 15.$$

**6. Factorise**

Dividing by  $(x - 3)$  gives  $2x^2 + 3x + 5$ , so  $f(x) = (x - 3)(2x^2 + 3x + 5)$ .

**7. Check the quadratic roots**

For  $2x^2 + 3x + 5$ , the discriminant is  $3^2 - 4(2)(5) = 9 - 40 = -31 < 0$ .

**8. Conclude**

The quadratic has no real roots, so  $f(x) = 0$  has only one real root,  $x = 3$ .

**Final answer**

(a)  $-21$ . (b) Proven.

(c)(i)  $f(x) = (x - 3)(2x^2 + 3x + 5)$ . (c)(ii) Only real root:  $x = 3$ .

## Question 5

## Arithmetic Sequences

5. A company that owned a silver mine
- extracted 480 tonnes of silver from the mine in year 1
  - extracted 465 tonnes of silver from the mine in year 2
  - extracted 450 tonnes of silver from the mine in year 3

and so on, forming an arithmetic sequence.

- (a) Find the mass of silver extracted in year 14 (2)

After a total of 7770 tonnes of silver was extracted, the company stopped mining.

Given that this occurred at the end of year  $N$ ,

- (b) show that

$$N^2 - 65N + 1036 = 0 \quad (3)$$

- (c) Hence, state the value of  $N$ . (1)

# Worked Solution - Question 5

## 1. Identify the arithmetic sequence

The first term is 480 and the common difference is  $465 - 480 = -15$ .

## 2. Find year fourteen

$u_{14} = 480 + 13(-15) = 285$ , so 285 tonnes were extracted in year 14.

## 3. Set up the total

The total after  $N$  years is  $S_N = \frac{N}{2} \{2(480) + (N - 1)(-15)\}$ .

## 4. Use the total extraction

$$\frac{N}{2}(975 - 15N) = 7770.$$

## 5. Show the quadratic

Multiplying by 2 and rearranging gives  $15N^2 - 975N + 15540 = 0$ , so  $N^2 - 65N + 1036 = 0$ .

## 6. Solve and choose N

$N = 28$  or  $37$ . Since extraction amounts would become negative before year 37, the stopping year is  $N = 28$ .

### Final answer

(a) 285 tonnes.

(b) Proven:  $N^2 - 65N + 1036 = 0$ .

(c)  $N = 28$ .

## Question 6

6. (i) The circle  $C_1$  has equation

$$x^2 + y^2 + 10x - 12y = k \quad \text{where } k \text{ is a constant}$$

- (a) Find the coordinates of the centre of  $C_1$  (2)
- (b) State the possible range in values for  $k$ . (2)
- (ii) The point  $P(p, 0)$ , the point  $Q(-2, 10)$  and the point  $R(8, -14)$  lie on a different circle,  $C_2$

Given that

- $p$  is a positive constant
- $QR$  is a diameter of  $C_2$

find the exact value of  $p$ .

(4)

## Worked Solution - Question 6

**1. Complete the square**

$$x^2 + 10x = (x + 5)^2 - 25 \text{ and } y^2 - 12y = (y - 6)^2 - 36.$$

**2. Write centre-radius form**

The circle becomes  $(x + 5)^2 + (y - 6)^2 = k + 61$ .

**3. Read the centre**

The centre is  $(-5, 6)$ .

**4. State the range for  $k$** 

For a circle with positive radius,  $k + 61 > 0$ , so  $k > -61$ .

**5. Use the diameter property**

If  $QR$  is a diameter and  $P$  lies on the circle, then  $\angle QPR = 90^\circ$ .

**6. Use perpendicular vectors**

So  $(Q - P) \cdot (R - P) = 0$ . With  $P = (p, 0)$ ,  $Q = (-2, 10)$ , and  $R = (8, -14)$ , this gives  $(-2 - p, 10) \cdot (8 - p, -14) = 0$ .

**7. Form the quadratic**

$$(-2 - p)(8 - p) - 140 = 0, \text{ so } p^2 - 6p - 156 = 0.$$

**8. Choose the positive value**

$$p = \frac{6 \pm \sqrt{660}}{2} = 3 \pm \sqrt{165}. \text{ Since } p \text{ is positive, } p = 3 + \sqrt{165}.$$

**Final answer**

*(i)(a)*  $(-5, 6)$ . *(i)(b)*  $k > -61$ . *(ii)*  $p = 3 + \sqrt{165}$ .

## Question 7

## Geometric Sequences

7. (i) A geometric sequence has first term 4 and common ratio 6  
Given that the  $n^{\text{th}}$  term is greater than  $10^{100}$ , find the minimum possible value of  $n$ . (3)

- (ii) A different geometric sequence has first term  $a$  and common ratio  $r$ .

Given that

- the second term of the sequence is  $-6$
- the sum to infinity of the series is 25

- (a) show that

$$25r^2 - 25r - 6 = 0 \quad (3)$$

- (b) Write down the solutions of

$$25r^2 - 25r - 6 = 0 \quad (1)$$

Hence,

- (c) state the value of  $r$ , giving a reason for your answer, (1)

- (d) find the sum of the first 4 terms of the series. (2)

# Worked Solution - Question 7

**1. Write the  $n$ th term**

The  $n$ th term of the first sequence is  $4 \cdot 6^{n-1}$ .

**2. Use logarithms**

We need  $4 \cdot 6^{n-1} > 10^{100}$ , so  $n - 1 > \frac{100 - \log_{10} 4}{\log_{10} 6} = 127.736 \dots$

**3. Choose the smallest integer**

Thus  $n$  must be greater than  $128.736 \dots$ , so the minimum possible value is **129**.

**4. Use the second term**

For the second sequence,  $ar = -6$ , so  $a = -\frac{6}{r}$ .

**5. Use the sum to infinity**

$\frac{a}{1-r} = 25$ . Substituting  $a = -\frac{6}{r}$  gives  $-6 = 25r(1-r)$ .

**6. Show the quadratic**

Rearranging gives  $25r^2 - 25r - 6 = 0$ .

**7. Solve for  $r$** 

$25r^2 - 25r - 6 = 0$  gives  $r = \frac{6}{5}$  or  $r = -\frac{1}{5}$ .

**8. Choose the valid ratio**

A sum to infinity exists only when  $|r| < 1$ , so  $r = -\frac{1}{5}$ .

**9. Find the first term**

Since  $ar = -6$ ,  $a = 30$ .

### 10. Sum four terms

$$S_4 = 30 \frac{1 - (-1/5)^4}{1 - (-1/5)} = \frac{624}{25}.$$

#### Final answer

(i)  $n = 129$ . (ii)(a) Proven.

(ii)(b)  $r = \frac{6}{5}$  or  $-\frac{1}{5}$ . (ii)(c)  $r = -\frac{1}{5}$ . (ii)(d)  $\frac{624}{25}$ .

## Question 8

## Applications of Differentiation

8. In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

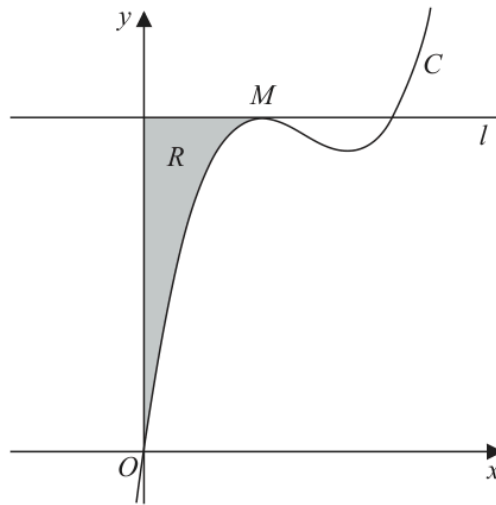


Figure 2

Figure 2 shows a sketch of part of the curve  $C$  with equation

$$y = \frac{4}{3}x^3 - 11x^2 + kx \quad \text{where } k \text{ is a constant}$$

The point  $M$  is the maximum turning point of  $C$  and is shown in Figure 2.

Given that the  $x$  coordinate of  $M$  is 2

(a) show that  $k = 28$  (3)

(b) Determine the range of values of  $x$  for which  $y$  is increasing. (2)

The line  $l$  passes through  $M$  and is parallel to the  $x$ -axis.

The region  $R$ , shown shaded in Figure 2, is bounded by the curve  $C$ , the line  $l$  and the  $y$ -axis.

(c) Find, by algebraic integration, the exact area of  $R$ . (5)

# Worked Solution - Question 8

Topic group

**1. Differentiate the curve**

$$\text{For } y = \frac{4}{3}x^3 - 11x^2 + kx, \quad \frac{dy}{dx} = 4x^2 - 22x + k.$$

**2. Use the turning point**

$$\text{At the turning point } M, \quad x = 2 \text{ and } \frac{dy}{dx} = 0.$$

**3. Show  $k$** 

$$4(2)^2 - 22(2) + k = 0, \text{ so } 16 - 44 + k = 0 \text{ and } k = 28.$$

**4. Factor the derivative**

$$\text{With } k = 28, \quad \frac{dy}{dx} = 4x^2 - 22x + 28 = 2(2x - 7)(x - 2).$$

**5. Find where  $y$  is increasing**

The derivative is positive outside its roots, so  $y$  is increasing for  $x < 2$  or  $x > \frac{7}{2}$ .

**6. Find the height of the line**

$$\text{At } x = 2, \quad y = \frac{4}{3}(8) - 11(4) + 28(2) = \frac{68}{3}. \text{ Hence line } l \text{ has equation } y = \frac{68}{3}.$$

**7. Set up the shaded area**

The region is between the line and the curve from  $x = 0$  to  $x = 2$ , so

$$A = \int_0^2 \left( \frac{68}{3} - \left( \frac{4}{3}x^3 - 11x^2 + 28x \right) \right) dx.$$

**8. Integrate**

$$A = \left[ \frac{68}{3}x - \frac{1}{3}x^4 + \frac{11}{3}x^3 - 14x^2 \right]_0^2.$$

### 9. Evaluate exactly

Substituting  $x = 2$  gives  $\frac{136}{3} - \frac{16}{3} + \frac{88}{3} - 56 = \frac{40}{3}$ .

#### Final answer

(a) Proven:  $k = 28$ .

(b)  $x < 2$  or  $x > \frac{7}{2}$ . (c)  $\frac{40}{3}$ .

## Question 9

Proof

9. (a) Prove that for all positive values of  $x$  and  $y$ ,

$$\frac{x + y}{2} \geq \sqrt{xy} \quad (3)$$

- (b) Prove by counter-example that this inequality does not hold when  $x$  and  $y$  are both negative.

(1)

# Worked Solution - Question 9

Topic group

## 1. Start with a square

For positive  $x$  and  $y$ , both  $\sqrt{x}$  and  $\sqrt{y}$  are real, so  $(\sqrt{x} - \sqrt{y})^2 \geq 0$ .

## 2. Expand

$$(\sqrt{x} - \sqrt{y})^2 = x - 2\sqrt{xy} + y \geq 0.$$

## 3. Rearrange

Thus  $x + y \geq 2\sqrt{xy}$ , and dividing by 2 gives  $\frac{x + y}{2} \geq \sqrt{xy}$ .

## 4. Give a negative counterexample

Take  $x = -1$  and  $y = -1$ . Then  $\frac{x + y}{2} = -1$ , while  $\sqrt{xy} = \sqrt{1} = 1$ .

## 5. Conclude

Since  $-1 \geq 1$  is false, the inequality does not hold for all negative  $x$  and  $y$ .

### Final answer

(a) Proven.

(b) For example,  $x = y = -1$  gives  $-1 \not\geq 1$ .

## Question 10

## Trigonometric Equations

10. In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

- (i) Solve, for  $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$$\tan^2\left(2x + \frac{\pi}{4}\right) = 3 \quad (5)$$

- (ii) Solve, for  $0 < \theta < 360^\circ$

$$(2 \sin \theta - \cos \theta)^2 = 1$$

giving your answers, as appropriate, to one decimal place.

(5)

**(Total 10 marks)**

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# Worked Solution - Question 10

**1. Take the square root**

Let  $u = 2x + \frac{\pi}{4}$ . From  $\tan^2 u = 3$ ,  $\tan u = \pm\sqrt{3}$ .

**2. Write the angle families**

Thus  $u = \frac{\pi}{3} + n\pi$  or  $u = -\frac{\pi}{3} + n\pi$ , where  $n$  is an integer.

**3. Return to  $x$** 

So  $x = \frac{\pi}{24} + \frac{n\pi}{2}$  or  $x = -\frac{7\pi}{24} + \frac{n\pi}{2}$ .

**4. Use the interval**

For  $-\frac{\pi}{2} < x < \frac{\pi}{2}$ , the solutions are  $x = -\frac{11\pi}{24}, -\frac{7\pi}{24}, \frac{\pi}{24}, \frac{5\pi}{24}$ .

**5. Expand the squared equation**

$(2 \sin \theta - \cos \theta)^2 = 1$  gives  $4 \sin^2 \theta - 4 \sin \theta \cos \theta + \cos^2 \theta = 1$ .

**6. Use the identity**

Since  $1 = \sin^2 \theta + \cos^2 \theta$ , this simplifies to  $3 \sin^2 \theta - 4 \sin \theta \cos \theta = 0$ .

**7. Factor**

$\sin \theta(3 \sin \theta - 4 \cos \theta) = 0$ .

**8. Solve the two cases**

$\sin \theta = 0$  gives  $\theta = 180^\circ$  in the interval. Also  $3 \sin \theta = 4 \cos \theta$  gives  $\tan \theta = \frac{4}{3}$ .

### 9. State theta

For  $0 < \theta < 360^\circ$ ,  $\tan \theta = \frac{4}{3}$  gives  $\theta = 53.1^\circ$  and  $233.1^\circ$ . Therefore  $\theta = 53.1^\circ, 180^\circ, 233.1^\circ$ .

#### Final answer

$$(i) x = -\frac{11\pi}{24}, -\frac{7\pi}{24}, \frac{\pi}{24}, \frac{5\pi}{24}. \quad (ii) \theta = 53.1^\circ, 180^\circ, 233.1^\circ.$$

**PAST PAPER**

# **WMA12/01 January 2022**

January 2022 | 10 questions | 75 marks

**10**

questions

**75**

marks

**Answers**

worked solution  
after each  
question

## Question 1

1. The table below shows corresponding values of  $x$  and  $y$  for

$$y = 2^{5-\sqrt{x}}$$

The values of  $y$  are given to 3 decimal places.

$x$	5	5.5	6	6.5	7
$y$	6.792	6.298	5.858	5.466	5.113

Using the trapezium rule with all the values of  $y$  in the given table,

- (a) obtain an estimate for

$$\int_5^7 2^{5-\sqrt{x}} dx$$

giving your answer to 2 decimal places.

(3)

- (b) Using your answer to part (a) and making your method clear, estimate

(i)  $\int_5^7 2^{6-\sqrt{x}} dx$

(ii)  $\int_5^7 (3 + 2^{5-\sqrt{x}}) dx$

(4)

# Worked Solution - Question 1

## 1. Identify the strip width

The  $x$  values increase by 0.5, so  $h = 0.5$ .

## 2. Apply the trapezium rule

$$\int_5^7 2^{5-\sqrt{x}} dx \approx \frac{0.5}{2} \{6.792 + 5.113 + 2(6.298 + 5.858 + 5.466)\}.$$

## 3. Calculate the estimate

This gives 11.78725, so the estimate is 11.79 to 2 decimal places.

## 4. Use the factor of two

$$2^{6-\sqrt{x}} = 2 \cdot 2^{5-\sqrt{x}}, \text{ so } \int_5^7 2^{6-\sqrt{x}} dx \approx 2(11.78725) = 23.5745.$$

## 5. Estimate part i

This is 23.57 to 2 decimal places.

## 6. Add the constant integral

$$\int_5^7 (3 + 2^{5-\sqrt{x}}) dx = \int_5^7 3 dx + \int_5^7 2^{5-\sqrt{x}} dx.$$

## 7. Estimate part ii

The estimate is  $3(7 - 5) + 11.78725 = 17.78725$ , so 17.79 to 2 decimal places.

### Final answer

(a) 11.79. (b)(i) 23.57. (b)(ii) 17.79.

## Question 2

## Applications of Differentiation

2. In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

The curve  $C$  has equation

$$y = 27x^{\frac{1}{2}} - x^{\frac{3}{2}} - 20 \quad x > 0$$

- (a) Find  $\frac{dy}{dx}$ , giving each term in simplest form. (2)
- (b) Hence find the coordinates of the stationary point of  $C$ . (4)
- (c) Find  $\frac{d^2y}{dx^2}$  and hence determine the nature of the stationary point of  $C$ . (2)

# Worked Solution - Question 2

Topic group

## 1. Differentiate

$$\text{For } y = 27x^{1/2} - x^{3/2} - 20, \frac{dy}{dx} = \frac{27}{2}x^{-1/2} - \frac{3}{2}x^{1/2}.$$

## 2. Set the gradient to zero

$$\frac{27}{2}x^{-1/2} - \frac{3}{2}x^{1/2} = 0.$$

## 3. Clear the powers

Multiplying by  $2x^{1/2}$  gives  $27 - 3x = 0$ , so  $x = 9$ .

## 4. Find $y$

$$\text{At } x = 9, y = 27\sqrt{9} - 9^{3/2} - 20 = 81 - 27 - 20 = 34.$$

## 5. State the stationary point

The stationary point is  $(9, 34)$ .

## 6. Differentiate again

$$\frac{d^2y}{dx^2} = -\frac{27}{4}x^{-3/2} - \frac{3}{4}x^{-1/2}.$$

## 7. Determine the nature

At  $x = 9$ , both terms in  $\frac{d^2y}{dx^2}$  are negative, so the stationary point is a local maximum.

**Final answer**

(a)  $\frac{dy}{dx} = \frac{27}{2}x^{-1/2} - \frac{3}{2}x^{1/2}$ . (b) (9, 34). (c) local maximum.

## Question 3

## Binomial Expansion

3. (a) Find the first 4 terms, in ascending powers of  $x$ , of the binomial expansion of

$$\left(2 - \frac{kx}{4}\right)^8$$

where  $k$  is a non-zero constant. Give each term in simplest form.

(4)

$$f(x) = (5 - 3x)\left(2 - \frac{kx}{4}\right)^8$$

In the expansion of  $f(x)$ , the constant term is 3 times the coefficient of  $x$ .

- (b) Find the value of  $k$ .

(3)

## Worked Solution - Question 3

**1. Write the first terms**

$$\left(2 - \frac{kx}{4}\right)^8 = 2^8 + \binom{8}{1}2^7\left(-\frac{kx}{4}\right) + \binom{8}{2}2^6\left(-\frac{kx}{4}\right)^2 + \binom{8}{3}2^5\left(-\frac{kx}{4}\right)^3 + \dots$$

**2. Simplify the expansion**

The first four terms are  $256 - 256kx + 112k^2x^2 - 28k^3x^3$ .

**3. Find the constant term in f**

For  $f(x) = (5 - 3x)\left(2 - \frac{kx}{4}\right)^8$ , the constant term is  $5(256) = 1280$ .

**4. Find the coefficient of x**

The coefficient of  $x$  is  $5(-256k) - 3(256) = -1280k - 768$ .

**5. Use the condition**

The constant term is 3 times the coefficient of  $x$ , so  $1280 = 3(-1280k - 768)$ .

**6. Solve for k**

$$1280 = -3840k - 2304, \text{ hence } k = -\frac{3584}{3840} = -\frac{14}{15}.$$

**Final answer**

$$(a) 256 - 256kx + 112k^2x^2 - 28k^3x^3. \quad (b) k = -\frac{14}{15}.$$

**Question 4****Laws of Logarithms**

4. Using the laws of logarithms, solve

$$\log_3(32 - 12x) = 2\log_3(1 - x) + 3$$

(5)

## Worked Solution - Question 4

**1. Rewrite the right side**

$$2\log_3(1-x) = \log_3(1-x)^2 \text{ and } 3 = \log_3 27.$$

**2. Combine the logarithms**

The equation becomes  $\log_3(32 - 12x) = \log_3\{27(1-x)^2\}$ .

**3. Equate the arguments**

$$32 - 12x = 27(1-x)^2.$$

**4. Form the quadratic**

$$32 - 12x = 27 - 54x + 27x^2, \text{ so } 27x^2 - 42x - 5 = 0.$$

**5. Solve**

$$x = \frac{42 \pm 48}{54}, \text{ giving } x = \frac{5}{3} \text{ or } x = -\frac{1}{9}.$$

**6. Check the domain**

The original logarithms require  $1-x > 0$ , so  $x < 1$ . Therefore  $x = \frac{5}{3}$  is invalid and  $x = -\frac{1}{9}$ .

**Final answer**

$$x = -\frac{1}{9}.$$

## Question 5

## Polynomials

5.

$$f(x) = 3x^3 + Ax^2 + Bx - 10$$

where  $A$  and  $B$  are integers.

Given that

- when  $f(x)$  is divided by  $(x - 1)$  the remainder is  $k$
- when  $f(x)$  is divided by  $(x + 1)$  the remainder is  $-10k$
- $k$  is a constant

(a) show that

$$11A + 9B = 83 \quad (3)$$

Given also that  $(3x - 2)$  is a factor of  $f(x)$ ,

(b) find the value of  $A$  and the value of  $B$ . (3)

(c) Hence find the quadratic expression  $g(x)$  such that

$$f(x) = (3x - 2)g(x) \quad (2)$$

## Worked Solution - Question 5

**1. Use  $x$  equals one**

$$f(1) = 3 + A + B - 10 = A + B - 7, \text{ so } k = A + B - 7.$$

**2. Use  $x$  equals negative one**

$$f(-1) = -3 + A - B - 10 = A - B - 13.$$

**3. Use the second remainder relation**

$$\text{Since } f(-1) = -10k, A - B - 13 = -10(A + B - 7).$$

**4. Show the relation**

$$\text{Rearranging gives } 11A + 9B = 83.$$

**5. Use the factor**

$$\text{Since } (3x - 2) \text{ is a factor, } f\left(\frac{2}{3}\right) = 0.$$

**6. Form the second equation**

$$3\left(\frac{2}{3}\right)^3 + A\left(\frac{2}{3}\right)^2 + B\left(\frac{2}{3}\right) - 10 = 0. \text{ Multiplying by 9 gives } 2A + 3B = 41$$

**7. Solve for A and B**

$$\text{Solving } 11A + 9B = 83 \text{ with } 2A + 3B = 41 \text{ gives } A = -8 \text{ and } B = 19.$$

**8. Divide by the factor**

$$\text{With these values, } f(x) = 3x^3 - 8x^2 + 19x - 10 = (3x - 2)(x^2 - 2x + 5).$$

**Final answer**

(a) Proven:  $11A + 9B = 83$ .

(b)  $A = -8$ ,  $B = 19$ . (c)  $g(x) = x^2 - 2x + 5$ .

## Question 6

6.

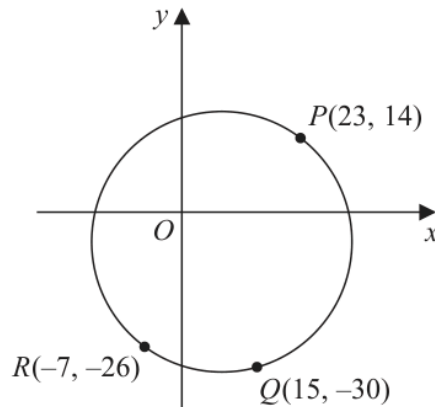


Figure 1

The points  $P(23, 14)$ ,  $Q(15, -30)$  and  $R(-7, -26)$  lie on the circle  $C$ , as shown in Figure 1.

- (a) Show that angle  $PQR = 90^\circ$  (2)
- (b) Hence, or otherwise, find
- (i) the centre of  $C$ ,
  - (ii) the radius of  $C$ . (3)

Given that the point  $S$  lies on  $C$  such that the distance  $QS$  is greatest,

- (c) find an equation of the tangent to  $C$  at  $S$ , giving your answer in the form  $ax + by + c = 0$ , where  $a$ ,  $b$  and  $c$  are integers to be found. (3)

## Worked Solution - Question 6

**1. Find the gradient  $QP$** 

The gradient of  $QP$  is  $\frac{14 - (-30)}{23 - 15} = \frac{44}{8} = \frac{11}{2}$ .

**2. Find the gradient  $QR$** 

The gradient of  $QR$  is  $\frac{-26 - (-30)}{-7 - 15} = \frac{4}{-22} = -\frac{2}{11}$ .

**3. Show the right angle**

The product of the two gradients is  $-1$ , so  $QP$  is perpendicular to  $QR$  and  $\angle PQR = 90^\circ$ .

**4. Use the diameter**

Since the angle at  $Q$  is a right angle,  $PR$  is a diameter of the circle.

**5. Find the centre**

The centre is the midpoint of  $P(23, 14)$  and  $R(-7, -26)$ , so it is  $(8, -6)$ .

**6. Find the radius**

$PR = \sqrt{30^2 + 40^2} = 50$ , so the radius is 25.

**7. Find the farthest point from  $Q$** 

The greatest distance from  $Q$  occurs at the point opposite  $Q$  on the diameter through the centre. Thus  $S = 2(8, -6) - (15, -30) = (1, 18)$ .

**8. Find the tangent gradient**

The radius from the centre to  $S$  has gradient  $\frac{18 - (-6)}{1 - 8} = -\frac{24}{7}$ , so the tangent gradient is  $\frac{7}{24}$ .

### 9. Write the tangent equation

$$y - 18 = \frac{7}{24}(x - 1), \text{ which rearranges to } 7x - 24y + 425 = 0.$$

#### Final answer

(a) Proven:  $\angle PQR = 90^\circ$ .

(b)(i)  $(8, -6)$ . (b)(ii) 25. (c)  $7x - 24y + 425 = 0$ .

**Question 7****Trigonometric Equations**

7. **In this question you must show all stages of your working.**

**Solutions relying entirely on calculator technology are not acceptable.**

- (i) Solve, for  $-90^\circ < x < 90^\circ$ , the equation

$$3 \sin(2x - 15^\circ) = \cos(2x - 15^\circ)$$

giving your answers to one decimal place.

**(4)**

- (ii) Solve, for  $0 < \theta < 2\pi$ , the equation

$$4 \sin^2 \theta + 8 \cos \theta = 3$$

giving your answers to 3 significant figures.

**(4)**

# Worked Solution - Question 7

**1. Solve the first tangent equation**

Let  $u = 2x - 15^\circ$ . Then  $3 \sin u = \cos u$ , so  $\tan u = \frac{1}{3}$ .

**2. Use the interval for  $u$** 

Since  $-90^\circ < x < 90^\circ$ ,  $-195^\circ < u < 165^\circ$ .

**3. Find  $x$** 

$u = 18.434\dots^\circ$  or  $u = -161.565\dots^\circ$ , so  $x = 16.7^\circ$  or  $x = -73.3^\circ$ .

**4. Rewrite the second equation**

$$4 \sin^2 \theta + 8 \cos \theta = 3.$$

**5. Use sin squared**

Substitute  $\sin^2 \theta = 1 - \cos^2 \theta$  to get  $4(1 - \cos^2 \theta) + 8 \cos \theta = 3$ .

**6. Solve for cos theta**

This gives  $4 \cos^2 \theta - 8 \cos \theta - 1 = 0$ , so  $\cos \theta = 1 \pm \frac{\sqrt{5}}{2}$ .

**7. Choose the valid cosine**

Only  $\cos \theta = 1 - \frac{\sqrt{5}}{2}$  lies between  $-1$  and  $1$ .

**8. State theta**

For  $0 < \theta < 2\pi$ ,  $\theta = 1.69$  or  $4.59$  to 3 significant figures.

**Final answer**

(i)  $x = -73.3^\circ, 16.7^\circ$ . (ii)  $\theta = 1.69, 4.59$ .

## Question 8

## Modelling with Sequences &amp; Series

8. A metal post is repeatedly hit in order to drive it into the ground.

Given that

- on the 1st hit, the post is driven 100 mm into the ground
- on the 2nd hit, the post is driven an **additional** 98 mm into the ground
- on the 3rd hit, the post is driven an **additional** 96 mm into the ground
- the **additional** distances the post travels on each subsequent hit form an arithmetic sequence

(a) show that the post is driven an **additional** 62 mm into the ground with the 20th hit. (1)

(b) Find the **total distance** that the post has been driven into the ground after 20 hits. (2)

Given that for each subsequent hit after the 20th hit

- the **additional** distances the post travels form a geometric sequence with common ratio  $r$
- on the 22nd hit, the post is driven an **additional** 60 mm into the ground

(c) find the value of  $r$ , giving your answer to 3 decimal places. (2)

After a total of  $N$  hits, the post will have been driven more than 3 m into the ground.

(d) Find, showing all steps in your working, the smallest possible value of  $N$ . (4)

# Worked Solution - Question 8

Topic group

## 1. Use the arithmetic rule

For the first 20 hits, the additional distances form an arithmetic sequence with first term 100 and common difference  $-2$ .

## 2. Show the twentieth hit

The twentieth additional distance is  $100 + 19(-2) = 62$  mm.

## 3. Sum the first twenty hits

$$S_{20} = \frac{20}{2}(100 + 62) = 1620 \text{ mm.}$$

## 4. Use the geometric information

After the twentieth hit, the distances follow the same ratio, so the twenty second distance is  $62r^2$ .

## 5. Find $r$

$$62r^2 = 60, \text{ hence } r = \sqrt{\frac{30}{31}} = 0.984 \text{ to 3 decimal places.}$$

## 6. Set up the remaining distance

More than 3 m means more than 3000 mm. After 20 hits, the remaining amount to exceed is  $3000 - 1620 = 1380$  mm.

## 7. Sum the geometric tail

If there are  $m$  further hits, their total is  $62r + 62r^2 + \dots + 62r^m = 62r \frac{1 - r^m}{1 - r}$ .

### 8. Find the first $m$ that works

Using  $r = \sqrt{30/31}$ ,  $62r \frac{1 - r^{27}}{1 - r}$  is not enough, but  $62r \frac{1 - r^{28}}{1 - r}$  makes the total exceed 3000 mm.

### 9. State $N$

Therefore  $m = 28$  further hits are needed after the first 20, so  
 $N = 20 + 28 = 48$ .

#### Final answer

(a) Proven: 62 mm.

(b) 1620 mm.

(c)  $r = 0.984$ . (d)  $N = 48$ .

## Question 9

9.

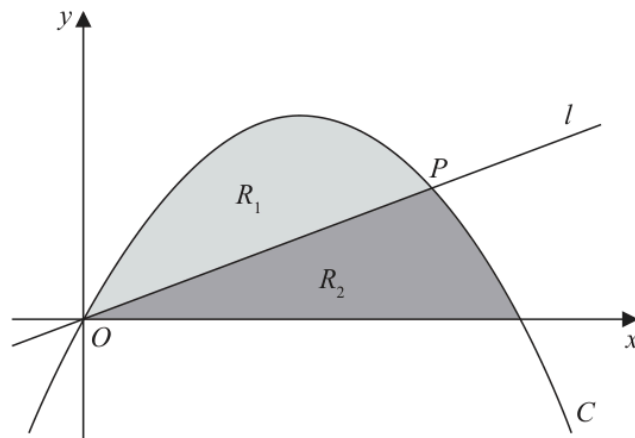


Figure 2

Figure 2 shows

- the curve  $C$  with equation  $y = x - x^2$
- the line  $l$  with equation  $y = mx$ , where  $m$  is a constant and  $0 < m < 1$

The line and the curve intersect at the origin  $O$  and at the point  $P$ .

- (a) Find, in terms of  $m$ , the coordinates of  $P$ . (2)

The region  $R_1$ , shown shaded in Figure 2, is bounded by  $C$  and  $l$ .

- (b) Show that the area of  $R_1$  is 
$$\frac{(1-m)^3}{6}$$
 (5)

The region  $R_2$ , also shown shaded in Figure 2, is bounded by  $C$ , the  $x$ -axis and  $l$ .

Given that the area of  $R_1$  is equal to the area of  $R_2$

- (c) find the exact value of  $m$ . (3)

## Worked Solution - Question 9

**1. Find the second intersection**

At an intersection,  $x - x^2 = mx$ , so  $x(1 - m - x) = 0$ .

**2. State P**

The non-zero intersection has  $x = 1 - m$ , and then  $y = m(1 - m)$ , so  $P = (1 - m, m(1 - m))$ .

**3. Set up area R one**

Region  $R_1$  is between the curve and the line from  $x = 0$  to  $x = 1 - m$ .

**4. Integrate the difference**

$$A_{R_1} = \int_0^{1-m} \{(x - x^2) - mx\} dx = \int_0^{1-m} \{(1 - m)x - x^2\} dx.$$

**5. Evaluate**

$$A_{R_1} = \left[ \frac{1 - m}{2} x^2 - \frac{1}{3} x^3 \right]_0^{1-m} = \frac{(1 - m)^3}{2} - \frac{(1 - m)^3}{3} = \frac{(1 - m)^3}{6}.$$

**6. Use the total curve area**

The total area under  $y = x - x^2$  from  $x = 0$  to  $x = 1$  is  $\int_0^1 (x - x^2) dx = \frac{1}{6}$ .

**7. Use equal regions**

Since  $R_1$  and  $R_2$  are equal and together make this total area, each has area  $\frac{1}{12}$ .

**8. Solve for m**

$$\frac{(1 - m)^3}{6} = \frac{1}{12}, \text{ so } (1 - m)^3 = \frac{1}{2} \text{ and } m = 1 - \sqrt[3]{\frac{1}{2}}.$$

**Final answer**

(a)  $P = (1 - m, m(1 - m))$ . (b) Proven:  $\frac{(1 - m)^3}{6}$ .

(c)  $m = 1 - \frac{1}{\sqrt[3]{2}}$ .

## Question 10

Proof

10. (i) Prove by counter example that the statement

“if  $p$  is a prime number then  $2p + 1$  is also a prime number”

is not true.

(1)

(ii) Use proof by exhaustion to prove that if  $n$  is an integer then

$$5n^2 + n + 12$$

is always even.

(4)

---

(Total 5 marks)

# Worked Solution - Question 10

Topic group

## 1. Give a counterexample

Take  $p = 7$ , which is prime.

## 2. Show the statement fails

$2p + 1 = 2(7) + 1 = 15$ , and 15 is not prime. So the statement is false.

## 3. Case one: $n$ even

If  $n$  is even, let  $n = 2k$ . Then

$5n^2 + n + 12 = 5(2k)^2 + 2k + 12 = 20k^2 + 2k + 12 = 2(10k^2 + k + 6)$ ,  
which is even.

## 4. Case two: $n$ odd

If  $n$  is odd, let  $n = 2k + 1$ . Then  $5n^2 + n + 12 = 5(2k + 1)^2 + (2k + 1) + 12$ .

## 5. Simplify the odd case

$5(2k + 1)^2 + (2k + 1) + 12 = 20k^2 + 22k + 18 = 2(10k^2 + 11k + 9)$ , which  
is even.

## 6. Conclude by exhaustion

Every integer is either even or odd, and both cases give an even result. Therefore  
 $5n^2 + n + 12$  is always even.

### Final answer

(i) For  $p = 7$ ,  $2p + 1 = 15$ , not prime.

(ii) Proven.

**PAST PAPER**

# **WMA12/01 May/June 2022**

May/June 2022 | 10 questions | 75 marks

**10**

questions

**75**

marks

**Answers**

worked solution  
after each  
question

# Question 1

## Binomial Expansion

1. Find the first four terms, in ascending powers of  $x$ , of the binomial expansion of

$$\left(2 + \frac{3}{8}x\right)^{10}$$

Give each coefficient as an integer.

(4)

## Worked Solution - Question 1

**1. Write the binomial terms**

$$\left(2 + \frac{3}{8}x\right)^{10} = 2^{10} + \binom{10}{1}2^9\left(\frac{3}{8}x\right) + \binom{10}{2}2^8\left(\frac{3}{8}x\right)^2 + \binom{10}{3}2^7\left(\frac{3}{8}x\right)^3 + \dots$$

**2. Simplify the constant and linear terms**

$$2^{10} = 1024 \text{ and } \binom{10}{1}2^9\left(\frac{3}{8}\right)x = 1920x.$$

**3. Simplify the quadratic term**

$$\binom{10}{2}2^8\left(\frac{3}{8}\right)^2x^2 = 1620x^2.$$

**4. Simplify the cubic term**

$$\binom{10}{3}2^7\left(\frac{3}{8}\right)^3x^3 = 810x^3.$$

**5. State the expansion**

The first four terms are  $1024 + 1920x + 1620x^2 + 810x^3$ .

**Final answer**

$$1024 + 1920x + 1620x^2 + 810x^3.$$

## Question 2

2.

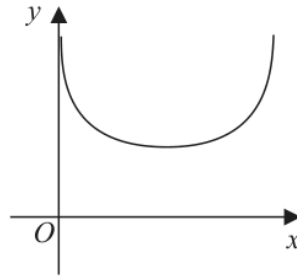


Figure 1

Figure 1 shows the graph of

$$y = 1 - \log_{10}(\sin x) \quad 0 < x < \pi$$

where  $x$  is in radians.

The table below shows some values of  $x$  and  $y$  for this graph, with values of  $y$  given to 3 decimal places.

$x$	0.5	1	1.5	2	2.5	3
$y$	1.319		1.001		1.223	1.850

- (a) Complete the table above, giving values of  $y$  to 3 decimal places. (2)
- (b) Use the trapezium rule with all the  $y$  values in the completed table to find, to 2 decimal places, an estimate for

$$\int_{0.5}^3 (1 - \log_{10}(\sin x)) dx \quad (3)$$

- (c) Use your answer to part (b) to find an estimate for

$$\int_{0.5}^3 (3 + \log_{10}(\sin x)) dx \quad (3)$$

# Worked Solution - Question 2

## 1. Complete the missing table values

Using  $y = 1 - \log_{10}(\sin x)$  gives  $y(1) = 1.075$  and  $y(2) = 1.041$  to 3 decimal places.

## 2. Identify the strip width

The  $x$  values increase by 0.5, so  $h = 0.5$ .

## 3. Apply the trapezium rule

$$A \approx \frac{0.5}{2} \{1.319 + 1.850 + 2(1.075 + 1.001 + 1.041 + 1.223)\}.$$

## 4. Calculate part b

This gives 2.9625..., so the estimate is 2.96 to 2 decimal places.

## 5. Rewrite the new integrand

Since  $y = 1 - \log_{10}(\sin x)$ , we have  $\log_{10}(\sin x) = 1 - y$ .

## 6. Use the interval length

$3 + \log_{10}(\sin x) = 4 - y$ , and the interval from 0.5 to 3 has length 2.5.

## 7. Estimate part c

$$\int_{0.5}^3 (3 + \log_{10}(\sin x)) dx \approx 4(2.5) - 2.9625\dots = 7.0374\dots, \text{ so } 7.04.$$

### Final answer

(a)  $y(1) = 1.075$ ,  $y(2) = 1.041$ . (b) 2.96. (c) 7.04.

## Question 3

Proof

3. (i) Show that the following statement is **false**:

“( $n + 1$ )<sup>3</sup> -  $n$ <sup>3</sup> is prime for all  $n \in \mathbb{N}$ ”

(2)

- (ii) Given that the points  $A(1, 0)$ ,  $B(3, -10)$  and  $C(7, -6)$  lie on a circle, prove that  $AB$  is a diameter of this circle.

(5)

# Worked Solution - Question 3

Topic group

**1. Choose a counterexample**

Take  $n = 5$ .

**2. Evaluate the expression**

$$(5 + 1)^3 - 5^3 = 216 - 125 = 91.$$

**3. Show it is not prime**

$91 = 7 \times 13$ , so the statement is false.

**4. Find the gradient CA**

For  $A(1, 0)$  and  $C(7, -6)$ , the gradient of  $CA$  is  $\frac{0 - (-6)}{1 - 7} = -1$ .

**5. Find the gradient CB**

For  $B(3, -10)$  and  $C(7, -6)$ , the gradient of  $CB$  is  $\frac{-10 - (-6)}{3 - 7} = 1$ .

**6. Show the right angle**

The product of the gradients is  $-1$ , so  $CA$  is perpendicular to  $CB$  and  $\angle ACB = 90^\circ$ .

**7. Conclude the diameter**

An angle of  $90^\circ$  at the circumference subtends a diameter, so  $AB$  is a diameter of the circle.

**Final answer**

(i) Counterexample:  $n = 5$  gives **91**, not prime.

(ii) Proven:  $AB$  is a diameter.

**Question 4****Laws of Logarithms**

4. In this question you must show all stages of your working.

Give your answers in fully simplified surd form.

Given that  $a$  and  $b$  are positive constants, solve the simultaneous equations

$$\begin{aligned}a - b &= 8 \\ \log_4 a + \log_4 b &= 3\end{aligned}$$

(6)

# Worked Solution - Question 4

Topic group

**1. Combine the logarithms**

$$\log_4 a + \log_4 b = \log_4(ab) = 3.$$

**2. Find  $ab$** 

$$\text{Therefore } ab = 4^3 = 64.$$

**3. Use  $a$  minus  $b$** 

$$\text{From } a - b = 8, b = a - 8.$$

**4. Form a quadratic**

$$a(a - 8) = 64, \text{ so } a^2 - 8a - 64 = 0.$$

**5. Solve for  $a$** 

$$a = \frac{8 \pm \sqrt{64 + 256}}{2} = 4 \pm 4\sqrt{5}.$$

**6. Choose positive constants**

$$\text{Since } b = a - 8 \text{ must also be positive, take } a = 4 + 4\sqrt{5}.$$

**7. Find  $b$** 

$$b = a - 8 = 4\sqrt{5} - 4.$$

**Final answer**

$$a = 4 + 4\sqrt{5}, \quad b = 4\sqrt{5} - 4.$$

**Question 5****Trigonometric Equations**

5. **In this question you must show all stages of your working.**

**Solutions relying entirely on calculator technology are not acceptable.**

Solve, for  $-180^\circ < \theta \leq 180^\circ$ , the equation

$$3 \tan(\theta + 43^\circ) = 2 \cos(\theta + 43^\circ)$$

**(6)**

# Worked Solution - Question 5

## 1. Substitute $u$

Let  $u = \theta + 43^\circ$ . The equation becomes  $3 \tan u = 2 \cos u$ .

## 2. Clear the cosine denominator

$$3 \frac{\sin u}{\cos u} = 2 \cos u, \text{ so } 3 \sin u = 2 \cos^2 u.$$

## 3. Use cos squared

$$3 \sin u = 2(1 - \sin^2 u), \text{ so } 2 \sin^2 u + 3 \sin u - 2 = 0.$$

## 4. Factor the quadratic

$$(2 \sin u - 1)(\sin u + 2) = 0.$$

## 5. Use the valid sine value

$$\sin u = -2 \text{ is impossible, so } \sin u = \frac{1}{2}.$$

## 6. Find $u$ in the interval

Since  $-180^\circ < \theta \leq 180^\circ$ ,  $-137^\circ < u \leq 223^\circ$ . Thus  $u = 30^\circ$  or  $150^\circ$ .

## 7. Return to theta

$$\theta = u - 43^\circ, \text{ so } \theta = -13^\circ \text{ or } 107^\circ.$$

### Final answer

$$\theta = -13^\circ, 107^\circ.$$

## Question 6

## Geometric Sequences

6. In a geometric sequence  $u_1, u_2, u_3, \dots$

- the common ratio is  $r$
- $u_2 + u_3 = 6$
- $u_4 = 8$

(a) Show that  $r$  satisfies

$$3r^2 - 4r - 4 = 0 \quad (3)$$

Given that the geometric sequence has a sum to infinity,

(b) find  $u_1$  (3)

(c) find  $S_\infty$  (2)

# Worked Solution - Question 6

**1. Write the terms**

$$u_2 = u_1r, u_3 = u_1r^2, \text{ and } u_4 = u_1r^3.$$

**2. Use u four**

$$\text{Since } u_4 = 8, u_1r^3 = 8, \text{ so } u_1r = \frac{8}{r^2}.$$

**3. Use u two plus u three**

$$u_2 + u_3 = u_1r(1 + r) = 6.$$

**4. Show the quadratic**

$$\text{Substitute } u_1r = \frac{8}{r^2} \text{ to get } \frac{8}{r^2}(1 + r) = 6. \text{ Hence } 8 + 8r = 6r^2, \text{ so } 3r^2 - 4r - 4 = 0.$$

**5. Solve for r**

$$3r^2 - 4r - 4 = 0 \text{ gives } r = 2 \text{ or } r = -\frac{2}{3}.$$

**6. Choose the sum-to-infinity ratio**

$$\text{A sum to infinity requires } |r| < 1, \text{ so } r = -\frac{2}{3}.$$

**7. Find u one**

$$u_1r^3 = 8, \text{ so } u_1 \left(-\frac{2}{3}\right)^3 = 8 \text{ and } u_1 = -27.$$

**8. Find the infinite sum**

$$S_\infty = \frac{u_1}{1 - r} = \frac{-27}{1 + 2/3} = -\frac{81}{5}.$$

**Final answer**

(a) Proven:  $3r^2 - 4r - 4 = 0$ .

(b)  $u_1 = -27$ . (c)  $S_\infty = -\frac{81}{5}$ .

## Question 7

7.  $f(x) = Ax^3 + 6x^2 - 4x + B$

where  $A$  and  $B$  are constants.

Given that

- $(x + 2)$  is a factor of  $f(x)$
- $\int_3^5 f(x) dx = 176$

find the value of  $A$  and the value of  $B$ .

(7)

# Worked Solution - Question 7

**1. Use the factor condition**

Since  $(x + 2)$  is a factor,  $f(-2) = 0$ .

**2. Form the first equation**

$A(-2)^3 + 6(-2)^2 - 4(-2) + B = 0$ , so  $-8A + 32 + B = 0$  and  $B = 8A - 32$ .

**3. Set up the integral**

$$\int_3^5 (Ax^3 + 6x^2 - 4x + B) dx = 176.$$

**4. Integrate**

$$\left[ \frac{A}{4}x^4 + 2x^3 - 2x^2 + Bx \right]_3^5 = 176.$$

**5. Simplify the second equation**

This gives  $136A + 164 + 2B = 176$ , so  $68A + B = 6$ .

**6. Solve simultaneously**

Using  $B = 8A - 32$  in  $68A + B = 6$  gives  $76A = 38$ , so  $A = \frac{1}{2}$ .

**7. Find B**

$$B = 8 \left( \frac{1}{2} \right) - 32 = -28.$$

**Final answer**

$$A = \frac{1}{2}, \quad B = -28.$$

**Question 8****Applications of Differentiation**

8. In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

A curve has equation

$$y = 256x^4 - 304x - 35 + \frac{27}{x^2} \quad x \neq 0$$

(a) Find  $\frac{dy}{dx}$

(3)

(b) Hence find the coordinates of the stationary points of the curve.

(5)

# Worked Solution - Question 8

Topic group

**1. Rewrite with powers**

$$y = 256x^4 - 304x - 35 + 27x^{-2}.$$

**2. Differentiate**

$$\frac{dy}{dx} = 1024x^3 - 304 - 54x^{-3} = 1024x^3 - 304 - \frac{54}{x^3}.$$

**3. Set the derivative to zero**

$$1024x^3 - 304 - \frac{54}{x^3} = 0.$$

**4. Use a substitution**

Multiplying by  $x^3$  gives  $1024x^6 - 304x^3 - 54 = 0$ . Let  $u = x^3$ .

**5. Solve for u**

$$1024u^2 - 304u - 54 = 0, \text{ or } 512u^2 - 152u - 27 = 0 = (8u + 1)(64u - 27).$$

**6. Find x**

$$\text{Thus } x^3 = -\frac{1}{8} \text{ or } x^3 = \frac{27}{64}, \text{ giving } x = -\frac{1}{2} \text{ or } x = \frac{3}{4}.$$

**7. Find the y values**

Substitution into  $y = 256x^4 - 304x - 35 + \frac{27}{x^2}$  gives  $y = 241$  at  $x = -\frac{1}{2}$  and  $y = -134$  at  $x = \frac{3}{4}$ .

**8. State the stationary points**

The stationary points are  $\left(-\frac{1}{2}, 241\right)$  and  $\left(\frac{3}{4}, -134\right)$ .

**Final answer**

$$(a) \frac{dy}{dx} = 1024x^3 - 304 - \frac{54}{x^3}. \quad (b) \left(-\frac{1}{2}, 241\right), \left(\frac{3}{4}, -134\right).$$

## Question 9

## Modelling with Sequences &amp; Series

9. A scientist is using carbon-14 dating to determine the age of some wooden items.

The equation for carbon-14 dating an item is given by

$$N = k\lambda^t$$

where

- $N$  grams is the amount of carbon-14 **currently** present in the item
- $k$  grams was the **initial** amount of carbon-14 present in the item
- $t$  is the number of years since the item was made
- $\lambda$  is a constant, with  $0 < \lambda < 1$

- (a) Sketch the graph of  $N$  against  $t$  for  $k = 1$  (2)

Given that it takes 5700 years for the amount of carbon-14 to reduce to half its initial value,

- (b) show that the value of the constant  $\lambda$  is 0.999878 to 6 decimal places. (2)

Given that Item  $A$

- is known to have had 15 grams of carbon-14 present initially
- is thought to be 3250 years old

- (c) calculate, to 3 significant figures, how much carbon-14 the equation predicts is currently in Item  $A$ . (2)

Item  $B$  is known to have initially had 25 grams of carbon-14 present, but only 18 grams now remain.

- (d) Use algebra to calculate the age of Item  $B$  to the nearest 100 years. (3)

# Worked Solution - Question 9

Topic group

## 1. Sketch the model for $k$ equals one

For  $k = 1$ ,  $N = \lambda^t$  with  $0 < \lambda < 1$ . The graph starts at  $(0, 1)$ , decreases as  $t$  increases, and tends towards 0.

## 2. Use the half-life information

After 5700 years,  $N = \frac{k}{2}$ , so  $k\lambda^{5700} = \frac{k}{2}$ .

## 3. Find lambda

$\lambda^{5700} = \frac{1}{2}$ , hence  $\lambda = \left(\frac{1}{2}\right)^{1/5700} = 0.999878\dots$ , so  $\lambda = 0.999878$  to 6 decimal places.

## 4. Model item A

For item A,  $k = 15$  and  $t = 3250$ , so  $N = 15\lambda^{3250}$ .

## 5. Calculate item A

$N = 15(0.9998784026\dots)^{3250} = 10.103\dots$ , so the current amount is 10.1 g to 3 significant figures.

## 6. Set up item B

For item B,  $18 = 25\lambda^t$ , so  $\lambda^t = \frac{18}{25}$ .

## 7. Take logarithms

$t \log \lambda = \log\left(\frac{18}{25}\right)$ , so  $t = \frac{\log(18/25)}{\log \lambda}$ .

### 8. Round the age

Using  $\lambda = (1/2)^{1/5700}$  gives  $t = 2701 \dots$ , so the age is **2700** years to the nearest 100 years.

#### Final answer

(a) Decreasing exponential through  $(0, 1)$ .

(b)  $\lambda = 0.999878$ .

(c) 10.1 g. (d) 2700 years.

## Question 10

10. The circle  $C$  has centre  $X(3, 5)$  and radius  $r$

The line  $l$  has equation  $y = 2x + k$ , where  $k$  is a constant.

(a) Show that  $l$  and  $C$  intersect when

$$5x^2 + (4k - 26)x + k^2 - 10k + 34 - r^2 = 0$$

(3)

Given that  $l$  is a tangent to  $C$ ,

(b) show that  $5r^2 = (k + p)^2$ , where  $p$  is a constant to be found.

(3)

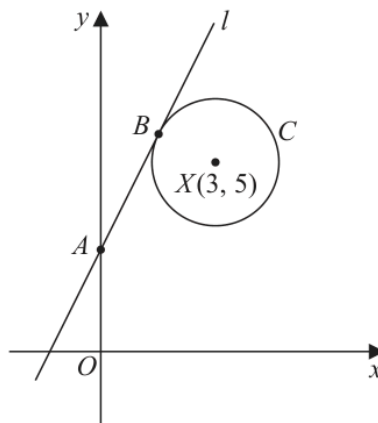


Figure 2

The line  $l$

- cuts the  $y$ -axis at the point  $A$
- touches the circle  $C$  at the point  $B$

as shown in Figure 2.

Given that  $AB = 2r$

(c) find the value of  $k$

(6)

(Total 12 marks)

## Worked Solution - Question 10

**1. Write the circle equation**

The circle has centre  $(3, 5)$  and radius  $r$ , so  $(x - 3)^2 + (y - 5)^2 = r^2$ .

**2. Substitute the line**

On the line,  $y = 2x + k$ . Substitution gives  $(x - 3)^2 + (2x + k - 5)^2 = r^2$ .

**3. Expand**

$$x^2 - 6x + 9 + 4x^2 + 4kx - 20x + k^2 - 10k + 25 - r^2 = 0.$$

**4. Show the intersection equation**

Collecting terms gives  $5x^2 + (4k - 26)x + k^2 - 10k + 34 - r^2 = 0$ .

**5. Use tangency**

For a tangent, the quadratic has one repeated root, so its discriminant is zero.

**6. Set the discriminant to zero**

$$(4k - 26)^2 - 20(k^2 - 10k + 34 - r^2) = 0.$$

**7. Simplify**

This reduces to  $20r^2 - 4k^2 - 8k - 4 = 0$ , hence  $5r^2 = (k + 1)^2$ . Therefore  $p = 1$ .

**8. Find the tangent point x coordinate**

For the repeated root,  $x_B = -\frac{4k - 26}{2 \cdot 5} = \frac{13 - 2k}{5}$ .

**9. Use distance AB**

The point  $A$  is  $(0, k)$  and  $B$  lies on a line of gradient  $2$ , so

$$AB = \sqrt{x_B^2 + (2x_B)^2} = \sqrt{5} x_B.$$

**10. Use the radius relation**

From  $5r^2 = (k + 1)^2$ ,  $r = \frac{k + 1}{\sqrt{5}}$  in the shown configuration.

**11. Apply AB equals two r**

$$\sqrt{5} \left( \frac{13 - 2k}{5} \right) = 2 \left( \frac{k + 1}{\sqrt{5}} \right).$$

**12. Solve for k**

Multiplying by  $\sqrt{5}$  gives  $13 - 2k = 2k + 2$ , so  $k = \frac{11}{4}$ .

**Final answer**

(a) Proven.

(b)  $5r^2 = (k + 1)^2$ , so  $p = 1$ .

(c)  $k = \frac{11}{4}$ .

**PAST PAPER**

# **WMA12/01 October 2022**

**October 2022 | 10 questions | 75 marks**

**10**

questions

**75**

marks

**Answers**

worked solution  
after each  
question

## Question 1

Proof

1. Given that  $a$ ,  $b$  and  $c$  are integers greater than 0 such that

- $c = b + 2$
- $a + b + c = 10$

Prove, by exhaustion, that the product of  $a$ ,  $b$  and  $c$  is always even.

You may use the table below to illustrate your answer.

(3)

*You may not need to use all rows of this table.*

$a$	$b$	$c$
	1	
	2	

# Worked Solution - Question 1

Topic group

## 1. Use the two conditions

Since  $c = b + 2$  and  $a + b + c = 10$ , substitute for  $c$ :  $a + b + (b + 2) = 10$ .

## 2. Write $a$ in terms of $b$

This gives  $a + 2b + 2 = 10$ , so  $a = 8 - 2b$ .

## 3. List the possible $b$ values

The integers  $a, b, c$  are all positive. Since  $b > 0$  and  $a = 8 - 2b > 0$ , the only possible values are  $b = 1, 2, 3$ .

## 4. Exhaust the cases

If  $b = 1$ , then  $a = 6$  and  $c = 3$ , so  $abc = 18$ . If  $b = 2$ , then  $a = 4$  and  $c = 4$ , so  $abc = 32$ . If  $b = 3$ , then  $a = 2$  and  $c = 5$ , so  $abc = 30$ .

## 5. Conclude the proof

The list contains every possible case, and in every case the product is even. Therefore  $abc$  is always even.

### Final answer

The possible triples are  $(6, 1, 3)$ ,  $(4, 2, 4)$  and  $(2, 3, 5)$ , giving products 18, 32 and 30. Each product is even, so  $abc$  is always even.

## Question 2

## Binomial Expansion

2. A curve  $C$  has equation  $y = f(x)$  where

$$f(x) = (2 - kx)^5$$

and  $k$  is a constant.

Given that when  $f(x)$  is divided by  $(4x - 5)$  the remainder is  $\frac{243}{32}$

(a) show that  $k = \frac{2}{5}$

(2)

(b) Find the first three terms, in ascending powers of  $x$ , of the binomial expansion of

$$\left(2 - \frac{2}{5}x\right)^5$$

giving each term in simplest form.

(3)

Using the solution to part (b) and making your method clear,

(c) find the gradient of  $C$  at the point where  $x = 0$

(2)

## Worked Solution - Question 2

**1. Use the remainder theorem**

Dividing by  $4x - 5$  means the remainder is  $f\left(\frac{5}{4}\right)$ .

**2. Substitute  $x$  equals five quarters**

$$f\left(\frac{5}{4}\right) = \left(2 - \frac{5k}{4}\right)^5 = \frac{243}{32}.$$

**3. Take the fifth root**

Since  $\frac{243}{32} = \left(\frac{3}{2}\right)^5$ , we have  $2 - \frac{5k}{4} = \frac{3}{2}$ .

**4. Find  $k$** 

$$\frac{5k}{4} = \frac{1}{2}, \text{ so } k = \frac{2}{5}.$$

**5. Expand the binomial**

With  $k = \frac{2}{5}$ ,  $f(x) = \left(2 - \frac{2}{5}x\right)^5$ .

**6. Find the first three terms**

$$2^5 + 5(2^4)\left(-\frac{2}{5}x\right) + 10(2^3)\left(-\frac{2}{5}x\right)^2 = 32 - 32x + \frac{64}{5}x^2.$$

**7. Use the coefficient of  $x$** 

For a polynomial, the gradient at  $x = 0$  is the coefficient of  $x$  in the expansion.

Therefore  $f'(0) = -32$ .

**Final answer**

(a)  $k = \frac{2}{5}$ .

(b)  $32 - 32x + \frac{64}{5}x^2$ .

(c)  $-32$ .

## Question 3

## Sequences &amp; Series

3. A sequence  $a_1, a_2, a_3, \dots$  is defined by

$$a_n = \cos^2\left(\frac{n\pi}{3}\right)$$

Find the exact values of

(a) (i)  $a_1$

(ii)  $a_2$

(iii)  $a_3$

(3)

(b) Hence find the exact value of

$$\sum_{n=1}^{50} \left\{ n + \cos^2\left(\frac{n\pi}{3}\right) \right\}$$

You must make your method clear.

(4)

## Worked Solution - Question 3

**1. Find the first term**

$$a_1 = \cos^2\left(\frac{\pi}{3}\right) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

**2. Find the second term**

$$a_2 = \cos^2\left(\frac{2\pi}{3}\right) = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}.$$

**3. Find the third term**

$$a_3 = \cos^2(\pi) = (-1)^2 = 1.$$

**4. Recognise the repeating pattern**

The values of  $\cos^2\left(\frac{n\pi}{3}\right)$  repeat every 3 terms as  $\frac{1}{4}, \frac{1}{4}, 1$ .

**5. Sum the n part**

$$\sum_{n=1}^{50} n = \frac{50 \cdot 51}{2} = 1275.$$

**6. Sum the cosine part**

There are **16** complete groups of 3 terms in the first **48** terms, then the first 2 terms repeat. Hence the cosine-square sum is

$$16\left(\frac{1}{4} + \frac{1}{4} + 1\right) + \frac{1}{4} + \frac{1}{4} = 24 + \frac{1}{2} = \frac{49}{2}.$$

**7. Add the two sums**

$$\sum_{n=1}^{50} \left(n + \cos^2\left(\frac{n\pi}{3}\right)\right) = 1275 + \frac{49}{2} = \frac{2599}{2}.$$

**Final answer**

$$(a) a_1 = \frac{1}{4}, a_2 = \frac{1}{4}, a_3 = 1. \quad (b) \frac{2599}{2}.$$

## Question 4

## Laws of Logarithms

4. The weight of a baby mammal is monitored over a 16-month period.

The weight of the mammal,  $w$ kg, is given by

$$w = \log_a(t + 5) - \log_a 4 \quad 2 \leq t \leq 18$$

where  $t$  is the age of the mammal in months and  $a$  is a constant.

Given that the weight of the mammal was 10kg when  $t = 3$

- (a) show that  $a = 1.072$  correct to 3 decimal places. (3)

Using  $a = 1.072$

- (b) find an equation for  $t$  in terms of  $w$  (3)

- (c) find the value of  $t$  when  $w = 15$ , giving your answer to 3 significant figures. (2)

## Worked Solution - Question 4

**1. Substitute the given information**

When  $t = 3$ ,  $w = 10$ . So  $10 = \log_a(3 + 5) - \log_a 4 = \log_a 8 - \log_a 4$ .

**2. Combine the logarithms**

$$\log_a 8 - \log_a 4 = \log_a \left( \frac{8}{4} \right) = \log_a 2, \text{ so } 10 = \log_a 2.$$

**3. Find a**

$10 = \log_a 2$  means  $a^{10} = 2$ , hence  $a = 2^{1/10} = 1.071773 \dots = 1.072$  to 3 decimal places.

**4. Start from the model**

Using  $a = 1.072$ ,  $w = \log_{1.072}(t + 5) - \log_{1.072} 4$ .

**5. Combine the logs again**

$$w = \log_{1.072} \left( \frac{t + 5}{4} \right).$$

**6. Make t the subject**

Changing from logarithmic form to exponential form gives  $(1.072)^w = \frac{t + 5}{4}$ , so  $t = 4(1.072)^w - 5$ .

**7. Substitute w equals fifteen**

When  $w = 15$ ,  $t = 4(1.072)^{15} - 5 = 6.3496 \dots$ , so  $t = 6.35$  months to 3 significant figures.

**Final answer**

(a)  $a = 1.072$  to 3 d.p.

(b)  $t = 4(1.072)^w - 5$ .

(c)  $t = 6.35$  months to 3 s.f.

## Question 5

## Trigonometric Equations

5.

In this question you must show detailed reasoning.

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that the equation

$$(3 \cos \theta - \tan \theta) \cos \theta = 2$$

can be written as

$$3 \sin^2 \theta + \sin \theta - 1 = 0 \quad (3)$$

(b) Hence solve for  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$$(3 \cos 2x - \tan 2x) \cos 2x = 2 \quad (5)$$

# Worked Solution - Question 5

## 1. Rewrite the left side

$(3 \cos \theta - \tan \theta) \cos \theta = 3 \cos^2 \theta - \sin \theta$  because  $\tan \theta \cos \theta = \sin \theta$ .

## 2. Move everything to one side

The equation becomes  $3 \cos^2 \theta - \sin \theta = 2$ .

## 3. Use cos squared

Since  $\cos^2 \theta = 1 - \sin^2 \theta$ ,  $3(1 - \sin^2 \theta) - \sin \theta = 2$ .

## 4. Show the required form

Expanding and rearranging gives  $3 \sin^2 \theta + \sin \theta - 1 = 0$ .

## 5. Apply the result to $2x$

For  $(3 \cos 2x - \tan 2x) \cos 2x = 2$ , take  $\theta = 2x$ . Then

$$3 \sin^2(2x) + \sin(2x) - 1 = 0.$$

## 6. Solve the quadratic

Let  $u = \sin(2x)$ . Then  $3u^2 + u - 1 = 0$ , so  $u = \frac{-1 \pm \sqrt{13}}{6}$ .

## 7. Use the interval

Since  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ , we have  $-\pi \leq 2x \leq \pi$ . Solving  $\sin(2x) = \frac{-1 + \sqrt{13}}{6}$

and  $\sin(2x) = \frac{-1 - \sqrt{13}}{6}$  in this interval gives

$$2x = -2.2665 \dots, -0.8751 \dots, 0.4492 \dots, 2.6924 \dots$$

## 8. Divide by two

Therefore  $x = -1.133 \dots, -0.4375 \dots, 0.2246 \dots, 1.346 \dots$ , giving

$$x = -1.13, -0.438, 0.225, 1.35 \text{ radians.}$$

**Final answer**

(a) Proven.

(b)  $x = -1.13, -0.438, 0.225, 1.35$  radians.

WMA12/01 OCTOBER 2022

7 marks

## Question 6

Integration

6. The curve  $C_1$  has equation  $y = f(x)$ .

A table of values of  $x$  and  $y$  for  $y = f(x)$  is shown below, with the  $y$  values rounded to 4 decimal places where appropriate.

$x$	0	0.5	1	1.5	2
$y$	3	2.6833	2.4	2.1466	1.92

(a) Use the trapezium rule with all the values of  $y$  in the table to find an approximation for

$$\int_0^2 f(x) \, dx$$

giving your answer to 3 decimal places.

(3)

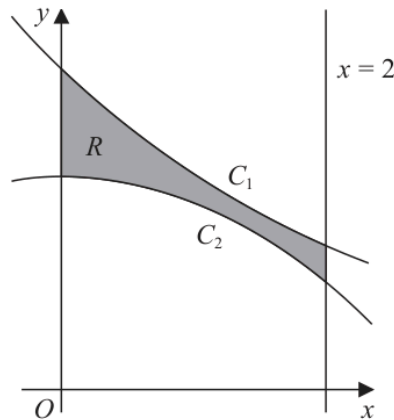


Figure 1

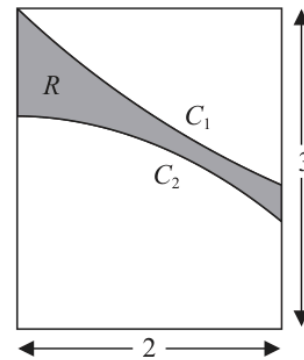


Figure 2

The region  $R$ , shown shaded in Figure 1, is bounded by

- the curve  $C_1$
- the curve  $C_2$  with equation  $y = 2 - \frac{1}{4}x^2$
- the line with equation  $x = 2$
- the  $y$ -axis

The region  $R$  forms part of the design for a logo shown in Figure 2.

The design consists of the shaded region  $R$  inside a rectangle of width 2 and height 3

Using calculus and the answer to part (a),

(b) calculate an estimate for the percentage of the logo which is shaded.

(4)

# Worked Solution - Question 6

## 1. Identify the strip width

The  $x$  values increase by  $0.5$ , so the trapezium-rule width is  $h = 0.5$ .

## 2. Apply the trapezium rule

$$\int_0^2 f(x) dx \approx \frac{0.5}{2} [3 + 1.92 + 2(2.6833 + 2.4 + 2.1466)].$$

## 3. Evaluate the estimate

This gives  $4.84495$ , so the approximation is  $4.845$  to 3 decimal places.

## 4. Find the area under $C_2$

$$\int_0^2 \left(2 - \frac{1}{4}x^2\right) dx = \left[2x - \frac{x^3}{12}\right]_0^2 = 4 - \frac{2}{3} = \frac{10}{3}.$$

## 5. Estimate the shaded area

The shaded region lies between  $C_1$  and  $C_2$ , so its estimated area is

$$4.845 - \frac{10}{3} = 1.5116\dots$$

## 6. Compare with the rectangle

The rectangle has area  $2 \times 3 = 6$ .

## 7. Convert to a percentage

Percentage shaded =  $\frac{1.5116\dots}{6} \times 100 = 25.19\dots\%$ , so the estimate is  $25.2\%$ .

**Final answer**

(a) 4.845.

(b) 25.2%.

## Question 7

## Applications of Differentiation

7. The curve  $C$  has equation

$$y = \frac{12x^3(x - 7) + 14x(13x - 15)}{21\sqrt{x}} \quad x > 0$$

(a) Write the equation of  $C$  in the form

$$y = ax^{\frac{7}{2}} + bx^{\frac{5}{2}} + cx^{\frac{3}{2}} + dx^{\frac{1}{2}}$$

where  $a$ ,  $b$ ,  $c$  and  $d$  are fully simplified constants.

(3)

The curve  $C$  has three turning points.

Using calculus,

(b) show that the  $x$  coordinates of the three turning points satisfy the equation

$$2x^3 - 10x^2 + 13x - 5 = 0$$

(3)

Given that the  $x$  coordinate of one of the turning points is 1

(c) find, using algebra, the exact  $x$  coordinates of the other two turning points.

*(Solutions based entirely on calculator technology are not acceptable.)*

(3)

# Worked Solution - Question 7

Topic group

## 1. Expand the numerator

$$12x^3(x - 7) + 14x(13x - 15) = 12x^4 - 84x^3 + 182x^2 - 210x.$$

## 2. Divide by 21 root $x$

Since  $\sqrt{x} = x^{1/2}$ , divide each term by  $21x^{1/2}$ .

## 3. Write in the required form

$$y = \frac{12}{21}x^{7/2} - \frac{84}{21}x^{5/2} + \frac{182}{21}x^{3/2} - \frac{210}{21}x^{1/2} = \frac{4}{7}x^{7/2} - 4x^{5/2} + \frac{26}{3}x^{3/2} - 10x^{1/2}$$

## 4. Differentiate

$$\frac{dy}{dx} = 2x^{5/2} - 10x^{3/2} + 13x^{1/2} - 5x^{-1/2}.$$

## 5. Factor the derivative

$$\frac{dy}{dx} = x^{-1/2}(2x^3 - 10x^2 + 13x - 5).$$

## 6. Use the turning point condition

At a turning point,  $\frac{dy}{dx} = 0$ . Since  $x > 0$ ,  $x^{-1/2} \neq 0$ , so the  $x$  coordinates satisfy  $2x^3 - 10x^2 + 13x - 5 = 0$ .

## 7. Use the known root

One root is  $x = 1$ , so divide  $2x^3 - 10x^2 + 13x - 5$  by  $(x - 1)$ .

## 8. Factor the cubic

$$2x^3 - 10x^2 + 13x - 5 = (x - 1)(2x^2 - 8x + 5).$$

### 9. Solve the quadratic

$$2x^2 - 8x + 5 = 0 \text{ gives } x = \frac{8 \pm \sqrt{64 - 40}}{4} = \frac{8 \pm 2\sqrt{6}}{4} = 2 \pm \frac{\sqrt{6}}{2}.$$

#### Final answer

(a)  $y = \frac{4}{7}x^{7/2} - 4x^{5/2} + \frac{26}{3}x^{3/2} - 10x^{1/2}.$

(b) Proven.

(c)  $x = 2 - \frac{\sqrt{6}}{2}$  and  $x = 2 + \frac{\sqrt{6}}{2}.$

## Question 8

## Geometric Sequences

8. A geometric sequence has first term  $a$  and common ratio  $r$

Given that  $S_{\infty} = 3a$

(a) show that  $r = \frac{2}{3}$

(2)

Given also that

$$u_2 - u_4 = 16$$

where  $u_k$  is the  $k^{\text{th}}$  term of this sequence,

(b) find the value of  $S_{10}$  giving your answer to one decimal place.

(5)

## Worked Solution - Question

## 8

**1. Use the sum to infinity formula**

For a geometric sequence,  $S_{\infty} = \frac{a}{1-r}$ . The question gives  $S_{\infty} = 3a$ .

**2. Find the common ratio**

$\frac{a}{1-r} = 3a$ . Since  $u_2 - u_4 = 16$ ,  $a \neq 0$ , so  $\frac{1}{1-r} = 3$  and  $r = \frac{2}{3}$ .

**3. Use  $u_2$  minus  $u_4$** 

$$u_2 - u_4 = ar - ar^3 = ar(1 - r^2) = 16.$$

**4. Substitute  $r$** 

$$a \left( \frac{2}{3} \right) \left( 1 - \frac{4}{9} \right) = 16.$$

**5. Solve for  $a$** 

$$a \cdot \frac{2}{3} \cdot \frac{5}{9} = 16, \text{ so } \frac{10a}{27} = 16 \text{ and } a = \frac{216}{5} = 43.2.$$

**6. Find  $S_{10}$** 

$$S_{10} = \frac{a(1-r^{10})}{1-r} = 3a \left( 1 - \left( \frac{2}{3} \right)^{10} \right).$$

**7. Evaluate**

$$S_{10} = 3 \left( \frac{216}{5} \right) \left( 1 - \left( \frac{2}{3} \right)^{10} \right) = \frac{92840}{729} = 127.352\dots, \text{ so } S_{10} = 127 \text{ to 3}$$

significant figures.

**Final answer**

(a)  $a = 43.2$ .

(b)  $S_{10} = 127.353\dots$ , so **127** to 3 s.f.

WMA12/01 OCTOBER 2022

12 marks

## Question 9

Circles

9.

In this question you must show detailed reasoning.

Solutions relying entirely on calculator technology are not acceptable.

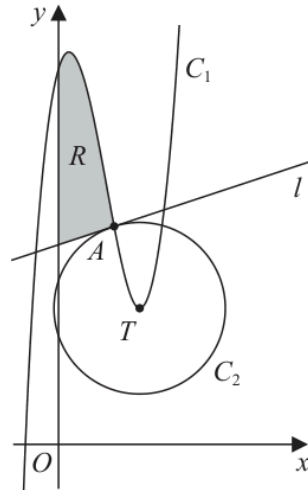


Figure 3

Figure 3 shows

- the curve  $C_1$  with equation  $y = x^3 - 5x^2 + 3x + 14$
- the circle  $C_2$  with centre  $T$

The point  $T$  is the minimum turning point of  $C_1$

Using Figure 3 and calculus,

- (a) find the coordinates of  $T$  (3)

The curve  $C_1$  intersects the circle  $C_2$  at the point  $A$  with  $x$  coordinate 2

- (b) Find an equation of the circle  $C_2$  (3)

The line  $l$  shown in Figure 3, is the tangent to circle  $C_2$  at  $A$

- (c) Show that an equation of  $l$  is 
$$y = \frac{1}{3}x + \frac{22}{3}$$
 (3)

The region  $R$ , shown shaded in Figure 3, is bounded by  $C_1$ ,  $l$  and the  $y$ -axis.

- (d) Find the exact area of  $R$ . (3)

## Worked Solution - Question 9

**1. Differentiate the cubic**

For  $y = x^3 - 5x^2 + 3x + 14$ ,  $\frac{dy}{dx} = 3x^2 - 10x + 3$ .

**2. Find stationary x values**

$3x^2 - 10x + 3 = 0$  factors as  $(3x - 1)(x - 3) = 0$ , so  $x = \frac{1}{3}$  or  $x = 3$ .

**3. Identify the minimum**

$\frac{d^2y}{dx^2} = 6x - 10$ . At  $x = 3$ , this is positive, so the minimum turning point has  $x = 3$ .

**4. Find T**

Substitute  $x = 3$ :  $y = 27 - 45 + 9 + 14 = 5$ . Hence  $T = (3, 5)$ .

**5. Find A**

At  $x = 2$  on  $C_1$ ,  $y = 8 - 20 + 6 + 14 = 8$ , so  $A = (2, 8)$ .

**6. Find the circle radius**

The centre of  $C_2$  is  $T = (3, 5)$  and it passes through  $A = (2, 8)$ . Thus  $r^2 = (2 - 3)^2 + (8 - 5)^2 = 10$ .

**7. Write the circle equation**

Therefore  $C_2$  has equation  $(x - 3)^2 + (y - 5)^2 = 10$ .

**8. Find the tangent gradient**

The radius  $TA$  has gradient  $\frac{8 - 5}{2 - 3} = -3$ , so the tangent at  $A$  has gradient  $\frac{1}{3}$ .

### 9. Write the tangent equation

Using  $A = (2, 8)$ ,  $y - 8 = \frac{1}{3}(x - 2)$ , so  $y = \frac{1}{3}x + \frac{22}{3}$ .

### 10. Set up the area

From  $x = 0$  to  $x = 2$ , the cubic is above the line. The required area is

$$\int_0^2 \left[ x^3 - 5x^2 + 3x + 14 - \left( \frac{1}{3}x + \frac{22}{3} \right) \right] dx.$$

### 11. Simplify the integrand

The integrand is  $x^3 - 5x^2 + \frac{8}{3}x + \frac{20}{3}$ .

### 12. Integrate

$$\left[ \frac{x^4}{4} - \frac{5x^3}{3} + \frac{4x^2}{3} + \frac{20x}{3} \right]_0^2 = 4 - \frac{40}{3} + \frac{16}{3} + \frac{40}{3} = \frac{28}{3}.$$

#### Final answer

(a)  $T = (3, 5)$ .

(b)  $(x - 3)^2 + (y - 5)^2 = 10$ .

(c)  $y = \frac{1}{3}x + \frac{22}{3}$ .

(d)  $\frac{28}{3}$ .

## Question 10

## Laws of Logarithms

10. Given  $a = \log_2 3$

(i) write, in simplest form, in terms of  $a$ ,

(a)  $\log_2 9$

(b)  $\log_2 \left( \frac{\sqrt{3}}{16} \right)$

(3)

(ii) Solve

$$3^x \times 2^{x+4} = 6$$

giving your answer, in simplest form, in terms of  $a$ .

(4)

# Worked Solution - Question 10

## 1. Use the power rule for log 2 of 9

$$\log_2 9 = \log_2(3^2) = 2 \log_2 3 = 2a.$$

## 2. Split the second logarithm

$$\log_2 \left( \frac{\sqrt{3}}{16} \right) = \log_2(3^{1/2}) - \log_2(16).$$

## 3. Simplify in terms of a

$$\log_2(3^{1/2}) = \frac{1}{2} \log_2 3 = \frac{a}{2} \text{ and } \log_2(16) = 4, \text{ so the expression is } \frac{a}{2} - 4.$$

## 4. Take log base 2 of both sides

$$\text{For } 3^x \times 2^{x+4} = 6, \text{ take } \log_2 \text{ of both sides: } \log_2(3^x) + \log_2(2^{x+4}) = \log_2 6.$$

## 5. Rewrite each term

$$\log_2(3^x) = xa, \log_2(2^{x+4}) = x + 4, \text{ and } \log_2 6 = \log_2(2 \cdot 3) = 1 + a.$$

## 6. Solve for x

$$\text{So } xa + x + 4 = a + 1. \text{ Hence } x(a + 1) = a - 3, \text{ giving } x = \frac{a - 3}{a + 1}.$$

### Final answer

$$(i)(a) \ 2a. \quad (i)(b) \ \frac{a}{2} - 4. \quad (ii) \ x = \frac{a - 3}{a + 1}.$$

**PAST PAPER**

# **WMA12/01 January 2023**

January 2023 | 10 questions | 75 marks

**10**

questions

**75**

marks

**Answers**

worked solution  
after each  
question

## Question 1

1.

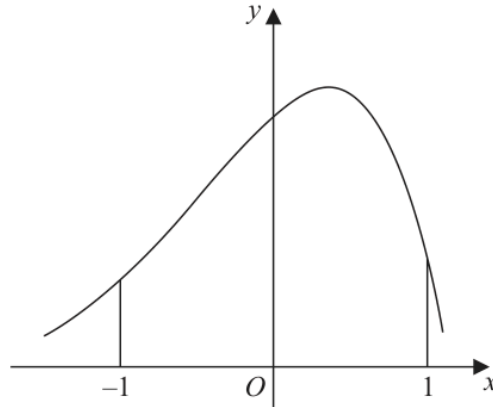


Figure 1

Figure 1 shows a sketch of part of the curve with equation  $y = f(x)$

The table below shows some corresponding values of  $x$  and  $y$  for this curve.

The values of  $y$  are given to 3 decimal places.

$x$	-1	-0.5	0	0.5	1
$y$	2.287	4.470	6.719	7.291	2.834

Using the trapezium rule with all the values of  $y$  in the given table,

(a) obtain an estimate for

$$\int_{-1}^1 f(x) dx$$

giving your answer to 2 decimal places.

(3)

(b) Use your answer to part (a) to estimate

(i)  $\int_{-1}^1 (f(x) - 2) dx$

(ii)  $\int_1^3 f(x-2) dx$

(3)

# Worked Solution - Question 1

## 1. Use the trapezium width

The  $x$  values run from  $-1$  to  $1$  in steps of  $0.5$ , so  $h = 0.5$ .

## 2. Apply the trapezium rule

$$\int_{-1}^1 f(x) dx \approx \frac{0.5}{2} [2.287 + 2.834 + 2(4.470 + 6.719 + 7.291)].$$

## 3. Evaluate the estimate

This gives  $10.52025$ , so the estimate is  $10.52$  to 2 decimal places.

## 4. Estimate the shifted-down area

$$\int_{-1}^1 (f(x) - 2) dx = \int_{-1}^1 f(x) dx - \int_{-1}^1 2 dx.$$

## 5. Subtract the rectangle

The interval has width  $2$ , so  $\int_{-1}^1 2 dx = 4$ . Hence the estimate is  $10.52 - 4 = 6.52$ .

## 6. Use a substitution

For  $\int_1^3 f(x-2) dx$ , let  $u = x - 2$ . When  $x = 1$ ,  $u = -1$ , and when  $x = 3$ ,  $u = 1$ .

## 7. Recognise the same integral

So  $\int_1^3 f(x-2) dx = \int_{-1}^1 f(u) du$ , which has the same estimate  $10.52$ .

**Final answer**

(a) 10.52.

(b)(i) 6.52. \quad (b)(ii) 10.52.

## Question 2

## Applications of Differentiation

2.

In this question you must show all stages of your working.

Solutions based entirely on calculator technology are not acceptable.

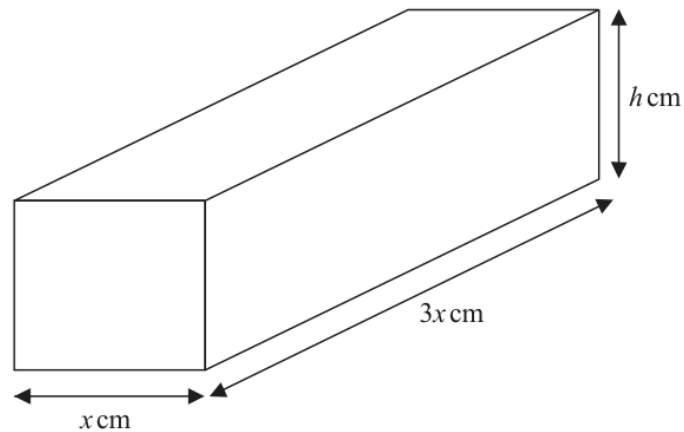


Figure 2

A brick is in the shape of a cuboid with width  $x$  cm, length  $3x$  cm and height  $h$  cm, as shown in Figure 2.

The volume of the brick is  $972 \text{ cm}^3$

(a) Show that the surface area of the brick,  $S \text{ cm}^2$ , is given by

$$S = 6x^2 + \frac{2592}{x}$$

(3)

(b) Find  $\frac{dS}{dx}$

(1)

(c) Hence find the value of  $x$  for which  $S$  is stationary.

(2)

(d) Find  $\frac{d^2S}{dx^2}$  and hence show that the value of  $x$  found in part (c) gives the minimum value of  $S$ .

(2)

(e) Hence find the minimum surface area of the brick.

(1)

# Worked Solution - Question 2

Topic group

**1. Use the volume**

The cuboid has dimensions  $x$ ,  $3x$  and  $h$ , so its volume is  $3x^2h = 972$ .

**2. Write  $h$  in terms of  $x$** 

$$\text{Hence } h = \frac{972}{3x^2} = \frac{324}{x^2}.$$

**3. Build the surface area**

The surface area is  $S = 2(3x^2 + 3xh + xh) = 6x^2 + 8xh$ .

**4. Substitute  $h$** 

$$S = 6x^2 + 8x \left( \frac{324}{x^2} \right) = 6x^2 + \frac{2592}{x}.$$

**5. Differentiate**

$$S = 6x^2 + 2592x^{-1}, \text{ so } \frac{dS}{dx} = 12x - 2592x^{-2} = 12x - \frac{2592}{x^2}.$$

**6. Set the derivative to zero**

$$\text{For a stationary value, } 12x - \frac{2592}{x^2} = 0.$$

**7. Solve for  $x$** 

Multiplying by  $x^2$  gives  $12x^3 = 2592$ , so  $x^3 = 216$  and  $x = 6$ .

**8. Use the second derivative**

$$\frac{d^2S}{dx^2} = 12 + \frac{5184}{x^3}.$$

### 9. Show it is a minimum

At  $x = 6$ ,  $\frac{d^2S}{dx^2} = 12 + \frac{5184}{216} = 36 > 0$ , so the stationary value is a minimum.

### 10. Find the minimum surface area

$S = 6(6)^2 + \frac{2592}{6} = 216 + 432 = 648$ , so the minimum surface area is  $648 \text{ cm}^2$ .

#### Final answer

(a)  $S = 6x^2 + \frac{2592}{x}$ .

(b)  $\frac{dS}{dx} = 12x - \frac{2592}{x^2}$ .

(c)  $x = 6$ .

(d) Minimum shown.

(e)  $648 \text{ cm}^2$ .

## Question 3

## Binomial Expansion

3. 
$$f(x) = \left(2 + \frac{kx}{8}\right)^7$$
 where  $k$  is a non-zero constant

- (a) Find the first 4 terms, in ascending powers of  $x$ , of the binomial expansion of  $f(x)$ .  
Give each term in simplest form.

(4)

Given that, in the binomial expansion of  $f(x)$ , the coefficients of  $x$ ,  $x^2$  and  $x^3$  are the first 3 terms of an arithmetic progression,

- (b) find, using algebra, the possible values of  $k$ .

*(Solutions relying entirely on calculator technology are not acceptable.)*

(3)

## Worked Solution - Question 3

**1. Start the binomial expansion**

$\left(2 + \frac{kx}{8}\right)^7$  begins with

$$2^7 + 7(2^6) \left(\frac{kx}{8}\right) + \binom{7}{2}(2^5) \left(\frac{kx}{8}\right)^2 + \binom{7}{3}(2^4) \left(\frac{kx}{8}\right)^3.$$

**2. Simplify the terms**

This gives  $128 + 56kx + \frac{21}{2}k^2x^2 + \frac{35}{32}k^3x^3$ .

**3. Use the arithmetic progression condition**

The coefficients of  $x$ ,  $x^2$  and  $x^3$  are  $56k$ ,  $\frac{21}{2}k^2$  and  $\frac{35}{32}k^3$ .

**4. Set middle twice**

For three terms in an arithmetic progression, twice the middle term equals the sum of the outer terms:  $21k^2 = 56k + \frac{35}{32}k^3$ .

**5. Use  $k$  non-zero**

Since  $k \neq 0$ , divide by  $k$  and multiply by 32:  $672k = 1792 + 35k^2$ .

**6. Solve the quadratic**

So  $35k^2 - 672k + 1792 = 0$ . The discriminant is  $672^2 - 4(35)(1792) = 200704 = 448^2$ .

**7. Find the possible values**

$$k = \frac{672 \pm 448}{70}, \text{ giving } k = 16 \text{ or } k = \frac{16}{5}.$$

**Final answer**

$$(a) 128 + 56kx + \frac{21}{2}k^2x^2 + \frac{35}{32}k^3x^3.$$

$$(b) k = \frac{16}{5} \text{ or } k = 16.$$

**Question 4****Laws of Logarithms**

4. (i) Using the laws of logarithms, solve

$$\log_3(4x) + 2 = \log_3(5x + 7) \quad (3)$$

- (ii) Given that

$$\sum_{r=1}^2 \log_a(y^r) = \sum_{r=1}^2 (\log_a y)^r \quad y > 1, a > 1, y \neq a$$

find  $y$  in terms of  $a$ , giving your answer in simplest form.

(3)

# Worked Solution - Question 4

## 1. Rewrite the constant

Since  $2 = \log_3 9$ , the equation becomes  $\log_3(4x) + \log_3 9 = \log_3(5x + 7)$ .

## 2. Combine the logarithms

$$\log_3(4x) + \log_3 9 = \log_3(36x).$$

## 3. Solve for x

Therefore  $36x = 5x + 7$ , so  $31x = 7$  and  $x = \frac{7}{31}$ .

## 4. Introduce L

Let  $L = \log_a y$ . Then  $\log_a(y^r) = r \log_a y = rL$ .

## 5. Evaluate the left sum

$$\sum_{r=1}^2 \log_a(y^r) = L + 2L = 3L.$$

## 6. Evaluate the right sum

$$\sum_{r=1}^2 (\log_a y)^r = L + L^2.$$

## 7. Solve for L

So  $3L = L + L^2$ , which gives  $L^2 - 2L = 0$  and  $L(L - 2) = 0$ .

## 8. Use the restrictions

Since  $a > 1$  and  $y > 1$ ,  $\log_a y > 0$ , so  $L \neq 0$ . Hence  $L = 2$  and  $y = a^2$ .

**Final answer**

$$(i) x = \frac{7}{31}.$$

$$(ii) y = a^2.$$

## Question 5

## Polynomials

5. 
$$f(x) = x^3 + (p + 3)x^2 - x + q$$

where  $p$  and  $q$  are constants and  $p > 0$

Given that  $(x - 3)$  is a factor of  $f(x)$

(a) show that

$$9p + q = -51 \quad (2)$$

Given also that when  $f(x)$  is divided by  $(x + p)$  the remainder is 9

(b) show that

$$3p^2 + p + q - 9 = 0 \quad (2)$$

(c) Hence find the value of  $p$  and the value of  $q$ .

(3)

(d) Hence find a quadratic expression  $g(x)$  such that

$$f(x) = (x - 3)g(x) \quad (2)$$

## Worked Solution - Question 5

**1. Use the factor theorem**

Since  $(x - 3)$  is a factor,  $f(3) = 0$ .

**2. Substitute x equals three**

$27 + 9(p + 3) - 3 + q = 0$ , so  $51 + 9p + q = 0$ .

**3. Show the first relation**

Therefore  $9p + q = -51$ .

**4. Use the remainder theorem**

Dividing by  $(x + p)$  means substituting  $x = -p$ . The remainder is  $f(-p)$ .

**5. Substitute x equals negative p**

$f(-p) = -p^3 + (p + 3)p^2 + p + q = 3p^2 + p + q$ .

**6. Show the second relation**

The remainder is 9, so  $3p^2 + p + q = 9$ , hence  $3p^2 + p + q - 9 = 0$ .

**7. Eliminate q**

From  $9p + q = -51$ ,  $q = -51 - 9p$ . Substitute into  $3p^2 + p + q - 9 = 0$ .

**8. Solve for p**

$3p^2 + p - 51 - 9p - 9 = 0$ , so  $3p^2 - 8p - 60 = 0$ . This gives  $p = 6$  or  $p = -\frac{10}{3}$ .

**9. Use p greater than zero**

Since  $p > 0$ ,  $p = 6$ . Then  $q = -51 - 9(6) = -105$ .

**10. Form f with p and q**

$$f(x) = x^3 + 9x^2 - x - 105.$$

**11. Divide by x minus three**

$$x^3 + 9x^2 - x - 105 = (x - 3)(x^2 + 12x + 35), \text{ so } g(x) = x^2 + 12x + 35.$$

**Final answer**

(a) Proven.

(b) Proven.

(c)  $p = 6$ ,  $q = -105$ .

(d)  $g(x) = x^2 + 12x + 35$ .

## Question 6

Circles

6. The circle  $C$  has equation

$$x^2 + y^2 + 8x - 4y = 0$$

(a) Find

- (i) the coordinates of the centre of  $C$ ,
- (ii) the exact radius of  $C$ .

(3)

The point  $P$  lies on  $C$ .

Given that the tangent to  $C$  at  $P$  has equation  $x + 2y + 10 = 0$

(b) find the coordinates of  $P$

(4)

(c) Find the equation of the normal to  $C$  at  $P$ , giving your answer in the form  $y = mx + c$  where  $m$  and  $c$  are integers to be found.

(3)

# Worked Solution - Question 6

**1. Complete the square**

$x^2 + y^2 + 8x - 4y = 0$  becomes  $(x + 4)^2 - 16 + (y - 2)^2 - 4 = 0$ .

**2. Write the circle form**

So  $(x + 4)^2 + (y - 2)^2 = 20$ .

**3. Read the centre and radius**

The centre is  $(-4, 2)$  and the radius is  $\sqrt{20} = 2\sqrt{5}$ .

**4. Use the tangent gradient**

The tangent  $x + 2y + 10 = 0$  has gradient  $-\frac{1}{2}$ .

**5. Find the radius gradient**

The radius to the point of contact is perpendicular to the tangent, so its gradient is 2.

**6. Write the normal through the centre**

The radius line through the centre  $(-4, 2)$  is  $y - 2 = 2(x + 4)$ , so  $y = 2x + 10$ .

**7. Intersect with the tangent**

Substitute  $y = 2x + 10$  into  $x + 2y + 10 = 0$ :  $x + 2(2x + 10) + 10 = 0$ .

**8. Find P**

$5x + 30 = 0$ , so  $x = -6$ . Then  $y = 2(-6) + 10 = -2$ , hence  $P = (-6, -2)$ .

**9. State the normal**

The normal at  $P$  is the radius line already found, so its equation is  $y = 2x + 10$ .

**Final answer**

$$(a)(i) (-4, 2).$$

$$(a)(ii) 2\sqrt{5}.$$

$$(b) P = (-6, -2).$$

$$(c) y = 2x + 10.$$

## Question 7

## Geometric Sequences

7. A geometric sequence has first term  $a$  and common ratio  $r$ , where  $r > 0$

Given that

- the 3rd term is 20
- the 5th term is 12.8

(a) show that  $r = 0.8$

(1)

(b) Hence find the value of  $a$ .

(2)

Given that the sum of the first  $n$  terms of this sequence is greater than 156

(c) find the smallest possible value of  $n$ .

*(Solutions based entirely on graphical or numerical methods are not acceptable.)*

(4)

# Worked Solution - Question 7

## 1. Write the given terms

For a geometric sequence, the third term is  $ar^2 = 20$  and the fifth term is  $ar^4 = 12.8$ .

## 2. Divide the equations

$$\frac{ar^4}{ar^2} = \frac{12.8}{20}, \text{ so } r^2 = 0.64.$$

## 3. Use $r$ positive

Since  $r > 0$ ,  $r = 0.8$ .

## 4. Find $a$

Using  $ar^2 = 20$ ,  $a(0.8)^2 = 20$ , so  $a = \frac{20}{0.64} = 31.25$ .

## 5. Write the sum formula

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{31.25(1 - 0.8^n)}{0.2} = 156.25(1 - 0.8^n).$$

## 6. Set up the inequality

We need  $156.25(1 - 0.8^n) > 156$ .

## 7. Isolate the power

$$1 - 0.8^n > \frac{156}{156.25} = \frac{624}{625}, \text{ so } 0.8^n < \frac{1}{625}.$$

## 8. Use logarithms

$n \log(0.8) < \log\left(\frac{1}{625}\right)$ . Since  $\log(0.8) < 0$ , dividing reverses the inequality:  
 $n > \frac{\log(1/625)}{\log(0.8)} = 28.85 \dots$

### 9. Choose the smallest integer

The smallest possible integer value is therefore  $n = 29$ .

#### Final answer

(a)  $r = 0.8$ .

(b)  $a = 31.25$ .

(c)  $n = 29$ .

## Question 8

## Trigonometric Equations

8. In this question you must show all stages of your working.

Solutions based entirely on calculator technology are not acceptable.

- (i) Solve, for  $-\frac{\pi}{2} < x < \pi$ , the equation

$$5 \sin(3x + 0.1) + 2 = 0$$

giving your answers, **in radians**, to 2 decimal places.

(4)

- (ii) Solve, for  $0 < \theta < 360^\circ$ , the equation

$$2 \tan \theta \sin \theta = 5 + \cos \theta$$

giving your answers, **in degrees**, to one decimal place.

(5)

# Worked Solution - Question 8

## 1. Isolate the sine term

$$5 \sin(3x + 0.1) + 2 = 0 \text{ gives } \sin(3x + 0.1) = -0.4.$$

## 2. Find the angle interval

Since  $-\frac{\pi}{2} < x < \pi$ , multiplying by 3 and adding 0.1 gives

$$-\frac{3\pi}{2} + 0.1 < 3x + 0.1 < 3\pi + 0.1.$$

## 3. Solve for the inner angle

In this interval,  $3x + 0.1 = -2.7301\dots, -0.4115\dots, 3.5531\dots, 5.8717\dots$

## 4. Find x values

Subtract 0.1 and divide by 3:

$$x = -0.9433\dots, -0.1705\dots, 1.1510\dots, 1.9239\dots$$

## 5. Round the radian answers

To 2 decimal places,  $x = -0.94, -0.17, 1.15, 1.92$ .

## 6. Rewrite the second equation

$$2 \tan \theta \sin \theta = 5 + \cos \theta. \text{ Using } \tan \theta = \frac{\sin \theta}{\cos \theta} \text{ gives } \frac{2 \sin^2 \theta}{\cos \theta} = 5 + \cos \theta.$$

## 7. Use sine squared

Multiplying by  $\cos \theta$  gives  $2 \sin^2 \theta = 5 \cos \theta + \cos^2 \theta$ . Substitute  $\sin^2 \theta = 1 - \cos^2 \theta$ .

## 8. Solve for cos theta

$$2(1 - \cos^2 \theta) = 5 \cos \theta + \cos^2 \theta, \text{ so } 3 \cos^2 \theta + 5 \cos \theta - 2 = 0.$$

### 9. Factor and reject impossible value

$(3 \cos \theta - 1)(\cos \theta + 2) = 0$ . Since  $\cos \theta = -2$  is impossible,  $\cos \theta = \frac{1}{3}$ .

### 10. Find theta

For  $0 < \theta < 360^\circ$ ,  $\cos \theta = \frac{1}{3}$  gives  $\theta = 70.5^\circ$  or  $289.5^\circ$  to one decimal place.

#### Final answer

(i)  $x = -0.94, -0.17, 1.15, 1.92$ .

(ii)  $\theta = 70.5^\circ, 289.5^\circ$ .

WMA12/01 JANUARY 2023

8 marks

## Question 9

Integration

9.

In this question you must show all stages of your working.

Solutions based entirely on calculator technology are not acceptable.

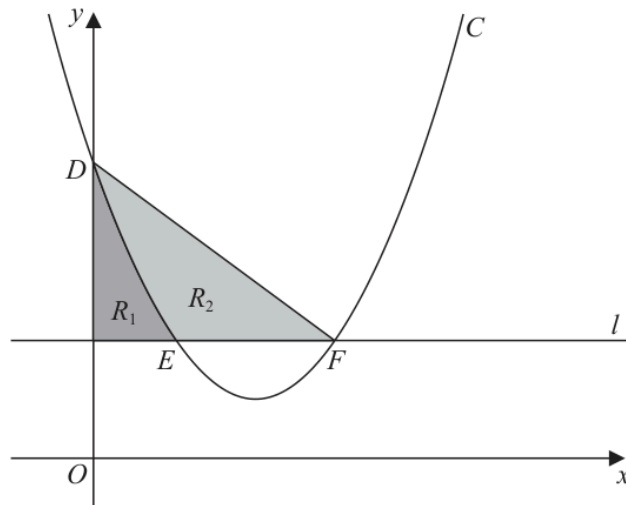


Figure 3

Figure 3 shows

- the curve  $C$  with equation  $y = x^2 - 4x + 5$
- the line  $l$  with equation  $y = 2$

The curve  $C$  intersects the  $y$ -axis at the point  $D$ .

(a) Write down the coordinates of  $D$ .

(1)

The curve  $C$  intersects the line  $l$  at the points  $E$  and  $F$ , as shown in Figure 3.

(b) Find the  $x$  coordinate of  $E$  and the  $x$  coordinate of  $F$ .

(2)

Shown shaded in Figure 3 is

- the region  $R_1$  which is bounded by  $C$ ,  $l$  and the  $y$ -axis
- the region  $R_2$  which is bounded by  $C$  and the line segments  $EF$  and  $DF$

Given that  $\frac{\text{area of } R_1}{\text{area of } R_2} = k$ , where  $k$  is a constant,

(c) use algebraic integration to find the exact value of  $k$ , giving your answer as a simplified fraction.

(5)

## Worked Solution - Question 9

**1. Find D**

At the  $y$ -axis,  $x = 0$ . On  $C$ ,  $y = 0^2 - 4(0) + 5 = 5$ , so  $D = (0, 5)$ .

**2. Find intersections with the line**

The line is  $y = 2$ , so solve  $x^2 - 4x + 5 = 2$ .

**3. Solve the quadratic**

$x^2 - 4x + 3 = 0$ , so  $(x - 1)(x - 3) = 0$ . Hence  $x_E = 1$  and  $x_F = 3$ .

**4. Area of R1**

$R_1$  lies between the curve and the line  $y = 2$  from  $x = 0$  to  $x = 1$ .

**5. Integrate R1**

$$R_1 = \int_0^1 (x^2 - 4x + 5 - 2) dx = \int_0^1 (x^2 - 4x + 3) dx.$$

**6. Evaluate R1**

$$R_1 = \left[ \frac{x^3}{3} - 2x^2 + 3x \right]_0^1 = \frac{1}{3} - 2 + 3 = \frac{4}{3}.$$

**7. Find the line DF**

The line through  $D = (0, 5)$  and  $F = (3, 2)$  has gradient  $-1$ , so its equation is  $y = 5 - x$ .

**8. Split R2**

For  $0 \leq x \leq 1$ ,  $R_2$  is between  $y = 5 - x$  and the curve. For  $1 \leq x \leq 3$ , it is between  $y = 5 - x$  and  $y = 2$ .

**9. Integrate R2**

$$R_2 = \int_0^1 [(5-x) - (x^2 - 4x + 5)] dx + \int_1^3 [(5-x) - 2] dx.$$

**10. Evaluate R2**

$$R_2 = \int_0^1 (-x^2 + 3x) dx + \int_1^3 (3-x) dx = \frac{7}{6} + 2 = \frac{19}{6}.$$

**11. Find the ratio**

$$k = \frac{\text{area of } R_1}{\text{area of } R_2} = \frac{4/3}{19/6} = \frac{8}{19}.$$

**Final answer**

(a)  $D = (0, 5)$ .

(b)  $x_E = 1, x_F = 3$ .

(c)  $k = \frac{8}{19}$ .

## Question 10

Proof

10. A student was asked to prove by exhaustion that

if  $n$  is an integer then  $2n^2 + n + 1$  is **not** divisible by 3

The start of the student's proof is shown in the box below.

Consider the case when  $n = 3k$

$$2n^2 + n + 1 = 18k^2 + 3k + 1 = 3(6k^2 + k) + 1$$

which is not divisible by 3

Complete this proof.

(4)

# Worked Solution - Question 10

Topic group

## 1. Use the three possible forms

Every integer is exactly one of the forms  $3k$ ,  $3k + 1$  or  $3k + 2$ , where  $k$  is an integer.

## 2. First case already given

For  $n = 3k$ , the expression becomes  $3(6k^2 + k) + 1$ , so it leaves remainder 1 when divided by 3.

## 3. Try $n$ equals $3k$ plus 1

If  $n = 3k + 1$ , then  $2n^2 + n + 1 = 2(3k + 1)^2 + (3k + 1) + 1$ .

## 4. Simplify the second case

$2(9k^2 + 6k + 1) + 3k + 2 = 18k^2 + 15k + 4 = 3(6k^2 + 5k + 1) + 1$ , which is not divisible by 3.

## 5. Try $n$ equals $3k$ plus 2

If  $n = 3k + 2$ , then  $2n^2 + n + 1 = 2(3k + 2)^2 + (3k + 2) + 1$ .

## 6. Simplify the third case

$2(9k^2 + 12k + 4) + 3k + 3 = 18k^2 + 27k + 11 = 3(6k^2 + 9k + 3) + 2$ , which is not divisible by 3.

## 7. Complete the exhaustion

All possible integer forms have been checked, and none gives a multiple of 3. Therefore  $2n^2 + n + 1$  is not divisible by 3.

**Final answer**

For  $n = 3k$ ,  $n = 3k + 1$  and  $n = 3k + 2$ , the expression has remainders **1**, **1** and **2** respectively when divided by **3**. Therefore  $2n^2 + n + 1$  is never divisible by **3**.

**PAST PAPER**

# **WMA12/01 May/June 2023**

**May/June 2023 | 11 questions | 75 marks**

**11**

questions

**75**

marks

**Answers**

worked solution  
after each  
question

## Question 1

1. The continuous curve  $C$  has equation  $y = f(x)$ .

A table of values of  $x$  and  $y$  for  $y = f(x)$  is shown below.

$x$	4.0	4.2	4.4	4.6	4.8	5.0
$y$	9.2	8.4556	3.8512	5.0342	7.8297	8.6

Use the trapezium rule with all the values of  $y$  in the table to find an approximation for

$$\int_4^5 f(x) dx$$

giving your answer to 3 decimal places.

(3)

# Worked Solution - Question 1

## 1. Identify the interval width

The  $x$  values go from 4.0 to 5.0 in steps of 0.2, so  $h = 0.2$ .

## 2. Apply the trapezium rule

$$\int_4^5 f(x) dx \approx \frac{0.2}{2} [9.2 + 8.6 + 2(8.4556 + 3.8512 + 5.0342 + 7.8297)].$$

## 3. Evaluate

The value is 6.81414, so the approximation is 6.814 to 3 decimal places.

**Final answer**

6.814.

**Question 2****Polynomials**

2.

**In this question you must show all stages of your working.****Solutions relying on calculator technology are not acceptable.**

$$f(x) = 4x^3 - 8x^2 + 5x + a$$

where  $a$  is a constant.Given that  $(2x - 3)$  is a factor of  $f(x)$ ,

(a) use the factor theorem to show that  $a = -3$  (2)

(b) Hence show that the equation  $f(x) = 0$  has only one real root. (4)

## Worked Solution - Question 2

**1. Use the factor theorem**

Since  $(2x - 3)$  is a factor,  $x = \frac{3}{2}$  is a root, so  $f\left(\frac{3}{2}\right) = 0$ .

**2. Substitute the root**

$$4\left(\frac{3}{2}\right)^3 - 8\left(\frac{3}{2}\right)^2 + 5\left(\frac{3}{2}\right) + a = 0.$$

**3. Find a**

$$13.5 - 18 + 7.5 + a = 0, \text{ so } 3 + a = 0 \text{ and } a = -3.$$

**4. Write the polynomial**

$$\text{With } a = -3, f(x) = 4x^3 - 8x^2 + 5x - 3.$$

**5. Factor using the known factor**

$$4x^3 - 8x^2 + 5x - 3 = (2x - 3)(2x^2 - x + 1).$$

**6. Check the quadratic**

The discriminant of  $2x^2 - x + 1$  is  $(-1)^2 - 4(2)(1) = 1 - 8 = -7$ .

**7. Conclude about real roots**

Since the quadratic has negative discriminant, it has no real roots. Therefore

$f(x) = 0$  has only the one real root from  $2x - 3 = 0$ , namely  $x = \frac{3}{2}$ .

**Final answer**

(a)  $a = -3$ .

(b)  $f(x) = 0$  has only one real root,  $x = \frac{3}{2}$ .

## Question 3

Circles

3. A circle  $C$  has centre  $(2, 5)$

Given that the point  $P(8, -3)$  lies on  $C$

(a) (i) find the radius of  $C$

(ii) find an equation for  $C$

(3)

(b) Find the equation of the tangent to  $C$  at  $P$  giving your answer in the form  $ax + by + c = 0$  where  $a$ ,  $b$  and  $c$  are integers to be found.

(4)

## Worked Solution - Question 3

**1. Use the distance formula**

The centre is  $(2, 5)$  and  $P = (8, -3)$ . The radius is  $\sqrt{(8 - 2)^2 + (-3 - 5)^2}$ .

**2. Find the radius**

$$r = \sqrt{6^2 + (-8)^2} = \sqrt{100} = 10.$$

**3. Write the circle equation**

With centre  $(2, 5)$  and radius 10, the circle is  $(x - 2)^2 + (y - 5)^2 = 100$ .

**4. Find the radius gradient**

The gradient of the radius from the centre to  $P$  is  $\frac{-3 - 5}{8 - 2} = \frac{-8}{6} = -\frac{4}{3}$ .

**5. Find the tangent gradient**

The tangent is perpendicular to the radius, so its gradient is  $\frac{3}{4}$ .

**6. Use point P**

The tangent through  $(8, -3)$  is  $y + 3 = \frac{3}{4}(x - 8)$ .

**7. Put into integer form**

Multiplying by 4 gives  $4y + 12 = 3x - 24$ , hence  $3x - 4y - 36 = 0$ .

**Final answer**

(a)(i) 10.

(a)(ii)  $(x - 2)^2 + (y - 5)^2 = 100$ .

(b)  $3x - 4y - 36 = 0$ .

## Question 4

## Binomial Expansion

4. The binomial expansion, in ascending powers of  $x$ , of

$$(3 + px)^5$$

where  $p$  is a constant, can be written in the form

$$A + Bx + Cx^2 + Dx^3 \dots$$

where  $A$ ,  $B$ ,  $C$  and  $D$  are constants.

- (a) Find the value of  $A$

(1)

Given that

- $B = 18D$
- $p < 0$

- (b) find

- the value of  $p$
- the value of  $C$

(6)

## Worked Solution - Question 4

**1. Find the constant term**

In  $(3 + px)^5$ , the constant term is  $3^5 = 243$ , so  $A = 243$ .

**2. Find B**

The coefficient of  $x$  is  $5(3^4)p = 405p$ , so  $B = 405p$ .

**3. Find D**

The coefficient of  $x^3$  is  $\binom{5}{3}(3^2)p^3 = 10 \cdot 9p^3 = 90p^3$ , so  $D = 90p^3$ .

**4. Use B equals 18D**

$$405p = 18(90p^3) = 1620p^3.$$

**5. Solve for p**

Since  $p < 0$ ,  $p \neq 0$ . Divide by  $405p$  to get  $1 = 4p^2$ , so  $p = \pm \frac{1}{2}$ . Therefore  $p = -\frac{1}{2}$ .

**6. Find C**

The coefficient of  $x^2$  is  $\binom{5}{2}(3^3)p^2 = 10 \cdot 27 \cdot \frac{1}{4} = \frac{135}{2}$ .

**Final answer**

(a)  $A = 243$ .

(b)(i)  $p = -\frac{1}{2}$ .

(b)(ii)  $C = \frac{135}{2}$ .

**Question 5****Laws of Logarithms**

5. Use the laws of logarithms to solve

$$\log_2(16x) + \log_2(x + 1) = 3 + \log_2(x + 6)$$

(5)

# Worked Solution - Question 5

Topic group

## 1. State the domain

The logarithms require  $16x > 0$ ,  $x + 1 > 0$  and  $x + 6 > 0$ , so  $x > 0$ .

## 2. Combine the left side

$$\log_2(16x) + \log_2(x + 1) = \log_2(16x(x + 1)).$$

## 3. Rewrite the right side

$$3 = \log_2 8, \text{ so } 3 + \log_2(x + 6) = \log_2(8(x + 6)).$$

## 4. Equate the arguments

$$\text{Therefore } 16x(x + 1) = 8(x + 6).$$

## 5. Solve the quadratic

$$16x^2 + 16x = 8x + 48, \text{ so } 16x^2 + 8x - 48 = 0. \text{ Dividing by 8 gives } 2x^2 + x - 6 = 0.$$

## 6. Factor and use the domain

$$(2x - 3)(x + 2) = 0, \text{ so } x = \frac{3}{2} \text{ or } x = -2. \text{ The domain } x > 0 \text{ rejects } -2, \\ \text{leaving } x = \frac{3}{2}.$$

**Final answer**

$$x = \frac{3}{2}.$$

## Question 6

## Geometric Sequences

6. In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

A software developer released an app to download.

The numbers of downloads of the app each month, in thousands, for the first three months after the app was released were

$$2k - 15 \quad k \quad k + 4$$

where  $k$  is a constant.

Given that the numbers of downloads each month are modelled as a geometric series,

(a) show that  $k^2 - 7k - 60 = 0$  (2)

(b) predict the number of downloads in the 4th month. (4)

The **total** number of all downloads of the app is predicted to exceed 3 million for the first time in the  $N$ th month.

(c) Calculate the value of  $N$  according to the model. (3)

# Worked Solution - Question 6

## 1. Use the geometric condition

For three consecutive terms of a geometric sequence, the middle term squared equals the product of the outer terms.

## 2. Set up the equation

$$\text{So } k^2 = (2k - 15)(k + 4).$$

## 3. Show the required quadratic

Expanding gives  $k^2 = 2k^2 - 7k - 60$ , hence  $k^2 - 7k - 60 = 0$ .

## 4. Find $k$

$k^2 - 7k - 60 = (k - 12)(k + 5) = 0$ , so  $k = 12$  or  $k = -5$ . The download numbers must be positive, so  $k = 12$ .

## 5. Find the common ratio

The first three monthly values are **9**, **12** and **16** thousand. The common ratio is  $\frac{12}{9} = \frac{4}{3}$ .

## 6. Predict month four

The 4th month is  $16 \cdot \frac{4}{3} = \frac{64}{3}$  thousand downloads, which is about **21300** downloads.

## 7. Write the cumulative total

In thousands, the total after  $N$  months is

$$S_N = \frac{9 \left( \left( \frac{4}{3} \right)^N - 1 \right)}{\frac{4}{3} - 1} = 27 \left( \left( \frac{4}{3} \right)^N - 1 \right).$$

### 8. Compare with three million

Three million downloads is **3000** thousand, so we need

$$27 \left( \left( \frac{4}{3} \right)^N - 1 \right) > 3000.$$

### 9. Solve the inequality

$$\left( \frac{4}{3} \right)^N > \frac{1009}{9}. \text{ Taking logarithms gives } N > \frac{\log(1009/9)}{\log(4/3)} = 16.405\dots$$

### 10. Choose the first month

The first integer month for which the total exceeds **3** million is  $N = 17$ .

#### Final answer

(a) Proven.

(b)  $\frac{64}{3}$  thousand downloads, about **21300** downloads.

(c)  $N = 17$ .

## Question 7

## Applications of Differentiation

7. The height of a river above a fixed point on the riverbed was monitored over a 7-day period.

The height of the river,  $H$  metres,  $t$  days after monitoring began, was given by

$$H = \frac{\sqrt{t}}{20}(20 + 6t - t^2) + 17 \quad 0 \leq t \leq 7$$

Given that  $H$  has a stationary value at  $t = \alpha$

- (a) use calculus to show that  $\alpha$  satisfies the equation

$$5\alpha^2 - 18\alpha - 20 = 0 \quad (5)$$

- (b) Hence find the value of  $\alpha$ , giving your answer to 3 decimal places. (1)

- (c) Use further calculus to prove that  $H$  is a maximum at this value of  $\alpha$ . (2)

# Worked Solution - Question 7

Topic group

**1. Rewrite H with powers**

$$H = \frac{1}{20}t^{1/2}(20 + 6t - t^2) + 17.$$

**2. Expand before differentiating**

$$H = \frac{1}{20}(20t^{1/2} + 6t^{3/2} - t^{5/2}) + 17.$$

**3. Differentiate**

$$\frac{dH}{dt} = \frac{1}{20} \left( 10t^{-1/2} + 9t^{1/2} - \frac{5}{2}t^{3/2} \right).$$

**4. Put over a common factor**

$$\frac{dH}{dt} = \frac{-5t^2 + 18t + 20}{40\sqrt{t}}.$$

**5. Use the stationary condition**

At  $t = \alpha$ ,  $\frac{dH}{dt} = 0$ . Since  $\alpha > 0$ , the denominator is not zero, so  
 $-5\alpha^2 + 18\alpha + 20 = 0$ .

**6. Show the required equation**

Multiplying by  $-1$  gives  $5\alpha^2 - 18\alpha - 20 = 0$ .

**7. Find alpha**

$$\alpha = \frac{18 \pm \sqrt{18^2 + 4(5)(20)}}{10} = \frac{9 \pm \sqrt{181}}{5}.$$

### 8. Use the time interval

Only the positive value lies in  $0 \leq t \leq 7$ , so  $\alpha = \frac{9 + \sqrt{181}}{5} = 4.491$  to 3 decimal places.

### 9. Differentiate again

$$\frac{d^2H}{dt^2} = \frac{-15t^2 + 18t - 20}{80t^{3/2}}.$$

### 10. Use the stationary equation

At  $t = \alpha$ ,  $5\alpha^2 = 18\alpha + 20$ . Therefore the numerator of  $\frac{d^2H}{dt^2}$  is  $-15\alpha^2 + 18\alpha - 20 = -3(18\alpha + 20) + 18\alpha - 20 = -36\alpha - 80$ .

### 11. Show it is negative

Since  $\alpha > 0$ ,  $-36\alpha - 80 < 0$ , and the denominator  $80\alpha^{3/2}$  is positive. Thus  $\frac{d^2H}{dt^2} < 0$ , so  $H$  is a maximum.

#### Final answer

- (a) Proven.
- (b)  $\alpha = 4.491$ .
- (c)  $H$  is a maximum at this value.

## Question 8

Proof

8. (i) A student writes the following statement:

“When  $a$  and  $b$  are consecutive **prime** numbers,  $a^2 + b^2$  is never a multiple of 10”

Prove by counter example that this statement is **not** true.

(2)

- (ii) Given that  $x$  and  $y$  are even integers greater than 0 and less than 6, prove by exhaustion, that

$$1 < x^2 - \frac{xy}{4} < 15$$

(3)

# Worked Solution - Question 8

Topic group

**1. Choose consecutive primes**

7 and 11 are consecutive prime numbers.

**2. Check the statement**

$7^2 + 11^2 = 49 + 121 = 170$ , which is a multiple of 10.

**3. Conclude the counterexample**

This single counterexample proves the student's statement is not true.

**4. List the possible even values**

The even integers greater than 0 and less than 6 are 2 and 4. Therefore the possible ordered pairs are (2, 2), (2, 4), (4, 2), (4, 4).

**5. Check x equals two**

If  $(x, y) = (2, 2)$ , then  $x^2 - \frac{xy}{4} = 4 - 1 = 3$ . If  $(x, y) = (2, 4)$ , then  $x^2 - \frac{xy}{4} = 4 - 2 = 2$ .

**6. Check x equals four**

If  $(x, y) = (4, 2)$ , then  $x^2 - \frac{xy}{4} = 16 - 2 = 14$ . If  $(x, y) = (4, 4)$ , then  $x^2 - \frac{xy}{4} = 16 - 4 = 12$ .

**7. Complete the exhaustion**

The possible values are 3, 2, 14, 12, and each lies strictly between 1 and 15. Hence  $1 < x^2 - \frac{xy}{4} < 15$  in every possible case.

**Final answer**

(i) A counterexample is  $a = 7, b = 11$ , since  $7^2 + 11^2 = 170$ .

(ii) Exhaustion proves the inequality for

$(x, y) = (2, 2), (2, 4), (4, 2), (4, 4)$ .

**Question 9****Trigonometric Equations**

9. **In this question you must show all stages of your working.**

**Solutions relying entirely on calculator technology are not acceptable.**

(a) Show that

$$3 \cos \theta (\tan \theta \sin \theta + 3) = 11 - 5 \cos \theta$$

may be written as

$$3 \cos^2 \theta - 14 \cos \theta + 8 = 0 \tag{3}$$

(b) Hence solve, for  $0 < x < 360^\circ$

$$3 \cos 2x (\tan 2x \sin 2x + 3) = 11 - 5 \cos 2x$$

giving your answers to one decimal place.

**(4)**

# Worked Solution - Question 9

Topic group

**1. Rewrite tan theta sin theta**

$$\tan \theta \sin \theta = \frac{\sin^2 \theta}{\cos \theta}.$$

**2. Simplify the left side**

$$3 \cos \theta (\tan \theta \sin \theta + 3) = 3 \sin^2 \theta + 9 \cos \theta.$$

**3. Use sine squared**

The equation becomes  $3(1 - \cos^2 \theta) + 9 \cos \theta = 11 - 5 \cos \theta$ .

**4. Rearrange**

$$3 - 3 \cos^2 \theta + 9 \cos \theta = 11 - 5 \cos \theta, \text{ so } 3 \cos^2 \theta - 14 \cos \theta + 8 = 0.$$

**5. Apply the result to 2x**

For the equation in part (b), take  $\theta = 2x$ . Then  $3 \cos^2(2x) - 14 \cos(2x) + 8 = 0$ .

**6. Solve the quadratic**

Let  $u = \cos(2x)$ . Then  $3u^2 - 14u + 8 = 0$ , so  $(3u - 2)(u - 4) = 0$ .

**7. Reject the impossible value**

$u = 4$  is impossible for cosine, so  $\cos(2x) = \frac{2}{3}$ .

**8. Use the interval**

Since  $0 < x < 360^\circ$ , we have  $0 < 2x < 720^\circ$ .

**9. Find x**

In this interval,  $2x = 48.189 \dots^\circ, 311.810 \dots^\circ, 408.189 \dots^\circ, 671.810 \dots^\circ$ .

Dividing by 2 gives  $x = 24.1^\circ, 155.9^\circ, 204.1^\circ, 335.9^\circ$  to one decimal place.

**Final answer**

(a) Proven.

(b)  $x = 24.1^\circ, 155.9^\circ, 204.1^\circ, 335.9^\circ$ .

## Question 10

10. The curve  $C$  has equation

$$y = \frac{(x - k)^2}{\sqrt{x}} \quad x > 0$$

where  $k$  is a **positive** constant.

(a) Show that

$$\int_1^{16} \frac{(x - k)^2}{\sqrt{x}} dx = ak^2 + bk + \frac{2046}{5}$$

where  $a$  and  $b$  are integers to be found.

(5)

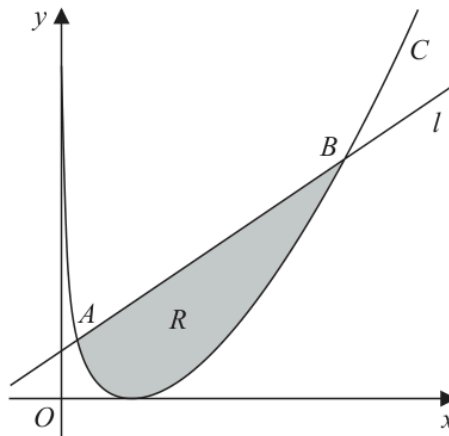


Figure 1

Figure 1 shows a sketch of the curve  $C$  and the line  $l$ .

Given that  $l$  intersects  $C$  at the point  $A(1, 9)$  and at the point  $B(16, q)$  where  $q$  is a constant,

(b) show that  $k = 4$

(2)

The region  $R$ , shown shaded in Figure 1, is bounded by  $C$  and  $l$

Using the answers to parts (a) and (b),

(c) find the area of region  $R$

(3)

## Worked Solution - Question 10

**1. Expand the integrand**

$$\frac{(x-k)^2}{\sqrt{x}} = \frac{x^2 - 2kx + k^2}{x^{1/2}} = x^{3/2} - 2kx^{1/2} + k^2x^{-1/2}.$$

**2. Integrate**

$$\int (x^{3/2} - 2kx^{1/2} + k^2x^{-1/2}) dx = \frac{2}{5}x^{5/2} - \frac{4k}{3}x^{3/2} + 2k^2x^{1/2}.$$

**3. Substitute the limits**

From 1 to 16, this gives  $\left(\frac{2048}{5} - \frac{256k}{3} + 8k^2\right) - \left(\frac{2}{5} - \frac{4k}{3} + 2k^2\right)$ .

**4. Simplify the integral**

The result is  $6k^2 - 84k + \frac{2046}{5}$ , so  $a = 6$  and  $b = -84$ .

**5. Use point A**

Since  $A(1, 9)$  lies on  $C$ ,  $9 = \frac{(1-k)^2}{\sqrt{1}} = (1-k)^2$ .

**6. Find  $k$** 

Thus  $1 - k = \pm 3$ , so  $k = 4$  or  $k = -2$ . Since  $k$  is positive,  $k = 4$ .

**7. Find point B**

When  $k = 4$  and  $x = 16$ ,  $q = \frac{(16-4)^2}{\sqrt{16}} = \frac{144}{4} = 36$ , so  $B = (16, 36)$ .

**8. Find the line**

The line through  $A(1, 9)$  and  $B(16, 36)$  has gradient  $\frac{36-9}{16-1} = \frac{9}{5}$ , so  $l$  has equation  $y = \frac{9}{5}x + \frac{36}{5}$ .

### 9. Area under the line

$$\int_1^{16} \left( \frac{9}{5}x + \frac{36}{5} \right) dx = \left[ \frac{9}{10}x^2 + \frac{36}{5}x \right]_1^{16} = \frac{675}{2}.$$

### 10. Area under the curve

From part (a), with  $k = 4$ ,  $\int_1^{16} \frac{(x-4)^2}{\sqrt{x}} dx = 6(4)^2 - 84(4) + \frac{2046}{5} = \frac{846}{5}.$

### 11. Subtract curve from line

The shaded area is  $\frac{675}{2} - \frac{846}{5} = \frac{1683}{10}.$

#### Final answer

(a)  $a = 6$ ,  $b = -84$ .

(b)  $k = 4$ .

(c) Area =  $\frac{1683}{10}.$

## Question 11

## Sequences &amp; Series

11. A sequence  $u_1, u_2, u_3, \dots$  is defined by

$$u_{n+1} = b - au_n$$

$$u_1 = 3$$

where  $a$  and  $b$  are constants.

(a) Find, in terms of  $a$  and  $b$ ,

(i)  $u_2$

(ii)  $u_3$

(2)

Given

- $\sum_{n=1}^3 u_n = 153$

- $b = a + 9$

(b) show that

$$a^2 - 5a - 66 = 0$$

(3)

(c) Hence find the larger possible value of  $u_2$

(3)

# Worked Solution - Question 11

**1. Find  $u_2$** 

Using  $u_{n+1} = b - au_n$  and  $u_1 = 3$ , we get  $u_2 = b - a(3) = b - 3a$ .

**2. Find  $u_3$** 

$$u_3 = b - au_2 = b - a(b - 3a) = b - ab + 3a^2.$$

**3. Use the sum condition**

$$u_1 + u_2 + u_3 = 153, \text{ so } 3 + (b - 3a) + (b - ab + 3a^2) = 153.$$

**4. Substitute  $b$  equals  $a$  plus nine**

Using  $b = a + 9$ , the left side becomes

$$3 + (a + 9) - 3a + (a + 9) - a(a + 9) + 3a^2.$$

**5. Simplify**

This is  $2a^2 - 10a + 21$ , so  $2a^2 - 10a + 21 = 153$ .

**6. Show the required equation**

$$2a^2 - 10a - 132 = 0. \text{ Dividing by } 2 \text{ gives } a^2 - 5a - 66 = 0.$$

**7. Solve for  $a$** 

$$a^2 - 5a - 66 = (a - 11)(a + 6) = 0, \text{ so } a = 11 \text{ or } a = -6.$$

**8. Find the corresponding  $u_2$  values**

If  $a = 11$ , then  $b = 20$  and  $u_2 = 20 - 3(11) = -13$ . If  $a = -6$ , then  $b = 3$  and  $u_2 = 3 - 3(-6) = 21$ .

**9. Choose the larger value**

The larger possible value of  $u_2$  is 21.

**Final answer**

(a)(i)  $u_2 = b - 3a$ .

(a)(ii)  $u_3 = b - ab + 3a^2$ .

(b) Proven.

(c) 21.

**PAST PAPER**

# **WMA12/01 October 2023**

**October 2023 | 10 questions | 75 marks**

**10**

questions

**75**

marks

**Answers**

worked solution  
after each  
question



# Worked Solution - Question 1

Topic group

## 1. Use the conditions

Since  $c = 3a + 1$  and  $a + b + c = 15$ , substitute for  $c$  to get  $a + b + 3a + 1 = 15$ .

## 2. Write $b$ in terms of $a$

This gives  $b = 14 - 4a$ .

## 3. List possible $a$ values

The integers  $a, b, c$  are positive. Since  $b = 14 - 4a > 0$ , we need  $a < 3.5$ . Therefore  $a = 1, 2, 3$ .

## 4. Check each case

If  $a = 1$ , then  $b = 10$  and  $c = 4$ , so  $abc = 40$ . If  $a = 2$ , then  $b = 6$  and  $c = 7$ , so  $abc = 84$ . If  $a = 3$ , then  $b = 2$  and  $c = 10$ , so  $abc = 60$ .

## 5. Conclude the exhaustion

The products **40**, **84** and **60** are all multiples of **4**. All possible cases have been checked, so  $abc$  is always a multiple of **4**.

### Final answer

The only possible triples are  $(1, 10, 4)$ ,  $(2, 6, 7)$  and  $(3, 2, 10)$ , giving products **40**, **84** and **60**. Each is a multiple of **4**, so  $abc$  is always a multiple of **4**.

## Question 2

## Sequences &amp; Series

2. A sequence  $u_1, u_2, u_3, \dots$  is defined by

$$u_1 = 3$$

$$u_{n+1} = 2 - \frac{4}{u_n}$$

(a) Find the value of  $u_2$ , the value of  $u_3$  and the value of  $u_4$

(3)

(b) Find the value of

$$\sum_{r=1}^{100} u_r$$

(2)

## Worked Solution - Question 2

**1. Find  $u_2$** 

$$u_2 = 2 - \frac{4}{u_1} = 2 - \frac{4}{3} = \frac{2}{3}.$$

**2. Find  $u_3$** 

$$u_3 = 2 - \frac{4}{u_2} = 2 - \frac{4}{2/3} = 2 - 6 = -4.$$

**3. Find  $u_4$** 

$$u_4 = 2 - \frac{4}{u_3} = 2 - \frac{4}{-4} = 3.$$

**4. Notice the cycle**

The sequence starts  $3, \frac{2}{3}, -4, 3, \frac{2}{3}, -4, \dots$ , so it repeats every 3 terms.

**5. Sum one cycle**

$$3 + \frac{2}{3} - 4 = -\frac{1}{3}.$$

**6. Use 100 terms**

There are **33** complete cycles in the first **99** terms, then one extra term  $u_{100} = 3$ .

**7. Find the sum**

$$\sum_{r=1}^{100} u_r = 33 \left( -\frac{1}{3} \right) + 3 = -11 + 3 = -8.$$

**Final answer**

(a)  $u_2 = \frac{2}{3}, u_3 = -4, u_4 = 3.$

(b)  $-8.$

**Question 3****Trigonometric Equations**

3. **In this question you must show all stages of your working.**

**Solutions relying entirely on calculator technology are not acceptable.**

(a) Solve, for  $0 < \theta \leq 360^\circ$  the equation

$$2 \tan \theta + 3 \sin \theta = 0$$

giving your answers, as appropriate, to one decimal place.

**(5)**

(b) Hence, or otherwise, find the smallest positive solution of

$$2 \tan(2x + 40^\circ) + 3 \sin(2x + 40^\circ) = 0$$

giving your answer to one decimal place.

**(2)**

# Worked Solution - Question 3

## 1. Factor the equation

$$2 \tan \theta + 3 \sin \theta = 0 \text{ becomes } \frac{2 \sin \theta}{\cos \theta} + 3 \sin \theta = 0.$$

## 2. Multiply by cos theta

Since  $\tan \theta$  is defined,  $\cos \theta \neq 0$ . Multiplying by  $\cos \theta$  gives  
 $2 \sin \theta + 3 \sin \theta \cos \theta = 0$ .

## 3. Factor

$$\sin \theta(2 + 3 \cos \theta) = 0.$$

## 4. Solve the sine case

For  $0 < \theta \leq 360^\circ$ ,  $\sin \theta = 0$  gives  $\theta = 180^\circ$  or  $360^\circ$ .

## 5. Solve the cosine case

$2 + 3 \cos \theta = 0$  gives  $\cos \theta = -\frac{2}{3}$ , so  $\theta = 131.8^\circ$  or  $228.2^\circ$  to one decimal place.

## 6. Apply to the new angle

For  $2 \tan(2x + 40^\circ) + 3 \sin(2x + 40^\circ) = 0$ , the angle  $2x + 40^\circ$  must be one of the solution angles, modulo  $360^\circ$ .

## 7. Find the smallest positive x

The smallest positive value comes from  $2x + 40^\circ = 131.8\dots^\circ$ , so  
 $x = \frac{131.8\dots - 40}{2} = 45.9^\circ$  to one decimal place.

**Final answer**

(a)  $\theta = 131.8^\circ, 180^\circ, 228.2^\circ, 360^\circ$ .

(b)  $x = 45.9^\circ$ .

## Question 4

## Polynomials

4. **In this question you must show all stages of your working.**  
**Solutions relying on calculator technology are not acceptable.**

$$f(x) = 4x^3 + ax^2 - 29x + b$$

where  $a$  and  $b$  are constants.

Given that  $(2x + 1)$  is a factor of  $f(x)$ ,

- (a) show that

$$a + 4b = -56 \quad (2)$$

Given also that when  $f(x)$  is divided by  $(x - 2)$  the remainder is  $-25$

- (b) find a second simplified equation linking  $a$  and  $b$ . (2)

- (c) Hence, using algebra and showing your working,

(i) find the value of  $a$  and the value of  $b$ ,

(ii) fully factorise  $f(x)$ .

(5)

## Worked Solution - Question 4

**1. Use the factor theorem**

Since  $(2x + 1)$  is a factor,  $x = -\frac{1}{2}$  is a root, so  $f\left(-\frac{1}{2}\right) = 0$ .

**2. Substitute  $x$  equals negative half**

$$4\left(-\frac{1}{2}\right)^3 + a\left(-\frac{1}{2}\right)^2 - 29\left(-\frac{1}{2}\right) + b = 0.$$

**3. Show the first relation**

$-\frac{1}{2} + \frac{a}{4} + \frac{29}{2} + b = 0$ . Multiplying by 4 gives  $a + 4b + 56 = 0$ , so  
 $a + 4b = -56$ .

**4. Use the remainder theorem**

Dividing by  $(x - 2)$  gives remainder  $f(2)$ .

**5. Substitute  $x$  equals two**

$$f(2) = 4(8) + 4a - 58 + b = 4a + b - 26.$$

**6. Find the second relation**

The remainder is  $-25$ , so  $4a + b - 26 = -25$ , hence  $4a + b = 1$ .

**7. Solve the simultaneous equations**

Solve  $a + 4b = -56$  and  $4a + b = 1$ . Multiplying the second equation by 4 gives  
 $16a + 4b = 4$ .

**8. Find  $a$  and  $b$** 

Subtracting  $a + 4b = -56$  from  $16a + 4b = 4$  gives  $15a = 60$ , so  $a = 4$ . Then  
 $16 + b = 1$ , so  $b = -15$ .

### 9. Write $f(x)$

Therefore  $f(x) = 4x^3 + 4x^2 - 29x - 15$ .

### 10. Factorise

Using the known factors  $(2x + 1)$  and  $(x - 2)$ , division gives

$$f(x) = (2x + 1)(x - 2)(2x + 15).$$

#### Final answer

(a) Proven.

(b)  $4a + b = 1$ .

(c)(i)  $a = 4$ ,  $b = -15$ .

(c)(ii)  $f(x) = (2x + 1)(x - 2)(2x + 15)$ .

**Question 5****Laws of Logarithms**

5.

**In this question you must show all stages of your working.****Solutions relying entirely on calculator technology are not acceptable.**

(i) Solve

$$3^a = 70$$

giving the answer to 3 decimal places.

**(2)**(ii) Find the exact value of  $b$  such that

$$4 + 3 \log_3 b = \log_3 5b$$

**(4)**

## Worked Solution - Question 5

Topic group

**1. Solve the exponential equation**

$$3^a = 70 \text{ gives } a = \log_3 70 = \frac{\log 70}{\log 3}.$$

**2. Round a**

$a = 3.867147\dots$ , so  $a = 3.867$  to 3 decimal places.

**3. Rewrite four as a logarithm**

$$4 = \log_3(3^4) = \log_3 81.$$

**4. Combine the left side**

$$4 + 3 \log_3 b = \log_3 81 + \log_3(b^3) = \log_3(81b^3).$$

**5. Equate arguments**

The equation becomes  $\log_3(81b^3) = \log_3(5b)$ , so  $81b^3 = 5b$ .

**6. Use the log domain**

Since  $\log_3 b$  is defined,  $b > 0$ . Divide by  $b$  to get  $81b^2 = 5$ .

**7. Find b**

$$b^2 = \frac{5}{81}, \text{ and } b > 0, \text{ so } b = \frac{\sqrt{5}}{9}.$$

**Final answer**

(i)  $a = 3.867$ .

(ii)  $b = \frac{\sqrt{5}}{9}$ .

## Question 6

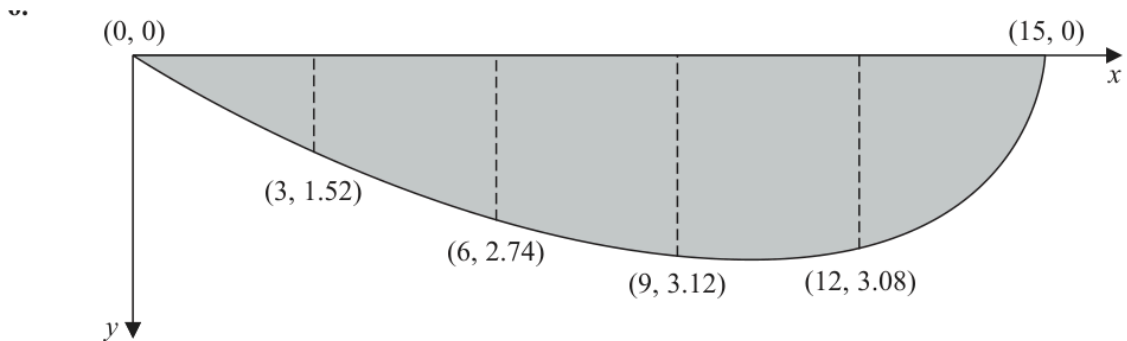


Figure 1

A river is being studied.

At one particular place, the river is 15 m wide.

The depth,  $y$  metres, of the river is measured at a point  $x$  metres from one side of the river.

Figure 1 shows a plot of the cross-section of the river and the coordinate values  $(x, y)$

- (a) Use the trapezium rule with all the  $y$  values given in Figure 1 to estimate the cross-sectional area of the river. (3)

The water in the river is modelled as flowing at a constant speed of  $1.5 \text{ m s}^{-1}$  across the whole of the cross-section.

- (b) Use the model and the answer to part (a) to estimate the volume of water flowing through this section of the river each minute, giving your answer in  $\text{m}^3$  to 2 significant figures. (2)

Assuming the model,

- (c) state, giving a reason for your answer, whether your answer for part (b) is an overestimate or an underestimate of the true volume of water flowing through this section of the river each minute. (1)

# Worked Solution - Question 6

## 1. Read the strip width

The  $x$  values are 0, 3, 6, 9, 12, 15, so the trapezium width is  $h = 3$ .

## 2. Apply the trapezium rule

$$\text{Area} \approx \frac{3}{2} [0 + 0 + 2(1.52 + 2.74 + 3.12 + 3.08)].$$

## 3. Find the cross-sectional area

This gives  $31.38 \text{ m}^2$ .

## 4. Convert flow speed to one minute

The speed is  $1.5 \text{ m s}^{-1}$ , so in one minute the water travels  $1.5 \times 60 = 90 \text{ m}$ .

## 5. Estimate volume

$$\text{Volume per minute} \approx 31.38 \times 90 = 2824.2 \text{ m}^3.$$

## 6. Round the volume

To 2 significant figures, this is  $2.8 \times 10^3 \text{ m}^3$ .

## 7. State over or under

It is an underestimate, because the curve between the plotted points lies below the straight-line trapezium chords in the diagram, so the true cross-sectional area is larger.

**Final answer**

(a)  $31.38 \text{ m}^2$ .

(b)  $2.8 \times 10^3 \text{ m}^3$ .

(c) Underestimate.

## Question 7

7.

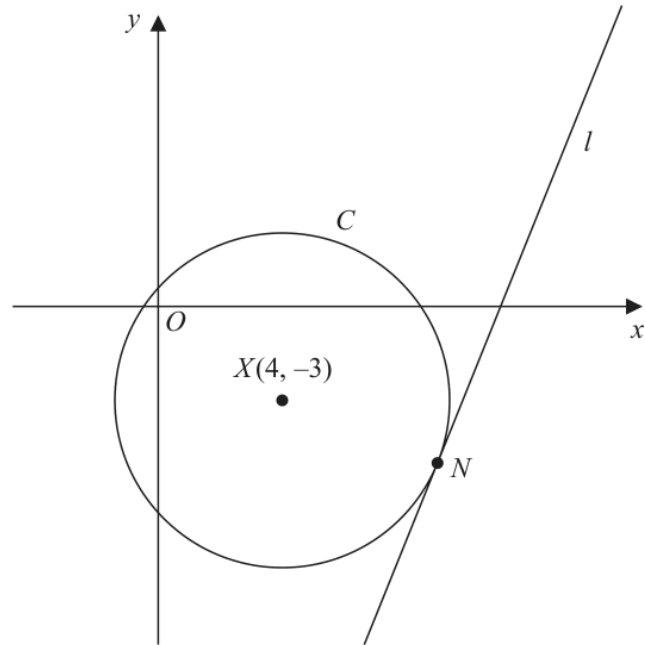


Figure 2

Figure 2 shows a sketch of

- the circle  $C$  with centre  $X(4, -3)$
- the line  $l$  with equation  $y = \frac{5}{2}x - \frac{55}{2}$

Given that  $l$  is the tangent to  $C$  at the point  $N$ ,

(a) show that an equation for the straight line passing through  $X$  and  $N$  is

$$2x + 5y + 7 = 0$$

(3)

(b) Hence find

- the coordinates of  $N$ ,
- an equation for  $C$ .

(5)

## Worked Solution - Question 7

**1. Use perpendicular gradients**

The tangent  $l$  has gradient  $\frac{5}{2}$ , so the radius  $XN$  is perpendicular to it and has gradient  $-\frac{2}{5}$ .

**2. Use the centre**

The line through  $X(4, -3)$  with gradient  $-\frac{2}{5}$  is  $y + 3 = -\frac{2}{5}(x - 4)$ .

**3. Show the required line**

Multiplying by 5 gives  $5y + 15 = -2x + 8$ , so  $2x + 5y + 7 = 0$ .

**4. Intersect the radius and tangent**

Use  $y = \frac{5}{2}x - \frac{55}{2}$  in  $2x + 5y + 7 = 0$ .

**5. Find x coordinate**

$2x + 5\left(\frac{5}{2}x - \frac{55}{2}\right) + 7 = 0$ . Multiplying by 2 gives  $4x + 25x - 275 + 14 = 0$ , so  $29x = 261$  and  $x = 9$ .

**6. Find y coordinate**

$y = \frac{5}{2}(9) - \frac{55}{2} = \frac{45 - 55}{2} = -5$ , so  $N = (9, -5)$ .

**7. Find the radius squared**

$XN^2 = (9 - 4)^2 + (-5 + 3)^2 = 5^2 + (-2)^2 = 29$ .

**8. Write the circle equation**

With centre  $(4, -3)$  and radius squared 29, the circle is  $(x - 4)^2 + (y + 3)^2 = 29$ .

**Final answer**

(a) Proven.

(b)(i)  $N = (9, -5)$ .

(b)(ii)  $(x - 4)^2 + (y + 3)^2 = 29$ .

## Question 8

## Arithmetic Sequences

8. In a large theatre there are  $n$  rows of seats, where  $n$  is a constant.

The number of seats in the first row is  $a$ , where  $a$  is a constant.

In each subsequent row there are 4 more seats than in the previous row so that

- in the 2nd row there are  $(a + 4)$  seats
- in the 3rd row there are  $(a + 8)$  seats
- the number of seats in each row form an **arithmetic** sequence

Given that the **total** number of seats in the first 10 rows is 360

(a) find the value of  $a$ .

(2)

Given also that the total number of seats in the  $n$  rows is 2146

(b) show that

$$n^2 + 8n - 1073 = 0$$

(2)

(c) Hence

- state the number of rows of seats in the theatre,
- find the maximum number of seats in any one row.

(3)

# Worked Solution - Question 8

**1. Use the arithmetic sum formula**

The first term is  $a$  and the common difference is 4.

**2. Use the first 10 rows**

$$S_{10} = \frac{10}{2}(2a + 9 \cdot 4) = 360.$$

**3. Find  $a$** 

$$5(2a + 36) = 360, \text{ so } 10a + 180 = 360 \text{ and } a = 18.$$

**4. Write  $S_n$** 

$$S_n = \frac{n}{2}(2(18) + (n - 1)4) = \frac{n}{2}(4n + 32) = 2n^2 + 16n.$$

**5. Use the total number of seats**

Since the total is 2146,  $2n^2 + 16n = 2146$ .

**6. Show the equation**

Dividing by 2 and rearranging gives  $n^2 + 8n - 1073 = 0$ .

**7. Solve for  $n$** 

$$n = \frac{-8 \pm \sqrt{8^2 + 4(1073)}}{2} = \frac{-8 \pm 66}{2}, \text{ so } n = 29 \text{ or } n = -37.$$

**8. Choose the row count**

The number of rows must be positive, so there are 29 rows.

**9. Find the maximum row size**

The largest row is the last row:  $18 + (29 - 1)4 = 18 + 112 = 130$  seats.

**Final answer**

(a)  $a = 18$ .

(b) Proven.

(c)(i) 29 rows.

(c)(ii) 130 seats.

## Question 9

## Applications of Differentiation

9.

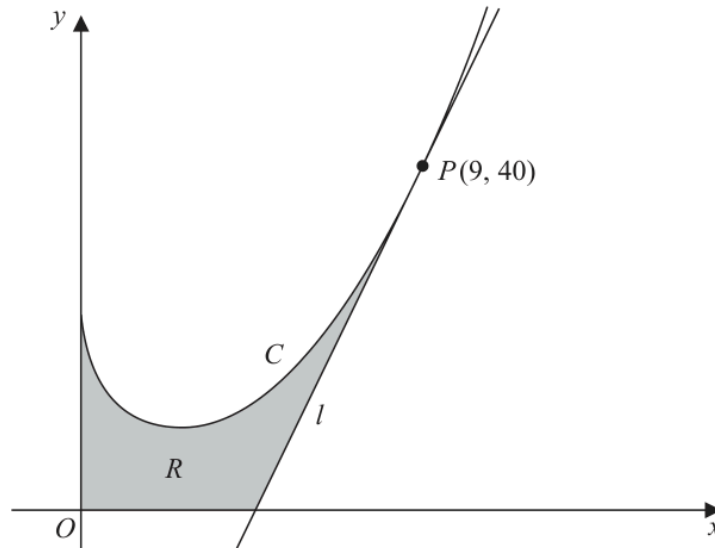


Figure 3

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 3 shows a sketch of part of the curve  $C$  with equation

$$y = \frac{2}{3}x^2 - 9\sqrt{x} + 13 \quad x \geq 0$$

(a) Find, using calculus, the range of values of  $x$  for which  $y$  is increasing.

(4)

The point  $P$  lies on  $C$  and has coordinates  $(9, 40)$ .

The line  $l$  is the tangent to  $C$  at the point  $P$ .

The finite region  $R$ , shown shaded in Figure 3, is bounded by the curve  $C$ , the line  $l$ , the  $x$ -axis and the  $y$ -axis.

(b) Find, using calculus, the exact area of  $R$ .

(8)

# Worked Solution - Question 9

Topic group

**1. Differentiate**

$$\text{For } y = \frac{2}{3}x^2 - 9\sqrt{x} + 13, \frac{dy}{dx} = \frac{4}{3}x - \frac{9}{2\sqrt{x}}.$$

**2. Set up increasing condition**

$$y \text{ is increasing when } \frac{dy}{dx} > 0.$$

**3. Multiply by a positive quantity**

$$\text{For } x > 0, \text{ multiply by } 6\sqrt{x}: 8x\sqrt{x} - 27 > 0.$$

**4. Solve the inequality**

$$x^{3/2} > \frac{27}{8} = \left(\frac{3}{2}\right)^3, \text{ so } \sqrt{x} > \frac{3}{2} \text{ and } x > \frac{9}{4}.$$

**5. Find the tangent gradient**

$$\text{At } P(9, 40), \frac{dy}{dx} = \frac{4}{3}(9) - \frac{9}{2 \cdot 3} = 12 - \frac{3}{2} = \frac{21}{2}.$$

**6. Write the tangent line**

$$y - 40 = \frac{21}{2}(x - 9), \text{ so } y = \frac{21}{2}x - \frac{109}{2}.$$

**7. Find where the tangent meets the x-axis**

$$\text{Set } y = 0: \frac{21}{2}x - \frac{109}{2} = 0, \text{ so } x = \frac{109}{21}.$$

**8. Split the area**

From  $x = 0$  to  $x = \frac{109}{21}$ , the lower boundary is the  $x$ -axis. From  $x = \frac{109}{21}$  to  $x = 9$ , the lower boundary is the tangent.

### 9. Set up the integral

$$R = \int_0^{109/21} \left( \frac{2}{3}x^2 - 9\sqrt{x} + 13 \right) dx + \int_{109/21}^9 \left( \frac{2}{3}x^2 - 9\sqrt{x} + 13 - \left( \frac{21}{2}x - \frac{1}{2} \right) \right) dx$$

### 10. Use antiderivatives

$$\int \left( \frac{2}{3}x^2 - 9\sqrt{x} + 13 \right) dx = \frac{2}{9}x^3 - 6x^{3/2} + 13x, \text{ and}$$
$$\int \left( \frac{21}{2}x - \frac{109}{2} \right) dx = \frac{21}{4}x^2 - \frac{109}{2}x.$$

### 11. Evaluate

Substituting the limits and simplifying gives  $R = \frac{857}{21}$ .

#### Final answer

(a)  $y$  is increasing for  $x > \frac{9}{4}$ .

(b) Area =  $\frac{857}{21}$ .

## Question 10

## Binomial Expansion

10. (i) (a) Find, in ascending powers of  $x$ , the 2nd, 3rd and 5th terms of the binomial expansion of

$$(3 + 2x)^6 \quad (3)$$

For a particular value of  $x$ , these three terms form consecutive terms in a geometric series.

- (b) Find this value of  $x$ . (3)

- (ii) In a **different** geometric series,

- the first term is  $\sin^2 \theta$
- the common ratio is  $2 \cos \theta$
- the sum to infinity is  $\frac{8}{5}$

- (a) Show that

$$5 \cos^2 \theta - 16 \cos \theta + 3 = 0 \quad (3)$$

- (b) Hence find the exact value of the 2nd term in the series. (3)

# Worked Solution - Question 10

## 1. Find the second term

The 2nd term of  $(3 + 2x)^6$  is  $\binom{6}{1}3^5(2x) = 6 \cdot 243 \cdot 2x = 2916x$ .

## 2. Find the third term

The 3rd term is  $\binom{6}{2}3^4(2x)^2 = 15 \cdot 81 \cdot 4x^2 = 4860x^2$ .

## 3. Find the fifth term

The 5th term is  $\binom{6}{4}3^2(2x)^4 = 15 \cdot 9 \cdot 16x^4 = 2160x^4$ .

## 4. Use the geometric condition

For three consecutive terms in a geometric series, the middle term squared equals the product of the two outer terms.

## 5. Set up the equation

$$(4860x^2)^2 = (2916x)(2160x^4).$$

## 6. Solve for x

For the non-zero geometric case, divide by  $x^4$ :  $4860^2 = 2916 \cdot 2160 x$ , so

$$x = \frac{4860^2}{2916 \cdot 2160} = \frac{15}{4}.$$

## 7. Use the sum to infinity

In the different geometric series, first term =  $\sin^2 \theta$ , common ratio =  $2 \cos \theta$ , and

$$S_{\infty} = \frac{8}{5}.$$

## 8. Form the equation

$$\frac{\sin^2 \theta}{1 - 2 \cos \theta} = \frac{8}{5}.$$

### 9. Use sine squared

$5 \sin^2 \theta = 8(1 - 2 \cos \theta)$ . With  $\sin^2 \theta = 1 - \cos^2 \theta$ , this becomes  
 $5(1 - \cos^2 \theta) = 8 - 16 \cos \theta$ .

### 10. Show the required equation

Rearranging gives  $5 \cos^2 \theta - 16 \cos \theta + 3 = 0$ .

### 11. Solve for cosine

$5 \cos^2 \theta - 16 \cos \theta + 3 = 0$  gives  $\cos \theta = 3$  or  $\cos \theta = \frac{1}{5}$ . The possible value is  
 $\cos \theta = \frac{1}{5}$ .

### 12. Find the second term

The 2nd term is first term times common ratio:

$$\sin^2 \theta (2 \cos \theta) = \left(1 - \frac{1}{25}\right) \cdot \frac{2}{5} = \frac{24}{25} \cdot \frac{2}{5} = \frac{48}{125}.$$

#### Final answer

(i)(a)  $2916x, 4860x^2, 2160x^4$ .

(i)(b)  $x = \frac{15}{4}$ .

(ii)(a) Proven.

(ii)(b)  $\frac{48}{125}$ .

**PAST PAPER**

# **WMA12/01 January 2024**

January 2024 | 10 questions | 75 marks

**10**

questions

**75**

marks

**Answers**

worked solution  
after each  
question

## Question 1

## Polynomials

1.  $f(x) = ax^3 + 3x^2 - 8x + 2$  where  $a$  is a constant

Given that when  $f(x)$  is divided by  $(x - 2)$  the remainder is 3, find the value of  $a$ .

(3)

## Worked Solution - Question 1

**1. Use the remainder theorem**

Dividing by  $(x - 2)$  means the remainder is  $f(2)$ .

**2. Substitute x equals two**

$$f(2) = a(2)^3 + 3(2)^2 - 8(2) + 2 = 8a + 12 - 16 + 2 = 8a - 2.$$

**3. Use the given remainder**

The remainder is 3, so  $8a - 2 = 3$ .

**4. Solve for a**

$$8a = 5, \text{ hence } a = \frac{5}{8}.$$

**Final answer**

$$a = \frac{5}{8}.$$

## Question 2

### Binomial Expansion

2. Find the coefficient of the term in  $x^7$  of the binomial expansion of

$$\left(\frac{3}{8} + 4x\right)^{12}$$

giving your answer in simplest form.

(3)

## Worked Solution - Question 2

**1. Choose the x power**

In  $\left(\frac{3}{8} + 4x\right)^{12}$ , the term in  $x^7$  comes from choosing  $(4x)^7$  and  $\left(\frac{3}{8}\right)^5$ .

**2. Write the term**

The coefficient is  $\binom{12}{7} \left(\frac{3}{8}\right)^5 4^7$ .

**3. Simplify the powers**

$4^7 = 2^{14}$  and  $8^5 = 2^{15}$ , so  $\left(\frac{3}{8}\right)^5 4^7 = \frac{3^5}{2}$ .

**4. Evaluate**

$\binom{12}{7} = 792$ , so the coefficient is  $792 \cdot \frac{243}{2} = 96228$ .

**Final answer**

96228.

## Question 3

Circles

3. The circle  $C$

- has centre  $A(3, 5)$
- passes through the point  $B(8, -7)$

(a) Find an equation for  $C$ .

(3)

The points  $M$  and  $N$  lie on  $C$  such that  $MN$  is a chord of  $C$ .

Given that  $MN$

- lies above the  $x$ -axis
- is parallel to the  $x$ -axis
- has length  $4\sqrt{22}$

(b) find an equation for the line passing through points  $M$  and  $N$ .

(3)

# Worked Solution - Question 3

Topic group

## 1. Find the radius squared

The centre is  $A(3, 5)$  and the circle passes through  $B(8, -7)$ , so  
 $r^2 = (8 - 3)^2 + (-7 - 5)^2 = 25 + 144 = 169$ .

## 2. Write the circle equation

Hence  $C$  has equation  $(x - 3)^2 + (y - 5)^2 = 169$ .

## 3. Use the horizontal chord

Since  $MN$  is parallel to the  $x$ -axis, its equation has the form  $y = c$ .

## 4. Use half the chord length

The chord length is  $4\sqrt{22}$ , so the half-length is  $2\sqrt{22}$ .

## 5. Relate chord length to the circle

At height  $y = c$ , the horizontal half-length satisfies  
 $169 - (c - 5)^2 = (2\sqrt{22})^2 = 88$ .

## 6. Solve for $c$

$(c - 5)^2 = 81$ , so  $c = 14$  or  $c = -4$ .

## 7. Use the position above the $x$ -axis

The chord lies above the  $x$ -axis, so  $c = 14$ . Therefore the line through  $M$  and  $N$  is  $y = 14$ .

**Final answer**

(a)  $(x - 3)^2 + (y - 5)^2 = 169$ .

(b)  $y = 14$ .

## Question 4

## Integration

4. (a) Sketch the curve with equation

$$y = a^{-x} + 4$$

where  $a$  is a constant and  $a > 1$

On your sketch show

- the coordinates of the point of intersection of the curve with the  $y$ -axis
- the equation of the asymptote to the curve.

(3)

$x$	-4	-1.5	1	3.5	6	8.5
$y$	13	6.280	4.577	4.146	4.037	4.009

The table above shows corresponding values of  $x$  and  $y$  for  $y = 3^{-\frac{1}{2}x} + 4$

The values of  $y$  are given to four significant figures, as appropriate.

Using the trapezium rule with all the values of  $y$  in the table,

- (b) find an approximate value for

$$\int_{-4}^{8.5} \left( 3^{-\frac{1}{2}x} + 4 \right) dx$$

giving your answer to two significant figures.

(3)

- (c) Using the answer to part (b), find an approximate value for

(i)  $\int_{-4}^{8.5} \left( 3^{-\frac{1}{2}x} \right) dx$

(ii)  $\int_{-4}^{8.5} \left( 3^{-\frac{1}{2}x} + 4 \right) dx + \int_{-8.5}^4 \left( 3^{\frac{1}{2}x} + 4 \right) dx$

(3)

## Worked Solution - Question 4

**1. Find the y-intercept**

For  $y = a^{-x} + 4$ , set  $x = 0$ :  $y = a^0 + 4 = 5$ , so the curve meets the  $y$ -axis at  $(0, 5)$ .

**2. Find the asymptote**

Since  $a > 1$ ,  $a^{-x} \rightarrow 0$  as  $x \rightarrow \infty$ , so the horizontal asymptote is  $y = 4$ .

**3. Describe the sketch**

The curve is decreasing, passes through  $(0, 5)$ , and approaches the line  $y = 4$  from above.

**4. Use the trapezium width**

The  $x$  values increase by 2.5, so  $h = 2.5$ .

**5. Apply the trapezium rule**

$$\int_{-4}^{8.5} (3^{-x/2} + 4) dx \approx \frac{2.5}{2} [13 + 4.009 + 2(6.280 + 4.577 + 4.146 + 4.037)].$$

**6. Round part b**

This gives 68.86125, so to two significant figures the estimate is 69.

**7. Remove the constant part**

$$\int_{-4}^{8.5} 4 dx = 4(12.5) = 50.$$

**8. Estimate part c i**

Using the answer to part (b),  $\int_{-4}^{8.5} 3^{-x/2} dx \approx 69 - 50 = 19$ .

### 9. Use symmetry by substitution

For  $\int_{-8.5}^4 (3^{x/2} + 4)dx$ , let  $u = -x$ . This changes it to the same integral as part (b).

### 10. Estimate part c ii

Therefore the two integrals in part (c)(ii) are approximately equal, so the total is  $69 + 69 = 138$ .

#### Final answer

(a)  $y$ -intercept  $(0, 5)$  and asymptote  $y = 4$ .

(b) 69.

(c)(i) 19.

(c)(ii) 138.

## Question 5

## Sequences &amp; Series

5. (i) Find the value of

$$\sum_{r=1}^{\infty} 6 \times (0.25)^r$$

(3)

(ii) A sequence  $u_1, u_2, u_3, \dots$  is defined by

$$\begin{aligned} u_1 &= 3 \\ u_{n+1} &= \frac{u_n - 3}{u_n - 2} \quad n \in \mathbb{N} \end{aligned}$$

(a) Show that this sequence is periodic.

(2)

(b) State the order of this sequence.

(1)

(c) Hence find

$$\sum_{n=1}^{70} u_n$$

(2)

## Worked Solution - Question 5

Topic group

**1. Identify the geometric series**

$\sum_{r=1}^{\infty} 6(0.25)^r$  has first term  $6(0.25) = 1.5$  and common ratio  $0.25$ .

**2. Sum to infinity**

$$S_{\infty} = \frac{1.5}{1 - 0.25} = \frac{1.5}{0.75} = 2.$$

**3. Find the first few terms**

$$u_1 = 3. \text{ Then } u_2 = \frac{3 - 3}{3 - 2} = 0.$$

**4. Continue the recurrence**

$$u_3 = \frac{0 - 3}{0 - 2} = \frac{3}{2}, \text{ and } u_4 = \frac{\frac{3}{2} - 3}{\frac{3}{2} - 2} = 3.$$

**5. Show periodicity**

Since  $u_4 = 3 = u_1$ , the terms repeat as  $3, 0, \frac{3}{2}, 3, 0, \frac{3}{2}, \dots$

**6. State the order**

The repeating cycle has length **3**, so the order is **3**.

**7. Sum one cycle**

$$\text{One full cycle sums to } 3 + 0 + \frac{3}{2} = \frac{9}{2}.$$

**8. Use 70 terms**

Since  $70 = 23 \cdot 3 + 1$ , the first **70** terms contain **23** complete cycles and one extra term  $u_1 = 3$ .

**9. Find the required sum**

$$\sum_{n=1}^{70} u_n = 23 \left( \frac{9}{2} \right) + 3 = \frac{207}{2} + \frac{6}{2} = \frac{213}{2}.$$

**Final answer**

(i) 2.

(ii)(a) Periodic.

(ii)(b) Order 3.

(ii)(c)  $\frac{213}{2}$ .

## Question 6

## Laws of Logarithms

6. (a) Given that

$$2\log_4(x+3) + \log_4 x = \log_4(4x+2) + \frac{1}{2}$$

show that

$$x^3 + 6x^2 + x - 4 = 0 \quad (4)$$

(b) Given also that  $-1$  is a root of the equation

$$x^3 + 6x^2 + x - 4 = 0$$

(i) use algebra to find the other two roots of the equation. (3)

(ii) Hence solve

$$2\log_4(x+3) + \log_4 x = \log_4(4x+2) + \frac{1}{2} \quad (1)$$

## Worked Solution - Question 6

**1. Combine the logarithms**

$2\log_4(x+3) = \log_4((x+3)^2)$ , so the left side is  $\log_4(x(x+3)^2)$ .

**2. Rewrite one half**

$\frac{1}{2} = \log_4 2$ , so the right side is  $\log_4(4x+2) + \log_4 2 = \log_4(8x+4)$ .

**3. Equate arguments**

Therefore  $x(x+3)^2 = 8x+4$ .

**4. Expand**

$x(x^2+6x+9) = 8x+4$ , so  $x^3+6x^2+9x = 8x+4$ .

**5. Show the cubic**

Rearranging gives  $x^3+6x^2+x-4 = 0$ .

**6. Use the given root**

Since  $-1$  is a root,  $(x+1)$  is a factor of  $x^3+6x^2+x-4$ .

**7. Factor the cubic**

$x^3+6x^2+x-4 = (x+1)(x^2+5x-4)$ .

**8. Solve the quadratic**

$x^2+5x-4 = 0$  gives  $x = \frac{-5 \pm \sqrt{25+16}}{2} = \frac{-5 \pm \sqrt{41}}{2}$ .

**9. Apply the logarithm domain**

The original log equation requires  $x > 0$ . The only positive root is

$$x = \frac{-5 + \sqrt{41}}{2}.$$

**Final answer**

(a) Proven.

$$(b)(i) \ x = \frac{-5 + \sqrt{41}}{2} \text{ and } x = \frac{-5 - \sqrt{41}}{2}.$$

$$(b)(ii) \ x = \frac{-5 + \sqrt{41}}{2}.$$

## Question 7

### Modelling with Sequences & Series

7. Wheat is grown on a farm.

- In year 1, the farm produced 300 tonnes of wheat.
- In year 12, the farm is predicted to produce 4000 tonnes of wheat.

Model *A* assumes that the amount of wheat produced on the farm will increase by the same amount each year.

- (a) Using model *A*, find the amount of wheat produced on the farm in year 4.  
Give your answer to the nearest 10 tonnes.

(3)

Model *B* assumes that the amount of wheat produced on the farm will increase by the same percentage each year.

- (b) Using model *B*, find the amount of wheat produced on the farm in year 2.  
Give your answer to the nearest 10 tonnes.

(3)

- (c) Calculate, according to the two models, the difference between the total amounts of wheat predicted to be produced on the farm from year 1 to year 12 inclusive.  
Give your answer to the nearest 10 tonnes.

(3)

# Worked Solution - Question 7

Topic group

## 1. Model A common difference

Model A is arithmetic. From year 1 to year 12 there are **11** equal increases, so

$$d = \frac{4000 - 300}{11} = \frac{3700}{11}.$$

## 2. Find year 4 in model A

Year 4 is  $300 + 3d = 300 + 3\left(\frac{3700}{11}\right) = 1309.09\dots$ , so the amount is **1310** tonnes to the nearest **10** tonnes.

## 3. Model B common ratio

Model B is geometric. If the ratio is  $r$ , then  $300r^{11} = 4000$ .

## 4. Find the ratio

$$r = \left(\frac{4000}{300}\right)^{1/11} = 1.2655\dots$$

## 5. Find year 2 in model B

Year 2 is  $300r = 379.65\dots$ , so the amount is **380** tonnes to the nearest **10** tonnes.

## 6. Total for model A

$$S_A = \frac{12}{2}(300 + 4000) = 25800 \text{ tonnes.}$$

## 7. Total for model B

$$S_B = 300 \frac{r^{12} - 1}{r - 1} = 17935.20\dots \text{ tonnes.}$$

### 8. Find the difference

The difference is  $25800 - 17935.20\dots = 7864.79\dots$ , which is **7860** tonnes to the nearest **10** tonnes.

#### Final answer

(a) 1310 tonnes.

(b) 380 tonnes.

(c) 7860 tonnes.

## Question 8

Proof

8. (i) Use a counter example to show that the following statement is **false**

“ $n^2 + 3n + 1$  is prime for all  $n \in \mathbb{N}$ ”

(2)

- (ii) Use algebra to prove by exhaustion that for all  $n \in \mathbb{N}$

“ $n^2 - 2$  is **not** a multiple of 4”

(4)

# Worked Solution - Question 8

Topic group

## 1. Choose a counterexample

Take  $n = 6$ .

## 2. Evaluate the expression

$$n^2 + 3n + 1 = 6^2 + 3(6) + 1 = 36 + 18 + 1 = 55.$$

## 3. Conclude false

55 is not prime, so the statement is false.

## 4. Use residues modulo four

Every natural number is one of  $4k$ ,  $4k + 1$ ,  $4k + 2$  or  $4k + 3$ .

## 5. Check $n$ equals $4k$

If  $n = 4k$ , then  $n^2 - 2 = 16k^2 - 2 = 4(4k^2 - 1) + 2$ , not a multiple of 4.

## 6. Check $n$ equals $4k$ plus one

If  $n = 4k + 1$ , then  $n^2 - 2 = 16k^2 + 8k - 1 = 4(4k^2 + 2k - 1) + 3$ , not a multiple of 4.

## 7. Check $n$ equals $4k$ plus two

If  $n = 4k + 2$ , then  $n^2 - 2 = 16k^2 + 16k + 2 = 4(4k^2 + 4k) + 2$ , not a multiple of 4.

## 8. Check $n$ equals $4k$ plus three

If  $n = 4k + 3$ , then  $n^2 - 2 = 16k^2 + 24k + 7 = 4(4k^2 + 6k + 1) + 3$ , not a multiple of 4.

### 9. Complete the proof

All possible cases leave remainder **2** or **3** when divided by **4**, so  $n^2 - 2$  is not a multiple of **4**.

#### Final answer

(i) Counterexample:  $n = 6$  gives **55**, which is not prime.

(ii) Proven by considering  $n = 4k, 4k + 1, 4k + 2, 4k + 3$ .

## Question 9

## Trigonometric Equations

9.

In this question you must show detailed reasoning.

Solutions relying entirely on calculator technology are not acceptable.

- (i) Solve, for
- $0 \leq x < 360^\circ$
- , the equation

$$\sin x \tan x = 5$$

giving your answers to one decimal place.

(6)

- (ii)

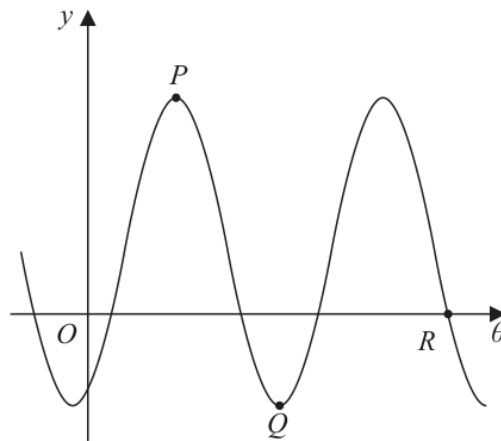


Figure 1

Figure 1 shows a sketch of part of the curve with equation

$$y = A \sin\left(2\theta - \frac{3\pi}{8}\right) + 2$$

where  $A$  is a constant and  $\theta$  is measured in radians.The points  $P$ ,  $Q$  and  $R$  lie on the curve and are shown in Figure 1.Given that the  $y$  coordinate of  $P$  is 7

- (a) state the value of
- $A$
- ,

(1)

- (b) find the exact coordinates of
- $Q$
- ,

(3)

- (c) find the value of
- $\theta$
- at
- $R$
- , giving your answer to 3 significant figures.

(4)

# Worked Solution - Question 9

## 1. Rewrite the equation

$\sin x \tan x = 5$  becomes  $\frac{\sin^2 x}{\cos x} = 5$ .

## 2. Use sine squared

Multiplying by  $\cos x$  gives  $\sin^2 x = 5 \cos x$ . Since  $\sin^2 x = 1 - \cos^2 x$ ,  
 $1 - \cos^2 x = 5 \cos x$ .

## 3. Solve for $\cos x$

$\cos^2 x + 5 \cos x - 1 = 0$ , so  $\cos x = \frac{-5 \pm \sqrt{29}}{2}$ .

## 4. Choose the possible cosine value

$\frac{-5 - \sqrt{29}}{2} < -1$ , so the valid value is  $\cos x = \frac{-5 + \sqrt{29}}{2}$ .

## 5. Find $x$

For  $0 \leq x < 360^\circ$ , this gives  $x = 78.9^\circ$  or  $281.1^\circ$  to one decimal place.

## 6. Find $A$ from $P$

The point  $P$  is a maximum point on the sine curve. Its  $y$  coordinate is  $7$ , so  
 $A + 2 = 7$  and  $A = 5$ .

## 7. Locate $Q$

Point  $Q$  is the next minimum after  $P$ , so  $\sin\left(2\theta - \frac{3\pi}{8}\right) = -1$ .

## 8. Find theta at $Q$

For this minimum,  $2\theta - \frac{3\pi}{8} = \frac{3\pi}{2}$ , so  $2\theta = \frac{15\pi}{8}$  and  $\theta = \frac{15\pi}{16}$ .

### 9. Find $y$ at $Q$

At the minimum,  $y = 2 - 5 = -3$ , so  $Q = \left(\frac{15\pi}{16}, -3\right)$ .

### 10. Set up $R$

At  $R$ ,  $y = 0$ , so  $5 \sin\left(2\theta - \frac{3\pi}{8}\right) + 2 = 0$ .

### 11. Choose the shown crossing

Thus  $\sin\left(2\theta - \frac{3\pi}{8}\right) = -\frac{2}{5}$ . The crossing  $R$  shown after the second maximum has  $2\theta - \frac{3\pi}{8} = 3\pi + \sin^{-1}\left(\frac{2}{5}\right)$ .

### 12. Find theta at $R$

$\theta = \frac{1}{2}\left(3\pi + \sin^{-1}\left(\frac{2}{5}\right) + \frac{3\pi}{8}\right) = 5.507\dots$ , so  $\theta = 5.51$  to 3 significant figures.

#### Final answer

(i)  $x = 78.9^\circ, 281.1^\circ$ .

(ii)(a)  $A = 5$ .

(ii)(b)  $Q = \left(\frac{15\pi}{16}, -3\right)$ .

(ii)(c)  $\theta = 5.51$ .

## Question 10

## Applications of Differentiation

10.

In this question you must show detailed reasoning.

Solutions relying entirely on calculator technology are not acceptable.

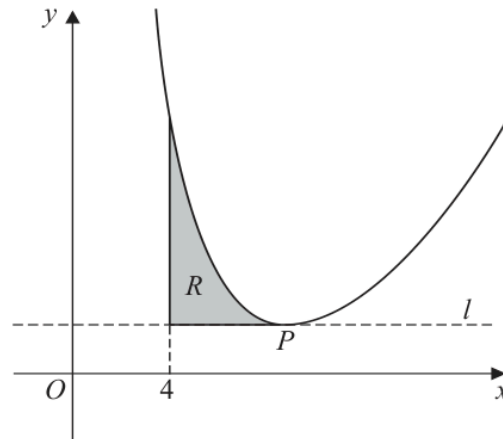


Figure 2

Figure 2 shows a sketch of the curve with equation

$$y = \frac{1}{2}x^2 + \frac{1458}{\sqrt{x^3}} - 74 \quad x > 0$$

The point  $P$  is the only stationary point on the curve.(a) Use calculus to show that the  $x$  coordinate of  $P$  is 9

(4)

The line  $l$  passes through the point  $P$  and is parallel to the  $x$ -axis.The region  $R$ , shown shaded in Figure 2, is bounded by the curve, the line  $l$  and the line with equation  $x = 4$ (b) Use algebraic integration to find the exact area of  $R$ .

(5)

# Worked Solution - Question 10

Topic group

**1. Rewrite the curve**

$$y = \frac{1}{2}x^2 + 1458x^{-3/2} - 74.$$

**2. Differentiate**

$$\frac{dy}{dx} = x + 1458 \left(-\frac{3}{2}\right)x^{-5/2} = x - 2187x^{-5/2}.$$

**3. Use the stationary condition**

At a stationary point,  $x - 2187x^{-5/2} = 0$ .

**4. Solve for  $x$** 

$$x = 2187x^{-5/2}, \text{ so } x^{7/2} = 2187 = 3^7.$$

**5. Show  $x$  coordinate**

Raising both sides to the power  $\frac{2}{7}$  gives  $x = 3^2 = 9$ . Since  $x > 0$ , this is the only stationary point.

**6. Find the horizontal line**

$$\text{At } x = 9, y = \frac{1}{2}(9)^2 + \frac{1458}{\sqrt{9^3}} - 74 = \frac{81}{2} + 54 - 74 = \frac{41}{2}.$$

**7. Set up the area**

The region lies from  $x = 4$  to  $x = 9$  between the curve and the horizontal line  $y = \frac{41}{2}$ .

**8. Integrate the difference**

$$R = \int_4^9 \left( \frac{1}{2}x^2 + 1458x^{-3/2} - 74 - \frac{41}{2} \right) dx.$$

**9. Find an antiderivative**

$$\int \left( \frac{1}{2}x^2 + 1458x^{-3/2} - \frac{189}{2} \right) dx = \frac{x^3}{6} - 2916x^{-1/2} - \frac{189}{2}x.$$

**10. Evaluate exactly**

$$R = \left[ \frac{x^3}{6} - 2916x^{-1/2} - \frac{189}{2}x \right]_4^9 = \frac{373}{3}.$$

**Final answer**

(a) Proven.

(b)  $\frac{373}{3}$ .

**PAST PAPER**

# **WMA12/01 May/June 2024**

**May/June 2024 | 10 questions | 75 marks**

**10**

questions

**75**

marks

**Answers**

worked solution  
after each  
question

# Question 1

## Binomial Expansion

1. (a) Find the first four terms, in ascending powers of  $x$ , of the binomial expansion of

$$\left(1 - \frac{1}{6}x\right)^9$$

giving each term in simplest form.

(3)

- (b) Hence find the coefficient of  $x^3$  in the expansion of

$$(10x + 3)\left(1 - \frac{1}{6}x\right)^9$$

giving the answer in simplest form.

(2)

## Worked Solution - Question 1

**1. Expand the first terms**

$$\left(1 - \frac{1}{6}x\right)^9 = 1 + 9\left(-\frac{1}{6}x\right) + \binom{9}{2}\left(-\frac{1}{6}x\right)^2 + \binom{9}{3}\left(-\frac{1}{6}x\right)^3 + \dots$$

**2. Simplify**

The first four terms are  $1 - \frac{3}{2}x + x^2 - \frac{7}{18}x^3$ .

**3. Identify contributions to x cubed**

In  $(10x + 3)\left(1 - \frac{1}{6}x\right)^9$ , the  $x^3$  coefficient comes from **3** times the  $x^3$  coefficient and **10x** times the  $x^2$  coefficient.

**4. Calculate the coefficient**

$$3\left(-\frac{7}{18}\right) + 10(1) = -\frac{7}{6} + 10 = \frac{53}{6}.$$

**Final answer**

(a)  $1 - \frac{3}{2}x + x^2 - \frac{7}{18}x^3$ .

(b)  $\frac{53}{6}$ .

**Question 2****Arithmetic Sequences**

2. **In this question you must show all stages of your working.  
Solutions relying entirely on calculator technology are not acceptable.**

In an arithmetic series,

- the sixth term is 2
- the sum of the first ten terms is  $-80$

For this series,

- (a) find the value of the first term and the value of the common difference. (4)

- (b) Hence find the smallest value of  $n$  for which

$$S_n > 8000 \quad (3)$$

# Worked Solution - Question 2

**1. Use the sixth term**

For an arithmetic series,  $u_6 = a + 5d = 2$ .

**2. Use the sum of ten terms**

$$S_{10} = \frac{10}{2}(2a + 9d) = -80, \text{ so } 2a + 9d = -16.$$

**3. Solve simultaneously**

From  $a + 5d = 2$ ,  $a = 2 - 5d$ . Substitute into  $2a + 9d = -16$ .

**4. Find a and d**

$2(2 - 5d) + 9d = -16$ , so  $4 - d = -16$  and  $d = 20$ . Then  
 $a = 2 - 5(20) = -98$ .

**5. Write  $S_n$** 

$$S_n = \frac{n}{2}(2a + (n-1)d) = \frac{n}{2}(-196 + 20n - 20) = 10n^2 - 108n.$$

**6. Set up the inequality**

We need  $10n^2 - 108n > 8000$ .

**7. Solve the boundary**

$$10n^2 - 108n - 8000 = 0 \text{ gives } 5n^2 - 54n - 4000 = 0.$$

**8. Find the positive threshold**

$$n = \frac{54 + \sqrt{82916}}{10} = 34.195\dots$$

**9. Choose the smallest integer**

Therefore the smallest integer value with  $S_n > 8000$  is  $n = 35$ .

**Final answer**

(a) First term =  $-98$ , common difference =  $20$ .

(b)  $n = 35$ .

**Question 3****Laws of Logarithms**

3.

**In this question you must show all stages of your working.  
Solutions relying entirely on calculator technology are not acceptable.**

- (i) Using the laws of logarithms, solve

$$2\log_2(2-x) = 4 + \log_2(x+10) \quad (5)$$

- (ii) Find the value of

$$\log_{\sqrt{a}} a^6$$

where  $a$  is a positive constant greater than 1

(1)

# Worked Solution - Question 3

Topic group

**1. State the domain**

The logarithms require  $2 - x > 0$  and  $x + 10 > 0$ , so  $-10 < x < 2$ .

**2. Combine powers**

$$2 \log_2(2 - x) = \log_2((2 - x)^2).$$

**3. Rewrite four**

$$4 = \log_2 16, \text{ so } 4 + \log_2(x + 10) = \log_2(16(x + 10)).$$

**4. Equate arguments**

$$(2 - x)^2 = 16(x + 10).$$

**5. Solve the quadratic**

$$x^2 - 4x + 4 = 16x + 160, \text{ so } x^2 - 20x - 156 = 0.$$

**6. Use the domain**

$x = 26$  or  $x = -6$ . The value  $26$  is outside the domain, so  $x = -6$ .

**7. Evaluate the logarithm**

$$\log_{\sqrt{a}}(a^6) = \log_{a^{1/2}}(a^6) = \frac{6}{1/2} = 12.$$

**Final answer**

(i)  $x = -6$ .

(ii) 12.

## Question 4

## Polynomials

4. 
$$f(x) = (x - 2)(2x^2 + 5x + k) + 21$$

where  $k$  is a constant.

- (a) State the remainder when  $f(x)$  is divided by  $(x - 2)$  (1)

Given that  $(2x - 1)$  is a factor of  $f(x)$

- (b) show that  $k = 11$  (2)

(c) Hence

- (i) fully factorise  $f(x)$ ,  
(ii) find the number of real solutions of the equation

$$f(x) = 0$$

giving a reason for your answer. (5)

## Worked Solution - Question 4

**1. Use the remainder theorem**

The remainder on division by  $(x - 2)$  is  $f(2)$ .

**2. Find f of two**

$$f(2) = (2 - 2)(2(2)^2 + 5(2) + k) + 21 = 21.$$

**3. Use the factor theorem**

Since  $(2x - 1)$  is a factor,  $x = \frac{1}{2}$  is a root, so  $f\left(\frac{1}{2}\right) = 0$ .

**4. Substitute x equals half**

$$\left(\frac{1}{2} - 2\right) \left(2\left(\frac{1}{2}\right)^2 + 5\left(\frac{1}{2}\right) + k\right) + 21 = 0.$$

**5. Solve for k**

$-\frac{3}{2}(3 + k) + 21 = 0$ . Multiplying by 2 gives  $-3(k + 3) + 42 = 0$ , so  $k = 11$ .

**6. Write f with k equals eleven**

$$f(x) = (x - 2)(2x^2 + 5x + 11) + 21 = 2x^3 + x^2 + x - 1.$$

**7. Factorise**

$$2x^3 + x^2 + x - 1 = (2x - 1)(x^2 + x + 1).$$

**8. Count real solutions**

The quadratic  $x^2 + x + 1$  has discriminant  $1 - 4 = -3$ , so it has no real roots.

Therefore  $f(x) = 0$  has one real solution.

**Final answer**

(a) 21.

(b)  $k = 11$ .

(c)(i)  $(2x - 1)(x^2 + x + 1)$ .

(c)(ii) One real solution.

**Question 5****Proof**

5. **In this question you must show detailed reasoning.**

(a) Given that  $x$  and  $y$  are positive numbers such that

$$(x - y)^3 > x^3 - y^3$$

prove that

$$y > x$$

**(4)**

(b) Using a counter example, show that the result in part (a) is not true for all real numbers.

**(2)**

# Worked Solution - Question 5

Topic group

**1. Start from the inequality**

$$(x - y)^3 > x^3 - y^3.$$

**2. Expand the left side**

$$x^3 - 3x^2y + 3xy^2 - y^3 > x^3 - y^3.$$

**3. Cancel common terms**

Subtract  $x^3 - y^3$  from both sides to get  $-3x^2y + 3xy^2 > 0$ .

**4. Factor**

$$3xy(y - x) > 0.$$

**5. Use positivity**

Since  $x$  and  $y$  are positive,  $3xy > 0$ . Therefore  $y - x > 0$ , so  $y > x$ .

**6. Give a real-number counterexample**

Take  $x = 1$  and  $y = -1$ . Then  $(x - y)^3 = 2^3 = 8$  and  $x^3 - y^3 = 1 - (-1) = 2$ , so the given inequality is true.

**7. Show the conclusion fails**

But  $y > x$  would mean  $-1 > 1$ , which is false. Therefore the result is not true for all real numbers.

**Final answer**

(a) Proven.

(b) For example,  $x = 1, y = -1$  satisfies the inequality but not  $y > x$ .

## Question 6

## Integration

6. (a) Sketch the curve with equation

$$y = a^x + 4$$

where  $a$  is a positive constant greater than 1

On your sketch, show

- the coordinates of the point of intersection of the curve with the  $y$ -axis
- the equation of the asymptote of the curve

(3)

$x$	2	2.3	2.6	2.9	3.2	3.5
$y$	0	0.3246	0.8629	1.6643	2.7896	4.3137

The table shows corresponding values of  $x$  and  $y$  for

$$y = 2^x - 2x$$

with the values of  $y$  given to 4 decimal places as appropriate.

Using the trapezium rule with all the values of  $y$  in the given table,

- (b) obtain an estimate for  $\int_2^{3.5} (2^x - 2x) dx$ , giving your answer to 2 decimal places.

(3)

- (c) Using your answer to part (b) and making your method clear, estimate

(i)  $\int_2^{3.5} (2^x + 2x) dx$

(ii)  $\int_2^{3.5} (2^{x+1} - 4x) dx$

(3)

## Worked Solution - Question 6

**1. Sketch key features**

For  $y = a^x + 4$  with  $a > 1$ , the curve is increasing.

**2. Find the y-intercept**

At  $x = 0$ ,  $y = a^0 + 4 = 5$ , so the  $y$ -intercept is  $(0, 5)$ .

**3. Find the asymptote**

As  $x \rightarrow -\infty$ ,  $a^x \rightarrow 0$ , so the horizontal asymptote is  $y = 4$ .

**4. Use the trapezium width**

The  $x$  values increase by  $0.3$ , so  $h = 0.3$ .

**5. Apply the trapezium rule**

$$\int_2^{3.5} (2^x - 2x) dx \approx \frac{0.3}{2} [0 + 4.3137 + 2(0.3246 + 0.8629 + 1.6643 + 2.7896)]$$

**6. Round part b**

This gives  $2.339475$ , so the estimate is  $2.34$  to 2 decimal places.

**7. Estimate part c i**

$$2^x + 2x = (2^x - 2x) + 4x, \text{ so } \int_2^{3.5} (2^x + 2x) dx \approx 2.34 + \int_2^{3.5} 4x dx.$$

**8. Add the extra integral**

$$\int_2^{3.5} 4x dx = [2x^2]_2^{3.5} = 16.5, \text{ so the estimate is } 2.34 + 16.5 = 18.84.$$

### 9. Estimate part c ii

$2^{x+1} - 4x = 2(2^x - 2x)$ , so the estimate is  $2(2.34) = 4.68$ .

#### Final answer

(a)  $y$ -intercept  $(0, 5)$  and asymptote  $y = 4$ .

(b) 2.34.

(c)(i) 18.84.

(c)(ii) 4.68.

## Question 7

7. The circle  $C_1$  has equation

$$x^2 + y^2 + 8x - 10y = 29$$

(a) (i) Find the coordinates of the centre of  $C_1$

(ii) Find the exact value of the radius of  $C_1$

(3)

**In part (b) you must show detailed reasoning.**

The circle  $C_2$  has equation

$$(x - 5)^2 + (y + 8)^2 = 52$$

(b) Prove that the circles  $C_1$  and  $C_2$  neither touch nor intersect.

(3)

# Worked Solution - Question 7

**1. Complete the square**

$x^2 + y^2 + 8x - 10y = 29$  becomes  $(x + 4)^2 - 16 + (y - 5)^2 - 25 = 29$ .

**2. Write circle C1 in standard form**

$$(x + 4)^2 + (y - 5)^2 = 70.$$

**3. Read centre and radius**

The centre of  $C_1$  is  $(-4, 5)$  and its radius is  $\sqrt{70}$ .

**4. Read C2**

$C_2$  has equation  $(x - 5)^2 + (y + 8)^2 = 52$ , so its centre is  $(5, -8)$  and its radius is  $\sqrt{52} = 2\sqrt{13}$ .

**5. Find distance between centres**

The distance between  $(-4, 5)$  and  $(5, -8)$  is  $\sqrt{9^2 + (-13)^2} = \sqrt{250} = 5\sqrt{10}$ .

**6. Compare with radii**

$$\sqrt{70} + 2\sqrt{13} = 15.58\dots, \text{ while } 5\sqrt{10} = 15.81\dots$$

**7. Conclude**

The distance between centres is greater than the sum of the radii, so the circles are separate. Therefore they neither touch nor intersect.

**Final answer**

(a)(i)  $(-4, 5)$ .

(a)(ii)  $\sqrt{70}$ .

(b) The circles neither touch nor intersect.

## Question 8

## Trigonometric Equations

8. **In this question you must show all stages of your working.**  
**Solutions relying entirely on calculator technology are not acceptable.**

- (i) Solve, for  $0 < x \leq \pi$ , the equation

$$5 \sin x \tan x + 13 = \cos x$$

giving your answer in radians to 3 significant figures.

(5)

- (ii) The temperature inside a greenhouse is monitored on one particular day.

The temperature,  $H^\circ\text{C}$ , inside the greenhouse,  $t$  hours after midnight, is modelled by the equation

$$H = 10 + 12 \sin(kt + 18)^\circ \quad 0 \leq t < 24$$

where  $k$  is a constant.

**Use the equation of the model to answer parts (a) to (c).**

Given that

- the temperature inside the greenhouse was  $20^\circ\text{C}$  at 6 am
- $0 < k < 20$

- (a) find all possible values for  $k$ , giving each answer to 2 decimal places.

(4)

Given further that  $0 < k < 10$

- (b) find the maximum temperature inside the greenhouse,

(1)

- (c) find the time of day at which this maximum temperature occurs.

Give your answer to the nearest minute.

(2)

# Worked Solution - Question 8

Topic group

## 1. Rewrite the trig equation

$$5 \sin x \tan x + 13 = \cos x \text{ becomes } \frac{5 \sin^2 x}{\cos x} + 13 = \cos x.$$

## 2. Multiply by $\cos x$

$$5 \sin^2 x + 13 \cos x = \cos^2 x.$$

## 3. Use sine squared

$$5(1 - \cos^2 x) + 13 \cos x = \cos^2 x, \text{ so } 6 \cos^2 x - 13 \cos x - 5 = 0.$$

## 4. Solve for $\cos x$

$$(2 \cos x - 5)(3 \cos x + 1) = 0, \text{ so } \cos x = \frac{5}{2} \text{ or } \cos x = -\frac{1}{3}.$$

## 5. Choose the valid root

$\cos x = \frac{5}{2}$  is impossible, so  $\cos x = -\frac{1}{3}$ . For  $0 < x \leq \pi$ ,  $x = 1.9106\dots$ , so  $x = 1.91$  radians to 3 significant figures.

## 6. Use the temperature at 6 am

$$\text{At } t = 6, 20 = 10 + 12 \sin(6k + 18)^\circ, \text{ so } \sin(6k + 18)^\circ = \frac{5}{6}.$$

## 7. Solve for $k$

$$\sin^{-1}\left(\frac{5}{6}\right) = 56.442\dots^\circ. \text{ Since } 0 < k < 20, 18 < 6k + 18 < 138, \text{ so}$$

$$6k + 18 = 56.442\dots \text{ or } 123.557\dots$$

## 8. Find possible $k$ values

This gives  $k = 6.407\dots$  or  $k = 17.592\dots$ , so  $k = 6.41$  or  $17.59$  to 2 decimal places.

### 9. Use the tighter condition

Given further that  $0 < k < 10$ , use  $k = 6.407\dots$

### 10. Find the maximum temperature

The maximum value of  $\sin$  is  $1$ , so the maximum temperature is  $10 + 12 = 22^\circ\text{C}$ .

### 11. Find the time of maximum

The maximum occurs when  $kt + 18 = 90$ , so  $t = \frac{72}{6.407\dots} = 11.2375\dots$  hours after midnight.

### 12. Convert to time of day

$0.2375\dots$  hours is  $14.25\dots$  minutes, so the maximum occurs at about **11:14** am.

#### Final answer

(i)  $x = 1.91$  radians.

(ii)(a)  $k = 6.41$  or  $17.59$ .

(ii)(b)  $22^\circ\text{C}$ .

(ii)(c) **11:14** am.

## Question 9

## Applications of Differentiation

9.

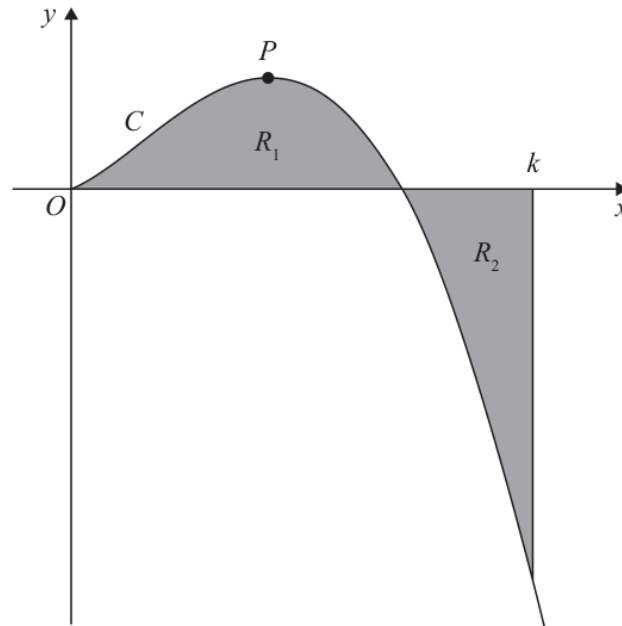


Figure 1

Figure 1 is a sketch of the curve  $C$  with equation

$$y = 2x^{\frac{3}{2}}(4-x) \quad x \geq 0$$

The point  $P$  is the stationary point of  $C$ .

(a) Find, using calculus, the  $x$  coordinate of  $P$ .

(4)

The region  $R_1$ , shown shaded in Figure 1, is bounded by  $C$  and the  $x$ -axis.

The region  $R_2$ , also shown shaded in Figure 1, is bounded by  $C$ , the  $x$ -axis and the line with equation  $x = k$ , where  $k$  is a constant.

Given that the area of  $R_1$  is equal to the area of  $R_2$

(b) find, using calculus, the exact value of  $k$ .

(4)

# Worked Solution - Question 9

Topic group

**1. Rewrite the curve**

$$y = 2x^{3/2}(4 - x) = 8x^{3/2} - 2x^{5/2}.$$

**2. Differentiate**

$$\frac{dy}{dx} = 12x^{1/2} - 5x^{3/2} = \sqrt{x}(12 - 5x).$$

**3. Find the stationary point**

For the interior stationary point  $P$ ,  $12 - 5x = 0$ , so  $x = \frac{12}{5}$ .

**4. Set up the area condition**

$R_1$  is the area from  $x = 0$  to  $x = 4$  above the  $x$ -axis.  $R_2$  is the area from  $x = 4$  to  $x = k$  below the  $x$ -axis.

**5. Use equal areas**

Equal areas mean  $\int_0^4 y \, dx = -\int_4^k y \, dx$ , so  $\int_0^k y \, dx = 0$ .

**6. Integrate  $y$** 

$$\int 2x^{3/2}(4 - x) \, dx = \int (8x^{3/2} - 2x^{5/2}) \, dx = \frac{16}{5}x^{5/2} - \frac{4}{7}x^{7/2}.$$

**7. Solve for  $k$** 

$$\frac{16}{5}k^{5/2} - \frac{4}{7}k^{7/2} = 0.$$

**8. Factor**

$$4k^{5/2} \left( \frac{4}{5} - \frac{k}{7} \right) = 0.$$

### 9. Choose the value beyond four

Since  $k > 4$ ,  $\frac{4}{5} - \frac{k}{7} = 0$ , so  $k = \frac{28}{5}$ .

**Final answer**

(a)  $x = \frac{12}{5}$ .

(b)  $k = \frac{28}{5}$ .

## Question 10

## Modelling with Sequences &amp; Series

10. **In this question you must show all stages of your working.**  
**Solutions relying entirely on calculator technology are not acceptable.**

The number of dormice and the number of voles on an island are being monitored.

Initially there are 2000 dormice on the island.

A model predicts that the number of dormice will increase by 3% each year, so that the numbers of dormice on the island at the end of each year form a geometric sequence.

- (a) Find, according to the model, the number of dormice on the island 6 years after monitoring began. Give your answer to 3 significant figures.

(2)

The number of voles on the island is being monitored over the same period of time.

Given that

- 4 years after monitoring began there were 3690 voles on the island
- 7 years after monitoring began there were 3470 voles on the island
- the number of voles on the island at the end of each year is modelled as a geometric sequence

- (b) find the equation of this model in the form

$$N = ab^t$$

where  $N$  is the number of voles,  $t$  years after monitoring began and  $a$  and  $b$  are constants. Give the value of  $a$  and the value of  $b$  to 2 significant figures.

(3)

When  $t = T$ , the number of dormice on the island is equal to the number of voles on the island.

- (c) Find, according to the models, the value of  $T$ , giving your answer to one decimal place.

(3)

# Worked Solution - Question 10

Topic group

## 1. Model the dormice

The dormice model is  $D = 2000(1.03)^t$ .

## 2. Find the dormice after six years

$D = 2000(1.03)^6 = 2388.10\dots$ , so there are **2390** dormice to 3 significant figures.

## 3. Set up the vole model

Let the vole model be  $N = ab^t$ .

## 4. Use the two data points

At  $t = 4$ ,  $ab^4 = 3690$ . At  $t = 7$ ,  $ab^7 = 3470$ .

## 5. Find b

Dividing gives  $b^3 = \frac{3470}{3690}$ , so  $b = 0.9797\dots$ , which is **0.98** to 2 significant figures.

## 6. Find a

$a = \frac{3690}{b^4} = 4005.18\dots$ , which is **4000** to 2 significant figures.

## 7. Write the vole model

So the model is approximately  $N = 4000(0.98)^t$ .

## 8. Set the two models equal

Use  $2000(1.03)^T = ab^T$  with the unrounded constants from the vole model.

### 9. Solve using logarithms

$$\left(\frac{1.03}{b}\right)^T = \frac{a}{2000}, \text{ so } T = \frac{\log(a/2000)}{\log(1.03/b)}.$$

### 10. Calculate T

Substituting  $a = 4005.18\dots$  and  $b = 0.9797\dots$  gives  $T = 13.875\dots$ , so  $T = 13.9$  to one decimal place.

#### Final answer

(a) 2390 dormice.

(b)  $N = 4000(0.98)^t$  to 2 s.f. constants.

(c)  $T = 13.9$ .

**PAST PAPER**

# **WMA12/01 May/June R 2024**

May/June R 2024 | 10 questions | 75 marks

**10**

questions

**75**

marks

**Answers**

worked solution  
after each  
question

## Question 1

## Sequences &amp; Series

1. The sequence  $u_1, u_2, u_3, \dots$  satisfies

$$u_{n+2} = 3u_{n+1} - 2u_n$$

Given that

•  $u_1 = 7$

•  $u_3 = 4$

(a) find the value of  $u_2$

(2)

(b) find  $\sum_{r=1}^4 (u_r + 2r)$

(3)

# Worked Solution - Question 1

Topic group

**1. Use the recurrence at  $n$  equals one**

$$u_3 = 3u_2 - 2u_1.$$

**2. Substitute given values**

$$4 = 3u_2 - 2(7), \text{ so } 4 = 3u_2 - 14.$$

**3. Find  $u_2$** 

$$3u_2 = 18, \text{ hence } u_2 = 6.$$

**4. Find  $u_4$** 

$$u_4 = 3u_3 - 2u_2 = 3(4) - 2(6) = 0.$$

**5. Evaluate the sum**

$$\sum_{r=1}^4 (u_r + 2r) = (7 + 2) + (6 + 4) + (4 + 6) + (0 + 8).$$

**6. Add the terms**

$$\text{The sum is } 9 + 10 + 10 + 8 = 37.$$

**Final answer**

(a)  $u_2 = 6.$

(b)  $37.$

## Question 2

2. The table shows corresponding values of  $x$  and  $y$  for a continuous curve with equation  $y = f(x)$  between  $x = -4$  and  $x = 5$ , where  $a$  is a constant.

$x$	-4	-2.5	-1	0.5	2	3.5	5
$y$	4.16	2.91	$a$	1.73	1.37	1.43	2.28

The trapezium rule is used with all the  $y$  values in the table to find an approximation for

$$\int_{-4}^5 f(x) \, dx$$

Given that the value of this approximation is 19.3

- (a) find the value of the constant  $a$  to 3 significant figures. (3)
- (b) Use the given answer of 19.3 to find an approximate value for

$$\int_{-4}^5 (2f(x) - 3) \, dx$$

(2)

## Worked Solution - Question 2

### 1. Use the trapezium width

The  $x$  values increase by 1.5, so  $h = 1.5$ .

### 2. Set up the trapezium estimate

$$19.3 = \frac{1.5}{2} [4.16 + 2.28 + 2(2.91 + a + 1.73 + 1.37 + 1.43)].$$

### 3. Solve for $a$

This simplifies to  $19.3 = 15.99 + 1.5a$ , so  $a = \frac{3.31}{1.5} = 2.206\dots$

### 4. Round $a$

Therefore  $a = 2.21$  to 3 significant figures.

### 5. Use the transformed integral

$$\int_{-4}^5 (2f(x) - 3)dx = 2 \int_{-4}^5 f(x)dx - \int_{-4}^5 3 dx.$$

### 6. Substitute the given estimate

This is  $2(19.3) - 3(9) = 38.6 - 27 = 11.6$ .

#### Final answer

(a)  $a = 2.21$  to 3 s.f.

(b) 11.6.

## Question 3

3. A circle has equation

$$x^2 + y^2 + 8x - 14y - 79 = 0$$

(a) Find

- (i) the coordinates of the centre of the circle,
- (ii) the radius of the circle.

(3)

Given that  $P$  is the point on the circle that is nearest the origin  $O$ ,

(b) find the exact length of  $OP$

(2)

## Worked Solution - Question 3

**1. Complete the square**

$$x^2 + y^2 + 8x - 14y - 79 = 0 \text{ becomes}$$
$$(x + 4)^2 - 16 + (y - 7)^2 - 49 - 79 = 0.$$

**2. Write standard form**

$$(x + 4)^2 + (y - 7)^2 = 144.$$

**3. Read centre and radius**

The centre is  $(-4, 7)$  and the radius is **12**.

**4. Find distance from centre to origin**

The distance from  $(-4, 7)$  to  $O(0, 0)$  is  $\sqrt{(-4)^2 + 7^2} = \sqrt{65}$ .

**5. Use origin inside the circle**

Since  $\sqrt{65} < 12$ , the origin is inside the circle.

**6. Find nearest distance to the circle**

The nearest point on the circle lies along the line from the centre to the origin, so  $OP = 12 - \sqrt{65}$ .

**Final answer**

(a)(i)  $(-4, 7)$ .

(a)(ii) **12**.

(b)  $12 - \sqrt{65}$ .

**Question 4****Binomial Expansion**

4. (a) Find the first 4 terms, in ascending powers of  $x$ , of the binomial expansion of

$$(3 + 2x)^6$$

giving each coefficient in simplest form.

(4)

- (b) Hence find the coefficient of  $x^2$  in the expansion of

$$\left(2x^2 - \frac{1}{6x}\right)(3 + 2x)^6$$

(3)

## Worked Solution - Question 4

**1. Write the expansion terms**

$$(3 + 2x)^6 = 3^6 + \binom{6}{1}3^5(2x) + \binom{6}{2}3^4(2x)^2 + \binom{6}{3}3^3(2x)^3 + \dots$$

**2. Simplify the first four terms**

This gives  $729 + 2916x + 4860x^2 + 4320x^3$ .

**3. Identify  $x$  squared contributions**

In  $\left(2x^2 - \frac{1}{6x}\right)(3 + 2x)^6$ , the  $x^2$  term comes from  $2x^2$  times the constant term and  $-\frac{1}{6x}$  times the  $x^3$  term.

**4. Calculate the coefficient**

The coefficient is  $2(729) - \frac{1}{6}(4320) = 1458 - 720 = 738$ .

**Final answer**

(a)  $729 + 2916x + 4860x^2 + 4320x^3$ .

(b) 738.

## Question 5

## Trigonometric Equations

5. **In this question you must show all stages of your working.  
Solutions relying entirely on calculator technology are not acceptable.**

The depth of water,  $D$  metres, in a harbour on a particular day is given by the equation

$$D = 8 + 5 \sin\left(\frac{\pi t}{6} + 3\right) \quad 0 \leq t < 24$$

where  $t$  is the number of hours after **midnight**.

- (a) Show that the depth of water in the harbour at 2 am is just over 4 metres. (1)
- (b) Find, to the nearest minute, the first time after **midday** when the depth of water in the harbour is exactly 6 metres. (5)

# Worked Solution - Question 5

Topic group

**1. Substitute  $t$  equals two**

$$D = 8 + 5 \sin \left( \frac{\pi(2)}{6} + 3 \right) = 8 + 5 \sin \left( \frac{\pi}{3} + 3 \right).$$

**2. Evaluate**

$D = 4.064\dots$ , so the depth at 2 am is just over 4 metres.

**3. Set depth equal to six**

$$6 = 8 + 5 \sin \left( \frac{\pi t}{6} + 3 \right), \text{ so } \sin \left( \frac{\pi t}{6} + 3 \right) = -\frac{2}{5}.$$

**4. Use the time interval**

After midday means  $t > 12$ . Then  $\frac{\pi t}{6} + 3$  is greater than  $2\pi + 3$ .

**5. Choose the first suitable angle**

The first angle after  $2\pi + 3$  with sine  $-\frac{2}{5}$  is  $3\pi + \sin^{-1} \left( \frac{2}{5} \right)$ .

**6. Solve for  $t$** 

$$\frac{\pi t}{6} + 3 = 3\pi + \sin^{-1} \left( \frac{2}{5} \right), \text{ so } t = 13.056\dots \text{ hours after midnight.}$$

**7. Convert to clock time**

$0.056\dots$  hours is about **3.38** minutes, so the time to the nearest minute is **13:03**, or **1:03 pm**.

**Final answer**

(a)  $D(2) = 4.06\dots$  m, just over 4 m.

(b) 13:03, or 1:03 pm.

## Question 6

## Applications of Differentiation

6.

$$f(x) = 4x^3 + px^2 + 8x + q$$

where  $p$  and  $q$  are constants.

Given that

- $(2x + 3)$  is a factor of  $f(x)$
- $f(x)$  has a remainder of  $-5$  when divided by  $(x + 2)$

(a) (i) show that  $p = 10$

(ii) find the value of  $q$ .

(5)

(b) Hence find the range of values of  $x$  for which  $f(x)$  is decreasing.

*(Solutions based entirely on calculator technology are not acceptable.)*

(4)

# Worked Solution - Question 6

Topic group

**1. Use the factor theorem**

Since  $(2x + 3)$  is a factor,  $x = -\frac{3}{2}$  is a root.

**2. Substitute  $x$  equals negative three halves**

$$4\left(-\frac{3}{2}\right)^3 + p\left(-\frac{3}{2}\right)^2 + 8\left(-\frac{3}{2}\right) + q = 0.$$

**3. Simplify the first condition**

$$-\frac{27}{2} + \frac{9}{4}p - 12 + q = 0, \text{ so } \frac{9}{4}p + q = \frac{51}{2}.$$

**4. Use the remainder theorem**

The remainder on division by  $(x + 2)$  is  $f(-2)$ .

**5. Substitute  $x$  equals negative two**

$$f(-2) = 4(-8) + 4p - 16 + q = 4p + q - 48.$$

**6. Use the given remainder**

$$4p + q - 48 = -5, \text{ so } 4p + q = 43.$$

**7. Solve for  $p$  and  $q$** 

Subtract  $\frac{9}{4}p + q = \frac{51}{2}$  from  $4p + q = 43$ :  $\frac{7}{4}p = \frac{35}{2}$ , so  $p = 10$ . Then  $4(10) + q = 43$ , so  $q = 3$ .

**8. Differentiate**

With  $p = 10$ ,  $f(x) = 4x^3 + 10x^2 + 8x + 3$ , so  $f'(x) = 12x^2 + 20x + 8$ .

### 9. Factor the derivative

$$f'(x) = 4(3x + 2)(x + 1).$$

### 10. Find where decreasing

The quadratic derivative is negative between its roots, so  $f(x)$  is decreasing for

$$-1 < x < -\frac{2}{3}.$$

#### Final answer

(a)(i)  $p = 10$ .

(a)(ii)  $q = 3$ .

(b)  $-1 < x < -\frac{2}{3}$ .

## Question 7

## Trigonometric Equations

7. **In this question you must show all stages of your working.  
Solutions relying entirely on calculator technology are not acceptable.**

- (i) Solve, for  $0 \leq x < 2\pi$ , the equation

$$3 \sin x \tan x = 11 + \cos x$$

giving the answers in radians to 3 decimal places.

(5)

- (ii) Given that

- $0 < \theta < 90^\circ$
- $\cos \theta = \frac{1}{3}$

find, in simplest form, the exact value of  $\tan \theta$

(2)

# Worked Solution - Question 7

Topic group

**1. Rewrite  $\tan x$** 

$$3 \sin x \tan x = 11 + \cos x \text{ becomes } \frac{3 \sin^2 x}{\cos x} = 11 + \cos x.$$

**2. Multiply by  $\cos x$** 

$$3 \sin^2 x = 11 \cos x + \cos^2 x.$$

**3. Use sine squared**

$$3(1 - \cos^2 x) = 11 \cos x + \cos^2 x, \text{ so } 4 \cos^2 x + 11 \cos x - 3 = 0.$$

**4. Solve for cosine**

$$(4 \cos x - 1)(\cos x + 3) = 0, \text{ so } \cos x = \frac{1}{4} \text{ or } \cos x = -3.$$

**5. Reject impossible value**

$$\cos x = -3 \text{ is impossible, so } \cos x = \frac{1}{4}.$$

**6. Find  $x$  values**

For  $0 \leq x < 2\pi$ , this gives  $x = 1.318\dots$  or  $x = 4.965\dots$ , so  $x = 1.318, 4.965$  radians.

**7. Use the right triangle identity**

$$\text{Given } 0 < \theta < 90^\circ \text{ and } \cos \theta = \frac{1}{3}, \sin \theta = \sqrt{1 - \frac{1}{9}} = \frac{2\sqrt{2}}{3}.$$

**8. Find  $\tan \theta$** 

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{2\sqrt{2}/3}{1/3} = 2\sqrt{2}.$$

**Final answer**

(i)  $x = 1.318, 4.965$ .

(ii)  $2\sqrt{2}$ .

## Question 8

## Arithmetic Sequences

8. (i) (a) In an **arithmetic** series the first term is  $a$  and the common difference is  $d$ .

Show that

$$S_n = \frac{n}{2}\{2a + (n-1)d\} \quad (3)$$

- (b) Hence find

$$900 + 892 + 884 + \dots + 500 \quad (3)$$

- (ii) Given that the first three terms of a **geometric** series are

$$k+4 \quad k-2 \quad 11-k$$

where  $k$  is a constant,

- (a) show that

$$2k^2 - 11k - 40 = 0 \quad (3)$$

Given also that this series is convergent,

- (b) find the value of  $S_\infty$  (4)

# Worked Solution - Question 8

**1. Write the first and last terms**

The first term is  $a$  and the  $n$ th term is  $a + (n - 1)d$ .

**2. Use the arithmetic sum formula**

$$S_n = \frac{n}{2}(\text{first term} + \text{last term}).$$

**3. Show the formula**

$$\text{Therefore } S_n = \frac{n}{2}\{a + [a + (n - 1)d]\} = \frac{n}{2}\{2a + (n - 1)d\}.$$

**4. Identify the arithmetic series**

For  $900 + 892 + 884 + \dots + 500$ , the first term is  $900$ , the common difference is  $-8$ , and the last term is  $500$ .

**5. Find the number of terms**

$$500 = 900 + (n - 1)(-8), \text{ so } n - 1 = 50 \text{ and } n = 51.$$

**6. Find the sum**

$$S = \frac{51}{2}(900 + 500) = 35700.$$

**7. Use the geometric condition**

For  $k + 4$ ,  $k - 2$ ,  $11 - k$  to be consecutive geometric terms,  
 $(k - 2)^2 = (k + 4)(11 - k)$ .

**8. Show the quadratic**

$$k^2 - 4k + 4 = -k^2 + 7k + 44, \text{ so } 2k^2 - 11k - 40 = 0.$$

### 9. Solve for $k$

$$2k^2 - 11k - 40 = 0 \text{ gives } k = 8 \text{ or } k = -\frac{5}{2}.$$

### 10. Use convergence

If  $k = 8$ , the terms are **12, 6, 3** and the common ratio is  $\frac{1}{2}$ , so the series is convergent. If  $k = -\frac{5}{2}$ , the common ratio is  $-3$ , so it is not convergent.

### 11. Find $S$ infinity

$$\text{Use } k = 8: S_{\infty} = \frac{12}{1 - 1/2} = 24.$$

#### Final answer

(i)(a) Proven.

(i)(b) 35700.

(ii)(a) Proven.

(ii)(b)  $S_{\infty} = 24$ .

## Question 9

## Laws of Logarithms

9. Given that

$$3 \log_2(t+4) - 2 \log_2(t-2) = 7$$

(a) verify that  $t = 4$  is a solution of the above equation,

(2)

(b) show that

$$t^3 - 116t^2 + 560t - 448 = 0$$

(3)

(c) Hence, using algebra and showing your working, solve

$$3 \log_2(t+4) - 2 \log_2(t-2) = 7$$

giving each answer in simplest form.

*(Solutions based entirely on calculator technology are not acceptable.)*

(4)

## Worked Solution - Question 9

**1. Verify  $t$  equals four**

Substitute  $t = 4$ :  $3 \log_2(8) - 2 \log_2(2) = 3(3) - 2(1) = 7$ .

**2. Combine the logarithms**

$$3 \log_2(t + 4) - 2 \log_2(t - 2) = \log_2 \left( \frac{(t + 4)^3}{(t - 2)^2} \right).$$

**3. Use seven as a logarithm**

$$7 = \log_2(128), \text{ so } \frac{(t + 4)^3}{(t - 2)^2} = 128.$$

**4. Clear the denominator**

$$(t + 4)^3 = 128(t - 2)^2.$$

**5. Expand**

$$t^3 + 12t^2 + 48t + 64 = 128t^2 - 512t + 512.$$

**6. Show the cubic**

Rearranging gives  $t^3 - 116t^2 + 560t - 448 = 0$ .

**7. Use the known root**

Since  $t = 4$  is a root,  $(t - 4)$  is a factor.

**8. Factor the cubic**

$$t^3 - 116t^2 + 560t - 448 = (t - 4)(t^2 - 112t + 112).$$

**9. Solve the quadratic**

$$t^2 - 112t + 112 = 0 \text{ gives } t = \frac{112 \pm \sqrt{112^2 - 4(112)}}{2} = 56 \pm 12\sqrt{21}.$$

### 10. Use the log domain

The original equation requires  $t > 2$ . The value  $56 - 12\sqrt{21}$  is less than  $2$ , so it is rejected. Hence  $t = 4$  or  $t = 56 + 12\sqrt{21}$ .

#### Final answer

(a) Verified.

(b) Proven.

(c)  $t = 4$  or  $t = 56 + 12\sqrt{21}$ .

## Question 10

## Applications of Differentiation

10.

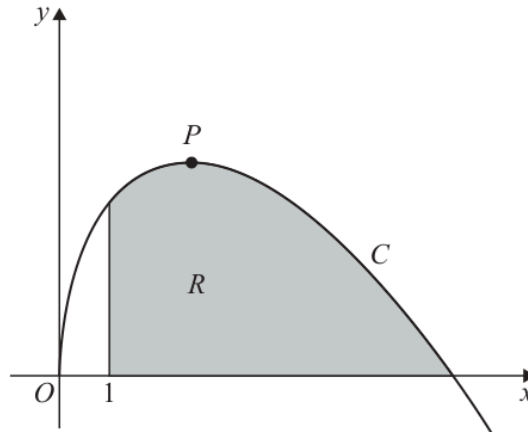


Figure 1

**In this question you must show all stages of your working.  
Solutions relying entirely on calculator technology are not acceptable.**

Figure 1 shows a sketch of part of the curve  $C$  with equation

$$y = \frac{9x - x^2}{2\sqrt{x}} \quad x > 0$$

The point  $P$  is the stationary point on  $C$ .

(a) Find, using calculus, the  $x$  coordinate of  $P$ .

(4)

The finite region  $R$ , shown shaded in Figure 1, is bounded by the curve  $C$ , the  $x$ -axis and the line with equation  $x = 1$

(b) Using calculus, calculate the exact area of  $R$ .

(5)

# Worked Solution - Question 10

Topic group

**1. Rewrite the curve**

$$y = \frac{9x - x^2}{2\sqrt{x}} = \frac{1}{2}(9x^{1/2} - x^{3/2}).$$

**2. Differentiate**

$$\frac{dy}{dx} = \frac{1}{2} \left( \frac{9}{2}x^{-1/2} - \frac{3}{2}x^{1/2} \right) = \frac{3(3-x)}{4\sqrt{x}}.$$

**3. Use the stationary condition**

For  $x > 0$ , the denominator is not zero. So  $\frac{dy}{dx} = 0$  when  $3 - x = 0$ .

**4. Find P**

Therefore the  $x$  coordinate of  $P$  is 3.

**5. Find where the curve meets the x-axis**

$y = 0$  when  $9x - x^2 = 0$ , so  $x = 9$  for the positive intercept.

**6. Set up the area**

The region is under the curve from  $x = 1$  to  $x = 9$ , so  $R = \int_1^9 \frac{9x - x^2}{2\sqrt{x}} dx$ .

**7. Integrate**

$$R = \int_1^9 \frac{1}{2}(9x^{1/2} - x^{3/2}) dx = \left[ 3x^{3/2} - \frac{1}{5}x^{5/2} \right]_1^9.$$

**8. Evaluate**

$$R = \left( 3(27) - \frac{1}{5}(243) \right) - \left( 3 - \frac{1}{5} \right) = \frac{148}{5}.$$

**Final answer**

(a)  $x = 3$ .

(b)  $\frac{148}{5}$ .

**PAST PAPER**

# **WMA12/01 October 2024**

**October 2024 | 11 questions | 75 marks**

**11**

questions

**75**

marks

**Answers**

worked solution  
after each  
question

## Question 1

1. A continuous curve has equation  $y = f(x)$ .

A table of values of  $x$  and  $y$  for  $y = f(x)$  is shown below.

$x$	0.5	1.75	3	4.25	5.5
$y$	3.479	6.101	7.448	6.823	5.182

Using the trapezium rule with all the values of  $y$  in the given table,

- (a) find an estimate for

$$\int_{0.5}^{5.5} f(x) \, dx$$

giving your answer to one decimal place.

(3)

- (b) Using your answer to part (a) and making your method clear, estimate

$$\int_{0.5}^{5.5} (f(x) + 4x) \, dx$$

(2)

# Worked Solution - Question 1

## 1. Find the strip width

The  $x$  values increase by 1.25, so the trapezium width is  $h = 1.25$ .

## 2. Apply the trapezium rule

$$\int_{0.5}^{5.5} f(x) dx \approx \frac{1.25}{2} \{3.479 + 5.182 + 2(6.101 + 7.448 + 6.823)\}.$$

## 3. Evaluate the estimate

This gives 30.878125, which is 30.9 to one decimal place.

## 4. Split the second integral

$$\int_{0.5}^{5.5} (f(x) + 4x) dx = \int_{0.5}^{5.5} f(x) dx + \int_{0.5}^{5.5} 4x dx.$$

## 5. Integrate $4x$

$$\int_{0.5}^{5.5} 4x dx = [2x^2]_{0.5}^{5.5} = 2(5.5)^2 - 2(0.5)^2 = 60.$$

## 6. Use part a

Using the estimate from part (a), the required value is  $30.9 + 60 = 90.9$ .

### Final answer

(a) 30.9.

(b) 90.9.

## Question 2

## Sequences &amp; Series

2. A sequence of numbers  $u_1, u_2, u_3, \dots$  is defined by

$$u_1 = 7$$

$$u_{n+1} = (-1)^n u_n + k$$

where  $k$  is a constant.

(a) Show that  $u_5 = 7$

(3)

Given that  $\sum_{r=1}^4 u_r = 30$

(b) find the value of  $k$ .

(2)

(c) Hence find the value of  $\sum_{r=1}^{150} u_r$

(2)

## Worked Solution - Question 2

**1. Find the first few terms**

$$u_1 = 7, u_2 = (-1)^1 u_1 + k = k - 7, u_3 = (-1)^2 u_2 + k = 2k - 7, \text{ and} \\ u_4 = (-1)^3 u_3 + k = 7 - k.$$

**2. Show u five**

$$u_5 = (-1)^4 u_4 + k = (7 - k) + k = 7.$$

**3. Use the given sum**

$$\sum_{r=1}^4 u_r = 7 + (k - 7) + (2k - 7) + (7 - k) = 2k.$$

**4. Find k**

$$\text{Since } \sum_{r=1}^4 u_r = 30, \text{ we have } 2k = 30, \text{ so } k = 15.$$

**5. Write one repeated block**

With  $k = 15$ , the first four terms are **7, 8, 23, -8**, and their sum is **30**. Since  $u_5 = u_1$ , the pattern repeats every four terms.

**6. Sum 150 terms**

There are **37** complete blocks in the first **150** terms, with the first two terms left over. Hence  $\sum_{r=1}^{150} u_r = 37(30) + 7 + 8 = 1125$ .

**Final answer**

(a) Proven.

(b)  $k = 15$ .

(c) 1125.

## Question 3

## Polynomials

3.

$$f(x) = 2x^3 - x^2 + Ax + B$$

where  $A$  and  $B$  are integers.

Given that when  $f(x)$  is divided by  $(x + 3)$  the remainder is 55

(a) show that

$$3A - B = -118$$

(2)

Given also that  $(2x - 5)$  is a factor of  $f(x)$ ,

(b) find the value of  $A$  and the value of  $B$ .

(3)

(c) Hence find the quotient when  $f(x)$  is divided by  $(x - 7)$

(2)

## Worked Solution - Question 3

**1. Use the remainder theorem**

Dividing by  $x + 3$  leaves remainder 55, so  $f(-3) = 55$ .

**2. Show the given equation**

$f(-3) = 2(-3)^3 - (-3)^2 - 3A + B = -63 - 3A + B$ . Therefore  $-63 - 3A + B = 55$ , so  $3A - B = -118$ .

**3. Use the factor theorem**

Since  $2x - 5$  is a factor,  $x = \frac{5}{2}$  is a root and  $f\left(\frac{5}{2}\right) = 0$ .

**4. Form a second equation**

$2\left(\frac{5}{2}\right)^3 - \left(\frac{5}{2}\right)^2 + \frac{5}{2}A + B = 0$ , so  $25 + \frac{5}{2}A + B = 0$ . Multiplying by 2 gives  $5A + 2B = -50$ .

**5. Solve simultaneously**

From  $3A - B = -118$ ,  $B = 3A + 118$ . Substitute into  $5A + 2B = -50$  to get  $11A = -286$ , so  $A = -26$  and  $B = 40$ .

**6. Write the polynomial**

$$f(x) = 2x^3 - x^2 - 26x + 40.$$

**7. Divide by x minus seven**

$2x^3 - x^2 - 26x + 40 = (x - 7)(2x^2 + 13x + 65) + 495$ , so the quotient is  $2x^2 + 13x + 65$ .

**Final answer**

(a) Proven.

(b)  $A = -26, B = 40$ .

(c)  $2x^2 + 13x + 65$ .

## Question 4

## Applications of Differentiation

4.

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

The curve  $C$  has equation

$$y = 4x^{\frac{1}{2}} + 9x^{-\frac{1}{2}} + 3 \quad x > 0$$

- (a) Find  $\frac{dy}{dx}$  giving each term in simplest form. (2)
- (b) Hence find the  $x$  coordinate of the stationary point of  $C$ . (2)
- (c) (i) Find  $\frac{d^2y}{dx^2}$  giving each term in simplest form.
- (ii) Hence determine the nature of the stationary point of  $C$ , giving a reason for your answer. (2)
- (d) State the range of values of  $x$  for which  $y$  is decreasing. (1)

# Worked Solution - Question 4

Topic group

**1. Rewrite using powers**

$$y = 4x^{1/2} + 9x^{-1/2} + 3.$$

**2. Differentiate**

$$\frac{dy}{dx} = 2x^{-1/2} - \frac{9}{2}x^{-3/2} = \frac{2}{\sqrt{x}} - \frac{9}{2x^{3/2}}.$$

**3. Set the derivative to zero**

$$\frac{dy}{dx} = \frac{4x - 9}{2x^{3/2}}. \text{ Since } x > 0, \text{ the stationary point occurs when } 4x - 9 = 0.$$

**4. Find the x coordinate**

$$\text{Therefore } x = \frac{9}{4}.$$

**5. Differentiate again**

$$\frac{d^2y}{dx^2} = -x^{-3/2} + \frac{27}{4}x^{-5/2} = -\frac{1}{x^{3/2}} + \frac{27}{4x^{5/2}}.$$

**6. Test the stationary point**

$$\frac{d^2y}{dx^2} = \frac{27 - 4x}{4x^{5/2}}. \text{ At } x = \frac{9}{4}, \text{ the numerator is } 27 - 9 = 18 > 0, \text{ so the stationary point is a minimum.}$$

**7. Find where y is decreasing**

$$\text{For } x > 0, 2x^{3/2} > 0, \text{ so } \frac{dy}{dx} < 0 \text{ when } 4x - 9 < 0. \text{ Hence } y \text{ is decreasing for } 0 < x < \frac{9}{4}.$$

**Final answer**

(a)  $\frac{2}{\sqrt{x}} - \frac{9}{2x^{3/2}}$ .

(b)  $x = \frac{9}{4}$ .

(c)(i)  $-\frac{1}{x^{3/2}} + \frac{27}{4x^{5/2}}$ .

(c)(ii) Minimum.

(d)  $0 < x < \frac{9}{4}$ .

## Question 5

## Binomial Expansion

5. (a) Find, in terms of  $a$ , the first 3 terms, in ascending powers of  $x$ , of the binomial expansion of

$$(2 + ax)^6$$

where  $a$  is a non-zero constant. Give each term in simplest form.

(3)

$$f(x) = \left(3 + \frac{1}{x}\right)^2 (2 + ax)^6$$

Given that the constant term in the expansion of  $f(x)$  is 576

- (b) find the value of  $a$ .

(4)

## Worked Solution - Question 5

**1. Write the first three binomial terms**

$$(2 + ax)^6 = 2^6 + 6(2^5)(ax) + \binom{6}{2}(2^4)(ax)^2 + \dots$$

**2. Simplify the terms**

The first three terms are  $64 + 192ax + 240a^2x^2$ .

**3. Expand the other factor**

$$\left(3 + \frac{1}{x}\right)^2 = 9 + 6x^{-1} + x^{-2}.$$

**4. Collect constant contributions**

The constant term in  $\left(3 + \frac{1}{x}\right)^2 (2 + ax)^6$  is  $9(64) + 6(192a) + 240a^2$ .

**5. Use the given constant term**

$$576 + 1152a + 240a^2 = 576, \text{ so } 240a^2 + 1152a = 0.$$

**6. Solve for a**

$$48a(5a + 24) = 0. \text{ Since } a \neq 0, 5a + 24 = 0, \text{ so } a = -\frac{24}{5}.$$

**Final answer**

$$(a) 64 + 192ax + 240a^2x^2.$$

$$(b) a = -\frac{24}{5}.$$

**Question 6****Laws of Logarithms**

6.

**In this question you must show all stages of your working.****Solutions relying entirely on calculator technology are not acceptable.**

Using the laws of logarithms, solve

$$\log_4(12 - 2x) = 2 + 2\log_4(x + 1)$$

**(5)**

# Worked Solution - Question 6

Topic group

**1. State the domain**

The logarithms require  $12 - 2x > 0$  and  $x + 1 > 0$ , so  $-1 < x < 6$ .

**2. Combine the right hand side**

$$2 = \log_4(16), \text{ so } 2 + 2\log_4(x + 1) = \log_4(16(x + 1)^2).$$

**3. Equate the arguments**

$$\log_4(12 - 2x) = \log_4(16(x + 1)^2) \text{ gives } 12 - 2x = 16(x + 1)^2.$$

**4. Form the quadratic**

$$12 - 2x = 16x^2 + 32x + 16, \text{ so } 8x^2 + 17x + 2 = 0.$$

**5. Factor and solve**

$$8x^2 + 17x + 2 = (8x + 1)(x + 2) = 0, \text{ so } x = -\frac{1}{8} \text{ or } x = -2.$$

**6. Check the domain**

The value  $x = -2$  is outside  $-1 < x < 6$ , so the solution is  $x = -\frac{1}{8}$ .

**Final answer**

$$x = -\frac{1}{8}.$$

## Question 7

## Modelling with Sequences &amp; Series

7. Jem pays money into a savings scheme,  $A$ , over a period of 300 months.

Jem pays £20 into scheme  $A$  in month 1, £20.50 in month 2, £21 in month 3 and so on, so that the amounts Jem pays each month form an arithmetic sequence.

(a) Show that Jem pays £69.50 into scheme  $A$  in month 100 (1)

(b) Find the **total** amount that Jem pays into scheme  $A$  over the period of 300 months. (2)

Kim pays money into a different savings scheme,  $B$ , over the same period of 300 months.

In a model, the amounts Kim pays into scheme  $B$  increase by the same percentage each month, so that the amounts Kim pays each month form a geometric sequence.

Given that Kim pays

- £20 into scheme  $B$  in month 1
- £250 into scheme  $B$  in month 300

(c) use the model to calculate, to the nearest £10, the difference between the total amount paid into scheme  $A$  and the total amount paid into scheme  $B$  over the period of 300 months. (3)

# Worked Solution - Question 7

Topic group

## 1. Identify Jem's sequence

Jem's payments are arithmetic with first term **20** and common difference **0.50**.

## 2. Show month 100

The month **100** payment is  $20 + 99(0.50) = 69.50$ , so Jem pays £69.50.

## 3. Sum Jem's payments

$$S_{300} = \frac{300}{2} \{2(20) + 299(0.50)\} = 150(189.5) = 28425.$$

## 4. Model Kim's payments

Let the common ratio be  $r$ . Since the first payment is **20** and the month **300** payment is **250**,  $20r^{299} = 250$ .

## 5. Find the ratio

Thus  $r^{299} = 12.5$  and  $r = 12.5^{1/299}$ .

## 6. Sum Kim's payments

$$S_B = 20 \frac{r^{300} - 1}{r - 1} \text{ with } r = 12.5^{1/299}, \text{ giving } S_B \approx 27362.95.$$

## 7. Find the difference

The difference is  $28425 - 27362.95 \approx 1062.05$ , which is £1,060 to the nearest £10.

**Final answer**

(a) Proven.

(b) £28,425.

(c) £1,060.

## Question 8

8.

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

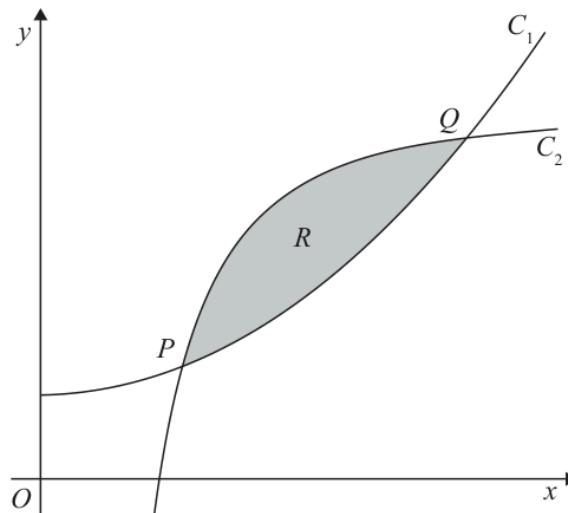


Figure 1

Figure 1 shows a sketch of part of the curve  $C_1$  with equation

$$y = x^2 + 3 \quad x > 0$$

and part of the curve  $C_2$  with equation

$$y = 13 - \frac{9}{x^2} \quad x > 0$$

The curves  $C_1$  and  $C_2$  intersect at the points  $P$  and  $Q$  as shown in Figure 1.

(a) Use algebra to find the  $x$  coordinate of  $P$  and the  $x$  coordinate of  $Q$ .

(4)

The finite region  $R$ , shown shaded in Figure 1, is bounded by  $C_1$  and  $C_2$

(b) Use algebraic integration to find the exact area of  $R$ .

(4)

## Worked Solution - Question 8

**1. Equate the curves**

$$x^2 + 3 = 13 - \frac{9}{x^2}.$$

**2. Clear the denominator**

Multiplying by  $x^2$  gives  $x^4 + 3x^2 = 13x^2 - 9$ , so  $x^4 - 10x^2 + 9 = 0$ .

**3. Factor**

Let  $u = x^2$ . Then  $u^2 - 10u + 9 = (u - 1)(u - 9) = 0$ , so  $x^2 = 1$  or  $x^2 = 9$ .

**4. Use  $x$  greater than zero**

Since  $x > 0$ , the intersections have  $x = 1$  and  $x = 3$ . Thus  $x_P = 1$  and  $x_Q = 3$ .

**5. Set up the area**

Between  $x = 1$  and  $x = 3$ , curve  $C_2$  is above curve  $C_1$ , so

$$R = \int_1^3 \left( 13 - \frac{9}{x^2} - (x^2 + 3) \right) dx.$$

**6. Simplify the integrand**

$$R = \int_1^3 (10 - x^2 - 9x^{-2}) dx.$$

**7. Integrate**

$$R = \left[ 10x - \frac{x^3}{3} + \frac{9}{x} \right]_1^3.$$

**8. Evaluate**

$$R = (30 - 9 + 3) - \left( 10 - \frac{1}{3} + 9 \right) = 24 - \frac{56}{3} = \frac{16}{3}.$$

**Final answer**

(a)  $x_P = 1, x_Q = 3.$

(b)  $\frac{16}{3}.$

**Question 9****Trigonometric Equations**

9.

**In this question you must show detailed reasoning.****Solutions relying entirely on calculator technology are not acceptable.**

(a) Show that the equation

$$2 \tan \theta = 3 \cos \theta$$

can be written as

$$3 \sin^2 \theta + 2 \sin \theta - 3 = 0 \quad (3)$$

(b) Hence solve, for  $-\pi < x < \pi$ , the equation

$$2 \tan \left( 2x + \frac{\pi}{3} \right) = 3 \cos \left( 2x + \frac{\pi}{3} \right)$$

giving your answers to 3 significant figures.

(4)

# Worked Solution - Question 9

## 1. Write $\tan$ in terms of $\sin$ and $\cos$

$$2 \tan \theta = 3 \cos \theta \text{ gives } 2 \frac{\sin \theta}{\cos \theta} = 3 \cos \theta.$$

## 2. Multiply by $\cos$ theta

Since  $\tan \theta$  is defined,  $\cos \theta \neq 0$ . Hence  $2 \sin \theta = 3 \cos^2 \theta$ .

## 3. Use the identity

$$2 \sin \theta = 3(1 - \sin^2 \theta), \text{ so } 3 \sin^2 \theta + 2 \sin \theta - 3 = 0.$$

## 4. Substitute the new angle

$$\text{Let } \theta = 2x + \frac{\pi}{3}. \text{ Then } 3 \sin^2 \theta + 2 \sin \theta - 3 = 0.$$

## 5. Solve for $\sin$ theta

$$\sin \theta = \frac{-2 \pm \sqrt{40}}{6} = \frac{-1 \pm \sqrt{10}}{3}. \text{ The root } \frac{-1 - \sqrt{10}}{3} \text{ is less than } -1, \text{ so reject it.}$$

## 6. Find the angle range

$$\text{Since } -\pi < x < \pi, -\frac{5\pi}{3} < \theta < \frac{7\pi}{3}.$$

## 7. Find the theta values

For  $\sin \theta = \frac{-1 + \sqrt{10}}{3}$  in this range,  $\theta \approx -3.94649, 0.804897, 2.33670,$  and  $7.08808$ .

## 8. Convert back to $x$

$$\text{Using } x = \frac{\theta - \pi/3}{2} \text{ gives } x \approx -2.49684, -0.121150, 0.644749, \text{ and } 3.02044.$$

## 9. Round the answers

To 3 significant figures,  $x = -2.50, -0.121, 0.645,$  and  $3.02$ .

### Final answer

(a) Proven.

(b)  $x = -2.50, -0.121, 0.645, 3.02$ .

## Question 10

10. The circle  $C$  has equation

$$x^2 + y^2 + 4x - 30y + 209 = 0$$

(a) Find

- (i) the coordinates of the centre of  $C$ ,
- (ii) the exact value of the radius of  $C$ .

(3)

The line  $L$  has equation  $y = mx + 1$ , where  $m$  is a constant.

Given that  $L$  is the tangent to  $C$  at the point  $P$ ,

(b) show that

$$2m^2 - 7m - 22 = 0$$

(3)

(c) Hence find the possible pairs of coordinates of  $P$ .

(4)

## Worked Solution - Question 10

**1. Complete the square**

$x^2 + y^2 + 4x - 30y + 209 = 0$  becomes  $(x + 2)^2 + (y - 15)^2 = 20$ .

**2. Read the centre and radius**

The centre is  $(-2, 15)$  and the radius is  $\sqrt{20} = 2\sqrt{5}$ .

**3. Use the tangent distance**

The line  $y = mx + 1$  can be written as  $mx - y + 1 = 0$ . Its distance from  $(-2, 15)$  must be  $2\sqrt{5}$ .

**4. Set up the equation for m**

$$\frac{|m(-2) - 15 + 1|}{\sqrt{m^2 + 1}} = 2\sqrt{5}, \text{ so } \frac{|-2m - 14|}{\sqrt{m^2 + 1}} = 2\sqrt{5}.$$

**5. Show the quadratic**

Squaring gives  $4(m + 7)^2 = 20(m^2 + 1)$ . Hence  $(m + 7)^2 = 5(m^2 + 1)$ , so  $2m^2 - 7m - 22 = 0$ .

**6. Find the possible gradients**

$$2m^2 - 7m - 22 = 0 \text{ gives } m = -2 \text{ or } m = \frac{11}{2}.$$

**7. Use perpendicular radii**

A tangent is perpendicular to the radius at the point of contact.

**8. Find the first point**

For  $m = -2$ , the radius gradient is  $\frac{1}{2}$ . The radius through the centre is  $y - 15 = \frac{1}{2}(x + 2)$ . Solving with  $y = -2x + 1$  gives  $x = -6$  and  $y = 13$ .

### 9. Find the second point

For  $m = \frac{11}{2}$ , the radius gradient is  $-\frac{2}{11}$ . The radius through the centre is  $y - 15 = -\frac{2}{11}(x + 2)$ . Solving with  $y = \frac{11}{2}x + 1$  gives  $x = \frac{12}{5}$  and  $y = \frac{71}{5}$ .

#### Final answer

(a)(i)  $(-2, 15)$ .

(a)(ii)  $2\sqrt{5}$ .

(b) Proven.

(c)  $(-6, 13)$  and  $\left(\frac{12}{5}, \frac{71}{5}\right)$ .

## Question 11

Proof

11. (i) Prove by counter example that the statement

“If  $n$  is a prime number then  $3^n + 2$  is also a prime number.”

is false.

(2)

(ii) Use proof by exhaustion to prove that if  $m$  is an integer that is **not** divisible by 3, then

$$m^2 - 1$$

is divisible by 3

(4)

# Worked Solution - Question 11

Topic group

## 1. Choose a prime value

Take  $n = 5$ , which is prime.

## 2. Evaluate the expression

$3^5 + 2 = 243 + 2 = 245 = 5 \times 49$ , so it is not prime. This disproves the statement.

## 3. List the possible forms of $m$

If integer  $m$  is not divisible by 3, then  $m$  has remainder 1 or 2 on division by 3.

## 4. Case one

If  $m = 3q + 1$ , then  $m^2 - 1 = (3q + 1)^2 - 1 = 9q^2 + 6q = 3(3q^2 + 2q)$ , which is divisible by 3.

## 5. Case two

If  $m = 3q + 2$ , then

$m^2 - 1 = (3q + 2)^2 - 1 = 9q^2 + 12q + 3 = 3(3q^2 + 4q + 1)$ , which is divisible by 3.

## 6. Conclude by exhaustion

Both possible cases for an integer not divisible by 3 have been checked, so  $m^2 - 1$  is divisible by 3.

### Final answer

(i) Counterexample:  $n = 5$  gives  $3^5 + 2 = 245$ , not prime.

(ii) Proven.

**PAST PAPER**

# **WMA12/01 January 2025**

January 2025 | 10 questions | 75 marks

**10**

questions

**75**

marks

**Answers**

worked solution  
after each  
question

# Question 1

## Arithmetic Sequences

1. The arithmetic series  $S$  is given by

$$S = 2 + 5 + 8 + 11 + \dots + 254$$

Find

- (a) the number of terms in the series, (2)
- (b) the sum of the series. (2)

## Worked Solution - Question

1

Topic group

**1. Identify the arithmetic series**

The first term is **2**, the common difference is **3**, and the last term is **254**.

**2. Find the number of terms**

Use  $254 = 2 + (n - 1)3$ . Then  $252 = 3(n - 1)$ , so  $n - 1 = 84$  and  $n = 85$ .

**3. Use the sum formula**

$S_n = \frac{n}{2}(a + l)$ , where  $a$  is the first term and  $l$  is the last term.

**4. Find the sum**

$$S = \frac{85}{2}(2 + 254) = 85 \times 128 = 10880.$$

**Final answer**

(a) 85.

(b) 10880.

**Question 2****Binomial Expansion**

2. (a) Find the first 4 terms, in ascending powers of  $x$ , of the binomial expansion of

$$(2 - 5x)^8$$

giving each term in simplest form.

**(4)**

This expansion is to be used to find an approximation for  $2.05^8$

- (b) State the value of  $x$  that should be used.

*(There is no need to carry out this calculation.)*

**(1)**

## Worked Solution - Question 2

**1. Write the first four terms**

$$(2 - 5x)^8 = 2^8 + 8(2^7)(-5x) + \binom{8}{2}(2^6)(-5x)^2 + \binom{8}{3}(2^5)(-5x)^3 + \dots$$

**2. Simplify each term**

The first four terms are  $256 - 5120x + 44800x^2 - 224000x^3$ .

**3. Connect the expansion to 2.05**

To use  $(2 - 5x)^8$  for  $2.05^8$ , set  $2 - 5x = 2.05$ .

**4. Find  $x$** 

$$-5x = 0.05, \text{ so } x = -0.01.$$

**Final answer**

$$(a) 256 - 5120x + 44800x^2 - 224000x^3.$$

$$(b) x = -0.01.$$

## Question 3

## Applications of Differentiation

3.

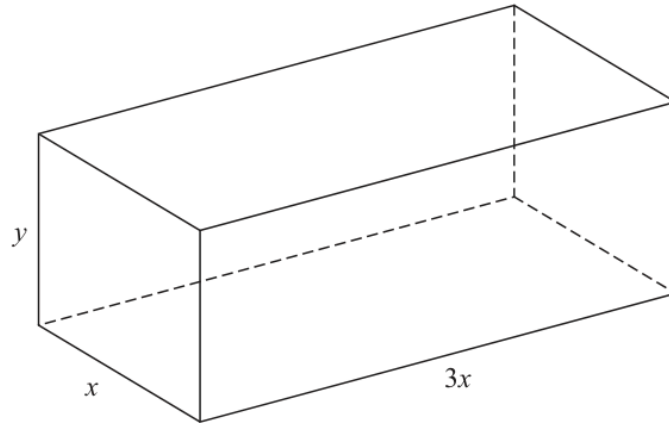


Figure 1

Figure 1 shows an open-topped container used for holding water.

The container is in the shape of a cuboid and is made of sheet metal.

The base of the container is a rectangle  $3x$  metres by  $x$  metres.

The height of the container is  $y$  metres as shown in Figure 1.

Given that the capacity of the container is  $120\text{ m}^3$

(a) show that the area  $A\text{ m}^2$  of the sheet metal used to make the container is given by

$$A = Px^2 + \frac{Q}{x}$$

where  $P$  and  $Q$  are positive constants to be found.

(4)

(b) Use calculus to find the value of  $x$  for which  $A$  has a stationary value, giving your answer to 3 significant figures.

(4)

(c) Find  $\frac{d^2A}{dx^2}$  and hence show that the value of  $x$  found in part (b) gives the minimum value of  $A$ .

(2)

# Worked Solution - Question 3

Topic group

## 1. Use the volume

The volume is  $(3x)(x)(y) = 3x^2y$ . Since the capacity is 120,  $3x^2y = 120$ , so  $y = \frac{40}{x^2}$ .

## 2. Write the metal area

The open top means the metal area is the base plus four sides:

$$A = 3x^2 + 2(3xy) + 2(xy).$$

## 3. Substitute for y

$$A = 3x^2 + 8xy = 3x^2 + 8x \left( \frac{40}{x^2} \right) = 3x^2 + \frac{320}{x}. \text{ Hence } P = 3 \text{ and } Q = 320.$$

## 4. Differentiate

$$\frac{dA}{dx} = 6x - 320x^{-2} = 6x - \frac{320}{x^2}.$$

## 5. Use the stationary condition

$$\text{Set } \frac{dA}{dx} = 0: 6x = \frac{320}{x^2}, \text{ so } 6x^3 = 320.$$

## 6. Find x

$$x^3 = \frac{160}{3}, \text{ so } x = \sqrt[3]{\frac{160}{3}} = 3.764\dots \text{ To 3 significant figures, } x = 3.76.$$

## 7. Differentiate again

$$\frac{d^2A}{dx^2} = 6 + 640x^{-3} = 6 + \frac{640}{x^3}.$$

## 8. Show the minimum

For  $x > 0$ ,  $6 + \frac{640}{x^3} > 0$ , so the stationary value found in part (b) is a minimum.

**Final answer**

(a)  $A = 3x^2 + \frac{320}{x}$ , so  $P = 3$  and  $Q = 320$ .

(b)  $x = 3.76$ .

(c)  $\frac{d^2A}{dx^2} = 6 + \frac{640}{x^3}$ , minimum.

## Question 4

### Geometric Sequences

4. **In this question you must show all stages of your working.  
Solutions relying entirely on calculator technology are not acceptable.**

Given that, in a particular geometric series,

- the sum of the first three terms is 70.2
- the sum to infinity is 75

find, for this series,

- (a) the common ratio,

**(4)**

- (b) the first term.

**(2)**

# Worked Solution - Question 4

**1. Write the two sums**

For first term  $a$  and common ratio  $r$ ,  $S_\infty = \frac{a}{1-r} = 75$  and

$$S_3 = \frac{a(1-r^3)}{1-r} = 70.2.$$

**2. Divide the equations**

$$\frac{S_3}{S_\infty} = 1 - r^3, \text{ so } \frac{70.2}{75} = 1 - r^3.$$

**3. Find the ratio**

$0.936 = 1 - r^3$ , so  $r^3 = 0.064$  and  $r = 0.4$ .

**4. Find the first term**

Use  $\frac{a}{1-0.4} = 75$ . Hence  $a = 75(0.6) = 45$ .

**Final answer**

(a)  $r = 0.4$ .

(b) 45.

## Question 5

## Polynomials

5.

$$f(x) = 3x^3 + ax^2 - 10x + b$$

where  $a$  and  $b$  are constants.

Given that  $(3x - 4)$  is a factor of  $f(x)$ ,

(a) show that  $16a + 9b = 56$

(2)

Given further that when  $f(x)$  is divided by  $(x - 2)$  the remainder is  $b$ ,

(b) find the value of  $a$  and the value of  $b$ .

(4)

(c) Hence, using algebra, fully factorise  $f(x)$ .

(3)

## Worked Solution - Question 5

**1. Use the factor theorem**

Since  $3x - 4$  is a factor,  $x = \frac{4}{3}$  is a root and  $f\left(\frac{4}{3}\right) = 0$ .

**2. Show the given equation**

$$3\left(\frac{4}{3}\right)^3 + a\left(\frac{4}{3}\right)^2 - 10\left(\frac{4}{3}\right) + b = 0, \text{ so } \frac{64}{9} + \frac{16a}{9} - \frac{120}{9} + b = 0.$$

Multiplying by 9 gives  $16a + 9b = 56$ .

**3. Use the remainder condition**

When dividing by  $x - 2$ , the remainder is  $f(2)$ . Given that the remainder is  $b$ ,  $f(2) = b$ .

**4. Find a**

$$f(2) = 24 + 4a - 20 + b = 4 + 4a + b. \text{ Thus } 4 + 4a + b = b, \text{ so } a = -1.$$

**5. Find b**

Substitute  $a = -1$  into  $16a + 9b = 56$ :  $-16 + 9b = 56$ , so  $b = 8$ .

**6. Write f of x**

$$f(x) = 3x^3 - x^2 - 10x + 8.$$

**7. Find another factor**

$$f(1) = 3 - 1 - 10 + 8 = 0, \text{ so } (x - 1) \text{ is a factor.}$$

**8. Fully factorise**

Dividing by  $(x - 1)$  gives  $3x^2 + 2x - 8 = (3x - 4)(x + 2)$ . Hence  $f(x) = (x - 1)(x + 2)(3x - 4)$ .

**Final answer**

(a) Proven.

(b)  $a = -1, b = 8$ .

(c)  $f(x) = (x - 1)(x + 2)(3x - 4)$ .

## Question 6

6.

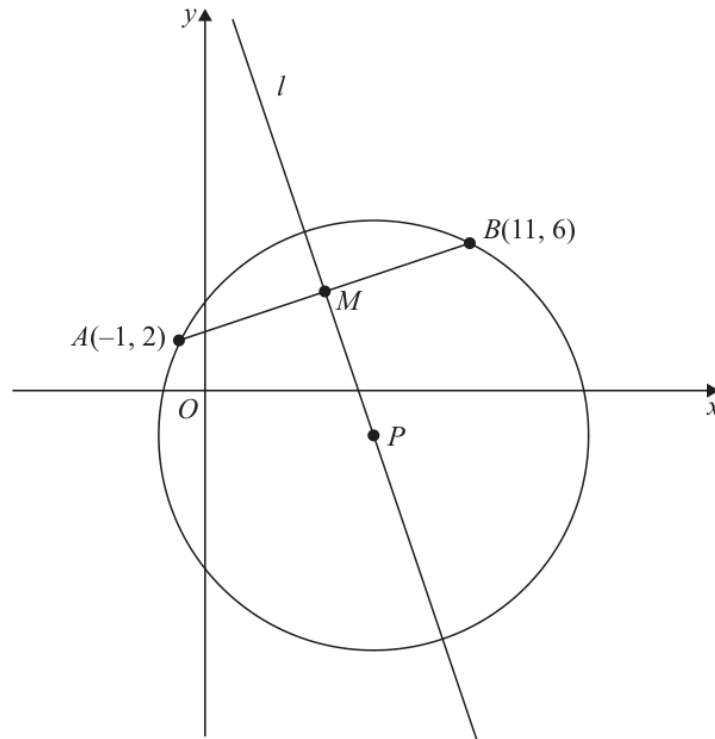


Figure 2

The point  $A(-1, 2)$  and the point  $B(11, 6)$  both lie on a circle with centre  $P$ .

The point  $M$  is the midpoint of  $AB$ .

Given that the line  $l$  passes through  $M$  and  $P$ , as shown in Figure 2,

- (a) find an equation for  $l$ , giving your answer in the form  $y = mx + c$ , where  $m$  and  $c$  are constants.

(4)

Given that  $P$  has coordinates  $(7, k)$ , where  $k$  is a constant,

- (b) find the value of  $k$ ,

(1)

- (c) find an equation for the circle.

(3)

## Worked Solution - Question 6

**1. Find the midpoint**

For  $A(-1, 2)$  and  $B(11, 6)$ , the midpoint is  $M\left(\frac{-1 + 11}{2}, \frac{2 + 6}{2}\right) = (5, 4)$ .

**2. Find the gradient of AB**

The gradient of  $AB$  is  $\frac{6 - 2}{11 - (-1)} = \frac{4}{12} = \frac{1}{3}$ .

**3. Use the perpendicular bisector**

The centre lies on the perpendicular bisector of the chord  $AB$ , so line  $l$  has gradient  $-3$  and passes through  $M(5, 4)$ .

**4. Find line  $l$** 

$y - 4 = -3(x - 5)$ , so  $y = -3x + 19$ .

**5. Use the centre coordinates**

Since  $P = (7, k)$  lies on  $l$ ,  $k = -3(7) + 19 = -2$ .

**6. Find the radius squared**

The centre is  $(7, -2)$ . Using point  $A$ ,  
 $r^2 = (7 - (-1))^2 + (-2 - 2)^2 = 8^2 + (-4)^2 = 80$ .

**7. Write the circle equation**

Therefore the circle is  $(x - 7)^2 + (y + 2)^2 = 80$ .

**Final answer**

(a)  $y = -3x + 19$ .

(b)  $k = -2$ .

(c)  $(x - 7)^2 + (y + 2)^2 = 80$ .

## Question 7

## Laws of Logarithms

7.

**In this question you must show all stages of your working.**  
**Solutions relying entirely on calculator technology are not acceptable.**

- (i) The table below shows values of  $x$  and  $y$ , where  $y = \log_{10}(x + 5)$ , for  $x$  values between  $-1$  and  $4$

$x$	$-1$	$0$	$1$	$2$	$3$	$4$
$y = \log_{10}(x + 5)$	$\log_{10} 4$	$\log_{10} 5$	$\log_{10} 6$	$\log_{10} 7$	$\log_{10} 8$	$\log_{10} 9$

Using the trapezium rule with all the  $y$  values in the given table, show that

$$\int_{-1}^4 \log_{10}(x + 5) dx \approx \log_{10} k$$

where  $k$  is an integer to be found.

(3)

- (ii) Find the value of  $a$  such that

$$2\log_5(5 - a) - \log_5(a + 25) = 1$$

(5)

## Worked Solution - Question 7

Topic group

**1. Use the trapezium rule**

The width is 1, so

$$\int_{-1}^4 \log_{10}(x+5) dx \approx \frac{1}{2} \{ \log_{10} 4 + \log_{10} 9 + 2(\log_{10} 5 + \log_{10} 6 + \log_{10} 7 + \log_{10} 8) \}$$

**2. Combine the logarithms**This is  $\log_{10} (\sqrt{4 \times 9} \times 5 \times 6 \times 7 \times 8)$ .**3. Find k**

$$\sqrt{4 \times 9} \times 5 \times 6 \times 7 \times 8 = 6 \times 1680 = 10080, \text{ so } k = 10080.$$

**4. State the domain**The logarithms require  $5 - a > 0$  and  $a + 25 > 0$ , so  $-25 < a < 5$ .**5. Combine the logarithms**

$$2 \log_5(5 - a) - \log_5(a + 25) = \log_5 \left( \frac{(5 - a)^2}{a + 25} \right).$$

**6. Use one as a logarithm**

$$1 = \log_5 5, \text{ so } \frac{(5 - a)^2}{a + 25} = 5.$$

**7. Form the quadratic**

$(5 - a)^2 = 5(a + 25)$ , so  $a^2 - 10a + 25 = 5a + 125$ . Hence  
 $a^2 - 15a - 100 = 0$ .

**8. Solve and check**

$(a - 20)(a + 5) = 0$ , so  $a = 20$  or  $a = -5$ . Only  $a = -5$  is in the domain, so  
 $a = -5$ .

**Final answer**

(i)  $k = 10080$ .

(ii)  $a = -5$ .

## Question 8

Proof

8. (i) A student states

“If  $x$  and  $y$  are irrational numbers,  $x \neq y$ , then  $xy$  is also irrational.”

Show, by counter example, that this statement is not always true.

(1)

- (ii) Prove, using algebra, that for all odd integers  $n$ , the value of the expression

$$n^3 + 3n + 2$$

is always even but never a multiple of 4

(4)

## Worked Solution - Question 8

Topic group

**1. Give a counterexample**

Let  $x = \sqrt{2}$  and  $y = \frac{1}{\sqrt{2}}$ . Both are irrational and  $x \neq y$ , but  $xy = 1$ , which is rational.

**2. Represent an odd integer**

Let  $n = 2q + 1$ , where  $q$  is an integer.

**3. Substitute into the expression**

$$n^3 + 3n + 2 = (2q + 1)^3 + 3(2q + 1) + 2.$$

**4. Expand and factor**

$$n^3 + 3n + 2 = 8q^3 + 12q^2 + 12q + 6 = 2(4q^3 + 6q^2 + 6q + 3).$$

**5. Show it is even**

The expression has a factor of **2**, so it is even.

**6. Show it is not a multiple of four**

The bracket  $4q^3 + 6q^2 + 6q + 3$  is odd, because the first three terms are even and **3** is odd. Therefore the expression is **2** times an odd integer, so it is not divisible by **4**.

**Final answer**

(i) For example,  $x = \sqrt{2}$  and  $y = \frac{1}{\sqrt{2}}$  give  $xy = 1$ .

(ii) Proven.

## Question 9

9.

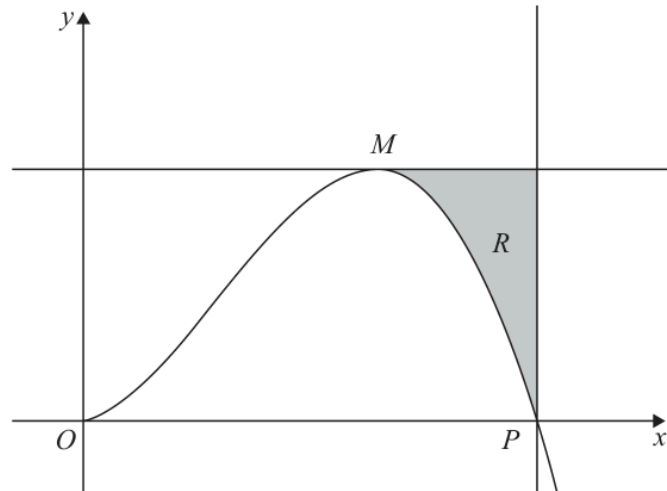


Figure 3

**In this question you must show all stages of your working.  
Solutions relying entirely on calculator technology are not acceptable.**

Figure 3 shows a sketch of part of the curve with equation

$$y = \frac{9x^2(5 - \sqrt{x})}{5} \quad x \geq 0$$

The curve has a turning point at the point  $M$ , as shown in Figure 3.

- (a) Using calculus, find the coordinates of  $M$ . (5)

The curve crosses the  $x$ -axis at the point  $P$ , as shown in Figure 3.

- (b) Use algebra to find the  $x$  coordinate of  $P$ . (2)

The finite region  $R$ , shown shaded in Figure 3, is bounded by the curve, the line through  $M$  parallel to the  $x$ -axis and the line through  $P$  parallel to the  $y$ -axis.

- (c) Use algebraic integration to find the area of  $R$ , giving your answer to one decimal place. (5)

## Worked Solution - Question 9

**1. Rewrite the curve**

$$y = \frac{9x^2(5 - \sqrt{x})}{5} = 9x^2 - \frac{9}{5}x^{5/2}.$$

**2. Differentiate**

$$\frac{dy}{dx} = 18x - \frac{9}{2}x^{3/2} = \frac{9}{2}x(4 - \sqrt{x}).$$

**3. Find the turning point x coordinate**

For the turning point shown away from the origin, set  $4 - \sqrt{x} = 0$ . Hence  $x = 16$ .

**4. Find the y coordinate**

$$\text{At } x = 16, y = \frac{9(16)^2(5 - 4)}{5} = \frac{2304}{5}.$$

**5. State M**

Therefore  $M \left( 16, \frac{2304}{5} \right)$ .

**6. Find the positive x intercept**

For the positive intercept,  $\frac{9x^2(5 - \sqrt{x})}{5} = 0$  gives  $5 - \sqrt{x} = 0$ , so  $x = 25$ .

**7. Set up the area**

The top boundary is  $y = \frac{2304}{5}$  and the curve is below it from  $x = 16$  to  $x = 25$ .

$$\text{So } R = \int_{16}^{25} \left( \frac{2304}{5} - 9x^2 + \frac{9}{5}x^{5/2} \right) dx.$$

### 8. Integrate

$$R = \left[ \frac{2304}{5}x - 3x^3 + \frac{18}{35}x^{7/2} \right]_{16}^{25}$$

### 9. Evaluate

$$R = \frac{9189}{7} = 1312.714\dots, \text{ so the area is } \mathbf{1312.7} \text{ to one decimal place.}$$

#### Final answer

(a)  $M \left( 16, \frac{2304}{5} \right)$ .

(b)  $x_P = 25$ .

(c) 1312.7.

## Question 10

## Trigonometric Equations

10.

In this question you must show all stages of your working.  
Solutions relying entirely on calculator technology are not acceptable.

(a) Show that

$$\cos \theta \left( 3 \tan \theta + \frac{2}{\tan \theta} \right) \equiv \sin \theta + \frac{2}{\sin \theta} \quad \theta \neq \frac{n\pi}{2} \quad (4)$$

(b) Hence solve, for  $0 < x < 2\pi$ , the equation

$$\cos x \left( 3 \tan x + \frac{2}{\tan x} \right) = 4 \sin x - 5$$

giving your answers to 3 significant figures.

(4)

# Worked Solution - Question 10

Topic group

**1. Expand the left side**

$$\cos \theta \left( 3 \tan \theta + \frac{2}{\tan \theta} \right) = 3 \sin \theta + \frac{2 \cos \theta}{\tan \theta}.$$

**2. Rewrite the second term**

$$\text{Since } \tan \theta = \frac{\sin \theta}{\cos \theta}, \quad \frac{2 \cos \theta}{\tan \theta} = \frac{2 \cos^2 \theta}{\sin \theta}.$$

**3. Use the identity**

$$3 \sin \theta + \frac{2 \cos^2 \theta}{\sin \theta} = 3 \sin \theta + \frac{2(1 - \sin^2 \theta)}{\sin \theta}.$$

**4. Simplify**

$$3 \sin \theta + \frac{2}{\sin \theta} - 2 \sin \theta = \sin \theta + \frac{2}{\sin \theta}, \text{ as required.}$$

**5. Use part a**

$$\text{The equation becomes } \sin x + \frac{2}{\sin x} = 4 \sin x - 5.$$

**6. Form a quadratic in  $\sin x$** 

$$\text{Multiplying by } \sin x \text{ gives } \sin^2 x + 2 = 4 \sin^2 x - 5 \sin x, \text{ so}$$
$$3 \sin^2 x - 5 \sin x - 2 = 0.$$

**7. Solve for  $\sin x$** 

$$(3 \sin x + 1)(\sin x - 2) = 0. \text{ Since } \sin x = 2 \text{ is impossible, } \sin x = -\frac{1}{3}.$$

### 8. Find $x$ in the interval

For  $0 < x < 2\pi$ ,  $\sin x = -\frac{1}{3}$  gives  $x = \pi + \sin^{-1}\left(\frac{1}{3}\right)$  or  
 $x = 2\pi - \sin^{-1}\left(\frac{1}{3}\right)$ .

### 9. Round the answers

Thus  $x = 3.481\dots$  or  $x = 5.943\dots$ , so to 3 significant figures  $x = 3.48$  or  
 $x = 5.94$ .

#### Final answer

(a) Proven.

(b)  $x = 3.48, 5.94$ .

**PAST PAPER**

# **WMA12/01 May/June 2025**

**May/June 2025 | 10 questions | 75 marks**

**10**

questions

**75**

marks

**Answers**

worked solution  
after each  
question

# Question 1

## Binomial Expansion

- 1: (a) Find the first 4 terms, in ascending powers of  $x$ , of the binomial expansion of

$$(1 - 4x)^7$$

giving each term in simplest form.

(4)

In the series expansion of

$$(5 + kx)(1 - 4x)^7 \quad \text{where } k \text{ is a constant}$$

the coefficient of the term in  $x^2$  is 1316

- (b) Use the answer to part (a) to find the value of  $k$ .

(2)

## Worked Solution - Question 1

**1. Write the binomial terms**

$$(1 - 4x)^7 = 1 + 7(-4x) + \binom{7}{2}(-4x)^2 + \binom{7}{3}(-4x)^3 + \dots$$

**2. Simplify**

The first four terms are  $1 - 28x + 336x^2 - 2240x^3$ .

**3. Find the  $x$  squared terms**

In  $(5 + kx)(1 - 4x)^7$ , the  $x^2$  coefficient comes from  $5(336x^2)$  and  $(kx)(-28x)$ .

**4. Use the given coefficient**

$$5(336) - 28k = 1316, \text{ so } 1680 - 28k = 1316.$$

**5. Find  $k$** 

$$28k = 364, \text{ hence } k = 13.$$

**Final answer**

$$(a) 1 - 28x + 336x^2 - 2240x^3.$$

$$(b) k = 13.$$

## Question 2

2: The line joining the points  $(-2, 5)$  and  $(4, 15)$  is the diameter of a circle  $C$ .

(a) Find an equation for  $C$ .

(5)

(b) Hence find the exact coordinates of the point on  $C$  that is nearest the  $x$ -axis.

(2)

## Worked Solution - Question 2

**1. Find the centre**

The centre is the midpoint of  $(-2, 5)$  and  $(4, 15)$ , so it is  $\left(\frac{-2+4}{2}, \frac{5+15}{2}\right) = (1, 10)$ .

**2. Find the radius squared**

Using  $(-2, 5)$ ,  $r^2 = (-2 - 1)^2 + (5 - 10)^2 = 9 + 25 = 34$ .

**3. Write the circle equation**

Therefore  $C$  has equation  $(x - 1)^2 + (y - 10)^2 = 34$ .

**4. Find the nearest point to the  $x$  axis**

The nearest point to the  $x$ -axis is vertically below the centre by one radius.

**5. Write the coordinates**

The radius is  $\sqrt{34}$ , so the point is  $(1, 10 - \sqrt{34})$ .

**Final answer**

(a)  $(x - 1)^2 + (y - 10)^2 = 34$ .

(b)  $(1, 10 - \sqrt{34})$ .

## Question 3

3:

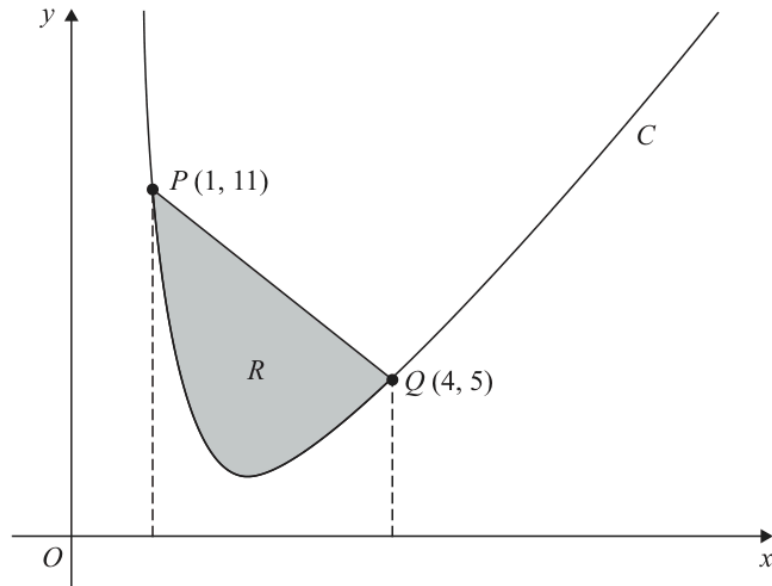


Figure 1

**In this question you must show all stages of your working.  
Solutions relying entirely on calculator technology are not acceptable.**

Figure 1 shows a sketch of the curve  $C$  with equation

$$y = 3x + \frac{16}{x^2} - 8 \quad x > 0$$

The points  $P(1, 11)$  and  $Q(4, 5)$  lie on  $C$  and are shown in Figure 1.

The region  $R$ , shown shaded in Figure 1, is bounded by  $C$  and line segment  $PQ$ .

Use algebraic integration to find the area of  $R$ .

(5)

## Worked Solution - Question 3

**1. Find the line PQ**

The gradient of  $PQ$  is  $\frac{5-11}{4-1} = -2$ , so the line through  $P(1, 11)$  is  $y - 11 = -2(x - 1)$ .

**2. Write the line equation**

Thus  $PQ$  has equation  $y = -2x + 13$ .

**3. Set up the area**

From  $x = 1$  to  $x = 4$ , the line segment is above the curve. Hence

$$R = \int_1^4 \left[ (-2x + 13) - \left( 3x + \frac{16}{x^2} - 8 \right) \right] dx.$$

**4. Simplify the integrand**

$$R = \int_1^4 (21 - 5x - 16x^{-2}) dx.$$

**5. Integrate**

$$R = \left[ 21x - \frac{5}{2}x^2 + \frac{16}{x} \right]_1^4.$$

**6. Evaluate**

$$R = (84 - 40 + 4) - \left( 21 - \frac{5}{2} + 16 \right) = 48 - \frac{69}{2} = \frac{27}{2}.$$

**Final answer**

$$\frac{27}{2}.$$

## Question 4

## Polynomials

4: The function  $f(x)$  is defined by

$$f(x) = ax^3 + bx^2 + 5x - 3$$

where  $a$  and  $b$  are constants.

Given that  $(x + 3)$  is a factor of  $f(x)$ ,

(a) show that

$$b = 3a + 2 \quad (2)$$

Given further that when  $f(x)$  is divided by  $(2x - 1)$  the remainder is  $\frac{7}{4}$

(b) find the value of  $a$  and the value of  $b$ . (4)

(c) Using algebra, find the quotient and the remainder when  $f(x)$  is divided by  $(x - 2)$  (3)

## Worked Solution - Question 4

**1. Use the factor theorem**

Since  $x + 3$  is a factor,  $f(-3) = 0$ .

**2. Show the relation**

$f(-3) = -27a + 9b - 15 - 3 = 0$ , so  $9b = 27a + 18$  and therefore  $b = 3a + 2$ .

**3. Use the remainder theorem**

When dividing by  $2x - 1$ , substitute  $x = \frac{1}{2}$ . The remainder is  $\frac{7}{4}$ , so

$$f\left(\frac{1}{2}\right) = \frac{7}{4}.$$
**4. Form the second equation**

$\frac{a}{8} + \frac{b}{4} + \frac{5}{2} - 3 = \frac{7}{4}$ . Multiplying by 8 gives  $a + 2b - 4 = 14$ , so  $a + 2b = 18$ .

**5. Solve for a and b**

Substitute  $b = 3a + 2$  into  $a + 2b = 18$ :  $a + 6a + 4 = 18$ , so  $a = 2$  and  $b = 8$ .

**6. Write f of x**

$$f(x) = 2x^3 + 8x^2 + 5x - 3.$$

**7. Divide by x minus two**

$$2x^3 + 8x^2 + 5x - 3 = (x - 2)(2x^2 + 12x + 29) + 55.$$

**8. State the result**

The quotient is  $2x^2 + 12x + 29$  and the remainder is 55.

**Final answer**

(a) Proven.

(b)  $a = 2, b = 8$ .

(c) Quotient  $2x^2 + 12x + 29$ , remainder 55.

## Question 5

$x$	-2	-0.5	1	2.5	4	5.5	7
$y$	12	4.243	1.5	0.530	0.188	0.066	0.023

The table above shows corresponding values of  $x$  and  $y$  for

$$y = 3\left(\frac{1}{2}\right)^x$$

The values of  $y$  are given to 3 decimal places as appropriate.

- (a) Using the trapezium rule with all the values of  $y$  in the given table, obtain an estimate for

$$\int_{-2}^7 3\left(\frac{1}{2}\right)^x dx$$

giving the answer to one decimal place.

(3)

Using the answer to part (a) and making your method clear, estimate

(b) (i)  $\int_{-2}^7 3\left(\frac{1}{2}\right)^{x+2} dx$

(ii)  $\int_{-2}^7 (2^{-x} + 2x) dx$

(3)

## Worked Solution - Question 5

**1. Find the strip width**

The  $x$  values increase by 1.5, so  $h = 1.5$ .

**2. Apply the trapezium rule**

$$\int_{-2}^7 3\left(\frac{1}{2}\right)^x dx \approx \frac{1.5}{2} \{12 + 0.023 + 2(4.243 + 1.5 + 0.530 + 0.188 + 0.066)\}$$

**3. Evaluate**

This gives 18.80775, which is 18.8 to one decimal place.

**4. Use the power shift**

$$3\left(\frac{1}{2}\right)^{x+2} = \frac{1}{4} \cdot 3\left(\frac{1}{2}\right)^x$$

**5. Estimate part b i**

$$\text{So } \int_{-2}^7 3\left(\frac{1}{2}\right)^{x+2} dx \approx \frac{1}{4}(18.80775) = 4.7019375, \text{ which is } 4.7.$$

**6. Rewrite part b ii**

$$\text{Since } 3\left(\frac{1}{2}\right)^x = 3 \cdot 2^{-x}, \text{ we have } 2^{-x} = \frac{1}{3} \cdot 3\left(\frac{1}{2}\right)^x.$$

**7. Estimate part b ii**

$$\int_{-2}^7 (2^{-x} + 2x) dx \approx \frac{18.80775}{3} + [x^2]_{-2}^7 = 6.26925 + 45 = 51.26925, \text{ which is } 51.3.$$

**Final answer**

(a) 18.8.

(b)(i) 4.7.

(b)(ii) 51.3.

## Question 6

## Laws of Logarithms

6:

In this question you must show all stages of your working.  
Solutions relying on calculator technology are not acceptable.

(i) Solve

$$2\log_2(4-x) = 3 + \log_2\left(\frac{x+11}{2}\right) \quad (5)$$

(ii) The curves  $C_1$  and  $C_2$  with equations

$$y = 3^{2x+1} \quad \text{and} \quad y = 6 \times 3^x$$

meet at the point  $P$ .Find the exact coordinates of  $P$ , writing your answer in the form  $(\log_3 a, b)$  where  $a$  and  $b$  are integers.

(5)

## Worked Solution - Question 6

**1. State the domain**

The logarithms require  $4 - x > 0$  and  $\frac{x + 11}{2} > 0$ , so  $-11 < x < 4$ .

**2. Use log laws**

$$2 \log_2(4 - x) = \log_2((4 - x)^2) \text{ and}$$
$$3 + \log_2\left(\frac{x + 11}{2}\right) = \log_2\left(8 \cdot \frac{x + 11}{2}\right).$$

**3. Equate the arguments**

Thus  $(4 - x)^2 = 4(x + 11)$ .

**4. Solve the quadratic**

$x^2 - 8x + 16 = 4x + 44$ , so  $x^2 - 12x - 28 = 0$ . Hence  $(x - 14)(x + 2) = 0$ .

**5. Check the domain**

The root 14 is outside the domain, so  $x = -2$ .

**6. Equate the curves**

At  $P$ ,  $3^{2x+1} = 6 \times 3^x$ .

**7. Let  $u$  equal 3 to the  $x$** 

Let  $u = 3^x$ . Then  $3^{2x+1} = 3u^2$ , so  $3u^2 = 6u$ .

**8. Find  $u$** 

Since  $u > 0$ , divide by  $3u$  to get  $u = 2$ . Hence  $3^x = 2$  and  $x = \log_3 2$ .

**9. Find  $y$** 

$y = 6 \times 3^x = 6 \times 2 = 12$ , so  $P = (\log_3 2, 12)$ .

**Final answer**

(i)  $x = -2$ .

(ii)  $P = (\log_3 2, 12)$ .

## Question 7

## Sequences &amp; Series

7: **In this question you must show all stages of your working.**  
**Solutions relying entirely on calculator technology are not acceptable.**

- (i) A **geometric** series begins

$$10 + 8 + 6.4 + \dots$$

- (a) Find the sum to infinity of this series.

(2)

Given that the  $k$ th term of this series is less than 0.0005

- (b) use algebra to find the smallest possible value of  $k$ .

(3)

- (ii) An **arithmetic** series begins

$$850 + 843 + 836 + \dots$$

Given that the sum of the first  $n$  terms of this series is  $S_n$   
find the greatest possible value of  $S_n$

(4)

## Worked Solution - Question 7

**1. Identify the geometric sequence**

The first term is 10 and the common ratio is  $\frac{8}{10} = 0.8$ .

**2. Find the sum to infinity**

$$S_{\infty} = \frac{10}{1 - 0.8} = 50.$$

**3. Write the  $k$ th term**

The  $k$ th term is  $10(0.8)^{k-1}$ .

**4. Use the inequality**

$$10(0.8)^{k-1} < 0.0005, \text{ so } (0.8)^{k-1} < 0.00005.$$

**5. Solve with logarithms**

Taking logarithms gives  $(k - 1) \log(0.8) < \log(0.00005)$ . Since  $\log(0.8) < 0$ ,

$$k - 1 > \frac{\log(0.00005)}{\log(0.8)} = 44.381 \dots$$
**6. Find the smallest integer**

Thus  $k > 45.381 \dots$ , so the smallest possible value is  $k = 46$ .

**7. Write the arithmetic sum**

For  $850 + 843 + 836 + \dots$ , the first term is 850 and the common difference is  $-7$ .

**8. Form  $S_n$** 

$$S_n = \frac{n}{2} \{2(850) + (n - 1)(-7)\} = \frac{n}{2} (1707 - 7n).$$

### 9. Find the best integer $n$

This quadratic has its maximum near  $n = \frac{1707}{14} = 121.928\dots$ , so check  $n = 122$ .

### 10. Find the greatest sum

$$S_{122} = \frac{122}{2}(1707 - 854) = 61(853) = 52033.$$

#### Final answer

(i)(a) 50.

(i)(b) 46.

(ii) 52033.

## Question 8

## Trigonometric Equations

**8:** In this question you must show all stages of your working.  
Solutions relying entirely on calculator technology are not acceptable.

(i) Solve, for  $0 \leq x < \pi$ , the equation

$$3 \tan \left( 2x + \frac{\pi}{5} \right) = \sqrt{3}$$

giving the answers in radians in the form  $k\pi$ , where  $k$  is a rational constant to be found.

(3)

(ii) Solve, for  $0 \leq \theta < 360^\circ$ , the equation

$$5 \sin \theta \tan \theta = \cos \theta + 4$$

giving your answers, in degrees, to one decimal place.

(5)

# Worked Solution - Question 8

Topic group

## 1. Solve the tangent equation

$$3 \tan \left( 2x + \frac{\pi}{5} \right) = \sqrt{3} \text{ gives } \tan \left( 2x + \frac{\pi}{5} \right) = \frac{1}{\sqrt{3}}.$$

## 2. Write the general angle

$$\text{Therefore } 2x + \frac{\pi}{5} = \frac{\pi}{6} + n\pi.$$

## 3. Find $x$

$$2x = -\frac{\pi}{30} + n\pi, \text{ so } x = -\frac{\pi}{60} + \frac{n\pi}{2}.$$

## 4. Use the interval

$$\text{For } 0 \leq x < \pi, \text{ the possible values are } x = \frac{29\pi}{60} \text{ and } x = \frac{59\pi}{60}.$$

## 5. Rewrite the second equation

$$\text{Using } \tan \theta = \frac{\sin \theta}{\cos \theta}, 5 \sin \theta \tan \theta = \cos \theta + 4 \text{ becomes } \frac{5 \sin^2 \theta}{\cos \theta} = \cos \theta + 4.$$

## 6. Form an equation in cos theta

Multiplying by  $\cos \theta$  gives  $5 \sin^2 \theta = \cos^2 \theta + 4 \cos \theta$ . Using  $\sin^2 \theta = 1 - \cos^2 \theta$  gives  $6 \cos^2 \theta + 4 \cos \theta - 5 = 0$ .

## 7. Solve for cos theta

$$\cos \theta = \frac{-2 \pm \sqrt{34}}{6}. \text{ The negative option is less than } -1, \text{ so reject it.}$$

## 8. Find theta

$$\cos \theta = \frac{-2 + \sqrt{34}}{6} \text{ gives } \theta = 50.320 \dots^\circ \text{ or } 309.679 \dots^\circ.$$

## 9. Round the answers

To one decimal place,  $\theta = 50.3^\circ$  or  $309.7^\circ$ .

### Final answer

$$(i) x = \frac{29\pi}{60}, \frac{59\pi}{60}.$$

$$(ii) \theta = 50.3^\circ, 309.7^\circ.$$

## Question 9

## Applications of Differentiation

9:

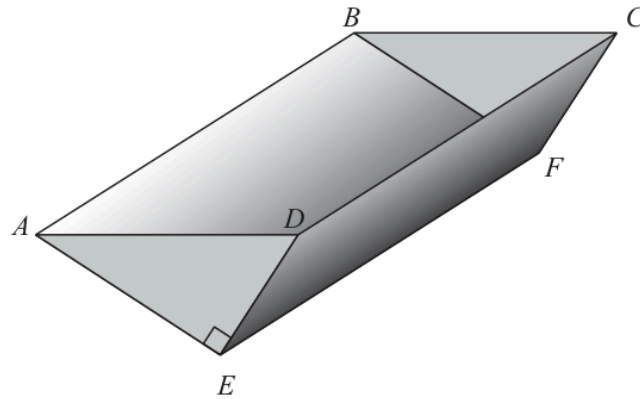


Figure 2

Figure 2 shows a sketch of an open container.

The sides  $CDEF$  and  $ABFE$  are rectangles.

The ends  $ADE$  and  $BCF$  are congruent (identical) right-angled triangles.

The container is made from metal of negligible thickness.

Given that

- $AE = BF = 3x$  metres
- $DE = CF = 2x$  metres
- $AB = DC = EF = L$  metres

and the capacity of the container is  $12 \text{ m}^3$

(a) show that the area of metal used to make the container,  $S \text{ m}^2$ , is given by

$$S = Px^2 + \frac{Q}{x}$$

where  $P$  and  $Q$  are positive integers to be found.

(4)

Given that  $x$  can vary,

(b) use algebraic calculus to find the minimum value of  $S$ , giving your answer to one decimal place.

(5)

(c) Justify that the value of  $S$  found in part (b) is a minimum.

(2)

# Worked Solution - Question 9

Topic group

## 1. Find the triangular end area

Each end is a right-angled triangle with perpendicular sides  $3x$  and  $2x$ , so its area is  $\frac{1}{2}(3x)(2x) = 3x^2$ .

## 2. Use the capacity

The volume is cross-section area times length, so  $3x^2L = 12$  and  $L = \frac{4}{x^2}$ .

## 3. Write the metal area

There are two triangular ends with total area  $6x^2$ , and two rectangular sides with total area  $3xL + 2xL = 5xL$ .

## 4. Substitute for L

$S = 6x^2 + 5x \left( \frac{4}{x^2} \right) = 6x^2 + \frac{20}{x}$ . Hence  $P = 6$  and  $Q = 20$ .

## 5. Differentiate

$$\frac{dS}{dx} = 12x - 20x^{-2} = 12x - \frac{20}{x^2}.$$

## 6. Find the stationary value of x

Set  $\frac{dS}{dx} = 0$ :  $12x = \frac{20}{x^2}$ , so  $x^3 = \frac{5}{3}$ .

## 7. Evaluate S

At  $x = \sqrt[3]{\frac{5}{3}}$ ,  $S = 6x^2 + \frac{20}{x} = 25.3029\dots$ , so the minimum value is **25.3** to one decimal place.

### 8. Differentiate again

$$\frac{d^2 S}{dx^2} = 12 + 40x^{-3} = 12 + \frac{40}{x^3}.$$

### 9. Justify the minimum

Since  $x > 0$ ,  $12 + \frac{40}{x^3} > 0$ , so the stationary value found is a minimum.

#### Final answer

(a)  $S = 6x^2 + \frac{20}{x}$ , so  $P = 6$  and  $Q = 20$ .

(b) 25.3.

(c) Minimum justified.

## Question 10

Proof

**10:** In this question you must show detailed reasoning.

Use algebra to prove by exhaustion that,

for all positive integers  $m$  that are **not** multiples of 3, the value of

$$m^2 + 3m + 2$$

is always a multiple of 3

(4)

# Worked Solution - Question 10

Topic group

## 1. List the possible cases

A positive integer that is not a multiple of 3 must have the form  $m = 3q + 1$  or  $m = 3q + 2$ .

## 2. Case one

If  $m = 3q + 1$ , then

$$m^2 + 3m + 2 = (3q + 1)^2 + 3(3q + 1) + 2 = 9q^2 + 15q + 6 = 3(3q^2 + 5q + 2)$$

## 3. Case two

If  $m = 3q + 2$ , then

$$m^2 + 3m + 2 = (3q + 2)^2 + 3(3q + 2) + 2 = 9q^2 + 21q + 12 = 3(3q^2 + 7q + 4)$$

## 4. Conclude by exhaustion

Both possible cases give a factor of 3, so  $m^2 + 3m + 2$  is always a multiple of 3.

### Final answer

Proven.

**PAST PAPER**

# **WMA12/01 October 2025**

**October 2025 | 9 questions | 75 marks**

**9**

questions

**75**

marks

**Answers**

worked solution  
after each  
question

# Question 1

## Arithmetic Sequences

1. An arithmetic series starts

$$78 + 75 + 72 + \dots$$

Find

- (a) the 18<sup>th</sup> term in the series, (2)
- (b) the sum of the first 100 terms of the series. (2)

## Worked Solution - Question

## 1

**1. Identify the sequence**

The first term is 78 and the common difference is  $75 - 78 = -3$ .

**2. Find the eighteenth term**

$$u_{18} = 78 + 17(-3) = 27.$$

**3. Use the sum formula**

$$S_{100} = \frac{100}{2} \{2(78) + 99(-3)\}.$$

**4. Evaluate the sum**

$$S_{100} = 50(156 - 297) = 50(-141) = -7050.$$

**Final answer**

(a) 27.

(b) -7050.

## Question 2

## Binomial Expansion

2. The first 4 terms, in ascending powers of  $x$ , in the binomial expansion of

$$(1 + px)^{10}$$

are

$$1 + 15x + qx^2 + rx^3$$

where  $p$ ,  $q$  and  $r$  are constants.

Find the value of  $p$ , the value of  $q$  and the value of  $r$ .

(6)

## Worked Solution - Question 2

**1. Write the first four terms**

$$(1 + px)^{10} = 1 + 10px + \binom{10}{2}p^2x^2 + \binom{10}{3}p^3x^3 + \dots$$

**2. Compare the  $x$  coefficient**

Since the coefficient of  $x$  is 15,  $10p = 15$ , so  $p = \frac{3}{2}$ .

**3. Find  $q$** 

$$q = \binom{10}{2}p^2 = 45\left(\frac{3}{2}\right)^2 = \frac{405}{4}.$$

**4. Find  $r$** 

$$r = \binom{10}{3}p^3 = 120\left(\frac{3}{2}\right)^3 = 405.$$

**Final answer**

$$p = \frac{3}{2}, q = \frac{405}{4}, r = 405.$$

## Question 3

3.

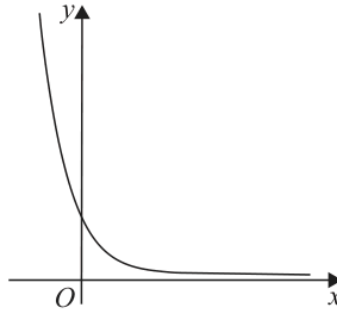


Figure 1

**In this question you must show all stages of your working.**  
**Solutions relying entirely on calculator technology are not acceptable.**

Figure 1 shows a sketch of the curve with equation  $y = 3 \times 2^{-x}$

The point  $P(k, 300\,000)$  lies on the curve.

(a) Use logarithms to find the value of  $k$  to 2 decimal places.

(2)

$x$	-0.5	1	2.5	4.0	5.5	7
$y$	4.243	1.5	0.530	0.188	0.066	0.023

The table shows corresponding values of  $x$  and  $y$  for  $y = 3 \times 2^{-x}$

The values of  $y$  are given to 3 decimal places where appropriate.

(b) (i) Use the trapezium rule, with all the values of  $y$  from the table, to find an approximate value, to 2 decimal places, for

$$\int_{-0.5}^7 3 \times 2^{-x} \, dx$$

(3)

(ii) Use your answer to part (b)(i) to estimate

$$\int_{-0.5}^7 2^{-x} \, dx + \int_{-7}^{0.5} 2^x \, dx$$

(2)

# Worked Solution - Question 3

## 1. Use the point on the curve

Since  $P(k, 300000)$  lies on  $y = 3 \times 2^{-x}$ ,  $300000 = 3 \times 2^{-k}$ .

## 2. Solve with logarithms

$100000 = 2^{-k}$ , so  $-k = \log_2(100000)$  and  $k = -\log_2(100000) = -16.609\dots$

## 3. Round $k$

To 2 decimal places,  $k = -16.61$ .

## 4. Find the strip width

The  $x$  values increase by 1.5, so  $h = 1.5$ .

## 5. Apply the trapezium rule

$$\int_{-0.5}^7 3 \times 2^{-x} dx \approx \frac{1.5}{2} \{4.243 + 0.023 + 2(1.5 + 0.530 + 0.188 + 0.066)\}.$$

## 6. Evaluate the estimate

This gives 6.6255, which is 6.63 to 2 decimal places.

## 7. Relate the first new integral

$$\int_{-0.5}^7 2^{-x} dx = \frac{1}{3} \int_{-0.5}^7 3 \times 2^{-x} dx.$$

## 8. Relate the second new integral

Using the substitution  $u = -x$ ,  $\int_{-7}^{0.5} 2^x dx = \int_{-0.5}^7 2^{-u} du.$

### 9. Estimate the total

The required estimate is therefore  $\frac{2}{3}(6.6255) = 4.417$ , which is **4.42** to 2 decimal places.

#### Final answer

(a)  $k = -16.61$ .

(b)(i) 6.63.

(b)(ii) 4.42.

## Question 4

## Polynomials

4. **In this question you must show all stages of your working.  
Solutions relying on calculator technology are not acceptable.**

$$f(x) = 4x^3 + 13x^2 - 10x + 8$$

- (a) When  $f(x)$  is divided by  $(x - 2)$  the remainder is  $R$  and the quotient is  $Q(x)$ .
- (i) Find  $Q(x)$ .
- (ii) Find  $R$ . (4)
- (b) (i) Use the factor theorem to show that  $(x + 4)$  is a factor of  $f(x)$ .
- (ii) Hence prove, using algebra, that the equation  $f(x) = 0$  has only one real solution. (5)
- (c) Find the range of values of  $x$  for which  $f(x)$  is decreasing. (3)

## Worked Solution - Question 4

**1. Divide by  $x$  minus two**

$$4x^3 + 13x^2 - 10x + 8 = (x - 2)(4x^2 + 21x + 32) + 72.$$

**2. State  $Q$  and  $R$** 

Hence  $Q(x) = 4x^2 + 21x + 32$  and  $R = 72$ .

**3. Use the factor theorem**

$$f(-4) = 4(-4)^3 + 13(-4)^2 - 10(-4) + 8 = -256 + 208 + 40 + 8 = 0.$$

**4. Show the factor**

Since  $f(-4) = 0$ ,  $(x + 4)$  is a factor of  $f(x)$ .

**5. Factor the cubic**

Dividing by  $(x + 4)$  gives  $f(x) = (x + 4)(4x^2 - 3x + 2)$ .

**6. Check the quadratic**

The discriminant of  $4x^2 - 3x + 2$  is  $(-3)^2 - 4(4)(2) = 9 - 32 = -23 < 0$ , so this quadratic has no real roots.

**7. Conclude the real solution**

Therefore  $f(x) = 0$  has only one real solution,  $x = -4$ .

**8. Differentiate**

$$f'(x) = 12x^2 + 26x - 10 = 2(3x - 1)(2x + 5).$$

**9. Find where  $f$  is decreasing**

$f(x)$  is decreasing when  $f'(x) < 0$ , which is between the roots. Hence

$$-\frac{5}{2} < x < \frac{1}{3}.$$

**Final answer**

(a)(i)  $Q(x) = 4x^2 + 21x + 32$ .

(a)(ii)  $R = 72$ .

(b) Proven; only real solution is  $x = -4$ .

(c)  $-\frac{5}{2} < x < \frac{1}{3}$ .

## Question 5

## Trigonometric Equations

5. **In this question you must show all stages of your working.**  
**Solutions relying entirely on calculator technology are not acceptable.**

- (i) Solve, for  $0 < \theta \leq 360^\circ$ , the equation

$$4 \tan \theta + 5 \sin \theta = 0$$

giving any non-exact answers to one decimal place.

(5)

- (ii) Solve, for  $0 < x < \pi$ , the equation

$$\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{5}{\cos x}$$

giving the answers, in radians, to 3 significant figures.

(4)

# Worked Solution - Question 5

**1. Rewrite the first equation**

$$4 \tan \theta + 5 \sin \theta = 0 \text{ gives } 4 \frac{\sin \theta}{\cos \theta} + 5 \sin \theta = 0.$$

**2. Factor sin theta**

$$\sin \theta \left( \frac{4}{\cos \theta} + 5 \right) = 0.$$

**3. Use sin theta equals zero**

$$\text{In } 0 < \theta \leq 360^\circ, \sin \theta = 0 \text{ gives } \theta = 180^\circ \text{ or } 360^\circ.$$

**4. Use the other factor**

$$\frac{4}{\cos \theta} + 5 = 0 \text{ gives } \cos \theta = -\frac{4}{5}.$$

**5. Find the remaining angles**

$$\text{For } 0 < \theta \leq 360^\circ, \cos \theta = -\frac{4}{5} \text{ gives } \theta = 143.130\dots^\circ \text{ or } 216.869\dots^\circ.$$

**6. Round part i**

The non-exact angles are  $143.1^\circ$  and  $216.9^\circ$  to one decimal place.

**7. Clear the fractions in part ii**

$$\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{5}{\cos x}. \text{ Multiplying by } \sin x \cos x \text{ gives}$$
$$\sin^2 x + \cos^2 x = 5 \sin x.$$

**8. Use the identity**

$$\sin^2 x + \cos^2 x = 1, \text{ so } 1 = 5 \sin x \text{ and } \sin x = \frac{1}{5}.$$

### 9. Solve in the interval

For  $0 < x < \pi$ ,  $x = \sin^{-1}\left(\frac{1}{5}\right)$  or  $x = \pi - \sin^{-1}\left(\frac{1}{5}\right)$ .

### 10. Round part ii

Thus  $x = 0.201357\dots$  or  $2.94023\dots$ , so  $x = 0.201$  or  $2.94$  to 3 significant figures.

#### Final answer

(i)  $\theta = 143.1^\circ, 180^\circ, 216.9^\circ, 360^\circ$ .

(ii)  $x = 0.201, 2.94$ .

## Question 6

6.

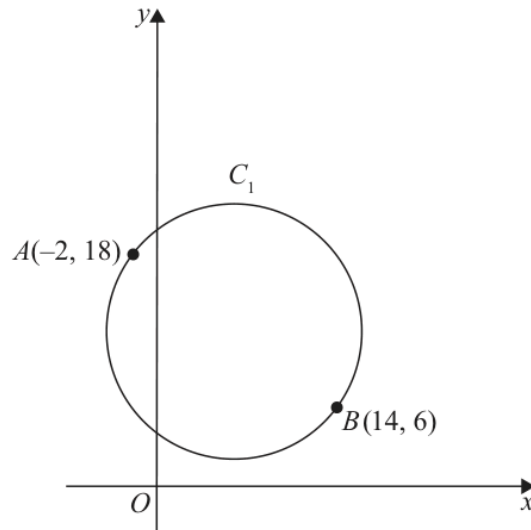


Figure 2

The points  $A(-2, 18)$  and  $B(14, 6)$  lie on a circle  $C_1$  as shown in Figure 2.

Given that  $AB$  is a diameter of the circle  $C_1$

(a) find an equation for  $C_1$  making your method clear.

(5)

A circle  $C_2$  has its centre at the origin.

Given that circles  $C_1$  and  $C_2$  touch,

(b) find possible equations for  $C_2$

(4)

## Worked Solution - Question 6

**1. Find the centre of C one**

Since  $AB$  is a diameter, the centre is the midpoint of  $A(-2, 18)$  and  $B(14, 6)$ .  
This is  $(6, 12)$ .

**2. Find the radius squared**

Using  $A(-2, 18)$ ,  $r^2 = (-2 - 6)^2 + (18 - 12)^2 = 64 + 36 = 100$ .

**3. Write C one**

Therefore  $C_1$  has equation  $(x - 6)^2 + (y - 12)^2 = 100$ .

**4. Find the centre distance**

The distance from the origin to  $(6, 12)$  is  $\sqrt{6^2 + 12^2} = 6\sqrt{5}$ .

**5. Find possible radii for C two**

Since  $C_1$  has radius 10, a circle centred at the origin can touch it with radius  $6\sqrt{5} - 10$  or  $6\sqrt{5} + 10$ .

**6. Write the equations**

Thus  $C_2$  can be  $x^2 + y^2 = (6\sqrt{5} - 10)^2 = 280 - 120\sqrt{5}$  or  
 $x^2 + y^2 = (6\sqrt{5} + 10)^2 = 280 + 120\sqrt{5}$ .

**Final answer**

(a)  $(x - 6)^2 + (y - 12)^2 = 100$ .

(b)  $x^2 + y^2 = 280 - 120\sqrt{5}$  or  $x^2 + y^2 = 280 + 120\sqrt{5}$ .

## Question 7

7.

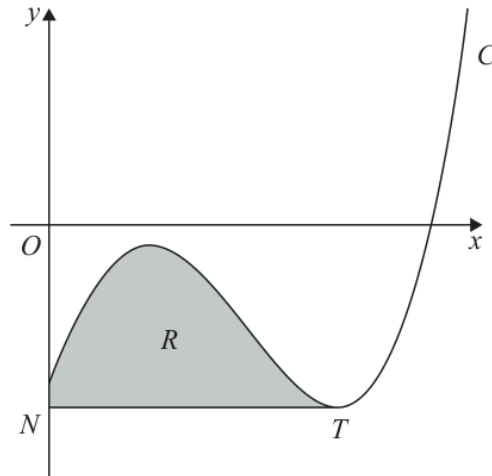


Figure 3

**In this question you must show all stages of your working.**  
**Solutions relying entirely on calculator technology are not acceptable.**

Figure 3 shows a sketch of part of the curve  $C$  with equation

$$y = x^3 - 4x^{\frac{5}{2}} - kx^{\frac{1}{2}} + 28x - 44 \quad x \geq 0$$

where  $k$  is a positive constant.

(a) Find  $\frac{dy}{dx}$  in simplest form.

(2)

The point  $T$ , shown in Figure 3, is a minimum stationary point on  $C$ .

Given that the  $x$  coordinate of  $T$  is 9

(b) show that  $k = 6$

(2)

The line through  $T$  parallel to the  $x$ -axis meets the  $y$ -axis at the point  $N$ .

The finite region  $R$ , shown shaded in Figure 3, is bounded by  $C$ , the  $y$ -axis and the line segment  $NT$ .

(c) Use algebraic integration to find the area of  $R$ , giving the answer to 3 significant figures.

(6)

## Worked Solution - Question 7

**1. Differentiate**

$$\text{For } y = x^3 - 4x^{5/2} - kx^{1/2} + 28x - 44, \frac{dy}{dx} = 3x^2 - 10x^{3/2} - \frac{k}{2}x^{-1/2} + 28.$$

**2. Write in simplest form**

$$\frac{dy}{dx} = 3x^2 - 10x^{3/2} - \frac{k}{2\sqrt{x}} + 28.$$

**3. Use the stationary point**

$$\text{At } T, x = 9 \text{ and } \frac{dy}{dx} = 0.$$

**4. Substitute x equals nine**

$$0 = 3(9)^2 - 10(9)^{3/2} - \frac{k}{2\sqrt{9}} + 28 = 243 - 270 - \frac{k}{6} + 28.$$

**5. Find k**

$$\text{This simplifies to } 0 = 1 - \frac{k}{6}, \text{ so } k = 6.$$

**6. Find the horizontal line**

$$\text{With } k = 6, \text{ the } y \text{ coordinate of } T \text{ is } 9^3 - 4(9)^{5/2} - 6\sqrt{9} + 28(9) - 44 = -53.$$

**7. Set up the area**

The line  $NT$  is  $y = -53$ . The region is between the curve and this line from  $x = 0$  to  $x = 9$ .

**8. Write the integral**

$$R = \int_0^9 (x^3 - 4x^{5/2} - 6x^{1/2} + 28x - 44 + 53) dx.$$

### 9. Integrate

$$R = \left[ \frac{x^4}{4} - \frac{8}{7}x^{7/2} - 4x^{3/2} + 14x^2 + 9x \right]_0^9.$$

### 10. Evaluate

$$R = \frac{6939}{28} = 247.821\dots, \text{ so the area is } \mathbf{248} \text{ to } \mathbf{3} \text{ significant figures.}$$

#### Final answer

(a)  $3x^2 - 10x^{3/2} - \frac{k}{2\sqrt{x}} + 28.$

(b) Proven.

(c) 248.

## Question 8

## Geometric Sequences

8. (i) A geometric series has first term  $a$  and common ratio  $r$ .

Prove that the sum of the first  $n$  terms of this series  $S_n$  is given by

$$S_n = \frac{a(1-r^n)}{1-r} \quad (3)$$

- (ii) A liquid is to be stored in a barrel.

Due to evaporation, the volume of the liquid in the barrel at the end of each year is 8% less than the volume of the liquid in the barrel at the start of the year.

At the start of the first year, the barrel is filled with 150 litres of the liquid.

- (a) Show that the amount of the liquid in the barrel at the end of 6 years is approximately 91 litres.

(2)

At the start of each year a new barrel is filled with 150 litres of the liquid so that, at the end of 40 years, there are 40 barrels containing the liquid.

- (b) Calculate the total amount of the liquid, to the nearest litre, in the 40 barrels at the end of 40 years.

(3)

# Worked Solution - Question 8

**1. Write  $S_n$** 

$$S_n = a + ar + ar^2 + \dots + ar^{n-1}.$$

**2. Multiply by  $r$** 

$$rS_n = ar + ar^2 + ar^3 + \dots + ar^n.$$

**3. Subtract**

$$S_n - rS_n = a - ar^n = a(1 - r^n).$$

**4. Rearrange**

$$(1 - r)S_n = a(1 - r^n), \text{ so } S_n = \frac{a(1 - r^n)}{1 - r}.$$

**5. Use the yearly multiplier**

An 8% decrease each year means the multiplier is **0.92**.

**6. Show the amount after six years**

After 6 years, the amount is  $150(0.92)^6 = 90.953\dots$ , which is approximately **91** litres.

**7. Set up the forty barrel total**

At the end of 40 years, the amounts are  $150(0.92)$ ,  $150(0.92)^2$ ,  $\dots$ ,  $150(0.92)^{40}$ .

**8. Sum the geometric sequence**

$$\text{The total is } 150(0.92) \frac{1 - 0.92^{40}}{1 - 0.92} = 1663.581\dots$$

**9. Round the total**

To the nearest litre, the total amount is **1664** litres.

**Final answer**

(i) Proven.

(ii)(a) Approximately **91** litres.

(ii)(b) **1664** litres.

## Question 9

## Laws of Logarithms

9.

In this question you must show detailed reasoning.  
Solutions relying on calculator technology are not acceptable.

(i) Solve

$$2\log_3(4x + 5) - \log_3(x + 3) = 2 \quad (5)$$

(ii) Given that  $a > 0$ ,  $b > 0$  and

$$\log_{10} a + \log_{10} b = \log_{10}(a + b)$$

(a) prove that  $a = \frac{b}{b-1}$  (3)

(b) Hence write down the full restriction on the value of  $b$ , giving a reason for your answer. (2)

## Worked Solution - Question 9

**1. State the domain**

The logarithms require  $4x + 5 > 0$  and  $x + 3 > 0$ , so  $x > -\frac{5}{4}$ .

**2. Combine the logarithms**

$$2 \log_3(4x + 5) - \log_3(x + 3) = \log_3 \left( \frac{(4x + 5)^2}{x + 3} \right).$$

**3. Use two as a logarithm**

$$2 = \log_3 9, \text{ so } \frac{(4x + 5)^2}{x + 3} = 9.$$

**4. Form the quadratic**

$(4x + 5)^2 = 9(x + 3)$ , so  $16x^2 + 40x + 25 = 9x + 27$ . Hence  $16x^2 + 31x - 2 = 0$ .

**5. Solve the quadratic**

$$16x^2 + 31x - 2 = (16x - 1)(x + 2) = 0, \text{ so } x = \frac{1}{16} \text{ or } x = -2.$$

**6. Check the domain**

$x = -2$  is outside the domain, so  $x = \frac{1}{16}$ .

**7. Use the logarithm equation**

$$\log_{10} a + \log_{10} b = \log_{10}(a + b) \text{ gives } \log_{10}(ab) = \log_{10}(a + b).$$

**8. Equate the arguments**

Since the logarithm function is one-to-one,  $ab = a + b$ .

### 9. Rearrange for $a$

$ab - a = b$ , so  $a(b - 1) = b$  and therefore  $a = \frac{b}{b - 1}$ .

### 10. Restrict $b$

Given  $a > 0$  and  $b > 0$ , the fraction  $\frac{b}{b - 1}$  must be positive. Since the numerator is positive, the denominator must also be positive, so  $b > 1$ .

#### Final answer

(i)  $x = \frac{1}{16}$ .

(ii)(a) Proven.

(ii)(b)  $b > 1$ .

**PAST PAPER**

# **WMA12/01 January 2026**

January 2026 | 10 questions | 75 marks

**10**

questions

**75**

marks

**Answers**

worked solution  
after each  
question

# Question 1

## Binomial Expansion

1. (a) Find the first 4 terms, in ascending powers of  $x$ , of the binomial expansion of

$$(3 + kx)^7$$

where  $k$  is a non-zero constant. Give each term in simplest form.

(4)

Given that, in this expansion, the coefficient of  $x^2$  is 3 times the coefficient of  $x^3$

- (b) find, in simplest form, the value of  $k$ .

(2)

## Worked Solution - Question 1

**1. Use the binomial expansion**

For  $(3 + kx)^7$ , the first four terms are

$$3^7 + \binom{7}{1}3^6(kx) + \binom{7}{2}3^5(kx)^2 + \binom{7}{3}3^4(kx)^3.$$

**2. Simplify the coefficients**

This gives  $2187 + 5103kx + 5103k^2x^2 + 2835k^3x^3$ .

**3. Compare the required coefficients**

The coefficient of  $x^2$  is  $5103k^2$  and the coefficient of  $x^3$  is  $2835k^3$ .

**4. Solve for k**

Since  $5103k^2 = 3(2835k^3)$  and  $k \neq 0$ , divide by  $k^2$  to get  $5103 = 8505k$ , hence  $k = \frac{5103}{8505} = \frac{3}{5}$ .

**Final answer**

(a)  $2187 + 5103kx + 5103k^2x^2 + 2835k^3x^3$ . (b)  $k = \frac{3}{5}$ .

## Question 2

2.

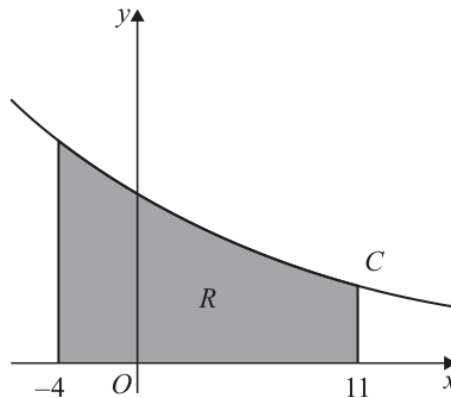


Figure 1

Figure 1 shows a sketch of part of the curve  $C$  with equation  $y = 2^{-0.1x}$

The table below shows corresponding values of  $x$  and  $y$  for  $y = 2^{-0.1x}$

The  $y$  values are given to 4 decimal places.

$x$	-4	-1	2	5	8	11
$y$	1.3195	1.0718	0.8706	0.7071	0.5743	0.4665

The region  $R$ , shown shaded in Figure 1, is bounded by  $C$ , the  $x$ -axis and the lines with equations  $x = -4$  and  $x = 11$

(a) Use the trapezium rule with all the values of  $y$  in the table to find an estimate for the area of  $R$ . Give the answer to 2 decimal places.

(3)

(b) State how you would use the trapezium rule to get a more accurate estimate for the true area of  $R$ .

(1)

Using the answer to part (a) and showing your working,

(c) estimate the value of

$$\int_{-4}^{11} 2^{3-0.1x} dx$$

(2)

## Worked Solution - Question 2

### 1. Find the strip width

The  $x$  values increase by 3, so  $h = 3$ .

### 2. Apply the trapezium rule

$$A \approx \frac{3}{2} \{1.3195 + 0.4665 + 2(1.0718 + 0.8706 + 0.7071 + 0.5743)\}.$$

### 3. Calculate the estimated area

$$A \approx \frac{3}{2} (8.2336) = 12.3504, \text{ so the area is } \mathbf{12.35} \text{ to 2 decimal places.}$$

### 4. Improve the estimate

A more accurate estimate would use more trapezia, so the strip width is smaller.

### 5. Use the scale factor in the integral

Since  $2^{3-0.1x} = 8 \cdot 2^{-0.1x}$ , the required estimate is  $8(12.35) = 98.8$ .

#### Final answer

(a) 12.35. (b) Use narrower strips, or more trapezia.

(c) 98.8.

## Question 3

## Polynomials

3. (i) Given that

- $f(x) = 4x^3 + 6x + k$ , where  $k$  is a constant
- $(x + 2)$  is a factor of  $f(x)$

find the remainder when  $f(x)$  is divided by  $(x - 5)$

(4)

(ii) Find the remainder,  $R$ , and the quotient,  $Q(x)$ , when

$$6x^3 - 15x^2 - 21x + 8$$

is divided by  $(2x + 3)$

(3)

## Worked Solution - Question 3

**1. Use the factor theorem**

Since  $(x + 2)$  is a factor of  $f(x) = 4x^3 + 6x + k$ , we have  $f(-2) = 0$ .

**2. Find  $k$** 

$4(-2)^3 + 6(-2) + k = 0$ , so  $-32 - 12 + k = 0$  and  $k = 44$ .

**3. Find the required remainder**

The remainder on division by  $(x - 5)$  is  $f(5) = 4(5)^3 + 6(5) + 44 = 574$ .

**4. Set up the division**

Write  $6x^3 - 15x^2 - 21x + 8 = (2x + 3)(Ax^2 + Bx + C) + R$ .

**5. Match coefficients**

Comparing coefficients gives  $A = 3$ ,  $B = -12$ , and  $C = \frac{15}{2}$ .

**6. Find the remainder**

Then  $(2x + 3) \left( 3x^2 - 12x + \frac{15}{2} \right) = 6x^3 - 15x^2 - 21x + \frac{45}{2}$ , so

$$R = 8 - \frac{45}{2} = -\frac{29}{2}.$$

**Final answer**

(i) 574. (ii)  $Q(x) = 3x^2 - 12x + \frac{15}{2}$ ,  $R = -\frac{29}{2}$ .

**Question 4****Integration**

4.

**In this question you must show all stages of your working.  
Solutions relying entirely on calculator technology are not acceptable.**

Given that  $k$  is a positive constant and

$$\int_k^{2k} \left( \frac{12}{x^2} + 4 \right) dx = 14$$

find the possible values of  $k$ .

**(5)**

## Worked Solution - Question 4

**1. Integrate first**

$$\int \left( \frac{12}{x^2} + 4 \right) dx = \int (12x^{-2} + 4) dx = -\frac{12}{x} + 4x.$$

**2. Apply the limits**

$$\left[ -\frac{12}{x} + 4x \right]_k^{2k} = \left( -\frac{6}{k} + 8k \right) - \left( -\frac{12}{k} + 4k \right) = \frac{6}{k} + 4k.$$

**3. Form a quadratic**

Given the integral is 14,  $\frac{6}{k} + 4k = 14$ . Multiplying by  $k$  gives  $4k^2 - 14k + 6 = 0$ , so  $2k^2 - 7k + 3 = 0$ .

**4. Solve for k**

$(2k - 1)(k - 3) = 0$ , hence  $k = \frac{1}{2}$  or  $k = 3$ . Both are positive, so both are valid.

**Final answer**

$$k = \frac{1}{2} \text{ or } k = 3.$$

## Question 5

5. The circle  $C_1$  has equation

$$x^2 + y^2 - 6x + 5y - 41 = 0$$

(a) Find

- (i) the coordinates of the centre of  $C_1$
- (ii) the radius of  $C_1$

(3)

The circle  $C_2$  has

- centre  $(-k, 0)$  where  $k$  is a positive constant
- radius 5

Given that circles  $C_1$  and  $C_2$  touch

(b) find the exact value of  $k$ .

(3)

## Worked Solution - Question 5

**1. Complete the square**

$$x^2 - 6x = (x - 3)^2 - 9 \text{ and } y^2 + 5y = \left(y + \frac{5}{2}\right)^2 - \frac{25}{4}.$$

**2. Write the circle in centre-radius form**

The equation becomes  $(x - 3)^2 + \left(y + \frac{5}{2}\right)^2 = \frac{225}{4}$ .

**3. Read the centre and radius**

The centre is  $\left(3, -\frac{5}{2}\right)$  and the radius is  $\sqrt{\frac{225}{4}} = \frac{15}{2}$ .

**4. Use the touching condition**

The centre of  $C_2$  is  $(-k, 0)$  and its radius is 5. Since the circles touch externally, the distance between centres is  $\frac{15}{2} + 5 = \frac{25}{2}$ .

**5. Solve for k**

$(k + 3)^2 + \left(\frac{5}{2}\right)^2 = \left(\frac{25}{2}\right)^2$ , so  $(k + 3)^2 = 150$ . Since  $k > 0$ ,  $k + 3 = 5\sqrt{6}$ , hence  $k = 5\sqrt{6} - 3$ .

**Final answer**

(a)(i)  $\left(3, -\frac{5}{2}\right)$ . (a)(ii)  $\frac{15}{2}$ . (b)  $k = 5\sqrt{6} - 3$ .

## Question 6

Proof

6. (i) Given that  $p$  and  $q$  are consecutive odd numbers, where  $p > q > 0$ , prove that

$$p^2 - q^2$$

is a multiple of 8

(4)

- (ii) The curve  $C$  has equation

$$y = x^3 + 12x^2 + 49x + 2$$

Prove that  $C$  has no stationary points.

(4)

# Worked Solution - Question 6

Topic group

## 1. Represent the consecutive odd numbers

Let  $q = 2n - 1$  and  $p = 2n + 1$ , where  $n$  is a positive integer. Then  $p > q > 0$  and the numbers are consecutive odd numbers.

## 2. Subtract the squares

$$p^2 - q^2 = (2n + 1)^2 - (2n - 1)^2.$$

## 3. Simplify the expression

$$p^2 - q^2 = (4n^2 + 4n + 1) - (4n^2 - 4n + 1) = 8n.$$

## 4. Conclude the proof

Since  $n$  is an integer,  $8n$  is a multiple of 8, so  $p^2 - q^2$  is a multiple of 8.

## 5. Differentiate the curve

For  $y = x^3 + 12x^2 + 49x + 2$ ,  $\frac{dy}{dx} = 3x^2 + 24x + 49$ .

## 6. Show the gradient is never zero

$$3x^2 + 24x + 49 = 3(x + 4)^2 + 1. \text{ This is always positive for real } x.$$

## 7. State the conclusion

Since  $\frac{dy}{dx}$  is never 0, the curve has no stationary points.

### Final answer

(i) Proven.

(ii)  $C$  has no stationary points.

**Question 7****Laws of Logarithms**

7. **In this question you must show all stages of your working.  
Solutions relying entirely on calculator technology are not acceptable.**

- (i) Find the exact solution of the equation

$$8^{2x-5} = 20$$

giving the answer in the form  $a + b \log_2 5$  where  $a$  and  $b$  are rational constants.

**(4)**

- (ii) Using the laws of logarithms, solve the equation

$$\log_3(13 + 2y) + 3 = 2\log_3(4 - y)$$

**(5)**

## Worked Solution - Question 7

**1. Take logarithms base 2**

From  $8^{2x-5} = 20$ , take  $\log_2$  of both sides to get  $(2x - 5) \log_2 8 = \log_2 20$ .

**2. Simplify the logarithms**

Since  $\log_2 8 = 3$  and  $\log_2 20 = \log_2(4 \cdot 5) = 2 + \log_2 5$ , we get  
 $3(2x - 5) = 2 + \log_2 5$ .

**3. Solve for x**

$6x - 15 = 2 + \log_2 5$ , so  $6x = 17 + \log_2 5$  and  $x = \frac{17}{6} + \frac{1}{6} \log_2 5$ .

**4. Use log laws**

$3 = \log_3 27$  and  $2 \log_3(4 - y) = \log_3(4 - y)^2$ .

**5. Remove the logarithms**

$\log_3(13 + 2y) + \log_3 27 = \log_3(4 - y)^2$ , so  $27(13 + 2y) = (4 - y)^2$ .

**6. Solve the quadratic**

$351 + 54y = y^2 - 8y + 16$ , hence  $y^2 - 62y - 335 = 0 = (y - 67)(y + 5)$ .

**7. Check the logarithm domains**

We need  $13 + 2y > 0$  and  $4 - y > 0$ . Therefore  $y = 67$  is not valid, and the solution is  $y = -5$ .

**Final answer**

(i)  $x = \frac{17}{6} + \frac{1}{6} \log_2 5$ . (ii)  $y = -5$ .

**Question 8****Trigonometric Equations**

8. **In this question you must show all stages of your working.  
Solutions relying entirely on calculator technology are not acceptable.**

- (i) Solve, for  $0 \leq \theta < 180$

$$4 \tan^2(2\theta - 30)^\circ + 1 = 49$$

giving the answers to one decimal place.

**(5)**

- (ii) Solve, for  $0 \leq x < 2\pi$

$$2 \tan x \sin x + 3 = 0$$

giving the answers, in radians, in the form  $k\pi$  where  $k$  is a rational constant.

**(5)**

# Worked Solution - Question 8

Topic group

## 1. Make the tangent expression the subject

$$4 \tan^2(2\theta - 30^\circ) + 1 = 49 \text{ gives } \tan^2(2\theta - 30^\circ) = 12.$$

## 2. Find the possible angle values

Let  $\alpha = 2\theta - 30^\circ$ . Since  $0 \leq \theta < 180^\circ$ ,  $-30^\circ \leq \alpha < 330^\circ$ . The solutions of  $\tan \alpha = \pm\sqrt{12}$  in this range are approximately  $73.9^\circ, 106.1^\circ, 253.9^\circ, 286.1^\circ$ .

## 3. Return to theta

$$\theta = \frac{\alpha + 30^\circ}{2}, \text{ giving } \theta = 51.9^\circ, 68.1^\circ, 141.9^\circ, 158.1^\circ \text{ to one decimal place.}$$

## 4. Rewrite tan x

$$2 \tan x \sin x + 3 = 0 \text{ becomes } 2 \left( \frac{\sin x}{\cos x} \right) \sin x + 3 = 0.$$

## 5. Form a quadratic in cos x

Multiplying by  $\cos x$  gives  $2 \sin^2 x + 3 \cos x = 0$ . Using  $\sin^2 x = 1 - \cos^2 x$ ,  $2(1 - \cos^2 x) + 3 \cos x = 0$ , so  $2 \cos^2 x - 3 \cos x - 2 = 0$ .

## 6. Solve for x

$$(2 \cos x + 1)(\cos x - 2) = 0. \text{ Since } \cos x = 2 \text{ is impossible, } \cos x = -\frac{1}{2}. \text{ In } 0 \leq x < 2\pi, x = \frac{2\pi}{3} \text{ or } x = \frac{4\pi}{3}.$$

### Final answer

$$(i) \theta = 51.9^\circ, 68.1^\circ, 141.9^\circ, 158.1^\circ. \quad (ii) x = \frac{2\pi}{3}, \frac{4\pi}{3}.$$

## Question 9

## Modelling with Sequences &amp; Series

9. A company mines lithium.

- in year 1, the company mined 12 400 tonnes of lithium
- in year 15, the company is predicted to mine 7 500 tonnes of lithium

Model *A* assumes that the mass of lithium the company mines will decrease by the same amount each year.

According to model *A*,

(a) find the mass of lithium the company will mine in year 5 (3)

(b) calculate the **total** mass of lithium the company will mine from year 1 to year 15 inclusive. (2)

Model *B* assumes that the mass of lithium the company mines will decrease by the same percentage each year.

(c) Find, according to model *B*, the mass of lithium the company will mine in year 10. Give the answer to the nearest 10 tonnes. (3)

According to model *B*, there is a limit to the **total** mass of lithium that can be mined.

(d) Calculate the value of this limit. Give the answer to the nearest 100 tonnes. (2)

# Worked Solution - Question 9

Topic group

## 1. Model A as an arithmetic sequence

For Model A, let the common difference be  $d$ . Since year 15 is the 15th term,  
 $7500 = 12400 + 14d$ .

## 2. Find year 5 for Model A

$$d = \frac{7500 - 12400}{14} = -350. \text{ Year 5 is } 12400 + 4(-350) = 11000 \text{ tonnes.}$$

## 3. Find the total from year 1 to year 15

$$S_{15} = \frac{15}{2}(12400 + 7500) = 149250 \text{ tonnes.}$$

## 4. Model B as a geometric sequence

Let the common ratio be  $r$ . Then  $7500 = 12400r^{14}$ , so

$$r = \left( \frac{7500}{12400} \right)^{1/14} \approx 0.964723.$$

## 5. Find year 10 for Model B

Year 10 is  $12400r^9 \approx 8975.26$ , which is **8980** tonnes to the nearest 10 tonnes.

## 6. Find the limiting total

Since  $0 < r < 1$ , the sum to infinity is  $S_{\infty} = \frac{12400}{1 - r} \approx 351508.1$ , so the limit is **351500** tonnes to the nearest 100 tonnes.

**Final answer**

(a) 11000 tonnes.

(b) 149250 tonnes.

(c) 8980 tonnes.

(d) 351500 tonnes.

## Question 10

## Applications of Differentiation

10.

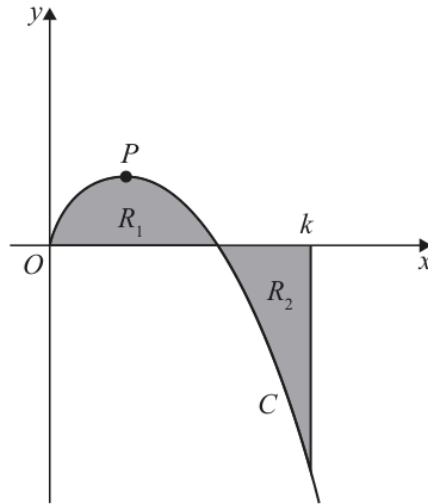


Figure 2

**In this question you must show all stages of your working.**  
**Solutions relying entirely on calculator technology are not acceptable.**

Figure 2 shows a sketch of the curve  $C$  with equation

$$y = \frac{\sqrt{x}(100 - x^2)}{40} \quad x \geq 0$$

The point  $P$  is a stationary point on  $C$ .

(a) Use algebraic differentiation to find the exact  $x$  coordinate of  $P$ .

(4)

The region  $R_1$ , shown shaded in Figure 2, is bounded by  $C$  and the  $x$ -axis.

The region  $R_2$ , also shown shaded in Figure 2, is bounded by  $C$ , the  $x$ -axis and the line with equation  $x = k$ , where  $k$  is a constant.

Given that the area of  $R_1$  is equal to the area of  $R_2$

(b) use algebraic integration to find the exact value of  $k$ .

(4)

# Worked Solution - Question 10

Topic group

## 1. Rewrite the curve

$$y = \frac{\sqrt{x}(100 - x^2)}{40} = \frac{5}{2}x^{1/2} - \frac{1}{40}x^{5/2}.$$

## 2. Differentiate

$$\frac{dy}{dx} = \frac{5}{4}x^{-1/2} - \frac{1}{16}x^{3/2}.$$

## 3. Use the stationary point

At  $P$ ,  $\frac{dy}{dx} = 0$ . Thus  $\frac{5}{4}x^{-1/2} = \frac{1}{16}x^{3/2}$ . Multiplying by  $16x^{1/2}$  gives  $20 = x^2$ .

## 4. Find the $x$ -coordinate

Since  $x \geq 0$ ,  $x = \sqrt{20} = 2\sqrt{5}$ .

## 5. Integrate the curve

$$\int y \, dx = \int \left( \frac{5}{2}x^{1/2} - \frac{1}{40}x^{5/2} \right) dx = \frac{5}{3}x^{3/2} - \frac{1}{140}x^{7/2}.$$

## 6. Use equal areas

The signed area from  $0$  to  $k$  is zero because the positive area  $R_1$  equals the negative area  $R_2$  in magnitude.

## 7. Solve for $k$

$$\frac{5}{3}k^{3/2} - \frac{1}{140}k^{7/2} = 0. \text{ Since } k > 0, \text{ divide by } k^{3/2} \text{ to get } \frac{5}{3} - \frac{k^2}{140} = 0, \text{ so}$$

$$k^2 = \frac{700}{3}.$$

## 8. Write the exact value

$$k = \sqrt{\frac{700}{3}} = \frac{10\sqrt{21}}{3}.$$

**Final answer**

$$(a) x = 2\sqrt{5}. \quad (b) k = \frac{10\sqrt{21}}{3}.$$