Elite Mathematics
Edexcel International A Level

IAL PURE 2 · WMA12

WMA12 Pure 2 Expertise With Answers

Questions 6 and above with worked-solution pages and cross-topic placements.

163	2019–2026	Questions with worked solutions
topic placements	19 exam sessions	checked by Dr Eslam



Dr Eslam Ahmed
Assistant Lecturer, Faculty of Engineering, Cairo University
eliteigcse.com · free for every student

TOPIC

Proof

Every past-paper question on this topic, October 2019 to January 2026.

WMA12/01 OCTOBER 2021

Question 9

4 marks | Proof

9. (a) Prove that for all positive values of x and y,

$$\frac{x+y}{2} \geqslant \sqrt{xy}$$

(3)

(b) Prove by counter-example that this inequality does not hold when x and y are both negative.

(1)

WORKED SOLUTION

Question 9 · WMA12/01 October 2021

1 Start with a square

For positive x and y, both $\sqrt{x}$ and $\sqrt{y}$ are real, so $(\sqrt{x}-\sqrt{y})^2 \geq 0$.

2 Expand

$(\sqrt{x}-\sqrt{y})^2 = x - 2\sqrt{xy} + y \geq 0$.

3 Rearrange

Thus $x+y \geq 2\sqrt{xy}$, and dividing by 2 gives $\dfrac{x+y}{2} \geq \sqrt{xy}$.

4 Give a negative counterexample

Take $x=-1$ and $y=-1$. Then $\dfrac{x+y}{2} = -1$, while $\sqrt{xy} = \sqrt{1} = 1$.

5 Conclude

Since $-1 \geq 1$ is false, the inequality does not hold for all negative x and y.

✓ **FINAL ANSWER**

(a) Proven.

(b) For example, $x=y=-1$ gives $-1 \not\geqslant 1$.

WMA12/01 JANUARY 2022

Question 10

5 marks | Proof

10. (i) Prove by counter example that the statement

"if p is a prime number then $2p + 1$ is also a prime number"

is not true.

(1)

(ii) Use proof by exhaustion to prove that if n is an integer then

$$5n^2 + n + 12$$

is always even.

(4)

(Total 5 marks)

WORKED SOLUTION

Question 10 · WMA12/01 January 2022

1 Give a counterexample

Take $p = 7$, which is prime.

2 Show the statement fails

$2p + 1 = 2(7) + 1 = 15$, and 15 is not prime. So the statement is false.

3 Case one: n even

If n is even, let $n = 2k$. Then

$5n^2 + n + 12 = 5(2k)^2 + 2k + 12 = 20k^2 + 2k + 12 = 2(10k^2 + k + 6)$, which is even.

4 Case two: n odd

If n is odd, let $n = 2k + 1$. Then $5n^2 + n + 12 = 5(2k + 1)^2 + (2k + 1) + 12$.

5 Simplify the odd case

$5(2k + 1)^2 + (2k + 1) + 12 = 20k^2 + 22k + 18 = 2(10k^2 + 11k + 9)$, which is even.

6 Conclude by exhaustion

Every integer is either even or odd, and both cases give an even result. Therefore $5n^2 + n + 12$ is always even.

FINAL ANSWER

(i) For $p = 7$, $2p + 1 = 15$, not prime.

(ii) Proven.

WMA12/01 JANUARY 2023

Question 10

4 marks | Proof

10. A student was asked to prove by exhaustion that

$$\text{if } n \text{ is an integer then } 2n^2 + n + 1 \text{ is } \textbf{not} \text{ divisible by } 3$$

The start of the student's proof is shown in the box below.

> Consider the case when $n = 3k$
>
> $$2n^2 + n + 1 = 18k^2 + 3k + 1 = 3(6k^2 + k) + 1$$
>
> which is not divisible by 3

Complete this proof.

(4)

WORKED SOLUTION

Question 10 · WMA12/01 January 2023

1 Use the three possible forms

Every integer is exactly one of the forms $3k$, $3k + 1$ or $3k + 2$, where k is an integer.

2 First case already given

For $n = 3k$, the expression becomes $3(6k^2 + k) + 1$, so it leaves remainder 1 when divided by 3.

3 Try n equals 3k plus 1

If $n = 3k + 1$, then $2n^2 + n + 1 = 2(3k + 1)^2 + (3k + 1) + 1$.

4 Simplify the second case

$2(9k^2 + 6k + 1) + 3k + 2 = 18k^2 + 15k + 4 = 3(6k^2 + 5k + 1) + 1$, which is not divisible by 3.

5 Try n equals 3k plus 2

If $n = 3k + 2$, then $2n^2 + n + 1 = 2(3k + 2)^2 + (3k + 2) + 1$.

6 Simplify the third case

$2(9k^2 + 12k + 4) + 3k + 3 = 18k^2 + 27k + 11 = 3(6k^2 + 9k + 3) + 2$, which is not divisible by 3.

7 **Complete the exhaustion**

All possible integer forms have been checked, and none gives a multiple of 3. Therefore $2n^2 + n + 1$ is not divisible by 3.

✓ **FINAL ANSWER**

For $n = 3k$, $n = 3k + 1$ and $n = 3k + 2$, the expression has remainders 1, 1 and 2 respectively when divided by 3. Therefore $2n^2 + n + 1$ is never divisible by 3.

WMA12/01 MAY/JUNE 2023

Question 8

5 marks | Proof

8. (i) A student writes the following statement:

"When a and b are consecutive **prime** numbers, $a^2 + b^2$ is never a multiple of 10"

Prove by counter example that this statement is **not** true.

(2)

(ii) Given that x and y are even integers greater than 0 and less than 6, prove by exhaustion, that

$$1 < x^2 - \frac{xy}{4} < 15$$

(3)

WORKED SOLUTION

Question 8 · WMA12/01 May/June 2023

1 Choose consecutive primes

7 and 11 are consecutive prime numbers.

2 Check the statement

$7^2 + 11^2 = 49 + 121 = 170$, which is a multiple of 10.

3 Conclude the counterexample

This single counterexample proves the student's statement is not true.

4 List the possible even values

The even integers greater than 0 and less than 6 are 2 and 4. Therefore the possible ordered pairs are $(2, 2), (2, 4), (4, 2), (4, 4)$.

5 Check x equals two

If $(x, y) = (2, 2)$, then $x^2 - \frac{xy}{4} = 4 - 1 = 3$. If $(x, y) = (2, 4)$, then $x^2 - \frac{xy}{4} = 4 - 2 = 2$.

6 Check x equals four

If $(x, y) = (4, 2)$, then $x^2 - \frac{xy}{4} = 16 - 2 = 14$. If $(x, y) = (4, 4)$, then $x^2 - \frac{xy}{4} = 16 - 4 = 12$.

7 **Complete the exhaustion**

The possible values are $3, 2, 14, 12$, and each lies strictly between 1 and 15. Hence $1 < x^2 - \dfrac{xy}{4} < 15$ in every possible case.

✓ **FINAL ANSWER**

(i) A counterexample is $a = 7, b = 11$, since $7^2 + 11^2 = 170$.

(ii) Exhaustion proves the inequality for $(x, y) = (2, 2), (2, 4), (4, 2), (4, 4)$.

WMA12/01 JANUARY 2024

Question 8

6 marks | Proof

8. (i) Use a counter example to show that the following statement is **false**

"$n^2 + 3n + 1$ is prime for all $n \in \mathbb{N}$"

(2)

(ii) Use algebra to prove by exhaustion that for all $n \in \mathbb{N}$

"$n^2 - 2$ is **not** a multiple of 4"

(4)

WORKED SOLUTION

Question 8 · WMA12/01 January 2024

1 **Choose a counterexample**

Take $n = 6$.

2 **Evaluate the expression**

$n^2 + 3n + 1 = 6^2 + 3(6) + 1 = 36 + 18 + 1 = 55$.

3 **Conclude false**

55 is not prime, so the statement is false.

4 **Use residues modulo four**

Every natural number is one of $4k$, $4k + 1$, $4k + 2$ or $4k + 3$.

5 **Check n equals 4k**

If $n = 4k$, then $n^2 - 2 = 16k^2 - 2 = 4(4k^2 - 1) + 2$, not a multiple of 4.

6 **Check n equals 4k plus one**

If $n = 4k + 1$, then $n^2 - 2 = 16k^2 + 8k - 1 = 4(4k^2 + 2k - 1) + 3$, not a multiple of 4.

7 **Check n equals 4k plus two**

If $n = 4k + 2$, then $n^2 - 2 = 16k^2 + 16k + 2 = 4(4k^2 + 4k) + 2$, not a multiple of 4.

8 **Check n equals 4k plus three**

If $n = 4k + 3$, then $n^2 - 2 = 16k^2 + 24k + 7 = 4(4k^2 + 6k + 1) + 3$, not a multiple of 4.

9 **Complete the proof**

All possible cases leave remainder 2 or 3 when divided by 4, so $n^2 - 2$ is not a multiple of 4.

✓ **FINAL ANSWER**

(i) Counterexample: $n = 6$ gives 55, which is not prime.

(ii) Proven by considering $n = 4k, 4k+1, 4k+2, 4k+3$.

WMA12/01 OCTOBER 2024

Question 11

6 marks | Proof

11. (i) Prove by counter example that the statement

"If n is a prime number then $3^n + 2$ is also a prime number."

is false.

(2)

(ii) Use proof by exhaustion to prove that if m is an integer that is **not** divisible by 3, then

$$m^2 - 1$$

is divisible by 3

(4)

WORKED SOLUTION

Question 11 · WMA12/01 October 2024

1 Choose a prime value

Take $n = 5$, which is prime.

2 Evaluate the expression

$3^5 + 2 = 243 + 2 = 245 = 5 \times 49$, so it is not prime. This disproves the statement.

3 List the possible forms of m

If integer m is not divisible by 3, then m has remainder 1 or 2 on division by 3.

4 Case one

If $m = 3q + 1$, then $m^2 - 1 = (3q + 1)^2 - 1 = 9q^2 + 6q = 3(3q^2 + 2q)$, which is divisible by 3.

5 Case two

If $m = 3q + 2$, then $m^2 - 1 = (3q + 2)^2 - 1 = 9q^2 + 12q + 3 = 3(3q^2 + 4q + 1)$, which is divisible by 3.

6 Conclude by exhaustion

Both possible cases for an integer not divisible by 3 have been checked, so $m^2 - 1$ is divisible by 3.

FINAL ANSWER

(i) Counterexample: $n = 5$ gives $3^5 + 2 = 245$, not prime.

(ii) Proven.

WMA12/01 JANUARY 2025

Question 8

5 marks | Proof

8. (i) A student states

"If x and y are irrational numbers, $x \neq y$, then xy is also irrational."

Show, by counter example, that this statement is not always true.

(1)

(ii) Prove, using algebra, that for all odd integers n, the value of the expression

$$n^3 + 3n + 2$$

is always even but never a multiple of 4

(4)

WORKED SOLUTION

Question 8 · WMA12/01 January 2025

1 **Give a counterexample**

Let $x = \sqrt{2}$ and $y = \dfrac{1}{\sqrt{2}}$. Both are irrational and $x \neq y$, but $xy = 1$, which is rational.

2 **Represent an odd integer**

Let $n = 2q + 1$, where q is an integer.

3 **Substitute into the expression**

$n^3 + 3n + 2 = (2q + 1)^3 + 3(2q + 1) + 2$.

4 **Expand and factor**

$n^3 + 3n + 2 = 8q^3 + 12q^2 + 12q + 6 = 2(4q^3 + 6q^2 + 6q + 3)$.

5 **Show it is even**

The expression has a factor of 2, so it is even.

6 **Show it is not a multiple of four**

The bracket $4q^3 + 6q^2 + 6q + 3$ is odd, because the first three terms are even and 3 is odd.

Therefore the expression is 2 times an odd integer, so it is not divisible by 4.

FINAL ANSWER

(i) For example, $x = \sqrt{2}$ and $y = \dfrac{1}{\sqrt{2}}$ give $xy = 1$.

(ii) Proven.

WMA12/01 MAY/JUNE 2025

Question 10

4 marks | Proof

10: **In this question you must show detailed reasoning.**

Use algebra to prove by exhaustion that,

for all positive integers m that are **not** multiples of 3, the value of

$$m^2 + 3m + 2$$

is always a multiple of 3

(4)

WORKED SOLUTION

Question 10 · WMA12/01 May/June 2025

1 **List the possible cases**

A positive integer that is not a multiple of 3 must have the form $m = 3q + 1$ or $m = 3q + 2$.

2 **Case one**

If $m = 3q + 1$, then

$m^2 + 3m + 2 = (3q + 1)^2 + 3(3q + 1) + 2 = 9q^2 + 15q + 6 = 3(3q^2 + 5q + 2)$.

3 **Case two**

If $m = 3q + 2$, then

$m^2 + 3m + 2 = (3q + 2)^2 + 3(3q + 2) + 2 = 9q^2 + 21q + 12 = 3(3q^2 + 7q + 4)$.

4 **Conclude by exhaustion**

Both possible cases give a factor of 3, so $m^2 + 3m + 2$ is always a multiple of 3.

✓ **FINAL ANSWER**

Proven.

WMA12/01 JANUARY 2026

Question 6

8 marks | Proof

6. (i) Given that p and q are consecutive odd numbers, where $p > q > 0$, prove that

$$p^2 - q^2$$

is a multiple of 8

(4)

(ii) The curve C has equation

$$y = x^3 + 12x^2 + 49x + 2$$

Prove that C has no stationary points.

(4)

WORKED SOLUTION

Question 6 · WMA12/01 January 2026

1 Represent the consecutive odd numbers

Let $q = 2n - 1$ and $p = 2n + 1$, where n is a positive integer. Then $p > q > 0$ and the numbers are consecutive odd numbers.

2 Subtract the squares

$p^2 - q^2 = (2n + 1)^2 - (2n - 1)^2$.

3 Simplify the expression

$p^2 - q^2 = (4n^2 + 4n + 1) - (4n^2 - 4n + 1) = 8n$.

4 Conclude the proof

Since n is an integer, $8n$ is a multiple of 8, so $p^2 - q^2$ is a multiple of 8.

5 Differentiate the curve

For $y = x^3 + 12x^2 + 49x + 2$, $\dfrac{dy}{dx} = 3x^2 + 24x + 49$.

6 Show the gradient is never zero

$3x^2 + 24x + 49 = 3(x + 4)^2 + 1$. This is always positive for real x.

7 State the conclusion

Since $\dfrac{dy}{dx}$ is never 0, the curve has no stationary points.

FINAL ANSWER

(i) Proven.

(ii) C has no stationary points.

WMA12/01 MAY/JUNE 2023

Question 7

8 marks | Proof | Main topic: Applications of Differentiation

7. The height of a river above a fixed point on the riverbed was monitored over a 7-day period.

The height of the river, H metres, t days after monitoring began, was given by

$$H = \frac{\sqrt{t}}{20}(20 + 6t - t^2) + 17 \qquad 0 \leqslant t \leqslant 7$$

Given that H has a stationary value at $t = \alpha$

(a) use calculus to show that α satisfies the equation

$$5\alpha^2 - 18\alpha - 20 = 0$$

(5)

(b) Hence find the value of α, giving your answer to 3 decimal places.

(1)

(c) Use further calculus to prove that H is a maximum at this value of α.

(2)

WORKED SOLUTION

Question 7 · WMA12/01 May/June 2023

1 Rewrite H with powers

$H = \frac{1}{20}t^{1/2}(20 + 6t - t^2) + 17.$

2 Expand before differentiating

$H = \frac{1}{20}(20t^{1/2} + 6t^{3/2} - t^{5/2}) + 17.$

3 Differentiate

$\frac{dH}{dt} = \frac{1}{20}\left(10t^{-1/2} + 9t^{1/2} - \frac{5}{2}t^{3/2}\right).$

4 Put over a common factor

$\frac{dH}{dt} = \frac{-5t^2 + 18t + 20}{40\sqrt{t}}.$

5 Use the stationary condition

At $t = \alpha$, $\frac{dH}{dt} = 0$. Since $\alpha > 0$, the denominator is not zero, so $-5\alpha^2 + 18\alpha + 20 = 0$.

6 **Show the required equation**

Multiplying by -1 gives $5\alpha^2 - 18\alpha - 20 = 0$.

7 **Find alpha**

$$\alpha = \frac{18 \pm \sqrt{18^2 + 4(5)(20)}}{10} = \frac{9 \pm \sqrt{181}}{5}.$$

8 **Use the time interval**

Only the positive value lies in $0 \le t \le 7$, so $\alpha = \dfrac{9 + \sqrt{181}}{5} = 4.491$ to 3 decimal places.

9 **Differentiate again**

$$\frac{d^2H}{dt^2} = \frac{-15t^2 + 18t - 20}{80t^{3/2}}.$$

10 **Use the stationary equation**

At $t = \alpha$, $5\alpha^2 = 18\alpha + 20$. Therefore the numerator of $\dfrac{d^2H}{dt^2}$ is $-15\alpha^2 + 18\alpha - 20 = -3(18\alpha + 20) + 18\alpha - 20 = -36\alpha - 80$.

11 **Show it is negative**

Since $\alpha > 0$, $-36\alpha - 80 < 0$, and the denominator $80\alpha^{3/2}$ is positive. Thus $\dfrac{d^2H}{dt^2} < 0$, so H is a maximum.

✓ **FINAL ANSWER**

(a) Proven.

(b) $\alpha = 4.491$.

(c) H is a maximum at this value.

WMA12/01 MAY/JUNE 2024

Question 7

6 marks | Proof | Main topic: Circles

7. The circle C_1 has equation

$$x^2 + y^2 + 8x - 10y = 29$$

(a) (i) Find the coordinates of the centre of C_1

(ii) Find the exact value of the radius of C_1

(3)

In part (b) you must show detailed reasoning.

The circle C_2 has equation

$$(x-5)^2 + (y+8)^2 = 52$$

(b) Prove that the circles C_1 and C_2 neither touch nor intersect.

(3)

WORKED SOLUTION

Question 7 · WMA12/01 May/June 2024

1 Complete the square

$x^2 + y^2 + 8x - 10y = 29$ becomes $(x+4)^2 - 16 + (y-5)^2 - 25 = 29$.

2 Write circle C1 in standard form

$(x+4)^2 + (y-5)^2 = 70$.

3 Read centre and radius

The centre of C_1 is $(-4, 5)$ and its radius is $\sqrt{70}$.

4 Read C2

C_2 has equation $(x-5)^2 + (y+8)^2 = 52$, so its centre is $(5, -8)$ and its radius is $\sqrt{52} = 2\sqrt{13}$.

5 Find distance between centres

The distance between $(-4, 5)$ and $(5, -8)$ is $\sqrt{9^2 + (-13)^2} = \sqrt{250} = 5\sqrt{10}$.

6 Compare with radii

$\sqrt{70} + 2\sqrt{13} = 15.58\ldots$, while $5\sqrt{10} = 15.81\ldots$.

7 **Conclude**

The distance between centres is greater than the sum of the radii, so the circles are separate.

Therefore they neither touch nor intersect.

✓ **FINAL ANSWER**

$(a)(i)$ $(-4, 5)$.

$(a)(ii)$ $\sqrt{70}$.

(b) The circles neither touch nor intersect.

WMA12/01 OCTOBER 2025

Question 9

10 marks | Proof | Main topic: Laws of Logarithms

9. **In this question you must show detailed reasoning.**
Solutions relying on calculator technology are not acceptable.

(i) Solve

$$2\log_3(4x+5) - \log_3(x+3) = 2$$

(5)

(ii) Given that $a > 0$, $b > 0$ and

$$\log_{10} a + \log_{10} b = \log_{10}(a+b)$$

(a) prove that $a = \dfrac{b}{b-1}$

(3)

(b) Hence write down the full restriction on the value of b, giving a reason for your answer.

(2)

WORKED SOLUTION

Question 9 · WMA12/01 October 2025

1 **State the domain**

The logarithms require $4x + 5 > 0$ and $x + 3 > 0$, so $x > -\dfrac{5}{4}$.

2 **Combine the logarithms**

$2\log_3(4x+5) - \log_3(x+3) = \log_3\left(\dfrac{(4x+5)^2}{x+3}\right)$.

3 **Use two as a logarithm**

$2 = \log_3 9$, so $\dfrac{(4x+5)^2}{x+3} = 9$.

4 **Form the quadratic**

$(4x+5)^2 = 9(x+3)$, so $16x^2 + 40x + 25 = 9x + 27$. Hence $16x^2 + 31x - 2 = 0$.

5 **Solve the quadratic**

$16x^2 + 31x - 2 = (16x - 1)(x + 2) = 0$, so $x = \dfrac{1}{16}$ or $x = -2$.

6 **Check the domain**

$x = -2$ is outside the domain, so $x = \frac{1}{16}$.

7 **Use the logarithm equation**

$\log_{10} a + \log_{10} b = \log_{10}(a + b)$ gives $\log_{10}(ab) = \log_{10}(a + b)$.

8 **Equate the arguments**

Since the logarithm function is one-to-one, $ab = a + b$.

9 **Rearrange for a**

$ab - a = b$, so $a(b - 1) = b$ and therefore $a = \frac{b}{b - 1}$.

10 **Restrict b**

Given $a > 0$ and $b > 0$, the fraction $\frac{b}{b - 1}$ must be positive. Since the numerator is positive, the denominator must also be positive, so $b > 1$.

✓ **FINAL ANSWER**

$(i)\ x = \frac{1}{16}$.

$(ii)(a)$ Proven.

$(ii)(b)\ b > 1$.

TOPIC

Polynomials

Every past-paper question on this topic, October 2019 to January 2026.

WMA12/01 MAY/JUNE 2021

Question 7

10 marks | Polynomials | Main topic: Laws of Logarithms

7. (a) Given that

$$3\log_3(2x-1) = 2 + \log_3(14x-25)$$

show that

$$2x^3 - 3x^2 - 30x + 56 = 0$$

(4)

(b) Show that -4 is a root of this cubic equation.

(2)

(c) Hence, using algebra and showing each step of your working, solve

$$3\log_3(2x-1) = 2 + \log_3(14x-25)$$

(4)

WORKED SOLUTION

Question 7 · WMA12/01 May/June 2021

1 Use the power law

$3\log_3(2x-1) = \log_3(2x-1)^3$.

2 Rewrite two as a logarithm

Since $2 = \log_3 9$, the right side is $\log_3 9 + \log_3(14x-25) = \log_3\{9(14x-25)\}$.

3 Equate the arguments

So $(2x-1)^3 = 9(14x-25)$.

4 Expand and simplify

$8x^3 - 12x^2 + 6x - 1 = 126x - 225$, which simplifies to $2x^3 - 3x^2 - 30x + 56 = 0$.

5 Check the given root

Substituting $x = -4$ gives $2(-64) - 3(16) - 30(-4) + 56 = 0$, so -4 is a root.

6 Factorise the cubic

Dividing by $(x+4)$ gives $2x^2 - 11x + 14 = (2x-7)(x-2)$.

7 **Apply the logarithm domain**

The possible roots are -4, 2, and $\frac{7}{2}$. The original logarithms require $2x-1>0$ and $14x-25>0$, so $x>\frac{25}{14}$.

8 **State the valid solutions**

Therefore the solutions are $x=2$ and $x=\frac{7}{2}$.

✓ **FINAL ANSWER**

(a) Proven.

(b) -4 is a root of the cubic.

(c) $x=2,\ \frac{7}{2}$.

WMA12/01 MAY/JUNE 2022

Question 7

7 marks | Polynomials | Main topic: Integration

7. $\mathrm{f}(x) = Ax^3 + 6x^2 - 4x + B$

where A and B are constants.

Given that

- $(x + 2)$ is a factor of $\mathrm{f}(x)$
- $\int_3^5 \mathrm{f}(x)\,\mathrm{d}x = 176$

find the value of A and the value of B.

(7)

WORKED SOLUTION

Question 7 · WMA12/01 May/June 2022

1 Use the factor condition

Since $(x + 2)$ is a factor, $f(-2) = 0$.

2 Form the first equation

$A(-2)^3 + 6(-2)^2 - 4(-2) + B = 0$, so $-8A + 32 + B = 0$ and $B = 8A - 32$.

3 Set up the integral

$\int_3^5 (Ax^3 + 6x^2 - 4x + B)\,dx = 176$.

4 Integrate

$$\left[\frac{A}{4}x^4 + 2x^3 - 2x^2 + Bx\right]_3^5 = 176.$$

5 Simplify the second equation

This gives $136A + 164 + 2B = 176$, so $68A + B = 6$.

6 Solve simultaneously

Using $B = 8A - 32$ in $68A + B = 6$ gives $76A = 38$, so $A = \frac{1}{2}$.

7 **Find B**

$$B = 8\left(\frac{1}{2}\right) - 32 = -28.$$

FINAL ANSWER

$$A = \frac{1}{2}, \quad B = -28.$$

Question 10

12 marks | Polynomials | Main topic: Circles

10. The circle C has centre $X(3, 5)$ and radius r

The line l has equation $y = 2x + k$, where k is a constant.

(a) Show that l and C intersect when

$$5x^2 + (4k - 26)x + k^2 - 10k + 34 - r^2 = 0$$

(3)

Given that l is a tangent to C,

(b) show that $5r^2 = (k + p)^2$, where p is a constant to be found.

(3)

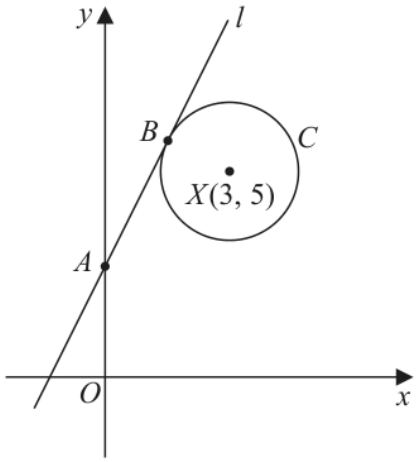


Figure 2

The line l

- cuts the y-axis at the point A
- touches the circle C at the point B

as shown in Figure 2.

Given that $AB = 2r$

(c) find the value of k

(6)

(Total 12 marks)

1 **Write the circle equation**

The circle has centre $(3, 5)$ and radius r, so $(x-3)^2 + (y-5)^2 = r^2$.

2 **Substitute the line**

On the line, $y = 2x + k$. Substitution gives $(x-3)^2 + (2x+k-5)^2 = r^2$.

3 **Expand**

$x^2 - 6x + 9 + 4x^2 + 4kx - 20x + k^2 - 10k + 25 - r^2 = 0$.

4 **Show the intersection equation**

Collecting terms gives $5x^2 + (4k-26)x + k^2 - 10k + 34 - r^2 = 0$.

5 **Use tangency**

For a tangent, the quadratic has one repeated root, so its discriminant is zero.

6 **Set the discriminant to zero**

$(4k-26)^2 - 20(k^2 - 10k + 34 - r^2) = 0$.

7 **Simplify**

This reduces to $20r^2 - 4k^2 - 8k - 4 = 0$, hence $5r^2 = (k+1)^2$. Therefore $p = 1$.

8 **Find the tangent point x coordinate**

For the repeated root, $x_B = -\dfrac{4k-26}{2 \cdot 5} = \dfrac{13-2k}{5}$.

9 **Use distance AB**

The point A is $(0, k)$ and B lies on a line of gradient 2, so $AB = \sqrt{x_B^2 + (2x_B)^2} = \sqrt{5}\,x_B$.

10 **Use the radius relation**

From $5r^2 = (k+1)^2$, $r = \dfrac{k+1}{\sqrt{5}}$ in the shown configuration.

11 **Apply AB equals two r**

$\sqrt{5}\left(\dfrac{13-2k}{5}\right) = 2\left(\dfrac{k+1}{\sqrt{5}}\right)$.

12 **Solve for k**

Multiplying by $\sqrt{5}$ gives $13 - 2k = 2k + 2$, so $k = \dfrac{11}{4}$.

FINAL ANSWER

(a) Proven.

(b) $5r^2 = (k+1)^2$, so $p = 1$.

(c) $k = \dfrac{11}{4}$.

Question 9

8 marks | Polynomials | Main topic: Integration

9. **In this question you must show all stages of your working.**

Solutions based entirely on calculator technology are not acceptable.

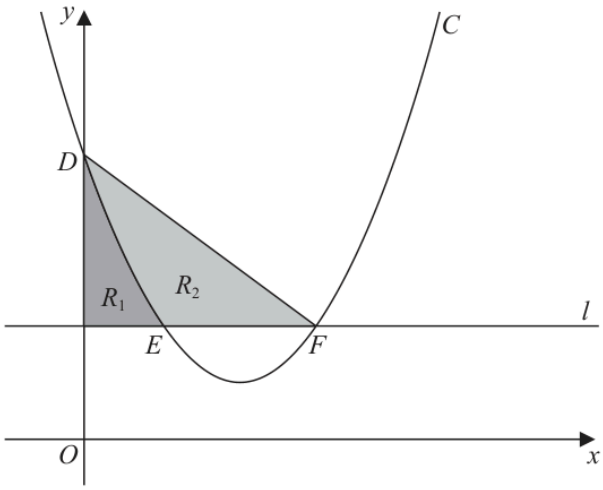


Figure 3

Figure 3 shows

- the curve C with equation $y = x^2 - 4x + 5$
- the line l with equation $y = 2$

The curve C intersects the y-axis at the point D.

(a) Write down the coordinates of D.

(1)

The curve C intersects the line l at the points E and F, as shown in Figure 3.

(b) Find the x coordinate of E and the x coordinate of F.

(2)

Shown shaded in Figure 3 is

- the region R_1 which is bounded by C, l and the y-axis
- the region R_2 which is bounded by C and the line segments EF and DF

Given that $\dfrac{\text{area of } R_1}{\text{area of } R_2} = k$, where k is a constant,

(c) use algebraic integration to find the exact value of k, giving your answer as a simplified fraction.

(5)

1 Find D

At the y-axis, $x = 0$. On C, $y = 0^2 - 4(0) + 5 = 5$, so $D = (0, 5)$.

2 Find intersections with the line

The line is $y = 2$, so solve $x^2 - 4x + 5 = 2$.

3 Solve the quadratic

$x^2 - 4x + 3 = 0$, so $(x - 1)(x - 3) = 0$. Hence $x_E = 1$ and $x_F = 3$.

4 Area of R1

R_1 lies between the curve and the line $y = 2$ from $x = 0$ to $x = 1$.

5 Integrate R1

$$R_1 = \int_0^1 \left(x^2 - 4x + 5 - 2\right) dx = \int_0^1 (x^2 - 4x + 3) dx.$$

6 Evaluate R1

$$R_1 = \left[\frac{x^3}{3} - 2x^2 + 3x\right]_0^1 = \frac{1}{3} - 2 + 3 = \frac{4}{3}.$$

7 Find the line DF

The line through $D = (0, 5)$ and $F = (3, 2)$ has gradient -1, so its equation is $y = 5 - x$.

8 Split R2

For $0 \le x \le 1$, R_2 is between $y = 5 - x$ and the curve. For $1 \le x \le 3$, it is between $y = 5 - x$ and $y = 2$.

9 Integrate R2

$$R_2 = \int_0^1 \left[(5 - x) - (x^2 - 4x + 5)\right] dx + \int_1^3 \left[(5 - x) - 2\right] dx.$$

10 Evaluate R2

$$R_2 = \int_0^1 (-x^2 + 3x) dx + \int_1^3 (3 - x) dx = \frac{7}{6} + 2 = \frac{19}{6}.$$

11 Find the ratio

$$k = \frac{\text{area of } R_1}{\text{area of } R_2} = \frac{4/3}{19/6} = \frac{8}{19}.$$

FINAL ANSWER

$(a)\ D = (0, 5).$

$(b)\ x_E = 1,\ x_F = 3.$

$(c)\ k = \dfrac{8}{19}.$

WMA12/01 JANUARY 2024

Question 6

8 marks | Polynomials | Main topic: Laws of Logarithms

6. (a) Given that

$$2\log_4(x+3) + \log_4 x = \log_4(4x+2) + \frac{1}{2}$$

show that

$$x^3 + 6x^2 + x - 4 = 0$$

(4)

(b) Given also that -1 is a root of the equation

$$x^3 + 6x^2 + x - 4 = 0$$

(i) use algebra to find the other two roots of the equation.

(3)

(ii) Hence solve

$$2\log_4(x+3) + \log_4 x = \log_4(4x+2) + \frac{1}{2}$$

(1)

WORKED SOLUTION

Question 6 · WMA12/01 January 2024

1 Combine the logarithms

$2\log_4(x+3) = \log_4((x+3)^2)$, so the left side is $\log_4(x(x+3)^2)$.

2 Rewrite one half

$\frac{1}{2} = \log_4 2$, so the right side is $\log_4(4x+2) + \log_4 2 = \log_4(8x+4)$.

3 Equate arguments

Therefore $x(x+3)^2 = 8x+4$.

4 Expand

$x(x^2+6x+9) = 8x+4$, so $x^3+6x^2+9x = 8x+4$.

5 **Show the cubic**

Rearranging gives $x^3 + 6x^2 + x - 4 = 0$.

6 **Use the given root**

Since -1 is a root, $(x + 1)$ is a factor of $x^3 + 6x^2 + x - 4$.

7 **Factor the cubic**

$x^3 + 6x^2 + x - 4 = (x + 1)(x^2 + 5x - 4)$.

8 **Solve the quadratic**

$x^2 + 5x - 4 = 0$ gives $x = \dfrac{-5 \pm \sqrt{25 + 16}}{2} = \dfrac{-5 \pm \sqrt{41}}{2}$.

9 **Apply the logarithm domain**

The original log equation requires $x > 0$. The only positive root is $x = \dfrac{-5 + \sqrt{41}}{2}$.

✓ **FINAL ANSWER**

(a) Proven.

$(b)(i)$ $x = \dfrac{-5 + \sqrt{41}}{2}$ and $x = \dfrac{-5 - \sqrt{41}}{2}$.

$(b)(ii)$ $x = \dfrac{-5 + \sqrt{41}}{2}$.

WMA12/01 MAY/JUNE R 2024

Question 6

9 marks | Polynomials | Main topic: Applications of Differentiation

6. $f(x) = 4x^3 + px^2 + 8x + q$

where p and q are constants.

Given that

- $(2x + 3)$ is a factor of $f(x)$
- $f(x)$ has a remainder of -5 when divided by $(x + 2)$

(a) (i) show that $p = 10$

(ii) find the value of q.

(5)

(b) Hence find the range of values of x for which $f(x)$ is decreasing.

(Solutions based entirely on calculator technology are not acceptable.)

(4)

WORKED SOLUTION

Question 6 · WMA12/01 May/June R 2024

1 Use the factor theorem

Since $(2x+3)$ is a factor, $x = -\frac{3}{2}$ is a root.

2 Substitute x equals negative three halves

$4\left(-\frac{3}{2}\right)^3 + p\left(-\frac{3}{2}\right)^2 + 8\left(-\frac{3}{2}\right) + q = 0.$

3 Simplify the first condition

$-\frac{27}{2} + \frac{9}{4}p - 12 + q = 0$, so $\frac{9}{4}p + q = \frac{51}{2}$.

4 Use the remainder theorem

The remainder on division by $(x+2)$ is $f(-2)$.

5 Substitute x equals negative two

$f(-2) = 4(-8) + 4p - 16 + q = 4p + q - 48.$

6 **Use the given remainder**

$4p + q - 48 = -5$, so $4p + q = 43$.

7 **Solve for p and q**

Subtract $\frac{9}{4}p + q = \frac{51}{2}$ from $4p + q = 43$: $\frac{7}{4}p = \frac{35}{2}$, so $p = 10$. Then $4(10) + q = 43$, so $q = 3$.

8 **Differentiate**

With $p = 10$, $f(x) = 4x^3 + 10x^2 + 8x + 3$, so $f'(x) = 12x^2 + 20x + 8$.

9 **Factor the derivative**

$f'(x) = 4(3x + 2)(x + 1)$.

10 **Find where decreasing**

The quadratic derivative is negative between its roots, so $f(x)$ is decreasing for $-1 < x < -\frac{2}{3}$.

✓ **FINAL ANSWER**

$(a)(i)\ p = 10$.

$(a)(ii)\ q = 3$.

$(b)\ -1 < x < -\frac{2}{3}$.

WMA12/01 MAY/JUNE R 2024

Question 9

9 marks | Polynomials | Main topic: Laws of Logarithms

9. Given that

$$3\log_2(t+4) - 2\log_2(t-2) = 7$$

(a) verify that $t = 4$ is a solution of the above equation,

(2)

(b) show that

$$t^3 - 116t^2 + 560t - 448 = 0$$

(3)

(c) Hence, using algebra and showing your working, solve

$$3\log_2(t+4) - 2\log_2(t-2) = 7$$

giving each answer in simplest form.

(*Solutions based entirely on calculator technology are not acceptable.*)

(4)

WORKED SOLUTION

Question 9 · WMA12/01 May/June R 2024

1 Verify t equals four

Substitute $t = 4$: $3\log_2(8) - 2\log_2(2) = 3(3) - 2(1) = 7$.

2 Combine the logarithms

$3\log_2(t+4) - 2\log_2(t-2) = \log_2\left(\dfrac{(t+4)^3}{(t-2)^2}\right)$.

3 Use seven as a logarithm

$7 = \log_2(128)$, so $\dfrac{(t+4)^3}{(t-2)^2} = 128$.

4 Clear the denominator

$(t+4)^3 = 128(t-2)^2$.

5 Expand

$t^3 + 12t^2 + 48t + 64 = 128t^2 - 512t + 512$.

6 **Show the cubic**

Rearranging gives $t^3 - 116t^2 + 560t - 448 = 0$.

7 **Use the known root**

Since $t = 4$ is a root, $(t - 4)$ is a factor.

8 **Factor the cubic**

$t^3 - 116t^2 + 560t - 448 = (t - 4)(t^2 - 112t + 112)$.

9 **Solve the quadratic**

$t^2 - 112t + 112 = 0$ gives $t = \dfrac{112 \pm \sqrt{112^2 - 4(112)}}{2} = 56 \pm 12\sqrt{21}$.

10 **Use the log domain**

The original equation requires $t > 2$. The value $56 - 12\sqrt{21}$ is less than 2, so it is rejected. Hence $t = 4$ or $t = 56 + 12\sqrt{21}$.

✓ **FINAL ANSWER**

(a) Verified.

(b) Proven.

(c) $t = 4$ or $t = 56 + 12\sqrt{21}$.

Question 8

8 marks | Polynomials | Main topic: Integration

8. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

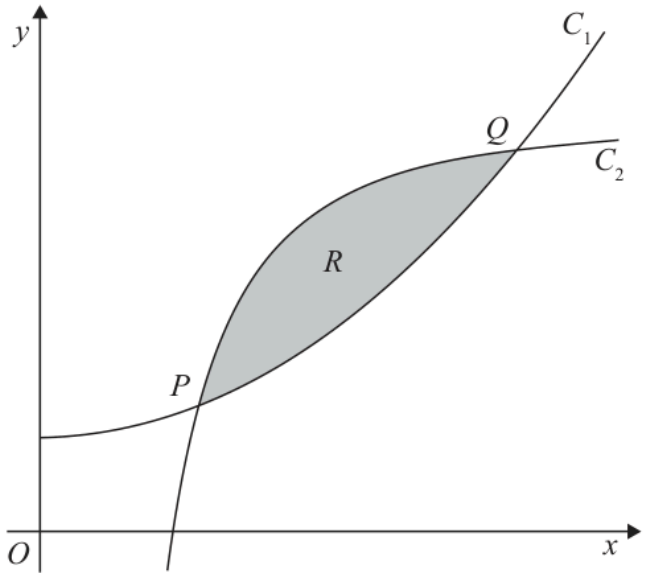


Figure 1

Figure 1 shows a sketch of part of the curve C_1 with equation

$$y = x^2 + 3 \qquad x > 0$$

and part of the curve C_2 with equation

$$y = 13 - \frac{9}{x^2} \qquad x > 0$$

The curves C_1 and C_2 intersect at the points P and Q as shown in Figure 1.

(a) Use algebra to find the x coordinate of P and the x coordinate of Q.

(4)

The finite region R, shown shaded in Figure 1, is bounded by C_1 and C_2

(b) Use algebraic integration to find the exact area of R.

(4)

1 **Equate the curves**

$x^2 + 3 = 13 - \dfrac{9}{x^2}$.

2 **Clear the denominator**

Multiplying by x^2 gives $x^4 + 3x^2 = 13x^2 - 9$, so $x^4 - 10x^2 + 9 = 0$.

3 **Factor**

Let $u = x^2$. Then $u^2 - 10u + 9 = (u - 1)(u - 9) = 0$, so $x^2 = 1$ or $x^2 = 9$.

4 **Use x greater than zero**

Since $x > 0$, the intersections have $x = 1$ and $x = 3$. Thus $x_P = 1$ and $x_Q = 3$.

5 **Set up the area**

Between $x = 1$ and $x = 3$, curve C_2 is above curve C_1, so $R = \displaystyle\int_1^3 \left(13 - \frac{9}{x^2} - (x^2 + 3)\right) dx$.

6 **Simplify the integrand**

$R = \displaystyle\int_1^3 (10 - x^2 - 9x^{-2})\, dx$.

7 **Integrate**

$R = \left[10x - \dfrac{x^3}{3} + \dfrac{9}{x}\right]_1^3$.

8 **Evaluate**

$R = (30 - 9 + 3) - \left(10 - \dfrac{1}{3} + 9\right) = 24 - \dfrac{56}{3} = \dfrac{16}{3}$.

✓ **FINAL ANSWER**

(a) $x_P = 1$, $x_Q = 3$.

(b) $\dfrac{16}{3}$.

TOPIC

Circles

Every past-paper question on this topic, October 2019 to January 2026.

WMA12/01 OCTOBER 2019

Question 6

7 marks | Circles

6.

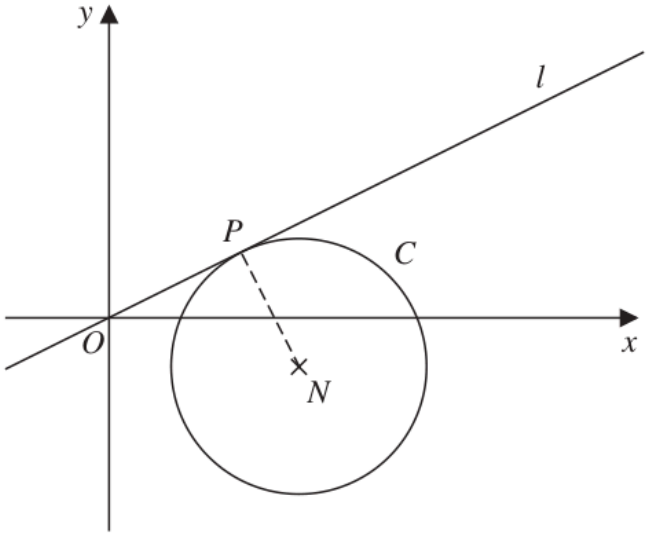


Figure 1

Figure 1 shows a sketch of a circle C with centre $N(4, -1)$.

The line l with equation $y = \frac{1}{2}x$ is a tangent to C at the point P.

Find

(a) the equation of line PN in the form $y = mx + c$, where m and c are constants,

(2)

(b) the equation of C.

(5)

WORKED SOLUTION

Question 6 · WMA12/01 October 2019

1 Use perpendicular gradients

The tangent has gradient $\frac{1}{2}$, so the radius PN has gradient -2.

2 Find the equation of PN

Line PN passes through $N(4, -1)$, so $y + 1 = -2(x - 4)$. Therefore $y = -2x + 7$.

3 **Find point P**

At P, the tangent $y = \frac{1}{2}x$ meets the radius $y = -2x + 7$. Thus $\frac{1}{2}x = -2x + 7$, so $x = \frac{14}{5}$ and $y = \frac{7}{5}$.

---

4 **Find the radius squared**

$$r^2 = \left(\frac{14}{5} - 4\right)^2 + \left(\frac{7}{5} + 1\right)^2 = \left(-\frac{6}{5}\right)^2 + \left(\frac{12}{5}\right)^2 = \frac{36}{5}.$$

---

5 **Write the circle equation**

With centre $(4, -1)$ and radius squared $\frac{36}{5}$, the circle is $(x-4)^2 + (y+1)^2 = \frac{36}{5}$.

✓ **FINAL ANSWER**

$(a)\ y = -2x + 7. \quad (b)\ (x-4)^2 + (y+1)^2 = \frac{36}{5}.$

WMA12/01 JANUARY 2020

Question 6

8 marks | Circles

6. The circle C has equation

$$x^2 + y^2 + 6x - 4y - 14 = 0$$

(a) Find

(i) the coordinates of the centre of C,

(ii) the exact radius of C.

(3)

The line with equation $y = k$, where k is a constant, is a tangent to C.

(b) Find the possible values of k.

(2)

The line with equation $y = p$, where p is a negative constant, is a chord of C.

Given that the length of this chord is 4 units,

(c) find the value of p.

(3)

WORKED SOLUTION

Question 6 · WMA12/01 January 2020

1 Complete the square

$x^2 + 6x = (x+3)^2 - 9$ and $y^2 - 4y = (y-2)^2 - 4$.

2 Find the centre and radius

The circle becomes $(x+3)^2 + (y-2)^2 = 27$. Hence the centre is $(-3, 2)$ and the radius is $\sqrt{27} = 3\sqrt{3}$.

3 Use horizontal tangents

A horizontal tangent has equation $y = k$. The possible tangent levels are one radius above and below the centre, so $k = 2 \pm 3\sqrt{3}$.

4 Use half the chord length

The chord has length 4, so half the chord is 2. If the chord is $y = p$, its perpendicular distance from the centre is $|p - 2|$.

5 **Apply Pythagoras**

$(p-2)^2+2^2=(3\sqrt{3})^2=27$, so $(p-2)^2=23$.

6 **Choose the negative value**

Since p is negative, $p=2-\sqrt{23}$.

✓ **FINAL ANSWER**

$(a)(i)\ (-3,2).\quad (a)(ii)\ 3\sqrt{3}.\quad (b)\ k=2\pm 3\sqrt{3}.\quad (c)\ p=2-\sqrt{23}.$

WMA12/01 JANUARY 2021

Question 9

10 marks | Circles

9. A circle C has equation

$$(x-k)^2+(y-2k)^2=k+7$$

where k is a positive constant.

(a) Write down, in terms of k,

(i) the coordinates of the centre of C,

(ii) the radius of C.

(2)

Given that the point $P(2,3)$ lies on C

(b) (i) show that $5k^2-17k+6=0$

(ii) hence find the possible values of k.

(3)

The tangent to the circle at P intersects the x-axis at point T.

Given that $k<2$

(c) calculate the exact area of triangle OPT.

(5)

WORKED SOLUTION

Question 9 · WMA12/01 January 2021

1 Read the centre and radius

From $(x-k)^2+(y-2k)^2=k+7$, the centre is $(k,2k)$ and the radius is $\sqrt{k+7}$.

2 Substitute point P

Since $P(2,3)$ lies on the circle, $(2-k)^2+(3-2k)^2=k+7$.

3 Show the quadratic

Expanding gives $k^2-4k+4+4k^2-12k+9=k+7$, hence $5k^2-17k+6=0$.

4 **Solve for k**

$5k^2 - 17k + 6 = (5k - 2)(k - 3)$, so $k = \dfrac{2}{5}$ or $k = 3$.

5 **Use the condition k less than two**

For the tangent calculation, $k < 2$, so the centre is $\left(\dfrac{2}{5}, \dfrac{4}{5}\right)$.

6 **Find the tangent gradient**

The radius from the centre to P has gradient $\dfrac{3 - 4/5}{2 - 2/5} = \dfrac{11}{8}$, so the tangent gradient is $-\dfrac{8}{11}$.

7 **Find the x intercept**

The tangent is $y - 3 = -\dfrac{8}{11}(x - 2)$. Setting $y = 0$ gives $x = \dfrac{49}{8}$, so $T = \left(\dfrac{49}{8}, 0\right)$.

8 **Calculate the area**

Triangle OPT has base $\dfrac{49}{8}$ on the x-axis and height 3, so its area is $\dfrac{1}{2} \cdot \dfrac{49}{8} \cdot 3 = \dfrac{147}{16}$.

✓ **FINAL ANSWER**

$(a)(i)\ (k, 2k).$ $(a)(ii)\ \sqrt{k + 7}.$ $(b)(ii)\ k = 3 \text{ or } \dfrac{2}{5}.$ $(c)\ \dfrac{147}{16}.$

WMA12/01 MAY/JUNE 2021

Question 6

7 marks | Circles

6. A circle has equation

$$x^2 - 6x + y^2 + 8y + k = 0$$

where k is a positive constant.

Given that the x-axis is a tangent to this circle,

(a) find the value of k.

(3)

The circle meets the coordinate axes at the points R, S and T.

(b) Find the exact area of the triangle RST.

(4)

WORKED SOLUTION

Question 6 · WMA12/01 May/June 2021

1 Complete the square

$x^2 - 6x = (x-3)^2 - 9$ and $y^2 + 8y = (y+4)^2 - 16$.

2 Write centre-radius form

The circle becomes $(x-3)^2 + (y+4)^2 = 25 - k$.

3 Use the tangent condition

The centre is $(3, -4)$. Since the x-axis is tangent, the radius is the distance from the centre to the x-axis, which is 4.

4 Find k

Thus $25 - k = 4^2 = 16$, so $k = 9$.

5 Find the intercepts

With $k = 9$, the circle is $(x-3)^2 + (y+4)^2 = 16$. On the x-axis, $y = 0$, giving $x = 3$.

6 Find the y-axis points

On the y-axis, $x = 0$, so $9 + (y+4)^2 = 16$. Hence $y = -4 \pm \sqrt{7}$.

7 **Calculate the triangle area**

The vertical base on the y-axis has length $2\sqrt{7}$, and its perpendicular distance to $(3, 0)$ is 3. The area is $\frac{1}{2}(2\sqrt{7})(3) = 3\sqrt{7}$.

✓ **FINAL ANSWER**

$(a)\ k = 9.\quad (b)\ 3\sqrt{7}.$

WMA12/01 OCTOBER 2021

Question 6

8 marks | Circles

6. (i) The circle C_1 has equation

$$x^2 + y^2 + 10x - 12y = k \qquad \text{where } k \text{ is a constant}$$

(a) Find the coordinates of the centre of C_1

(2)

(b) State the possible range in values for k.

(2)

(ii) The point $P(p, 0)$, the point $Q(-2, 10)$ and the point $R(8, -14)$ lie on a different circle, C_2

Given that

- p is a positive constant
- QR is a diameter of C_2

find the exact value of p.

(4)

WORKED SOLUTION

Question 6 · WMA12/01 October 2021

1 Complete the square

$x^2 + 10x = (x+5)^2 - 25$ and $y^2 - 12y = (y-6)^2 - 36$.

2 Write centre-radius form

The circle becomes $(x+5)^2 + (y-6)^2 = k + 61$.

3 Read the centre

The centre is $(-5, 6)$.

4 State the range for k

For a circle with positive radius, $k + 61 > 0$, so $k > -61$.

5 Use the diameter property

If QR is a diameter and P lies on the circle, then $\angle QPR = 90°$.

6 **Use perpendicular vectors**

So $(Q-P)\cdot(R-P)=0$. With $P=(p,0)$, $Q=(-2,10)$, and $R=(8,-14)$, this gives $(-2-p,10)\cdot(8-p,-14)=0$.

7 **Form the quadratic**

$(-2-p)(8-p)-140=0$, so $p^2-6p-156=0$.

8 **Choose the positive value**

$p=\dfrac{6\pm\sqrt{660}}{2}=3\pm\sqrt{165}$. Since p is positive, $p=3+\sqrt{165}$.

✓ **FINAL ANSWER**

$(i)(a)\ (-5,6).\quad (i)(b)\ k>-61.\quad (ii)\ p=3+\sqrt{165}.$

WMA12/01 JANUARY 2022

Question 6

8 marks | Circles

6.

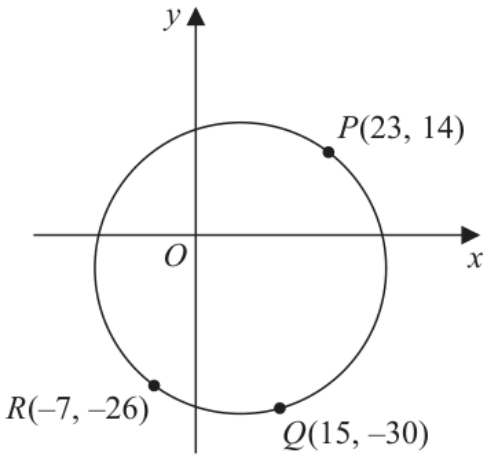


Figure 1

The points $P(23, 14)$, $Q(15, -30)$ and $R(-7, -26)$ lie on the circle C, as shown in Figure 1.

(a) Show that angle $PQR = 90°$

(2)

(b) Hence, or otherwise, find

(i) the centre of C,

(ii) the radius of C.

(3)

Given that the point S lies on C such that the distance QS is greatest,

(c) find an equation of the tangent to C at S, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers to be found.

(3)

WORKED SOLUTION

Question 6 · WMA12/01 January 2022

1 **Find the gradient QP**

The gradient of QP is $\dfrac{14-(-30)}{23-15} = \dfrac{44}{8} = \dfrac{11}{2}$.

2 **Find the gradient QR**

The gradient of QR is $\dfrac{-26-(-30)}{-7-15} = \dfrac{4}{-22} = -\dfrac{2}{11}$.

3 **Show the right angle**

The product of the two gradients is -1, so QP is perpendicular to QR and $\angle PQR = 90°$.

4 **Use the diameter**

Since the angle at Q is a right angle, PR is a diameter of the circle.

5 **Find the centre**

The centre is the midpoint of $P(23, 14)$ and $R(-7, -26)$, so it is $(8, -6)$.

6 **Find the radius**

$PR = \sqrt{30^2 + 40^2} = 50$, so the radius is 25.

7 **Find the farthest point from Q**

The greatest distance from Q occurs at the point opposite Q on the diameter through the centre. Thus $S = 2(8, -6) - (15, -30) = (1, 18)$.

8 **Find the tangent gradient**

The radius from the centre to S has gradient $\dfrac{18 - (-6)}{1 - 8} = -\dfrac{24}{7}$, so the tangent gradient is $\dfrac{7}{24}$.

9 **Write the tangent equation**

$y - 18 = \dfrac{7}{24}(x - 1)$, which rearranges to $7x - 24y + 425 = 0$.

✓ **FINAL ANSWER**

(a) Proven: $\angle PQR = 90°$.

$(b)(i)$ $(8, -6)$. $(b)(ii)$ 25. (c) $7x - 24y + 425 = 0$.

Question 10

12 marks | Circles

10. The circle C has centre $X(3, 5)$ and radius r

The line l has equation $y = 2x + k$, where k is a constant.

(a) Show that l and C intersect when

$$5x^2 + (4k - 26)x + k^2 - 10k + 34 - r^2 = 0$$

(3)

Given that l is a tangent to C,

(b) show that $5r^2 = (k + p)^2$, where p is a constant to be found.

(3)

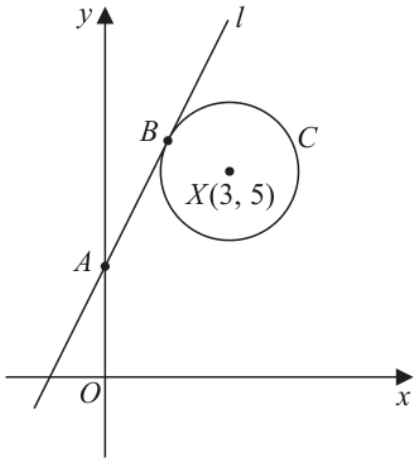


Figure 2

The line l

- cuts the y-axis at the point A
- touches the circle C at the point B

as shown in Figure 2.

Given that $AB = 2r$

(c) find the value of k

(6)

(Total 12 marks)

1 **Write the circle equation**

The circle has centre $(3, 5)$ and radius r, so $(x-3)^2 + (y-5)^2 = r^2$.

2 **Substitute the line**

On the line, $y = 2x + k$. Substitution gives $(x-3)^2 + (2x+k-5)^2 = r^2$.

3 **Expand**

$x^2 - 6x + 9 + 4x^2 + 4kx - 20x + k^2 - 10k + 25 - r^2 = 0$.

4 **Show the intersection equation**

Collecting terms gives $5x^2 + (4k-26)x + k^2 - 10k + 34 - r^2 = 0$.

5 **Use tangency**

For a tangent, the quadratic has one repeated root, so its discriminant is zero.

6 **Set the discriminant to zero**

$(4k-26)^2 - 20(k^2 - 10k + 34 - r^2) = 0$.

7 **Simplify**

This reduces to $20r^2 - 4k^2 - 8k - 4 = 0$, hence $5r^2 = (k+1)^2$. Therefore $p = 1$.

8 **Find the tangent point x coordinate**

For the repeated root, $x_B = -\dfrac{4k-26}{2 \cdot 5} = \dfrac{13-2k}{5}$.

9 **Use distance AB**

The point A is $(0, k)$ and B lies on a line of gradient 2, so $AB = \sqrt{x_B^2 + (2x_B)^2} = \sqrt{5}\,x_B$.

10 **Use the radius relation**

From $5r^2 = (k+1)^2$, $r = \dfrac{k+1}{\sqrt{5}}$ in the shown configuration.

11 **Apply AB equals two r**

$\sqrt{5}\left(\dfrac{13-2k}{5}\right) = 2\left(\dfrac{k+1}{\sqrt{5}}\right)$.

12 **Solve for k**

Multiplying by $\sqrt{5}$ gives $13 - 2k = 2k + 2$, so $k = \dfrac{11}{4}$.

FINAL ANSWER

(a) Proven.

(b) $5r^2 = (k+1)^2$, so $p = 1$.

(c) $k = \dfrac{11}{4}$.

Question 9

12 marks | Circles

9. **In this question you must show detailed reasoning.**

Solutions relying entirely on calculator technology are not acceptable.

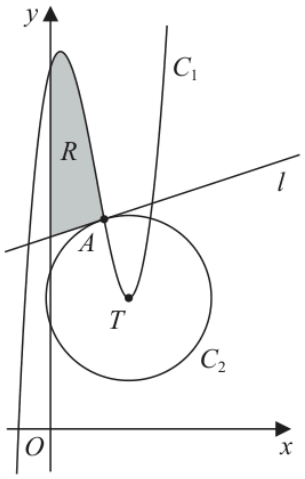


Figure 3

Figure 3 shows

- the curve C_1 with equation $y = x^3 - 5x^2 + 3x + 14$
- the circle C_2 with centre T

The point T is the minimum turning point of C_1

Using Figure 3 and calculus,

(a) find the coordinates of T

(3)

The curve C_1 intersects the circle C_2 at the point A with x coordinate 2

(b) Find an equation of the circle C_2

(3)

The line l shown in Figure 3, is the tangent to circle C_2 at A

(c) Show that an equation of l is

$$y = \frac{1}{3}x + \frac{22}{3}$$

(3)

The region R, shown shaded in Figure 3, is bounded by C_1, l and the y-axis.

(d) Find the exact area of R.

(3)

1 **Differentiate the cubic**

For $y = x^3 - 5x^2 + 3x + 14$, $\dfrac{dy}{dx} = 3x^2 - 10x + 3$.

2 **Find stationary x values**

$3x^2 - 10x + 3 = 0$ factors as $(3x - 1)(x - 3) = 0$, so $x = \dfrac{1}{3}$ or $x = 3$.

3 **Identify the minimum**

$\dfrac{d^2y}{dx^2} = 6x - 10$. At $x = 3$, this is positive, so the minimum turning point has $x = 3$.

4 **Find T**

Substitute $x = 3$: $y = 27 - 45 + 9 + 14 = 5$. Hence $T = (3, 5)$.

5 **Find A**

At $x = 2$ on C_1, $y = 8 - 20 + 6 + 14 = 8$, so $A = (2, 8)$.

6 **Find the circle radius**

The centre of C_2 is $T = (3, 5)$ and it passes through $A = (2, 8)$. Thus $r^2 = (2 - 3)^2 + (8 - 5)^2 = 10$.

7 **Write the circle equation**

Therefore C_2 has equation $(x - 3)^2 + (y - 5)^2 = 10$.

8 **Find the tangent gradient**

The radius TA has gradient $\dfrac{8 - 5}{2 - 3} = -3$, so the tangent at A has gradient $\dfrac{1}{3}$.

9 **Write the tangent equation**

Using $A = (2, 8)$, $y - 8 = \dfrac{1}{3}(x - 2)$, so $y = \dfrac{1}{3}x + \dfrac{22}{3}$.

10 **Set up the area**

From $x = 0$ to $x = 2$, the cubic is above the line. The required area is $\displaystyle\int_0^2 \left[x^3 - 5x^2 + 3x + 14 - \left(\frac{1}{3}x + \frac{22}{3}\right)\right] dx$.

11 **Simplify the integrand**

The integrand is $x^3 - 5x^2 + \dfrac{8}{3}x + \dfrac{20}{3}$.

12 **Integrate**

$$\left[\frac{x^4}{4}-\frac{5x^3}{3}+\frac{4x^2}{3}+\frac{20x}{3}\right]_0^2 = 4-\frac{40}{3}+\frac{16}{3}+\frac{40}{3}=\frac{28}{3}.$$

✓ **FINAL ANSWER**

$(a)\ T=(3,5).$

$(b)\ (x-3)^2+(y-5)^2=10.$

$(c)\ y=\frac{1}{3}x+\frac{22}{3}.$

$(d)\ \frac{28}{3}.$

WMA12/01 JANUARY 2023

Question 6

10 marks | Circles

6. The circle C has equation

$$x^2 + y^2 + 8x - 4y = 0$$

(a) Find

(i) the coordinates of the centre of C,

(ii) the exact radius of C.

(3)

The point P lies on C.

Given that the tangent to C at P has equation $x + 2y + 10 = 0$

(b) find the coordinates of P

(4)

(c) Find the equation of the normal to C at P, giving your answer in the form $y = mx + c$ where m and c are integers to be found.

(3)

WORKED SOLUTION

Question 6 · WMA12/01 January 2023

1 Complete the square

$x^2 + y^2 + 8x - 4y = 0$ becomes $(x + 4)^2 - 16 + (y - 2)^2 - 4 = 0$.

2 Write the circle form

So $(x + 4)^2 + (y - 2)^2 = 20$.

3 Read the centre and radius

The centre is $(-4, 2)$ and the radius is $\sqrt{20} = 2\sqrt{5}$.

4 Use the tangent gradient

The tangent $x + 2y + 10 = 0$ has gradient $-\frac{1}{2}$.

5 Find the radius gradient

The radius to the point of contact is perpendicular to the tangent, so its gradient is 2.

6 **Write the normal through the centre**

The radius line through the centre $(-4, 2)$ is $y - 2 = 2(x + 4)$, so $y = 2x + 10$.

7 **Intersect with the tangent**

Substitute $y = 2x + 10$ into $x + 2y + 10 = 0$: $x + 2(2x + 10) + 10 = 0$.

8 **Find P**

$5x + 30 = 0$, so $x = -6$. Then $y = 2(-6) + 10 = -2$, hence $P = (-6, -2)$.

9 **State the normal**

The normal at P is the radius line already found, so its equation is $y = 2x + 10$.

✓ **FINAL ANSWER**

$(a)(i)$ $(-4, 2)$.

$(a)(ii)$ $2\sqrt{5}$.

(b) $P = (-6, -2)$.

(c) $y = 2x + 10$.

Question 7

8 marks | Circles

7.

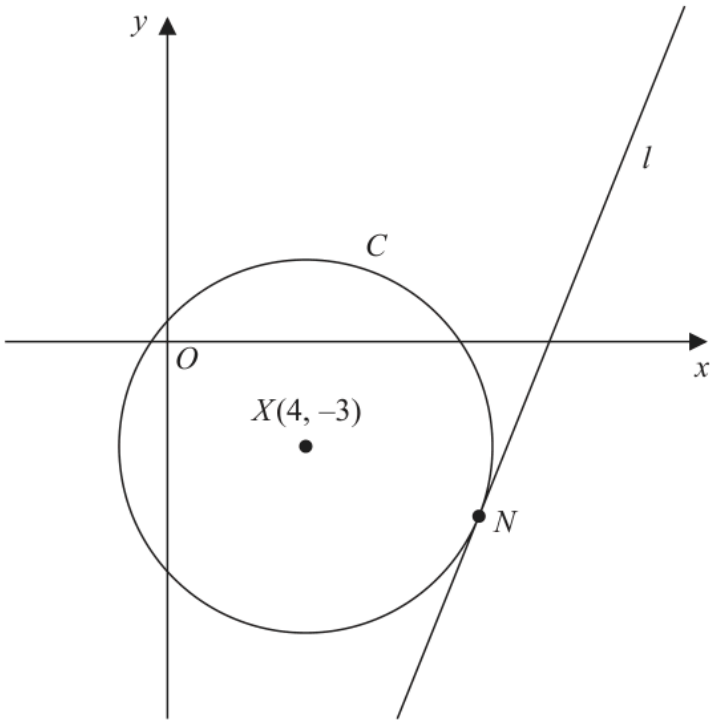


Figure 2

Figure 2 shows a sketch of

- the circle C with centre $X(4, -3)$
- the line l with equation $y = \frac{5}{2}x - \frac{55}{2}$

Given that l is the tangent to C at the point N,

(a) show that an equation for the straight line passing through X and N is

$$2x + 5y + 7 = 0$$

(3)

(b) Hence find

(i) the coordinates of N,

(ii) an equation for C.

(5)

1 Use perpendicular gradients

The tangent l has gradient $\frac{5}{2}$, so the radius XN is perpendicular to it and has gradient $-\frac{2}{5}$.

2 Use the centre

The line through $X(4,-3)$ with gradient $-\frac{2}{5}$ is $y+3=-\frac{2}{5}(x-4)$.

3 Show the required line

Multiplying by 5 gives $5y+15=-2x+8$, so $2x+5y+7=0$.

4 Intersect the radius and tangent

Use $y=\frac{5}{2}x-\frac{55}{2}$ in $2x+5y+7=0$.

5 Find x coordinate

$2x+5\left(\frac{5}{2}x-\frac{55}{2}\right)+7=0$. Multiplying by 2 gives $4x+25x-275+14=0$, so $29x=261$ and $x=9$.

6 Find y coordinate

$y=\frac{5}{2}(9)-\frac{55}{2}=\frac{45-55}{2}=-5$, so $N=(9,-5)$.

7 Find the radius squared

$XN^2=(9-4)^2+(-5+3)^2=5^2+(-2)^2=29$.

8 Write the circle equation

With centre $(4,-3)$ and radius squared 29, the circle is $(x-4)^2+(y+3)^2=29$.

FINAL ANSWER

(a) Proven.

$(b)(i)$ $N=(9,-5)$.

$(b)(ii)$ $(x-4)^2+(y+3)^2=29$.

WMA12/01 MAY/JUNE 2024

Question 7

6 marks | Circles

7. The circle C_1 has equation

$$x^2 + y^2 + 8x - 10y = 29$$

(a) (i) Find the coordinates of the centre of C_1

(ii) Find the exact value of the radius of C_1

(3)

In part (b) you must show detailed reasoning.

The circle C_2 has equation

$$(x-5)^2 + (y+8)^2 = 52$$

(b) Prove that the circles C_1 and C_2 neither touch nor intersect.

(3)

WORKED SOLUTION

Question 7 · WMA12/01 May/June 2024

1 Complete the square

$x^2 + y^2 + 8x - 10y = 29$ becomes $(x+4)^2 - 16 + (y-5)^2 - 25 = 29$.

2 Write circle C1 in standard form

$(x+4)^2 + (y-5)^2 = 70$.

3 Read centre and radius

The centre of C_1 is $(-4, 5)$ and its radius is $\sqrt{70}$.

4 Read C2

C_2 has equation $(x-5)^2 + (y+8)^2 = 52$, so its centre is $(5, -8)$ and its radius is $\sqrt{52} = 2\sqrt{13}$.

5 Find distance between centres

The distance between $(-4, 5)$ and $(5, -8)$ is $\sqrt{9^2 + (-13)^2} = \sqrt{250} = 5\sqrt{10}$.

6 Compare with radii

$\sqrt{70} + 2\sqrt{13} = 15.58\ldots$, while $5\sqrt{10} = 15.81\ldots$.

7 **Conclude**

The distance between centres is greater than the sum of the radii, so the circles are separate.

Therefore they neither touch nor intersect.

✓ **FINAL ANSWER**

$(a)(i)$ $(-4, 5)$.

$(a)(ii)$ $\sqrt{70}$.

(b) The circles neither touch nor intersect.

WMA12/01 OCTOBER 2024

Question 10

10 marks | Circles

10. The circle C has equation

$$x^2 + y^2 + 4x - 30y + 209 = 0$$

(a) Find

(i) the coordinates of the centre of C,

(ii) the exact value of the radius of C.

(3)

The line L has equation $y = mx + 1$, where m is a constant.

Given that L is the tangent to C at the point P,

(b) show that

$$2m^2 - 7m - 22 = 0$$

(3)

(c) Hence find the possible pairs of coordinates of P.

(4)

WORKED SOLUTION

Question 10 · WMA12/01 October 2024

1 Complete the square

$x^2 + y^2 + 4x - 30y + 209 = 0$ becomes $(x+2)^2 + (y-15)^2 = 20$.

2 Read the centre and radius

The centre is $(-2, 15)$ and the radius is $\sqrt{20} = 2\sqrt{5}$.

3 Use the tangent distance

The line $y = mx + 1$ can be written as $mx - y + 1 = 0$. Its distance from $(-2, 15)$ must be $2\sqrt{5}$.

4 Set up the equation for m

$\dfrac{|m(-2) - 15 + 1|}{\sqrt{m^2+1}} = 2\sqrt{5}$, so $\dfrac{|-2m - 14|}{\sqrt{m^2+1}} = 2\sqrt{5}$.

5 Show the quadratic

Squaring gives $4(m+7)^2 = 20(m^2+1)$. Hence $(m+7)^2 = 5(m^2+1)$, so $2m^2 - 7m - 22 = 0$.

6 **Find the possible gradients**

$2m^2 - 7m - 22 = 0$ gives $m = -2$ or $m = \frac{11}{2}$.

7 **Use perpendicular radii**

A tangent is perpendicular to the radius at the point of contact.

8 **Find the first point**

For $m = -2$, the radius gradient is $\frac{1}{2}$. The radius through the centre is $y - 15 = \frac{1}{2}(x + 2)$. Solving with $y = -2x + 1$ gives $x = -6$ and $y = 13$.

9 **Find the second point**

For $m = \frac{11}{2}$, the radius gradient is $-\frac{2}{11}$. The radius through the centre is $y - 15 = -\frac{2}{11}(x + 2)$. Solving with $y = \frac{11}{2}x + 1$ gives $x = \frac{12}{5}$ and $y = \frac{71}{5}$.

✓ **FINAL ANSWER**

$(a)(i)$ $(-2, 15)$.

$(a)(ii)$ $2\sqrt{5}$.

(b) Proven.

(c) $(-6, 13)$ and $\left(\frac{12}{5}, \frac{71}{5}\right)$.

Question 6

8 marks | Circles

6.

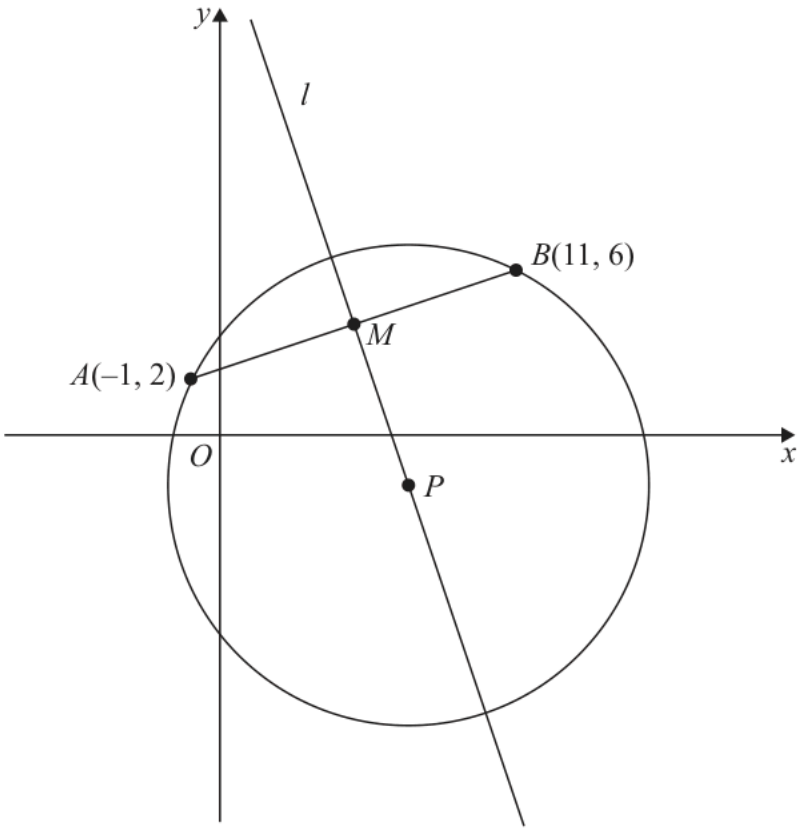


Figure 2

The point A $(-1, 2)$ and the point B $(11, 6)$ both lie on a circle with centre P.

The point M is the midpoint of AB.

Given that the line l passes through M and P, as shown in Figure 2,

(a) find an equation for l, giving your answer in the form $y = mx + c$, where m and c are constants.

(4)

Given that P has coordinates $(7, k)$, where k is a constant,

(b) find the value of k,

(1)

(c) find an equation for the circle.

(3)

1 Find the midpoint

For $A(-1,2)$ and $B(11,6)$, the midpoint is $M\left(\frac{-1+11}{2},\frac{2+6}{2}\right)=(5,4)$.

2 Find the gradient of AB

The gradient of AB is $\frac{6-2}{11-(-1)}=\frac{4}{12}=\frac{1}{3}$.

3 Use the perpendicular bisector

The centre lies on the perpendicular bisector of the chord AB, so line l has gradient -3 and passes through $M(5,4)$.

4 Find line l

$y-4=-3(x-5)$, so $y=-3x+19$.

5 Use the centre coordinates

Since $P=(7,k)$ lies on l, $k=-3(7)+19=-2$.

6 Find the radius squared

The centre is $(7,-2)$. Using point A, $r^2=(7-(-1))^2+(-2-2)^2=8^2+(-4)^2=80$.

7 Write the circle equation

Therefore the circle is $(x-7)^2+(y+2)^2=80$.

FINAL ANSWER

(a) $y=-3x+19$.

(b) $k=-2$.

(c) $(x-7)^2+(y+2)^2=80$.

WMA12/01 OCTOBER 2025

Question 6

9 marks | Circles

6.

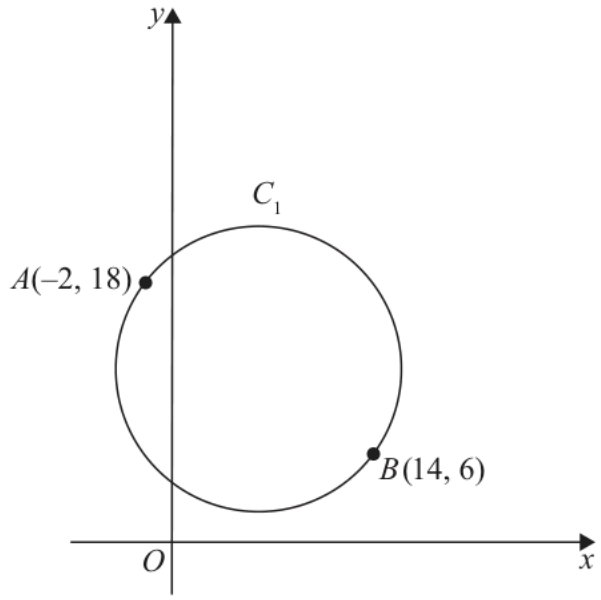


Figure 2

The points $A(-2, 18)$ and $B(14, 6)$ lie on a circle C_1 as shown in Figure 2.

Given that AB is a diameter of the circle C_1

(a) find an equation for C_1 making your method clear.

(5)

A circle C_2 has its centre at the origin.

Given that circles C_1 and C_2 touch,

(b) find possible equations for C_2

(4)

WORKED SOLUTION

Question 6 · WMA12/01 October 2025

1 Find the centre of C one

Since AB is a diameter, the centre is the midpoint of $A(-2, 18)$ and $B(14, 6)$. This is $(6, 12)$.

2 Find the radius squared

Using $A(-2, 18)$, $r^2 = (-2-6)^2 + (18-12)^2 = 64 + 36 = 100$.

3 **Write C one**

Therefore C_1 has equation $(x-6)^2+(y-12)^2=100$.

4 **Find the centre distance**

The distance from the origin to $(6,12)$ is $\sqrt{6^2+12^2}=6\sqrt{5}$.

5 **Find possible radii for C two**

Since C_1 has radius 10, a circle centred at the origin can touch it with radius $6\sqrt{5}-10$ or $6\sqrt{5}+10$.

6 **Write the equations**

Thus C_2 can be $x^2+y^2=(6\sqrt{5}-10)^2=280-120\sqrt{5}$ or $x^2+y^2=(6\sqrt{5}+10)^2=280+120\sqrt{5}$.

✓ **FINAL ANSWER**

(a) $(x-6)^2+(y-12)^2=100$.

(b) $x^2+y^2=280-120\sqrt{5}$ or $x^2+y^2=280+120\sqrt{5}$.

TOPIC

Binomial Expansion

Every past-paper question on this topic, October 2019 to January 2026.

WMA12/01 OCTOBER 2023

Question 10

12 marks | Binomial Expansion

10. (i) (a) Find, in ascending powers of x, the 2nd, 3rd and 5th terms of the binomial expansion of

$$(3 + 2x)^6$$

(3)

For a particular value of x, these three terms form consecutive terms in a geometric series.

(b) Find this value of x.

(3)

(ii) In a **different** geometric series,

- the first term is $\sin^2\theta$
- the common ratio is $2\cos\theta$
- the sum to infinity is $\frac{8}{5}$

(a) Show that

$$5\cos^2\theta - 16\cos\theta + 3 = 0$$

(3)

(b) Hence find the exact value of the 2nd term in the series.

(3)

WORKED SOLUTION

Question 10 · WMA12/01 October 2023

1 Find the second term

The 2nd term of $(3+2x)^6$ is $\binom{6}{1}3^5(2x) = 6 \cdot 243 \cdot 2x = 2916x$.

2 Find the third term

The 3rd term is $\binom{6}{2}3^4(2x)^2 = 15 \cdot 81 \cdot 4x^2 = 4860x^2$.

3 Find the fifth term

The 5th term is $\binom{6}{4}3^2(2x)^4 = 15 \cdot 9 \cdot 16x^4 = 2160x^4$.

4 Use the geometric condition

For three consecutive terms in a geometric series, the middle term squared equals the product of the two outer terms.

5 **Set up the equation**

$(4860x^2)^2 = (2916x)(2160x^4)$.

6 **Solve for x**

For the non-zero geometric case, divide by x^4: $4860^2 = 2916 \cdot 2160\, x$, so $x = \dfrac{4860^2}{2916 \cdot 2160} = \dfrac{15}{4}$.

7 **Use the sum to infinity**

In the different geometric series, first term $= \sin^2\theta$, common ratio $= 2\cos\theta$, and $S_\infty = \dfrac{8}{5}$.

8 **Form the equation**

$\dfrac{\sin^2\theta}{1 - 2\cos\theta} = \dfrac{8}{5}$.

9 **Use sine squared**

$5\sin^2\theta = 8(1 - 2\cos\theta)$. With $\sin^2\theta = 1 - \cos^2\theta$, this becomes $5(1 - \cos^2\theta) = 8 - 16\cos\theta$.

10 **Show the required equation**

Rearranging gives $5\cos^2\theta - 16\cos\theta + 3 = 0$.

11 **Solve for cosine**

$5\cos^2\theta - 16\cos\theta + 3 = 0$ gives $\cos\theta = 3$ or $\cos\theta = \dfrac{1}{5}$. The possible value is $\cos\theta = \dfrac{1}{5}$.

12 **Find the second term**

The 2nd term is first term times common ratio: $\sin^2\theta(2\cos\theta) = \left(1 - \dfrac{1}{25}\right) \cdot \dfrac{2}{5} = \dfrac{24}{25} \cdot \dfrac{2}{5} = \dfrac{48}{125}$

.

✓ **FINAL ANSWER**

$(i)(a)$ $2916x$, $4860x^2$, $2160x^4$.

$(i)(b)$ $x = \dfrac{15}{4}$.

$(ii)(a)$ Proven.

$(ii)(b)$ $\dfrac{48}{125}$.

TOPIC

Arithmetic Sequences

Every past-paper question on this topic, October 2019 to January 2026.

WMA12/01 OCTOBER 2023

Question 8

7 marks | Arithmetic Sequences

8. In a large theatre there are n rows of seats, where n is a constant.

The number of seats in the first row is a, where a is a constant.

In each subsequent row there are 4 more seats than in the previous row so that

- in the 2nd row there are $(a + 4)$ seats
- in the 3rd row there are $(a + 8)$ seats
- the number of seats in each row form an **arithmetic** sequence

Given that the **total** number of seats in the first 10 rows is 360

(a) find the value of a.

(2)

Given also that the total number of seats in the n rows is 2146

(b) show that

$$n^2 + 8n - 1073 = 0$$

(2)

(c) Hence

(i) state the number of rows of seats in the theatre,

(ii) find the maximum number of seats in any one row.

(3)

WORKED SOLUTION

Question 8 · WMA12/01 October 2023

1 Use the arithmetic sum formula

The first term is a and the common difference is 4.

2 Use the first 10 rows

$S_{10} = \frac{10}{2}(2a + 9 \cdot 4) = 360$.

3 Find a

$5(2a + 36) = 360$, so $10a + 180 = 360$ and $a = 18$.

4 Write S n

$S_n = \frac{n}{2}(2(18) + (n-1)4) = \frac{n}{2}(4n + 32) = 2n^2 + 16n$.

5 **Use the total number of seats**

Since the total is 2146, $2n^2 + 16n = 2146$.

6 **Show the equation**

Dividing by 2 and rearranging gives $n^2 + 8n - 1073 = 0$.

7 **Solve for n**

$n = \dfrac{-8 \pm \sqrt{8^2 + 4(1073)}}{2} = \dfrac{-8 \pm 66}{2}$, so $n = 29$ or $n = -37$.

8 **Choose the row count**

The number of rows must be positive, so there are 29 rows.

9 **Find the maximum row size**

The largest row is the last row: $18 + (29 - 1)4 = 18 + 112 = 130$ seats.

✓ **FINAL ANSWER**

(a) $a = 18$.

(b) Proven.

$(c)(i)$ 29 rows.

$(c)(ii)$ 130 seats.

WMA12/01 MAY/JUNE R 2024

Question 8

13 marks | Arithmetic Sequences

8. (i) (a) In an **arithmetic** series the first term is a and the common difference is d.

Show that

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

(3)

(b) Hence find

$$900 + 892 + 884 + \ldots + 500$$

(3)

(ii) Given that the first three terms of a **geometric** series are

$$k+4 \qquad k-2 \qquad 11-k$$

where k is a constant,

(a) show that

$$2k^2 - 11k - 40 = 0$$

(3)

Given also that this series is convergent,

(b) find the value of S_∞

(4)

WORKED SOLUTION

Question 8 · WMA12/01 May/June R 2024

1 Write the first and last terms

The first term is a and the nth term is $a + (n-1)d$.

2 Use the arithmetic sum formula

$S_n = \frac{n}{2}(\text{first term} + \text{last term})$.

3 Show the formula

Therefore $S_n = \frac{n}{2}\{a + [a + (n-1)d]\} = \frac{n}{2}\{2a + (n-1)d\}$.

4 Identify the arithmetic series

For $900 + 892 + 884 + \cdots + 500$, the first term is 900, the common difference is -8, and the last term is 500.

5 **Find the number of terms**

$500 = 900 + (n-1)(-8)$, so $n - 1 = 50$ and $n = 51$.

6 **Find the sum**

$S = \frac{51}{2}(900 + 500) = 35700$.

7 **Use the geometric condition**

For $k + 4$, $k - 2$, $11 - k$ to be consecutive geometric terms, $(k-2)^2 = (k+4)(11-k)$.

8 **Show the quadratic**

$k^2 - 4k + 4 = -k^2 + 7k + 44$, so $2k^2 - 11k - 40 = 0$.

9 **Solve for k**

$2k^2 - 11k - 40 = 0$ gives $k = 8$ or $k = -\frac{5}{2}$.

10 **Use convergence**

If $k = 8$, the terms are $12, 6, 3$ and the common ratio is $\frac{1}{2}$, so the series is convergent. If $k = -\frac{5}{2}$, the common ratio is -3, so it is not convergent.

11 **Find S infinity**

Use $k = 8$: $S_\infty = \frac{12}{1 - 1/2} = 24$.

FINAL ANSWER

$(i)(a)$ Proven.

$(i)(b)$ 35700.

$(ii)(a)$ Proven.

$(ii)(b)$ $S_\infty = 24$.

WMA12/01 OCTOBER 2019

Question 9

12 marks | Arithmetic Sequences | Main topic: Trigonometric Equations

9. Solutions based entirely on graphical or numerical methods are not acceptable in this question.

(i) Solve, for $0 \leqslant \theta < 180°$, the equation

$$3\sin(2\theta - 10°) = 1$$

giving your answers to one decimal place.

(4)

(ii) The first three terms of an arithmetic sequence are

$$\sin\alpha, \quad \frac{1}{\tan\alpha} \quad \text{and} \quad 2\sin\alpha$$

where α is a constant.

(a) Show that $2\cos\alpha = 3\sin^2\alpha$

(3)

Given that $\pi < \alpha < 2\pi$,

(b) find, showing all working, the value of α to 3 decimal places.

(5)

WORKED SOLUTION

Question 9 · WMA12/01 October 2019

1 Solve the sine equation

Let $u = 2\theta - 10°$. Then $\sin u = \frac{1}{3}$.

2 Find the valid angle values

Since $0 \leq \theta < 180°$, $-10° \leq u < 350°$. In this interval, $u = 19.471\ldots°$ or $u = 160.529\ldots°$.

3 Return to theta

$\theta = \frac{u + 10°}{2}$, giving $\theta = 14.7°$ and $85.3°$ to one decimal place.

4 **Use the arithmetic sequence condition**

For three consecutive terms of an arithmetic sequence, second minus first equals third minus second. Hence $\dfrac{1}{\tan\alpha} - \sin\alpha = 2\sin\alpha - \dfrac{1}{\tan\alpha}$.

5 **Prove the required identity**

This gives $\dfrac{2}{\tan\alpha} = 3\sin\alpha$. Since $\dfrac{1}{\tan\alpha} = \dfrac{\cos\alpha}{\sin\alpha}$, we get $\dfrac{2\cos\alpha}{\sin\alpha} = 3\sin\alpha$, so $2\cos\alpha = 3\sin^2\alpha$.

6 **Form a quadratic in cos alpha**

Using $\sin^2\alpha = 1 - \cos^2\alpha$, $2\cos\alpha = 3(1-\cos^2\alpha)$, so $3\cos^2\alpha + 2\cos\alpha - 3 = 0$.

7 **Solve in the given interval**

$\cos\alpha = \dfrac{-1+\sqrt{10}}{3}$; the other root is less than -1. Since $\pi < \alpha < 2\pi$ and cosine is positive, α is in the fourth quadrant.

8 **State alpha**

$\alpha = 2\pi - \cos^{-1}\left(\dfrac{-1+\sqrt{10}}{3}\right) = 5.517$ radians to 3 decimal places.

✓ **FINAL ANSWER**

$(i)\ \theta = 14.7°,\ 85.3°.$ $\quad (ii)(a)$ Proven.

$(ii)(b)\ \alpha = 5.517$ radians.

WMA12/01 JANUARY 2020

Question 8

7 marks | Arithmetic Sequences | Main topic: Sequences & Series

8. (i) An arithmetic series has first term a and common difference d.

Prove that the sum to n terms of this series is

$$\frac{n}{2}\{2a + (n-1)d\}$$

(3)

(ii) A sequence $u_1, u_2, u_3, \ldots$ is given by

$$u_n = 5n + 3(-1)^n$$

Find the value of

(a) u_5

(1)

(b) $\sum_{n=1}^{59} u_n$

(3)

WORKED SOLUTION

Question 8 · WMA12/01 January 2020

1 Write the series forwards

$S_n = a + (a+d) + (a+2d) + \cdots + (a+(n-1)d)$.

2 Write the series backwards

$S_n = (a+(n-1)d) + (a+(n-2)d) + \cdots + a$.

3 Add the two forms

Each pair adds to $2a + (n-1)d$, and there are n pairs. Hence $2S_n = n\{2a + (n-1)d\}$.

4 Complete the proof

Dividing by 2 gives $S_n = \frac{n}{2}\{2a + (n-1)d\}$.

5 Find u5

$u_5 = 5(5) + 3(-1)^5 = 25 - 3 = 22$.

6 **Sum the sequence**

$\sum_{n=1}^{59} u_n = 5\sum_{n=1}^{59} n + 3\sum_{n=1}^{59}(-1)^n$.

7 **Calculate the total**

$5\sum_{n=1}^{59} n = 5 \cdot \dfrac{59 \cdot 60}{2} = 8850$, and $\sum_{n=1}^{59}(-1)^n = -1$. Therefore the total is $8850 - 3 = 8847$.

✓ **FINAL ANSWER**

(i) Proven.

$(ii)(a)$ 22. $(ii)(b)$ 8847.

WMA12/01 JANUARY 2021

Question 10

11 marks | Arithmetic Sequences | Main topic: Modelling with Sequences & Series

10. In this question you must show detailed reasoning.

Owen wants to train for 12 weeks in preparation for running a marathon.

During the 12-week period he will run every Sunday and every Wednesday.

- On Sunday in week 1 he will run 15 km
- On Sunday in week 12 he will run 37 km

He considers two different 12-week training plans.

In training plan A, he will increase the distance he runs each Sunday by the same amount.

(a) Calculate the distance he will run on Sunday in week 5 under training plan A.

(3)

In training plan B, he will increase the distance he runs each Sunday by the same percentage.

(b) Calculate the distance he will run on Sunday in week 5 under training plan B.
Give your answer in km to one decimal place.

(3)

Owen will also run a fixed distance, x km, each Wednesday over the 12-week period.

Given that

- x is an integer
- the total distance that Owen will run on Sundays and Wednesdays over the 12 weeks will not exceed 360 km

(c) (i) find the maximum value of x, if he uses training plan A,

(ii) find the maximum value of x, if he uses training plan B.

(5)

(Total 11 marks)

WORKED SOLUTION

Question 10 · WMA12/01 January 2021

1 **Find the arithmetic difference**

Plan A has 12 Sunday runs from 15 km to 37 km, so there are 11 equal increases. The common difference is $\frac{37-15}{11}=2$.

2 **Find week five for plan A**

Week 5 is the fifth term, so the distance is $15 + 4(2) = 23\ \text{km}$.

3 **Find the geometric ratio**

For plan B, $15r^{11} = 37$, so $r = \left(\frac{37}{15}\right)^{1/11}$.

4 **Find week five for plan B**

The fifth Sunday distance is $15r^4 = 15\left(\frac{37}{15}\right)^{4/11} = 20.829\ldots$, so it is $20.8\ \text{km}$.

5 **Sum the Sunday runs for plan A**

The arithmetic total is $\frac{12}{2}(15 + 37) = 312$ km.

6 **Find the Wednesday limit for plan A**

The total distance condition gives $312 + 12x \leq 360$, so $x \leq 4$. The maximum integer is 4.

7 **Sum the Sunday runs for plan B**

The geometric total is $15\frac{r^{12} - 1}{r - 1} = 294.185\ldots$ km.

8 **Find the Wednesday limit for plan B**

Now $294.185\ldots + 12x \leq 360$, so $x \leq 5.484\ldots$. The maximum integer is 5.

✓ **FINAL ANSWER**

$(a)\ 23\ \text{km}.$ $(b)\ 20.8\ \text{km}.$ $(c)(i)\ x = 4.$ $(c)(ii)\ x = 5.$

WMA12/01 JANUARY 2022

Question 8

9 marks | Arithmetic Sequences | Main topic: Modelling with Sequences & Series

8. A metal post is repeatedly hit in order to drive it into the ground.

Given that

- on the 1st hit, the post is driven 100 mm into the ground
- on the 2nd hit, the post is driven an **additional** 98 mm into the ground
- on the 3rd hit, the post is driven an **additional** 96 mm into the ground
- the **additional** distances the post travels on each subsequent hit form an arithmetic sequence

(a) show that the post is driven an **additional** 62 mm into the ground with the 20th hit.

(1)

(b) Find the **total distance** that the post has been driven into the ground after 20 hits.

(2)

Given that for each subsequent hit after the 20th hit

- the **additional** distances the post travels form a geometric sequence with common ratio r
- on the 22nd hit, the post is driven an **additional** 60 mm into the ground

(c) find the value of r, giving your answer to 3 decimal places.

(2)

After a total of N hits, the post will have been driven more than 3 m into the ground.

(d) Find, showing all steps in your working, the smallest possible value of N.

(4)

WORKED SOLUTION

Question 8 · WMA12/01 January 2022

1 **Use the arithmetic rule**

For the first 20 hits, the additional distances form an arithmetic sequence with first term 100 and common difference -2.

2 **Show the twentieth hit**

The twentieth additional distance is $100 + 19(-2) = 62$ mm.

3 **Sum the first twenty hits**

$S_{20} = \frac{20}{2}(100 + 62) = 1620$ mm.

4 **Use the geometric information**

After the twentieth hit, the distances follow the same ratio, so the twenty second distance is $62r^2$.

5 **Find r**

$62r^2 = 60$, hence $r = \sqrt{\dfrac{30}{31}} = 0.984$ to 3 decimal places.

6 **Set up the remaining distance**

More than 3 m means more than 3000 mm. After 20 hits, the remaining amount to exceed is $3000 - 1620 = 1380$ mm.

7 **Sum the geometric tail**

If there are m further hits, their total is $62r + 62r^2 + \cdots + 62r^m = 62r\dfrac{1 - r^m}{1 - r}$.

8 **Find the first m that works**

Using $r = \sqrt{30/31}$, $62r\dfrac{1 - r^{27}}{1 - r}$ is not enough, but $62r\dfrac{1 - r^{28}}{1 - r}$ makes the total exceed 3000 mm.

9 **State N**

Therefore $m = 28$ further hits are needed after the first 20, so $N = 20 + 28 = 48$.

✓ **FINAL ANSWER**

(a) Proven: 62 mm.

(b) 1620 mm.

(c) $r = 0.984$. (d) $N = 48$.

WMA12/01 JANUARY 2024

Question 7

9 marks | Arithmetic Sequences | Main topic: Modelling with Sequences & Series

7. Wheat is grown on a farm.

- In year 1, the farm produced 300 tonnes of wheat.
- In year 12, the farm is predicted to produce 4 000 tonnes of wheat.

Model A assumes that the amount of wheat produced on the farm will increase by the same amount each year.

(a) Using model A, find the amount of wheat produced on the farm in year 4.
Give your answer to the nearest 10 tonnes.

(3)

Model B assumes that the amount of wheat produced on the farm will increase by the same percentage each year.

(b) Using model B, find the amount of wheat produced on the farm in year 2.
Give your answer to the nearest 10 tonnes.

(3)

(c) Calculate, according to the two models, the difference between the total amounts of wheat predicted to be produced on the farm from year 1 to year 12 inclusive.
Give your answer to the nearest 10 tonnes.

(3)

WORKED SOLUTION

Question 7 · WMA12/01 January 2024

1 **Model A common difference**

Model A is arithmetic. From year 1 to year 12 there are 11 equal increases, so $d = \dfrac{4000 - 300}{11} = \dfrac{3700}{11}$.

2 **Find year 4 in model A**

Year 4 is $300 + 3d = 300 + 3\left(\dfrac{3700}{11}\right) = 1309.09\ldots$, so the amount is 1310 tonnes to the nearest 10 tonnes.

3 **Model B common ratio**

Model B is geometric. If the ratio is r, then $300r^{11} = 4000$.

4 **Find the ratio**

$r = \left(\frac{4000}{300}\right)^{1/11} = 1.2655\ldots$.

5 **Find year 2 in model B**

Year 2 is $300r = 379.65\ldots$, so the amount is 380 tonnes to the nearest 10 tonnes.

6 **Total for model A**

$S_A = \frac{12}{2}(300 + 4000) = 25800$ tonnes.

7 **Total for model B**

$S_B = 300\frac{r^{12} - 1}{r - 1} = 17935.20\ldots$ tonnes.

8 **Find the difference**

The difference is $25800 - 17935.20\ldots = 7864.79\ldots$, which is 7860 tonnes to the nearest 10 tonnes.

✓ **FINAL ANSWER**

(a) 1310 tonnes.

(b) 380 tonnes.

(c) 7860 tonnes.

WMA12/01 OCTOBER 2024

Question 7

6 marks | Arithmetic Sequences | Main topic: Modelling with Sequences & Series

7. Jem pays money into a savings scheme, A, over a period of 300 months.

Jem pays £20 into scheme A in month 1, £20.50 in month 2, £21 in month 3 and so on, so that the amounts Jem pays each month form an arithmetic sequence.

(a) Show that Jem pays £69.50 into scheme A in month 100

(1)

(b) Find the **total** amount that Jem pays into scheme A over the period of 300 months.

(2)

Kim pays money into a different savings scheme, B, over the same period of 300 months.

In a model, the amounts Kim pays into scheme B increase by the same percentage each month, so that the amounts Kim pays each month form a geometric sequence.

Given that Kim pays

- £20 into scheme B in month 1
- £250 into scheme B in month 300

(c) use the model to calculate, to the nearest £10, the difference between the total amount paid into scheme A and the total amount paid into scheme B over the period of 300 months.

(3)

WORKED SOLUTION

Question 7 · WMA12/01 October 2024

1 Identify Jem's sequence

Jem's payments are arithmetic with first term 20 and common difference 0.50.

2 Show month 100

The month 100 payment is $20 + 99(0.50) = 69.50$, so Jem pays £69.50.

3 Sum Jem's payments

$S_{300} = \frac{300}{2}\{2(20) + 299(0.50)\} = 150(189.5) = 28425$.

4 Model Kim's payments

Let the common ratio be r. Since the first payment is 20 and the month 300 payment is 250, $20r^{299} = 250$.

5 **Find the ratio**

Thus $r^{299} = 12.5$ and $r = 12.5^{1/299}$.

6 **Sum Kim's payments**

$S_B = 20\dfrac{r^{300} - 1}{r - 1}$ with $r = 12.5^{1/299}$, giving $S_B \approx 27362.95$.

7 **Find the difference**

The difference is $28425 - 27362.95 \approx 1062.05$, which is £1,060 to the nearest £10.

✓ **FINAL ANSWER**

(a) Proven.

(b) £28,425.

(c) £1,060.

WMA12/01 MAY/JUNE 2025

Question 7

9 marks | Arithmetic Sequences | Main topic: Sequences & Series

7: **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(i) A **geometric** series begins

$$10 + 8 + 6.4 + \ldots$$

(a) Find the sum to infinity of this series.

(2)

Given that the kth term of this series is less than 0.0005

(b) use algebra to find the smallest possible value of k.

(3)

(ii) An **arithmetic** series begins

$$850 + 843 + 836 + \ldots$$

Given that the sum of the first n terms of this series is S_n
find the greatest possible value of S_n

(4)

WORKED SOLUTION

Question 7 · WMA12/01 May/June 2025

1 **Identify the geometric sequence**

The first term is 10 and the common ratio is $\frac{8}{10} = 0.8$.

2 **Find the sum to infinity**

$S_\infty = \frac{10}{1 - 0.8} = 50$.

3 **Write the kth term**

The kth term is $10(0.8)^{k-1}$.

4 **Use the inequality**

$10(0.8)^{k-1} < 0.0005$, so $(0.8)^{k-1} < 0.00005$.

5 **Solve with logarithms**

Taking logarithms gives $(k-1)\log(0.8) < \log(0.00005)$. Since $\log(0.8) < 0$,

$k - 1 > \dfrac{\log(0.00005)}{\log(0.8)} = 44.381\ldots.$

6 **Find the smallest integer**

Thus $k > 45.381\ldots$, so the smallest possible value is $k = 46$.

7 **Write the arithmetic sum**

For $850 + 843 + 836 + \cdots$, the first term is 850 and the common difference is -7.

8 **Form S n**

$S_n = \dfrac{n}{2}\{2(850) + (n-1)(-7)\} = \dfrac{n}{2}(1707 - 7n).$

9 **Find the best integer n**

This quadratic has its maximum near $n = \dfrac{1707}{14} = 121.928\ldots$, so check $n = 122$.

10 **Find the greatest sum**

$S_{122} = \dfrac{122}{2}(1707 - 854) = 61(853) = 52033.$

✓ **FINAL ANSWER**

$(i)(a)\ 50.$

$(i)(b)\ 46.$

$(ii)\ 52033.$

WMA12/01 JANUARY 2026

Question 9

10 marks | Arithmetic Sequences | Main topic: Modelling with Sequences & Series

9. A company mines lithium.

- in year 1, the company mined 12 400 tonnes of lithium
- in year 15, the company is predicted to mine 7 500 tonnes of lithium

Model A assumes that the mass of lithium the company mines will decrease by the same amount each year.

According to model A,

(a) find the mass of lithium the company will mine in year 5

(3)

(b) calculate the **total** mass of lithium the company will mine from year 1 to year 15 inclusive.

(2)

Model B assumes that the mass of lithium the company mines will decrease by the same percentage each year.

(c) Find, according to model B, the mass of lithium the company will mine in year 10. Give the answer to the nearest 10 tonnes.

(3)

According to model B, there is a limit to the **total** mass of lithium that can be mined.

(d) Calculate the value of this limit. Give the answer to the nearest 100 tonnes.

(2)

WORKED SOLUTION

Question 9 · WMA12/01 January 2026

1 **Model A as an arithmetic sequence**

For Model A, let the common difference be d. Since year 15 is the 15th term, $7500 = 12400 + 14d$.

2 **Find year 5 for Model A**

$d = \dfrac{7500 - 12400}{14} = -350$. Year 5 is $12400 + 4(-350) = 11000$ tonnes.

3 **Find the total from year 1 to year 15**

$S_{15} = \dfrac{15}{2}(12400 + 7500) = 149250$ tonnes.

4 Model B as a geometric sequence

Let the common ratio be r. Then $7500 = 12400r^{14}$, so $r = \left(\frac{7500}{12400}\right)^{1/14} \approx 0.964723$.

5 Find year 10 for Model B

Year 10 is $12400r^{9} \approx 8975.26$, which is 8980 tonnes to the nearest 10 tonnes.

6 Find the limiting total

Since $0 < r < 1$, the sum to infinity is $S_{\infty} = \frac{12400}{1 - r} \approx 351508.1$, so the limit is 351500 tonnes to the nearest 100 tonnes.

FINAL ANSWER

(a) 11000 tonnes.

(b) 149250 tonnes.

(c) 8980 tonnes.

(d) 351500 tonnes.

TOPIC

Geometric Sequences

Every past-paper question on this topic, October 2019 to January 2026.

WMA12/01 OCTOBER 2021

Question 7

10 marks | Geometric Sequences

7. (i) A geometric sequence has first term 4 and common ratio 6
Given that the n^{th} term is greater than 10^{100}, find the minimum possible value of n.

(3)

(ii) A different geometric sequence has first term a and common ratio r.

Given that

- the second term of the sequence is -6
- the sum to infinity of the series is 25

(a) show that

$$25r^2 - 25r - 6 = 0$$

(3)

(b) Write down the solutions of

$$25r^2 - 25r - 6 = 0$$

(1)

Hence,

(c) state the value of r, giving a reason for your answer,

(1)

(d) find the sum of the first 4 terms of the series.

(2)

WORKED SOLUTION

Question 7 · WMA12/01 October 2021

1 **Write the nth term**

The nth term of the first sequence is $4 \cdot 6^{n-1}$.

2 **Use logarithms**

We need $4 \cdot 6^{n-1} > 10^{100}$, so $n - 1 > \dfrac{100 - \log_{10} 4}{\log_{10} 6} = 127.736\ldots$.

3 **Choose the smallest integer**

Thus n must be greater than $128.736\ldots$, so the minimum possible value is 129.

4 **Use the second term**

For the second sequence, $ar = -6$, so $a = -\dfrac{6}{r}$.

5 **Use the sum to infinity**

$\dfrac{a}{1-r} = 25$. Substituting $a = -\dfrac{6}{r}$ gives $-6 = 25r(1-r)$.

6 **Show the quadratic**

Rearranging gives $25r^2 - 25r - 6 = 0$.

7 **Solve for r**

$25r^2 - 25r - 6 = 0$ gives $r = \dfrac{6}{5}$ or $r = -\dfrac{1}{5}$.

8 **Choose the valid ratio**

A sum to infinity exists only when $|r| < 1$, so $r = -\dfrac{1}{5}$.

9 **Find the first term**

Since $ar = -6$, $a = 30$.

10 **Sum four terms**

$S_4 = 30\dfrac{1-(-1/5)^4}{1-(-1/5)} = \dfrac{624}{25}$.

✓ **FINAL ANSWER**

$(i)\ n = 129.$ $(ii)(a)$ Proven.

$(ii)(b)\ r = \dfrac{6}{5}$ or $-\dfrac{1}{5}.$ $(ii)(c)\ r = -\dfrac{1}{5}.$ $(ii)(d)\ \dfrac{624}{25}.$

WMA12/01 MAY/JUNE 2022

Question 6

8 marks | Geometric Sequences

6. In a geometric sequence $u_1, u_2, u_3, \ldots$

- the common ratio is r
- $u_2 + u_3 = 6$
- $u_4 = 8$

(a) Show that r satisfies

$$3r^2 - 4r - 4 = 0$$

(3)

Given that the geometric sequence has a sum to infinity,

(b) find u_1

(3)

(c) find S_∞

(2)

WORKED SOLUTION

Question 6 · WMA12/01 May/June 2022

1 Write the terms

$u_2 = u_1 r$, $u_3 = u_1 r^2$, and $u_4 = u_1 r^3$.

2 Use u four

Since $u_4 = 8$, $u_1 r^3 = 8$, so $u_1 r = \dfrac{8}{r^2}$.

3 Use u two plus u three

$u_2 + u_3 = u_1 r(1 + r) = 6$.

4 Show the quadratic

Substitute $u_1 r = \dfrac{8}{r^2}$ to get $\dfrac{8}{r^2}(1 + r) = 6$. Hence $8 + 8r = 6r^2$, so $3r^2 - 4r - 4 = 0$.

5 Solve for r

$3r^2 - 4r - 4 = 0$ gives $r = 2$ or $r = -\dfrac{2}{3}$.

6 **Choose the sum-to-infinity ratio**

A sum to infinity requires $|r| < 1$, so $r = -\frac{2}{3}$.

7 **Find u one**

$u_1 r^3 = 8$, so $u_1\left(-\frac{2}{3}\right)^3 = 8$ and $u_1 = -27$.

8 **Find the infinite sum**

$$S_\infty = \frac{u_1}{1-r} = \frac{-27}{1+2/3} = -\frac{81}{5}.$$

FINAL ANSWER

(a) Proven: $3r^2 - 4r - 4 = 0$.

(b) $u_1 = -27$. (c) $S_\infty = -\frac{81}{5}$.

WMA12/01 OCTOBER 2022

Question 8

7 marks | Geometric Sequences

8. A geometric sequence has first term a and common ratio r

Given that $S_\infty = 3a$

(a) show that $r = \frac{2}{3}$

(2)

Given also that

$$u_2 - u_4 = 16$$

where u_k is the k^{th} term of this sequence,

(b) find the value of S_{10} giving your answer to one decimal place.

(5)

WORKED SOLUTION

Question 8 · WMA12/01 October 2022

1 Use the sum to infinity formula

For a geometric sequence, $S_\infty = \frac{a}{1-r}$. The question gives $S_\infty = 3a$.

2 Find the common ratio

$\frac{a}{1-r} = 3a$. Since $u_2 - u_4 = 16$, $a \neq 0$, so $\frac{1}{1-r} = 3$ and $r = \frac{2}{3}$.

3 Use u2 minus u4

$u_2 - u_4 = ar - ar^3 = ar(1 - r^2) = 16$.

4 Substitute r

$a\left(\frac{2}{3}\right)\left(1 - \frac{4}{9}\right) = 16$.

5 Solve for a

$a \cdot \frac{2}{3} \cdot \frac{5}{9} = 16$, so $\frac{10a}{27} = 16$ and $a = \frac{216}{5} = 43.2$.

6 Find S10

$S_{10} = \frac{a(1 - r^{10})}{1 - r} = 3a\left(1 - \left(\frac{2}{3}\right)^{10}\right)$.

7 **Evaluate**

$S_{10} = 3\left(\frac{216}{5}\right)\left(1-\left(\frac{2}{3}\right)^{10}\right) = \frac{92840}{729} = 127.352\ldots$, so $S_{10} = 127$ to 3 significant figures.

FINAL ANSWER

(a) $a = 43.2$.

(b) $S_{10} = 127.353\ldots$, so 127 to 3 s.f.

WMA12/01 JANUARY 2023

Question 7

7 marks | Geometric Sequences

7. A geometric sequence has first term a and common ratio r, where $r > 0$

Given that

- the 3rd term is 20
- the 5th term is 12.8

(a) show that $r = 0.8$

(1)

(b) Hence find the value of a.

(2)

Given that the sum of the first n terms of this sequence is greater than 156

(c) find the smallest possible value of n.

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(4)

WORKED SOLUTION

Question 7 · WMA12/01 January 2023

1 Write the given terms

For a geometric sequence, the third term is $ar^2 = 20$ and the fifth term is $ar^4 = 12.8$.

2 Divide the equations

$\frac{ar^4}{ar^2} = \frac{12.8}{20}$, so $r^2 = 0.64$.

3 Use r positive

Since $r > 0$, $r = 0.8$.

4 Find a

Using $ar^2 = 20$, $a(0.8)^2 = 20$, so $a = \frac{20}{0.64} = 31.25$.

5 Write the sum formula

$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{31.25(1 - 0.8^n)}{0.2} = 156.25(1 - 0.8^n)$.

6 **Set up the inequality**

We need $156.25(1 - 0.8^n) > 156$.

7 **Isolate the power**

$1 - 0.8^n > \dfrac{156}{156.25} = \dfrac{624}{625}$, so $0.8^n < \dfrac{1}{625}$.

8 **Use logarithms**

$n \log(0.8) < \log\left(\dfrac{1}{625}\right)$. Since $\log(0.8) < 0$, dividing reverses the inequality:

$n > \dfrac{\log(1/625)}{\log(0.8)} = 28.85\ldots$.

9 **Choose the smallest integer**

The smallest possible integer value is therefore $n = 29$.

✓ **FINAL ANSWER**

(a) $r = 0.8$.

(b) $a = 31.25$.

(c) $n = 29$.

WMA12/01 MAY/JUNE 2023

Question 6

9 marks | Geometric Sequences

6. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

A software developer released an app to download.

The numbers of downloads of the app each month, in thousands, for the first three months after the app was released were

$$2k - 15 \qquad k \qquad k + 4$$

where k is a constant.

Given that the numbers of downloads each month are modelled as a geometric series,

(a) show that $k^2 - 7k - 60 = 0$

(2)

(b) predict the number of downloads in the 4th month.

(4)

The **total** number of all downloads of the app is predicted to exceed 3 million for the first time in the Nth month.

(c) Calculate the value of N according to the model.

(3)

WORKED SOLUTION

Question 6 · WMA12/01 May/June 2023

1 **Use the geometric condition**

For three consecutive terms of a geometric sequence, the middle term squared equals the product of the outer terms.

2 **Set up the equation**

So $k^2 = (2k - 15)(k + 4)$.

3 **Show the required quadratic**

Expanding gives $k^2 = 2k^2 - 7k - 60$, hence $k^2 - 7k - 60 = 0$.

4 **Find k**

$k^2 - 7k - 60 = (k - 12)(k + 5) = 0$, so $k = 12$ or $k = -5$. The download numbers must be positive, so $k = 12$.

5 **Find the common ratio**

The first three monthly values are 9, 12 and 16 thousand. The common ratio is $\frac{12}{9} = \frac{4}{3}$.

6 **Predict month four**

The 4th month is $16 \cdot \frac{4}{3} = \frac{64}{3}$ thousand downloads, which is about 21300 downloads.

7 **Write the cumulative total**

In thousands, the total after N months is $S_N = \frac{9\left(\left(\frac{4}{3}\right)^N - 1\right)}{\frac{4}{3} - 1} = 27\left(\left(\frac{4}{3}\right)^N - 1\right)$.

8 **Compare with three million**

Three million downloads is 3000 thousand, so we need $27\left(\left(\frac{4}{3}\right)^N - 1\right) > 3000$.

9 **Solve the inequality**

$\left(\frac{4}{3}\right)^N > \frac{1009}{9}$. Taking logarithms gives $N > \frac{\log(1009/9)}{\log(4/3)} = 16.405\ldots$.

10 **Choose the first month**

The first integer month for which the total exceeds 3 million is $N = 17$.

✓ **FINAL ANSWER**

(a) Proven.

(b) $\frac{64}{3}$ thousand downloads, about 21300 downloads.

(c) $N = 17$.

WMA12/01 OCTOBER 2025

Question 8

8 marks | Geometric Sequences

8. (i) A geometric series has first term a and common ratio r.

Prove that the sum of the first n terms of this series S_n is given by

$$S_n = \frac{a(1-r^n)}{1-r}$$

(3)

(ii) A liquid is to be stored in a barrel.

Due to evaporation, the volume of the liquid in the barrel at the end of each year is 8% less than the volume of the liquid in the barrel at the start of the year.

At the start of the first year, the barrel is filled with 150 litres of the liquid.

(a) Show that the amount of the liquid in the barrel at the end of 6 years is approximately 91 litres.

(2)

At the start of each year a new barrel is filled with 150 litres of the liquid so that, at the end of 40 years, there are 40 barrels containing the liquid.

(b) Calculate the total amount of the liquid, to the nearest litre, in the 40 barrels at the end of 40 years.

(3)

WORKED SOLUTION

Question 8 · WMA12/01 October 2025

1 Write S n

$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$.

2 Multiply by r

$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n$.

3 Subtract

$S_n - rS_n = a - ar^n = a(1 - r^n)$.

4 Rearrange

$(1 - r)S_n = a(1 - r^n)$, so $S_n = \dfrac{a(1 - r^n)}{1 - r}$.

5 **Use the yearly multiplier**

An 8% decrease each year means the multiplier is 0.92.

6 **Show the amount after six years**

After 6 years, the amount is $150(0.92)^6 = 90.953\ldots$, which is approximately 91 litres.

7 **Set up the forty barrel total**

At the end of 40 years, the amounts are $150(0.92), 150(0.92)^2, \ldots, 150(0.92)^{40}$.

8 **Sum the geometric sequence**

The total is $150(0.92)\dfrac{1-0.92^{40}}{1-0.92} = 1663.581\ldots$.

9 **Round the total**

To the nearest litre, the total amount is 1664 litres.

✓ **FINAL ANSWER**

(i) Proven.

$(ii)(a)$ Approximately 91 litres.

$(ii)(b)$ 1664 litres.

WMA12/01 JANUARY 2021

Question 10

11 marks | Geometric Sequences | Main topic: Modelling with Sequences & Series

10. In this question you must show detailed reasoning.

Owen wants to train for 12 weeks in preparation for running a marathon.

During the 12-week period he will run every Sunday and every Wednesday.

- On Sunday in week 1 he will run 15 km
- On Sunday in week 12 he will run 37 km

He considers two different 12-week training plans.

In training plan A, he will increase the distance he runs each Sunday by the same amount.

(a) Calculate the distance he will run on Sunday in week 5 under training plan A.

(3)

In training plan B, he will increase the distance he runs each Sunday by the same percentage.

(b) Calculate the distance he will run on Sunday in week 5 under training plan B.
Give your answer in km to one decimal place.

(3)

Owen will also run a fixed distance, x km, each Wednesday over the 12-week period.

Given that

- x is an integer
- the total distance that Owen will run on Sundays and Wednesdays over the 12 weeks will not exceed 360 km

(c) (i) find the maximum value of x, if he uses training plan A,

(ii) find the maximum value of x, if he uses training plan B.

(5)

(Total 11 marks)

WORKED SOLUTION

Question 10 · WMA12/01 January 2021

1 **Find the arithmetic difference**

Plan A has 12 Sunday runs from 15 km to 37 km, so there are 11 equal increases. The common difference is $\frac{37-15}{11}=2$.

2 **Find week five for plan A**

Week 5 is the fifth term, so the distance is $15 + 4(2) = 23\ \text{km}$.

3 **Find the geometric ratio**

For plan B, $15r^{11} = 37$, so $r = \left(\dfrac{37}{15}\right)^{1/11}$.

4 **Find week five for plan B**

The fifth Sunday distance is $15r^4 = 15\left(\dfrac{37}{15}\right)^{4/11} = 20.829\ldots$, so it is $20.8\ \text{km}$.

5 **Sum the Sunday runs for plan A**

The arithmetic total is $\dfrac{12}{2}(15 + 37) = 312$ km.

6 **Find the Wednesday limit for plan A**

The total distance condition gives $312 + 12x \leq 360$, so $x \leq 4$. The maximum integer is 4.

7 **Sum the Sunday runs for plan B**

The geometric total is $15\dfrac{r^{12} - 1}{r - 1} = 294.185\ldots$ km.

8 **Find the Wednesday limit for plan B**

Now $294.185\ldots + 12x \leq 360$, so $x \leq 5.484\ldots$. The maximum integer is 5.

✓ **FINAL ANSWER**

$(a)\ 23\ \text{km}.$ $(b)\ 20.8\ \text{km}.$ $(c)(i)\ x = 4.$ $(c)(ii)\ x = 5.$

WMA12/01 JANUARY 2022

Question 8

9 marks | Geometric Sequences | Main topic: Modelling with Sequences & Series

8. A metal post is repeatedly hit in order to drive it into the ground.

Given that

- on the 1st hit, the post is driven 100 mm into the ground
- on the 2nd hit, the post is driven an **additional** 98 mm into the ground
- on the 3rd hit, the post is driven an **additional** 96 mm into the ground
- the **additional** distances the post travels on each subsequent hit form an arithmetic sequence

(a) show that the post is driven an **additional** 62 mm into the ground with the 20th hit.

(1)

(b) Find the **total distance** that the post has been driven into the ground after 20 hits.

(2)

Given that for each subsequent hit after the 20th hit

- the **additional** distances the post travels form a geometric sequence with common ratio r
- on the 22nd hit, the post is driven an **additional** 60 mm into the ground

(c) find the value of r, giving your answer to 3 decimal places.

(2)

After a total of N hits, the post will have been driven more than 3 m into the ground.

(d) Find, showing all steps in your working, the smallest possible value of N.

(4)

WORKED SOLUTION

Question 8 · WMA12/01 January 2022

1 **Use the arithmetic rule**

For the first 20 hits, the additional distances form an arithmetic sequence with first term 100 and common difference -2.

2 **Show the twentieth hit**

The twentieth additional distance is $100 + 19(-2) = 62$ mm.

3 **Sum the first twenty hits**

$S_{20} = \frac{20}{2}(100 + 62) = 1620$ mm.

4 **Use the geometric information**

After the twentieth hit, the distances follow the same ratio, so the twenty second distance is $62r^2$.

5 **Find r**

$62r^2 = 60$, hence $r = \sqrt{\dfrac{30}{31}} = 0.984$ to 3 decimal places.

6 **Set up the remaining distance**

More than 3 m means more than 3000 mm. After 20 hits, the remaining amount to exceed is $3000 - 1620 = 1380$ mm.

7 **Sum the geometric tail**

If there are m further hits, their total is $62r + 62r^2 + \cdots + 62r^m = 62r\dfrac{1 - r^m}{1 - r}$.

8 **Find the first m that works**

Using $r = \sqrt{30/31}$, $62r\dfrac{1 - r^{27}}{1 - r}$ is not enough, but $62r\dfrac{1 - r^{28}}{1 - r}$ makes the total exceed 3000 mm.

9 **State N**

Therefore $m = 28$ further hits are needed after the first 20, so $N = 20 + 28 = 48$.

✓ **FINAL ANSWER**

(a) Proven: 62 mm.

(b) 1620 mm.

(c) $r = 0.984$. (d) $N = 48$.

WMA12/01 OCTOBER 2023

Question 10

12 marks | Geometric Sequences | Main topic: Binomial Expansion

10. (i) (a) Find, in ascending powers of x, the 2nd, 3rd and 5th terms of the binomial expansion of

$$(3 + 2x)^6$$

(3)

For a particular value of x, these three terms form consecutive terms in a geometric series.

(b) Find this value of x.

(3)

(ii) In a **different** geometric series,

- the first term is $\sin^2\theta$
- the common ratio is $2\cos\theta$
- the sum to infinity is $\frac{8}{5}$

(a) Show that

$$5\cos^2\theta - 16\cos\theta + 3 = 0$$

(3)

(b) Hence find the exact value of the 2nd term in the series.

(3)

WORKED SOLUTION

Question 10 · WMA12/01 October 2023

1 Find the second term

The 2nd term of $(3+2x)^6$ is $\binom{6}{1}3^5(2x) = 6 \cdot 243 \cdot 2x = 2916x$.

2 Find the third term

The 3rd term is $\binom{6}{2}3^4(2x)^2 = 15 \cdot 81 \cdot 4x^2 = 4860x^2$.

3 Find the fifth term

The 5th term is $\binom{6}{4}3^2(2x)^4 = 15 \cdot 9 \cdot 16x^4 = 2160x^4$.

4 Use the geometric condition

For three consecutive terms in a geometric series, the middle term squared equals the product of the two outer terms.

5 **Set up the equation**

$(4860x^2)^2 = (2916x)(2160x^4)$.

6 **Solve for x**

For the non-zero geometric case, divide by x^4: $4860^2 = 2916 \cdot 2160\,x$, so $x = \dfrac{4860^2}{2916 \cdot 2160} = \dfrac{15}{4}$.

7 **Use the sum to infinity**

In the different geometric series, first term $= \sin^2\theta$, common ratio $= 2\cos\theta$, and $S_\infty = \dfrac{8}{5}$.

8 **Form the equation**

$\dfrac{\sin^2\theta}{1 - 2\cos\theta} = \dfrac{8}{5}$.

9 **Use sine squared**

$5\sin^2\theta = 8(1 - 2\cos\theta)$. With $\sin^2\theta = 1 - \cos^2\theta$, this becomes $5(1 - \cos^2\theta) = 8 - 16\cos\theta$.

10 **Show the required equation**

Rearranging gives $5\cos^2\theta - 16\cos\theta + 3 = 0$.

11 **Solve for cosine**

$5\cos^2\theta - 16\cos\theta + 3 = 0$ gives $\cos\theta = 3$ or $\cos\theta = \dfrac{1}{5}$. The possible value is $\cos\theta = \dfrac{1}{5}$.

12 **Find the second term**

The 2nd term is first term times common ratio: $\sin^2\theta(2\cos\theta) = \left(1 - \dfrac{1}{25}\right) \cdot \dfrac{2}{5} = \dfrac{24}{25} \cdot \dfrac{2}{5} = \dfrac{48}{125}$

.

✓ **FINAL ANSWER**

$(i)(a)$ $2916x$, $4860x^2$, $2160x^4$.

$(i)(b)$ $x = \dfrac{15}{4}$.

$(ii)(a)$ Proven.

$(ii)(b)$ $\dfrac{48}{125}$.

WMA12/01 JANUARY 2024

Question 7

9 marks | Geometric Sequences | Main topic: Modelling with Sequences & Series

7. Wheat is grown on a farm.

- In year 1, the farm produced 300 tonnes of wheat.
- In year 12, the farm is predicted to produce 4 000 tonnes of wheat.

Model A assumes that the amount of wheat produced on the farm will increase by the same amount each year.

(a) Using model A, find the amount of wheat produced on the farm in year 4.
Give your answer to the nearest 10 tonnes.

(3)

Model B assumes that the amount of wheat produced on the farm will increase by the same percentage each year.

(b) Using model B, find the amount of wheat produced on the farm in year 2.
Give your answer to the nearest 10 tonnes.

(3)

(c) Calculate, according to the two models, the difference between the total amounts of wheat predicted to be produced on the farm from year 1 to year 12 inclusive.
Give your answer to the nearest 10 tonnes.

(3)

WORKED SOLUTION

Question 7 · WMA12/01 January 2024

1 **Model A common difference**

Model A is arithmetic. From year 1 to year 12 there are 11 equal increases, so $d = \dfrac{4000 - 300}{11} = \dfrac{3700}{11}$.

2 **Find year 4 in model A**

Year 4 is $300 + 3d = 300 + 3\left(\dfrac{3700}{11}\right) = 1309.09\ldots$, so the amount is 1310 tonnes to the nearest 10 tonnes.

3 **Model B common ratio**

Model B is geometric. If the ratio is r, then $300r^{11} = 4000$.

4 **Find the ratio**

$r = \left(\frac{4000}{300}\right)^{1/11} = 1.2655\ldots$.

5 **Find year 2 in model B**

Year 2 is $300r = 379.65\ldots$, so the amount is 380 tonnes to the nearest 10 tonnes.

6 **Total for model A**

$S_A = \frac{12}{2}(300 + 4000) = 25800$ tonnes.

7 **Total for model B**

$S_B = 300\frac{r^{12} - 1}{r - 1} = 17935.20\ldots$ tonnes.

8 **Find the difference**

The difference is $25800 - 17935.20\ldots = 7864.79\ldots$, which is 7860 tonnes to the nearest 10 tonnes.

✓ **FINAL ANSWER**

(a) 1310 tonnes.

(b) 380 tonnes.

(c) 7860 tonnes.

WMA12/01 MAY/JUNE 2024

Question 10

8 marks | Geometric Sequences | Main topic: Modelling with Sequences & Series

10. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The number of dormice and the number of voles on an island are being monitored.

Initially there are 2000 dormice on the island.

A model predicts that the number of dormice will increase by 3% each year, so that the numbers of dormice on the island at the end of each year form a geometric sequence.

(a) Find, according to the model, the number of dormice on the island 6 years after monitoring began. Give your answer to 3 significant figures.

(2)

The number of voles on the island is being monitored over the same period of time.

Given that
- 4 years after monitoring began there were 3690 voles on the island
- 7 years after monitoring began there were 3470 voles on the island
- the number of voles on the island at the end of each year is modelled as a geometric sequence

(b) find the equation of this model in the form

$$N = ab^t$$

where N is the number of voles, t years after monitoring began and a and b are constants. Give the value of a and the value of b to 2 significant figures.

(3)

When $t = T$, the number of dormice on the island is equal to the number of voles on the island.

(c) Find, according to the models, the value of T, giving your answer to one decimal place.

(3)

WORKED SOLUTION

Question 10 · WMA12/01 May/June 2024

1 **Model the dormice**

The dormice model is $D = 2000(1.03)^t$.

2 **Find the dormice after six years**

$D = 2000(1.03)^6 = 2388.10\ldots$, so there are $\mathbf{2390}$ dormice to 3 significant figures.

3 **Set up the vole model**

Let the vole model be $N = ab^t$.

4 **Use the two data points**

At $t = 4$, $ab^4 = 3690$. At $t = 7$, $ab^7 = 3470$.

5 **Find b**

Dividing gives $b^3 = \dfrac{3470}{3690}$, so $b = 0.9797\ldots$, which is 0.98 to 2 significant figures.

6 **Find a**

$a = \dfrac{3690}{b^4} = 4005.18\ldots$, which is 4000 to 2 significant figures.

7 **Write the vole model**

So the model is approximately $N = 4000(0.98)^t$.

8 **Set the two models equal**

Use $2000(1.03)^T = ab^T$ with the unrounded constants from the vole model.

9 **Solve using logarithms**

$\left(\dfrac{1.03}{b}\right)^T = \dfrac{a}{2000}$, so $T = \dfrac{\log(a/2000)}{\log(1.03/b)}$.

10 **Calculate T**

Substituting $a = 4005.18\ldots$ and $b = 0.9797\ldots$ gives $T = 13.875\ldots$, so $T = 13.9$ to one decimal place.

✓ **FINAL ANSWER**

(a) 2390 dormice.

(b) $N = 4000(0.98)^t$ to 2 s.f. constants.

(c) $T = 13.9$.

WMA12/01 MAY/JUNE R 2024

Question 8

13 marks | Geometric Sequences | Main topic: Arithmetic Sequences

8. (i) (a) In an **arithmetic** series the first term is a and the common difference is d.

Show that

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

(3)

(b) Hence find

$$900 + 892 + 884 + \ldots + 500$$

(3)

(ii) Given that the first three terms of a **geometric** series are

$$k+4 \qquad k-2 \qquad 11-k$$

where k is a constant,

(a) show that

$$2k^2 - 11k - 40 = 0$$

(3)

Given also that this series is convergent,

(b) find the value of S_∞

(4)

WORKED SOLUTION

Question 8 · WMA12/01 May/June R 2024

1 **Write the first and last terms**

The first term is a and the nth term is $a + (n-1)d$.

2 **Use the arithmetic sum formula**

$S_n = \frac{n}{2}(\text{first term} + \text{last term})$.

3 **Show the formula**

Therefore $S_n = \frac{n}{2}\{a + [a + (n-1)d]\} = \frac{n}{2}\{2a + (n-1)d\}$.

4 **Identify the arithmetic series**

For $900 + 892 + 884 + \cdots + 500$, the first term is 900, the common difference is -8, and the last term is 500.

5 **Find the number of terms**

$500 = 900 + (n-1)(-8)$, so $n - 1 = 50$ and $n = 51$.

6 **Find the sum**

$S = \frac{51}{2}(900 + 500) = 35700$.

7 **Use the geometric condition**

For $k + 4$, $k - 2$, $11 - k$ to be consecutive geometric terms, $(k-2)^2 = (k+4)(11-k)$.

8 **Show the quadratic**

$k^2 - 4k + 4 = -k^2 + 7k + 44$, so $2k^2 - 11k - 40 = 0$.

9 **Solve for k**

$2k^2 - 11k - 40 = 0$ gives $k = 8$ or $k = -\frac{5}{2}$.

10 **Use convergence**

If $k = 8$, the terms are $12, 6, 3$ and the common ratio is $\frac{1}{2}$, so the series is convergent. If $k = -\frac{5}{2}$, the common ratio is -3, so it is not convergent.

11 **Find S infinity**

Use $k = 8$: $S_\infty = \frac{12}{1 - 1/2} = 24$.

✓ **FINAL ANSWER**

$(i)(a)$ Proven.

$(i)(b)$ 35700.

$(ii)(a)$ Proven.

$(ii)(b)$ $S_\infty = 24$.

WMA12/01 OCTOBER 2024

Question 7

6 marks | Geometric Sequences | Main topic: Modelling with Sequences & Series

7. Jem pays money into a savings scheme, A, over a period of 300 months.

Jem pays £20 into scheme A in month 1, £20.50 in month 2, £21 in month 3 and so on, so that the amounts Jem pays each month form an arithmetic sequence.

(a) Show that Jem pays £69.50 into scheme A in month 100

(1)

(b) Find the **total** amount that Jem pays into scheme A over the period of 300 months.

(2)

Kim pays money into a different savings scheme, B, over the same period of 300 months.

In a model, the amounts Kim pays into scheme B increase by the same percentage each month, so that the amounts Kim pays each month form a geometric sequence.

Given that Kim pays

- £20 into scheme B in month 1
- £250 into scheme B in month 300

(c) use the model to calculate, to the nearest £10, the difference between the total amount paid into scheme A and the total amount paid into scheme B over the period of 300 months.

(3)

WORKED SOLUTION

Question 7 · WMA12/01 October 2024

1 Identify Jem's sequence

Jem's payments are arithmetic with first term 20 and common difference 0.50.

2 Show month 100

The month 100 payment is $20 + 99(0.50) = 69.50$, so Jem pays £69.50.

3 Sum Jem's payments

$S_{300} = \frac{300}{2}\{2(20) + 299(0.50)\} = 150(189.5) = 28425$.

4 Model Kim's payments

Let the common ratio be r. Since the first payment is 20 and the month 300 payment is 250, $20r^{299} = 250$.

5 **Find the ratio**

Thus $r^{299} = 12.5$ and $r = 12.5^{1/299}$.

6 **Sum Kim's payments**

$S_B = 20\dfrac{r^{300} - 1}{r - 1}$ with $r = 12.5^{1/299}$, giving $S_B \approx 27362.95$.

7 **Find the difference**

The difference is $28425 - 27362.95 \approx 1062.05$, which is £1,060 to the nearest £10.

✓ **FINAL ANSWER**

(a) Proven.

(b) £28,425.

(c) £1,060.

WMA12/01 MAY/JUNE 2025

Question 7

9 marks | Geometric Sequences | Main topic: Sequences & Series

7: **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(i) A **geometric** series begins

$$10 + 8 + 6.4 + \ldots$$

(a) Find the sum to infinity of this series.

(2)

Given that the kth term of this series is less than 0.0005

(b) use algebra to find the smallest possible value of k.

(3)

(ii) An **arithmetic** series begins

$$850 + 843 + 836 + \ldots$$

Given that the sum of the first n terms of this series is S_n
find the greatest possible value of S_n

(4)

WORKED SOLUTION

Question 7 · WMA12/01 May/June 2025

1 **Identify the geometric sequence**

The first term is 10 and the common ratio is $\frac{8}{10} = 0.8$.

2 **Find the sum to infinity**

$S_\infty = \frac{10}{1 - 0.8} = 50$.

3 **Write the kth term**

The kth term is $10(0.8)^{k-1}$.

4 **Use the inequality**

$10(0.8)^{k-1} < 0.0005$, so $(0.8)^{k-1} < 0.00005$.

5 **Solve with logarithms**

Taking logarithms gives $(k-1)\log(0.8) < \log(0.00005)$. Since $\log(0.8) < 0$,
$k - 1 > \dfrac{\log(0.00005)}{\log(0.8)} = 44.381\ldots$.

6 **Find the smallest integer**

Thus $k > 45.381\ldots$, so the smallest possible value is $k = 46$.

7 **Write the arithmetic sum**

For $850 + 843 + 836 + \cdots$, the first term is 850 and the common difference is -7.

8 **Form S n**

$S_n = \dfrac{n}{2}\{2(850) + (n-1)(-7)\} = \dfrac{n}{2}(1707 - 7n)$.

9 **Find the best integer n**

This quadratic has its maximum near $n = \dfrac{1707}{14} = 121.928\ldots$, so check $n = 122$.

10 **Find the greatest sum**

$S_{122} = \dfrac{122}{2}(1707 - 854) = 61(853) = 52033$.

✓ **FINAL ANSWER**

$(i)(a)\ 50$.

$(i)(b)\ 46$.

$(ii)\ 52033$.

WMA12/01 JANUARY 2026

Question 9

10 marks | Geometric Sequences | Main topic: Modelling with Sequences & Series

9. A company mines lithium.

- in year 1, the company mined 12 400 tonnes of lithium
- in year 15, the company is predicted to mine 7 500 tonnes of lithium

Model A assumes that the mass of lithium the company mines will decrease by the same amount each year.

According to model A,

(a) find the mass of lithium the company will mine in year 5

(3)

(b) calculate the **total** mass of lithium the company will mine from year 1 to year 15 inclusive.

(2)

Model B assumes that the mass of lithium the company mines will decrease by the same percentage each year.

(c) Find, according to model B, the mass of lithium the company will mine in year 10. Give the answer to the nearest 10 tonnes.

(3)

According to model B, there is a limit to the **total** mass of lithium that can be mined.

(d) Calculate the value of this limit. Give the answer to the nearest 100 tonnes.

(2)

WORKED SOLUTION

Question 9 · WMA12/01 January 2026

1 **Model A as an arithmetic sequence**

For Model A, let the common difference be d. Since year 15 is the 15th term, $7500 = 12400 + 14d$.

2 **Find year 5 for Model A**

$d = \dfrac{7500 - 12400}{14} = -350$. Year 5 is $12400 + 4(-350) = 11000$ tonnes.

3 **Find the total from year 1 to year 15**

$S_{15} = \dfrac{15}{2}(12400 + 7500) = 149250$ tonnes.

4 **Model B as a geometric sequence**

Let the common ratio be r. Then $7500 = 12400r^{14}$, so $r = \left(\frac{7500}{12400}\right)^{1/14} \approx 0.964723$.

5 **Find year 10 for Model B**

Year 10 is $12400r^{9} \approx 8975.26$, which is 8980 tonnes to the nearest 10 tonnes.

6 **Find the limiting total**

Since $0 < r < 1$, the sum to infinity is $S_{\infty} = \frac{12400}{1-r} \approx 351508.1$, so the limit is 351500 tonnes to the nearest 100 tonnes.

✓ **FINAL ANSWER**

(a) 11000 tonnes.

(b) 149250 tonnes.

(c) 8980 tonnes.

(d) 351500 tonnes.

TOPIC

Sequences & Series

Every past-paper question on this topic, October 2019 to January 2026.

WMA12/01 JANUARY 2020

Question 8

7 marks | Sequences & Series

8. (i) An arithmetic series has first term a and common difference d.

Prove that the sum to n terms of this series is

$$\frac{n}{2}\{2a + (n-1)d\}$$

(3)

(ii) A sequence $u_1, u_2, u_3, \ldots$ is given by

$$u_n = 5n + 3(-1)^n$$

Find the value of

(a) u_5

(1)

(b) $\sum_{n=1}^{59} u_n$

(3)

WORKED SOLUTION

Question 8 · WMA12/01 January 2020

1 Write the series forwards

$S_n = a + (a+d) + (a+2d) + \cdots + (a+(n-1)d)$.

2 Write the series backwards

$S_n = (a+(n-1)d) + (a+(n-2)d) + \cdots + a$.

3 Add the two forms

Each pair adds to $2a + (n-1)d$, and there are n pairs. Hence $2S_n = n\{2a + (n-1)d\}$.

4 Complete the proof

Dividing by 2 gives $S_n = \frac{n}{2}\{2a + (n-1)d\}$.

5 Find u5

$u_5 = 5(5) + 3(-1)^5 = 25 - 3 = 22$.

6 **Sum the sequence**

$\sum_{n=1}^{59} u_n = 5\sum_{n=1}^{59} n + 3\sum_{n=1}^{59}(-1)^n$.

7 **Calculate the total**

$5\sum_{n=1}^{59} n = 5 \cdot \dfrac{59 \cdot 60}{2} = 8850$, and $\sum_{n=1}^{59}(-1)^n = -1$. Therefore the total is $8850 - 3 = 8847$.

✓ **FINAL ANSWER**

(i) Proven.

$(ii)(a)$ 22. $(ii)(b)$ 8847.

WMA12/01 JANUARY 2021

Question 8

7 marks | Sequences & Series

8. A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_{n+1} = 2(a_n + 3)^2 - 7$$
$$a_1 = p - 3$$

where p is a constant.

(a) Find an expression for a_2 in terms of p, giving your answer in simplest form.

(1)

Given that $\sum_{n=1}^{3} a_n = p + 15$

(b) find the possible values of a_2

(6)

WORKED SOLUTION

Question 8 · WMA12/01 January 2021

1 Find a two

Since $a_1 = p - 3$, we have $a_1 + 3 = p$. Therefore $a_2 = 2(a_1 + 3)^2 - 7 = 2p^2 - 7$.

2 Find a three

$a_2 + 3 = 2p^2 - 4$, so $a_3 = 2(2p^2 - 4)^2 - 7 = 8(p^2 - 2)^2 - 7$.

3 Use the given sum

$a_1 + a_2 + a_3 = p + 15$.

4 Substitute the expressions

$(p - 3) + (2p^2 - 7) + 8(p^2 - 2)^2 - 7 = p + 15$.

5 Simplify

Cancelling p and expanding gives $8p^4 - 30p^2 = 0$, so $2p^2(4p^2 - 15) = 0$.

6 Find the possible a two values

If $p = 0$, then $a_2 = -7$. If $p^2 = \frac{15}{4}$, then $a_2 = 2 \cdot \frac{15}{4} - 7 = \frac{1}{2}$.

FINAL ANSWER

$(a)\ a_2 = 2p^2 - 7. \quad (b)\ a_2 = -7 \text{ or } \dfrac{1}{2}.$

WMA12/01 MAY/JUNE 2023

Question 11

8 marks | Sequences & Series

11. A sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_{n+1} = b - au_n$$

$$u_1 = 3$$

where a and b are constants.

(a) Find, in terms of a and b,

(i) u_2

(ii) u_3

(2)

Given

- $\sum_{n=1}^{3} u_n = 153$
- $b = a + 9$

(b) show that

$$a^2 - 5a - 66 = 0$$

(3)

(c) Hence find the larger possible value of u_2

(3)

WORKED SOLUTION

Question 11 · WMA12/01 May/June 2023

1 Find u2

Using $u_{n+1} = b - au_n$ and $u_1 = 3$, we get $u_2 = b - a(3) = b - 3a$.

2 Find u3

$u_3 = b - au_2 = b - a(b - 3a) = b - ab + 3a^2$.

3 Use the sum condition

$u_1 + u_2 + u_3 = 153$, so $3 + (b - 3a) + (b - ab + 3a^2) = 153$.

4 Substitute b equals a plus nine

Using $b = a + 9$, the left side becomes $3 + (a + 9) - 3a + (a + 9) - a(a + 9) + 3a^2$.

5 **Simplify**

This is $2a^2 - 10a + 21$, so $2a^2 - 10a + 21 = 153$.

6 **Show the required equation**

$2a^2 - 10a - 132 = 0$. Dividing by 2 gives $a^2 - 5a - 66 = 0$.

7 **Solve for a**

$a^2 - 5a - 66 = (a - 11)(a + 6) = 0$, so $a = 11$ or $a = -6$.

8 **Find the corresponding u2 values**

If $a = 11$, then $b = 20$ and $u_2 = 20 - 3(11) = -13$. If $a = -6$, then $b = 3$ and $u_2 = 3 - 3(-6) = 21$.

9 **Choose the larger value**

The larger possible value of u_2 is 21.

✓ **FINAL ANSWER**

$(a)(i)\ u_2 = b - 3a$.

$(a)(ii)\ u_3 = b - ab + 3a^2$.

(b) Proven.

$(c)\ 21$.

WMA12/01 MAY/JUNE 2025

Question 7

9 marks | Sequences & Series

7: **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(i) A **geometric** series begins

$$10 + 8 + 6.4 + \ldots$$

(a) Find the sum to infinity of this series.

(2)

Given that the kth term of this series is less than 0.0005

(b) use algebra to find the smallest possible value of k.

(3)

(ii) An **arithmetic** series begins

$$850 + 843 + 836 + \ldots$$

Given that the sum of the first n terms of this series is S_n
find the greatest possible value of S_n

(4)

WORKED SOLUTION

Question 7 · WMA12/01 May/June 2025

1 **Identify the geometric sequence**

The first term is 10 and the common ratio is $\frac{8}{10} = 0.8$.

2 **Find the sum to infinity**

$S_\infty = \frac{10}{1 - 0.8} = 50$.

3 **Write the kth term**

The kth term is $10(0.8)^{k-1}$.

4 **Use the inequality**

$10(0.8)^{k-1} < 0.0005$, so $(0.8)^{k-1} < 0.00005$.

5 **Solve with logarithms**

Taking logarithms gives $(k-1)\log(0.8) < \log(0.00005)$. Since $\log(0.8) < 0$,

$k - 1 > \dfrac{\log(0.00005)}{\log(0.8)} = 44.381\ldots.$

6 **Find the smallest integer**

Thus $k > 45.381\ldots$, so the smallest possible value is $k = 46$.

7 **Write the arithmetic sum**

For $850 + 843 + 836 + \cdots$, the first term is 850 and the common difference is -7.

8 **Form S n**

$S_n = \dfrac{n}{2}\{2(850) + (n-1)(-7)\} = \dfrac{n}{2}(1707 - 7n).$

9 **Find the best integer n**

This quadratic has its maximum near $n = \dfrac{1707}{14} = 121.928\ldots$, so check $n = 122$.

10 **Find the greatest sum**

$S_{122} = \dfrac{122}{2}(1707 - 854) = 61(853) = 52033.$

✓ **FINAL ANSWER**

$(i)(a)\ 50.$

$(i)(b)\ 46.$

$(ii)\ 52033.$

WMA12/01 OCTOBER 2019

Question 7

7 marks | Sequences & Series | Main topic: Laws of Logarithms

7. Given $\log_a b = k$, find, in simplest form in terms of k,

(i) $\log_a\left(\frac{\sqrt{a}}{b}\right)$ **(2)**

(ii) $\frac{\log_a a^2 b}{\log_a b^3}$ **(2)**

(iii) $\sum_{n=1}^{50}(k + \log_a b^n)$ **(3)**

WORKED SOLUTION

Question 7 · WMA12/01 October 2019

1 Use the given logarithm

Since $\log_a b = k$, we can replace every $\log_a b$ by k.

2 Simplify part (i)

$\log_a\left(\frac{\sqrt{a}}{b}\right) = \log_a(a^{1/2}) - \log_a b = \frac{1}{2} - k$.

3 Simplify the numerator and denominator

$\log_a(a^2 b) = \log_a a^2 + \log_a b = 2 + k$, and $\log_a(b^3) = 3\log_a b = 3k$.

4 Write part (ii)

Therefore $\frac{\log_a(a^2 b)}{\log_a(b^3)} = \frac{k+2}{3k}$.

5 Simplify the summand

$\log_a(b^n) = n\log_a b = nk$, so $k + \log_a(b^n) = k + nk = (n+1)k$.

6 Sum the series

$\sum_{n=1}^{50}(n+1)k = k\left(\sum_{n=1}^{50} n + 50\right) = k(1275 + 50) = 1325k$.

FINAL ANSWER

$(i)\ \frac{1}{2} - k.\quad (ii)\ \frac{k+2}{3k}.\quad (iii)\ 1325k.$

WMA12/01 OCTOBER 2021

Question 7

10 marks | Sequences & Series | Main topic: Geometric Sequences

7. (i) A geometric sequence has first term 4 and common ratio 6
Given that the n^{th} term is greater than 10^{100}, find the minimum possible value of n.

(3)

(ii) A different geometric sequence has first term a and common ratio r.

Given that

- the second term of the sequence is -6
- the sum to infinity of the series is 25

(a) show that

$$25r^2 - 25r - 6 = 0$$

(3)

(b) Write down the solutions of

$$25r^2 - 25r - 6 = 0$$

(1)

Hence,

(c) state the value of r, giving a reason for your answer,

(1)

(d) find the sum of the first 4 terms of the series.

(2)

WORKED SOLUTION

Question 7 · WMA12/01 October 2021

1 Write the nth term

The nth term of the first sequence is $4 \cdot 6^{n-1}$.

2 Use logarithms

We need $4 \cdot 6^{n-1} > 10^{100}$, so $n - 1 > \dfrac{100 - \log_{10} 4}{\log_{10} 6} = 127.736\ldots$.

3 Choose the smallest integer

Thus n must be greater than $128.736\ldots$, so the minimum possible value is 129.

4 **Use the second term**

For the second sequence, $ar = -6$, so $a = -\dfrac{6}{r}$.

5 **Use the sum to infinity**

$\dfrac{a}{1-r} = 25$. Substituting $a = -\dfrac{6}{r}$ gives $-6 = 25r(1-r)$.

6 **Show the quadratic**

Rearranging gives $25r^2 - 25r - 6 = 0$.

7 **Solve for r**

$25r^2 - 25r - 6 = 0$ gives $r = \dfrac{6}{5}$ or $r = -\dfrac{1}{5}$.

8 **Choose the valid ratio**

A sum to infinity exists only when $|r| < 1$, so $r = -\dfrac{1}{5}$.

9 **Find the first term**

Since $ar = -6$, $a = 30$.

10 **Sum four terms**

$S_4 = 30\dfrac{1-(-1/5)^4}{1-(-1/5)} = \dfrac{624}{25}$.

✓ **FINAL ANSWER**

$(i)\ n = 129.$ $(ii)(a)$ Proven.

$(ii)(b)\ r = \dfrac{6}{5}$ or $-\dfrac{1}{5}.$ $(ii)(c)\ r = -\dfrac{1}{5}.$ $(ii)(d)\ \dfrac{624}{25}.$

WMA12/01 MAY/JUNE 2022

Question 6

8 marks | Sequences & Series | Main topic: Geometric Sequences

6. In a geometric sequence $u_1, u_2, u_3, \ldots$

- the common ratio is r
- $u_2 + u_3 = 6$
- $u_4 = 8$

(a) Show that r satisfies

$$3r^2 - 4r - 4 = 0$$

(3)

Given that the geometric sequence has a sum to infinity,

(b) find u_1

(3)

(c) find S_∞

(2)

WORKED SOLUTION

Question 6 · WMA12/01 May/June 2022

1 Write the terms

$u_2 = u_1 r$, $u_3 = u_1 r^2$, and $u_4 = u_1 r^3$.

2 Use u four

Since $u_4 = 8$, $u_1 r^3 = 8$, so $u_1 r = \dfrac{8}{r^2}$.

3 Use u two plus u three

$u_2 + u_3 = u_1 r(1 + r) = 6$.

4 Show the quadratic

Substitute $u_1 r = \dfrac{8}{r^2}$ to get $\dfrac{8}{r^2}(1 + r) = 6$. Hence $8 + 8r = 6r^2$, so $3r^2 - 4r - 4 = 0$.

5 Solve for r

$3r^2 - 4r - 4 = 0$ gives $r = 2$ or $r = -\dfrac{2}{3}$.

6 **Choose the sum-to-infinity ratio**

A sum to infinity requires $|r| < 1$, so $r = -\frac{2}{3}$.

7 **Find u one**

$u_1 r^3 = 8$, so $u_1\left(-\frac{2}{3}\right)^3 = 8$ and $u_1 = -27$.

8 **Find the infinite sum**

$$S_\infty = \frac{u_1}{1-r} = \frac{-27}{1+2/3} = -\frac{81}{5}.$$

FINAL ANSWER

(a) Proven: $3r^2 - 4r - 4 = 0$.

(b) $u_1 = -27$. (c) $S_\infty = -\frac{81}{5}$.

WMA12/01 OCTOBER 2022

Question 8

7 marks | Sequences & Series | Main topic: Geometric Sequences

8. A geometric sequence has first term a and common ratio r

Given that $S_\infty = 3a$

(a) show that $r = \frac{2}{3}$

(2)

Given also that

$$u_2 - u_4 = 16$$

where u_k is the k^{th} term of this sequence,

(b) find the value of S_{10} giving your answer to one decimal place.

(5)

WORKED SOLUTION

Question 8 · WMA12/01 October 2022

1 Use the sum to infinity formula

For a geometric sequence, $S_\infty = \frac{a}{1-r}$. The question gives $S_\infty = 3a$.

2 Find the common ratio

$\frac{a}{1-r} = 3a$. Since $u_2 - u_4 = 16$, $a \neq 0$, so $\frac{1}{1-r} = 3$ and $r = \frac{2}{3}$.

3 Use u2 minus u4

$u_2 - u_4 = ar - ar^3 = ar(1-r^2) = 16$.

4 Substitute r

$a\left(\frac{2}{3}\right)\left(1 - \frac{4}{9}\right) = 16$.

5 Solve for a

$a \cdot \frac{2}{3} \cdot \frac{5}{9} = 16$, so $\frac{10a}{27} = 16$ and $a = \frac{216}{5} = 43.2$.

6 Find S10

$S_{10} = \frac{a(1-r^{10})}{1-r} = 3a\left(1 - \left(\frac{2}{3}\right)^{10}\right)$.

7 **Evaluate**

$S_{10} = 3\left(\frac{216}{5}\right)\left(1-\left(\frac{2}{3}\right)^{10}\right) = \frac{92840}{729} = 127.352\ldots$, so $S_{10} = 127$ to 3 significant figures.

✓ **FINAL ANSWER**

(a) $a = 43.2$.

(b) $S_{10} = 127.353\ldots$, so 127 to 3 s.f.

WMA12/01 JANUARY 2023

Question 7

7 marks | Sequences & Series | Main topic: Geometric Sequences

7. A geometric sequence has first term a and common ratio r, where $r > 0$

Given that

- the 3rd term is 20
- the 5th term is 12.8

(a) show that $r = 0.8$

(1)

(b) Hence find the value of a.

(2)

Given that the sum of the first n terms of this sequence is greater than 156

(c) find the smallest possible value of n.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(4)

WORKED SOLUTION

Question 7 · WMA12/01 January 2023

1 Write the given terms

For a geometric sequence, the third term is $ar^2 = 20$ and the fifth term is $ar^4 = 12.8$.

2 Divide the equations

$\frac{ar^4}{ar^2} = \frac{12.8}{20}$, so $r^2 = 0.64$.

3 Use r positive

Since $r > 0$, $r = 0.8$.

4 Find a

Using $ar^2 = 20$, $a(0.8)^2 = 20$, so $a = \frac{20}{0.64} = 31.25$.

5 Write the sum formula

$S_n = \frac{a(1-r^n)}{1-r} = \frac{31.25(1-0.8^n)}{0.2} = 156.25(1-0.8^n)$.

6 **Set up the inequality**

We need $156.25(1-0.8^n) > 156$.

7 **Isolate the power**

$1-0.8^n > \dfrac{156}{156.25} = \dfrac{624}{625}$, so $0.8^n < \dfrac{1}{625}$.

8 **Use logarithms**

$n\log(0.8) < \log\left(\dfrac{1}{625}\right)$. Since $\log(0.8) < 0$, dividing reverses the inequality:

$n > \dfrac{\log(1/625)}{\log(0.8)} = 28.85\ldots$.

9 **Choose the smallest integer**

The smallest possible integer value is therefore $n = 29$.

✓ **FINAL ANSWER**

(a) $r = 0.8$.

(b) $a = 31.25$.

(c) $n = 29$.

WMA12/01 OCTOBER 2023

Question 8

7 marks | Sequences & Series | Main topic: Arithmetic Sequences

8. In a large theatre there are n rows of seats, where n is a constant.

The number of seats in the first row is a, where a is a constant.

In each subsequent row there are 4 more seats than in the previous row so that

- in the 2nd row there are $(a + 4)$ seats
- in the 3rd row there are $(a + 8)$ seats
- the number of seats in each row form an **arithmetic** sequence

Given that the **total** number of seats in the first 10 rows is 360

(a) find the value of a.

(2)

Given also that the total number of seats in the n rows is 2146

(b) show that

$$n^2 + 8n - 1073 = 0$$

(2)

(c) Hence

(i) state the number of rows of seats in the theatre,

(ii) find the maximum number of seats in any one row.

(3)

WORKED SOLUTION

Question 8 · WMA12/01 October 2023

1 **Use the arithmetic sum formula**

The first term is a and the common difference is 4.

2 **Use the first 10 rows**

$S_{10} = \frac{10}{2}(2a + 9 \cdot 4) = 360$.

3 **Find a**

$5(2a + 36) = 360$, so $10a + 180 = 360$ and $a = 18$.

4 **Write S n**

$S_n = \frac{n}{2}(2(18) + (n-1)4) = \frac{n}{2}(4n + 32) = 2n^2 + 16n$.

5 **Use the total number of seats**

Since the total is 2146, $2n^2 + 16n = 2146$.

6 **Show the equation**

Dividing by 2 and rearranging gives $n^2 + 8n - 1073 = 0$.

7 **Solve for n**

$n = \frac{-8 \pm \sqrt{8^2 + 4(1073)}}{2} = \frac{-8 \pm 66}{2}$, so $n = 29$ or $n = -37$.

8 **Choose the row count**

The number of rows must be positive, so there are 29 rows.

9 **Find the maximum row size**

The largest row is the last row: $18 + (29 - 1)4 = 18 + 112 = 130$ seats.

✓ **FINAL ANSWER**

(a) $a = 18$.

(b) Proven.

$(c)(i)$ 29 rows.

$(c)(ii)$ 130 seats.

TOPIC

Modelling with Sequences & Series

Every past-paper question on this topic, October 2019 to January 2026.

WMA12/01 JANUARY 2021

Question 10

11 marks | Modelling with Sequences & Series

10. In this question you must show detailed reasoning.

Owen wants to train for 12 weeks in preparation for running a marathon.

During the 12-week period he will run every Sunday and every Wednesday.

- On Sunday in week 1 he will run 15 km
- On Sunday in week 12 he will run 37 km

He considers two different 12-week training plans.

In training plan A, he will increase the distance he runs each Sunday by the same amount.

(a) Calculate the distance he will run on Sunday in week 5 under training plan A.

(3)

In training plan B, he will increase the distance he runs each Sunday by the same percentage.

(b) Calculate the distance he will run on Sunday in week 5 under training plan B.
Give your answer in km to one decimal place.

(3)

Owen will also run a fixed distance, x km, each Wednesday over the 12-week period.

Given that

- x is an integer
- the total distance that Owen will run on Sundays and Wednesdays over the 12 weeks will not exceed 360 km

(c) (i) find the maximum value of x, if he uses training plan A,

(ii) find the maximum value of x, if he uses training plan B.

(5)

(Total 11 marks)

WORKED SOLUTION

Question 10 · WMA12/01 January 2021

1 **Find the arithmetic difference**

Plan A has 12 Sunday runs from 15 km to 37 km, so there are 11 equal increases. The common difference is $\frac{37-15}{11}=2$.

2 **Find week five for plan A**

Week 5 is the fifth term, so the distance is $15 + 4(2) = 23\ \text{km}$.

3 **Find the geometric ratio**

For plan B, $15r^{11} = 37$, so $r = \left(\dfrac{37}{15}\right)^{1/11}$.

4 **Find week five for plan B**

The fifth Sunday distance is $15r^4 = 15\left(\dfrac{37}{15}\right)^{4/11} = 20.829\ldots$, so it is $20.8\ \text{km}$.

5 **Sum the Sunday runs for plan A**

The arithmetic total is $\dfrac{12}{2}(15 + 37) = 312$ km.

6 **Find the Wednesday limit for plan A**

The total distance condition gives $312 + 12x \leq 360$, so $x \leq 4$. The maximum integer is 4.

7 **Sum the Sunday runs for plan B**

The geometric total is $15\dfrac{r^{12} - 1}{r - 1} = 294.185\ldots$ km.

8 **Find the Wednesday limit for plan B**

Now $294.185\ldots + 12x \leq 360$, so $x \leq 5.484\ldots$. The maximum integer is 5.

✓ **FINAL ANSWER**

$(a)\ 23\ \text{km}.\quad (b)\ 20.8\ \text{km}.\quad (c)(i)\ x = 4.\quad (c)(ii)\ x = 5.$

WMA12/01 JANUARY 2022

Question 8

9 marks | Modelling with Sequences & Series

8. A metal post is repeatedly hit in order to drive it into the ground.

Given that

- on the 1st hit, the post is driven 100 mm into the ground
- on the 2nd hit, the post is driven an **additional** 98 mm into the ground
- on the 3rd hit, the post is driven an **additional** 96 mm into the ground
- the **additional** distances the post travels on each subsequent hit form an arithmetic sequence

(a) show that the post is driven an **additional** 62 mm into the ground with the 20th hit.

(1)

(b) Find the **total distance** that the post has been driven into the ground after 20 hits.

(2)

Given that for each subsequent hit after the 20th hit

- the **additional** distances the post travels form a geometric sequence with common ratio r
- on the 22nd hit, the post is driven an **additional** 60 mm into the ground

(c) find the value of r, giving your answer to 3 decimal places.

(2)

After a total of N hits, the post will have been driven more than 3 m into the ground.

(d) Find, showing all steps in your working, the smallest possible value of N.

(4)

WORKED SOLUTION

Question 8 · WMA12/01 January 2022

1 **Use the arithmetic rule**

For the first 20 hits, the additional distances form an arithmetic sequence with first term 100 and common difference -2.

2 **Show the twentieth hit**

The twentieth additional distance is $100 + 19(-2) = 62$ mm.

3 **Sum the first twenty hits**

$S_{20} = \frac{20}{2}(100 + 62) = 1620$ mm.

4 **Use the geometric information**

After the twentieth hit, the distances follow the same ratio, so the twenty second distance is $62r^2$.

5 **Find r**

$62r^2 = 60$, hence $r = \sqrt{\dfrac{30}{31}} = 0.984$ to 3 decimal places.

6 **Set up the remaining distance**

More than 3 m means more than 3000 mm. After 20 hits, the remaining amount to exceed is $3000 - 1620 = 1380$ mm.

7 **Sum the geometric tail**

If there are m further hits, their total is $62r + 62r^2 + \cdots + 62r^m = 62r\dfrac{1 - r^m}{1 - r}$.

8 **Find the first m that works**

Using $r = \sqrt{30/31}$, $62r\dfrac{1 - r^{27}}{1 - r}$ is not enough, but $62r\dfrac{1 - r^{28}}{1 - r}$ makes the total exceed 3000 mm.

9 **State N**

Therefore $m = 28$ further hits are needed after the first 20, so $N = 20 + 28 = 48$.

✓ **FINAL ANSWER**

(a) Proven: 62 mm.

(b) 1620 mm.

(c) $r = 0.984$. (d) $N = 48$.

Question 9

9 marks | Modelling with Sequences & Series

9. A scientist is using carbon-14 dating to determine the age of some wooden items.

The equation for carbon-14 dating an item is given by

$$N = k\lambda^t$$

where

- N grams is the amount of carbon-14 **currently** present in the item
- k grams was the **initial** amount of carbon-14 present in the item
- t is the number of years since the item was made
- λ is a constant, with $0 < \lambda < 1$

(a) Sketch the graph of N against t for $k = 1$

(2)

Given that it takes 5700 years for the amount of carbon-14 to reduce to half its initial value,

(b) show that the value of the constant λ is 0.999878 to 6 decimal places.

(2)

Given that Item A

- is known to have had 15 grams of carbon-14 present initially
- is thought to be 3250 years old

(c) calculate, to 3 significant figures, how much carbon-14 the equation predicts is currently in Item A.

(2)

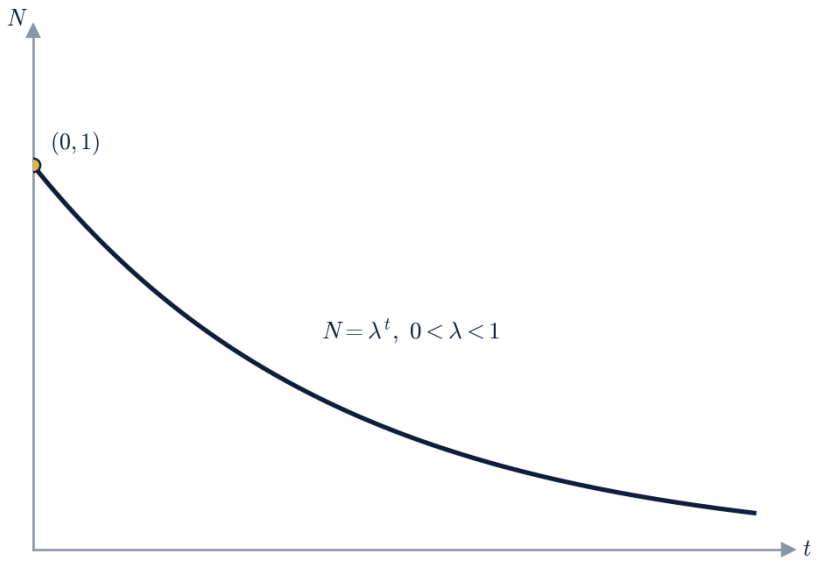
Item B is known to have initially had 25 grams of carbon-14 present, but only 18 grams now remain.

(d) Use algebra to calculate the age of Item B to the nearest 100 years.

(3)

1 **Sketch the model for k equals one**

For $k = 1$, $N = \lambda^t$ with $0 < \lambda < 1$. The graph starts at $(0, 1)$, decreases as t increases, and tends towards 0.



2 **Use the half-life information**

After 5700 years, $N = \dfrac{k}{2}$, so $k\lambda^{5700} = \dfrac{k}{2}$.

3 **Find lambda**

$\lambda^{5700} = \dfrac{1}{2}$, hence $\lambda = \left(\dfrac{1}{2}\right)^{1/5700} = 0.999878\ldots$, so $\lambda = 0.999878$ to 6 decimal places.

4 **Model item A**

For item A, $k = 15$ and $t = 3250$, so $N = 15\lambda^{3250}$.

5 **Calculate item A**

$N = 15(0.9998784026\ldots)^{3250} = 10.103\ldots$, so the current amount is 10.1 g to 3 significant figures.

6 **Set up item B**

For item B, $18 = 25\lambda^t$, so $\lambda^t = \dfrac{18}{25}$.

7 **Take logarithms**

$t \log \lambda = \log\left(\dfrac{18}{25}\right)$, so $t = \dfrac{\log(18/25)}{\log \lambda}$.

8 **Round the age**

Using $\lambda = (1/2)^{1/5700}$ gives $t = 2701\ldots$, so the age is 2700 years to the nearest 100 years.

FINAL ANSWER

(a) Decreasing exponential through $(0, 1)$.

(b) $\lambda = 0.999878$.

(c) $10.1\ \text{g}$. (d) 2700 years.

WMA12/01 JANUARY 2024

Question 7

9 marks | Modelling with Sequences & Series

7. Wheat is grown on a farm.

- In year 1, the farm produced 300 tonnes of wheat.
- In year 12, the farm is predicted to produce 4000 tonnes of wheat.

Model A assumes that the amount of wheat produced on the farm will increase by the same amount each year.

(a) Using model A, find the amount of wheat produced on the farm in year 4.
Give your answer to the nearest 10 tonnes.

(3)

Model B assumes that the amount of wheat produced on the farm will increase by the same percentage each year.

(b) Using model B, find the amount of wheat produced on the farm in year 2.
Give your answer to the nearest 10 tonnes.

(3)

(c) Calculate, according to the two models, the difference between the total amounts of wheat predicted to be produced on the farm from year 1 to year 12 inclusive.
Give your answer to the nearest 10 tonnes.

(3)

WORKED SOLUTION

Question 7 · WMA12/01 January 2024

1 Model A common difference

Model A is arithmetic. From year 1 to year 12 there are 11 equal increases, so $d = \dfrac{4000 - 300}{11} = \dfrac{3700}{11}$.

2 Find year 4 in model A

Year 4 is $300 + 3d = 300 + 3\left(\dfrac{3700}{11}\right) = 1309.09\ldots$, so the amount is 1310 tonnes to the nearest 10 tonnes.

3 Model B common ratio

Model B is geometric. If the ratio is r, then $300r^{11} = 4000$.

4 **Find the ratio**

$r = \left(\frac{4000}{300}\right)^{1/11} = 1.2655\ldots$.

---

5 **Find year 2 in model B**

Year 2 is $300r = 379.65\ldots$, so the amount is 380 tonnes to the nearest 10 tonnes.

---

6 **Total for model A**

$S_A = \frac{12}{2}(300 + 4000) = 25800$ tonnes.

---

7 **Total for model B**

$S_B = 300\frac{r^{12} - 1}{r - 1} = 17935.20\ldots$ tonnes.

---

8 **Find the difference**

The difference is $25800 - 17935.20\ldots = 7864.79\ldots$, which is 7860 tonnes to the nearest 10 tonnes.

✓ **FINAL ANSWER**

(a) 1310 tonnes.

(b) 380 tonnes.

(c) 7860 tonnes.

WMA12/01 MAY/JUNE 2024

Question 10

8 marks | Modelling with Sequences & Series

10. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The number of dormice and the number of voles on an island are being monitored.

Initially there are 2000 dormice on the island.

A model predicts that the number of dormice will increase by 3% each year, so that the numbers of dormice on the island at the end of each year form a geometric sequence.

(a) Find, according to the model, the number of dormice on the island 6 years after monitoring began. Give your answer to 3 significant figures.

(2)

The number of voles on the island is being monitored over the same period of time.

Given that
- 4 years after monitoring began there were 3690 voles on the island
- 7 years after monitoring began there were 3470 voles on the island
- the number of voles on the island at the end of each year is modelled as a geometric sequence

(b) find the equation of this model in the form

$$N = ab^t$$

where N is the number of voles, t years after monitoring began and a and b are constants. Give the value of a and the value of b to 2 significant figures.

(3)

When $t = T$, the number of dormice on the island is equal to the number of voles on the island.

(c) Find, according to the models, the value of T, giving your answer to one decimal place.

(3)

WORKED SOLUTION

Question 10 · WMA12/01 May/June 2024

1 **Model the dormice**

The dormice model is $D = 2000(1.03)^t$.

2 **Find the dormice after six years**

$D = 2000(1.03)^6 = 2388.10\ldots$, so there are 2390 dormice to 3 significant figures.

3 **Set up the vole model**

Let the vole model be $N = ab^t$.

4 **Use the two data points**

At $t = 4$, $ab^4 = 3690$. At $t = 7$, $ab^7 = 3470$.

5 **Find b**

Dividing gives $b^3 = \dfrac{3470}{3690}$, so $b = 0.9797\ldots$, which is 0.98 to 2 significant figures.

6 **Find a**

$a = \dfrac{3690}{b^4} = 4005.18\ldots$, which is 4000 to 2 significant figures.

7 **Write the vole model**

So the model is approximately $N = 4000(0.98)^t$.

8 **Set the two models equal**

Use $2000(1.03)^T = ab^T$ with the unrounded constants from the vole model.

9 **Solve using logarithms**

$\left(\dfrac{1.03}{b}\right)^T = \dfrac{a}{2000}$, so $T = \dfrac{\log(a/2000)}{\log(1.03/b)}$.

10 **Calculate T**

Substituting $a = 4005.18\ldots$ and $b = 0.9797\ldots$ gives $T = 13.875\ldots$, so $T = 13.9$ to one decimal place.

✓ **FINAL ANSWER**

(a) 2390 dormice.

(b) $N = 4000(0.98)^t$ to 2 s.f. constants.

(c) $T = 13.9$.

WMA12/01 OCTOBER 2024

Question 7

6 marks | Modelling with Sequences & Series

7. Jem pays money into a savings scheme, A, over a period of 300 months.

Jem pays £20 into scheme A in month 1, £20.50 in month 2, £21 in month 3 and so on, so that the amounts Jem pays each month form an arithmetic sequence.

(a) Show that Jem pays £69.50 into scheme A in month 100

(1)

(b) Find the **total** amount that Jem pays into scheme A over the period of 300 months.

(2)

Kim pays money into a different savings scheme, B, over the same period of 300 months.

In a model, the amounts Kim pays into scheme B increase by the same percentage each month, so that the amounts Kim pays each month form a geometric sequence.

Given that Kim pays

- £20 into scheme B in month 1
- £250 into scheme B in month 300

(c) use the model to calculate, to the nearest £10, the difference between the total amount paid into scheme A and the total amount paid into scheme B over the period of 300 months.

(3)

WORKED SOLUTION

Question 7 · WMA12/01 October 2024

1 Identify Jem's sequence

Jem's payments are arithmetic with first term 20 and common difference 0.50.

2 Show month 100

The month 100 payment is $20 + 99(0.50) = 69.50$, so Jem pays £69.50.

3 Sum Jem's payments

$S_{300} = \frac{300}{2}\{2(20) + 299(0.50)\} = 150(189.5) = 28425$.

4 Model Kim's payments

Let the common ratio be r. Since the first payment is 20 and the month 300 payment is 250, $20r^{299} = 250$.

5 **Find the ratio**

Thus $r^{299} = 12.5$ and $r = 12.5^{1/299}$.

6 **Sum Kim's payments**

$S_B = 20\dfrac{r^{300} - 1}{r - 1}$ with $r = 12.5^{1/299}$, giving $S_B \approx 27362.95$.

7 **Find the difference**

The difference is $28425 - 27362.95 \approx 1062.05$, which is £1,060 to the nearest £10.

✓ **FINAL ANSWER**

(a) Proven.

(b) £28,425.

(c) £1,060.

WMA12/01 JANUARY 2026

Question 9

10 marks | Modelling with Sequences & Series

9. A company mines lithium.

- in year 1, the company mined 12 400 tonnes of lithium
- in year 15, the company is predicted to mine 7 500 tonnes of lithium

Model A assumes that the mass of lithium the company mines will decrease by the same amount each year.

According to model A,

(a) find the mass of lithium the company will mine in year 5

(3)

(b) calculate the **total** mass of lithium the company will mine from year 1 to year 15 inclusive.

(2)

Model B assumes that the mass of lithium the company mines will decrease by the same percentage each year.

(c) Find, according to model B, the mass of lithium the company will mine in year 10. Give the answer to the nearest 10 tonnes.

(3)

According to model B, there is a limit to the **total** mass of lithium that can be mined.

(d) Calculate the value of this limit. Give the answer to the nearest 100 tonnes.

(2)

WORKED SOLUTION

Question 9 · WMA12/01 January 2026

1 **Model A as an arithmetic sequence**

For Model A, let the common difference be d. Since year 15 is the 15th term, $7500 = 12400 + 14d$.

2 **Find year 5 for Model A**

$d = \dfrac{7500 - 12400}{14} = -350$. Year 5 is $12400 + 4(-350) = 11000$ tonnes.

3 **Find the total from year 1 to year 15**

$S_{15} = \dfrac{15}{2}(12400 + 7500) = 149250$ tonnes.

4 **Model B as a geometric sequence**

Let the common ratio be r. Then $7500 = 12400r^{14}$, so $r = \left(\frac{7500}{12400}\right)^{1/14} \approx 0.964723$.

5 **Find year 10 for Model B**

Year 10 is $12400r^{9} \approx 8975.26$, which is 8980 tonnes to the nearest 10 tonnes.

6 **Find the limiting total**

Since $0 < r < 1$, the sum to infinity is $S_{\infty} = \frac{12400}{1 - r} \approx 351508.1$, so the limit is 351500 tonnes to the nearest 100 tonnes.

✓ **FINAL ANSWER**

(a) 11000 tonnes.

(b) 149250 tonnes.

(c) 8980 tonnes.

(d) 351500 tonnes.

Question 7

6 marks | Modelling with Sequences & Series | Main topic: Integration

7.

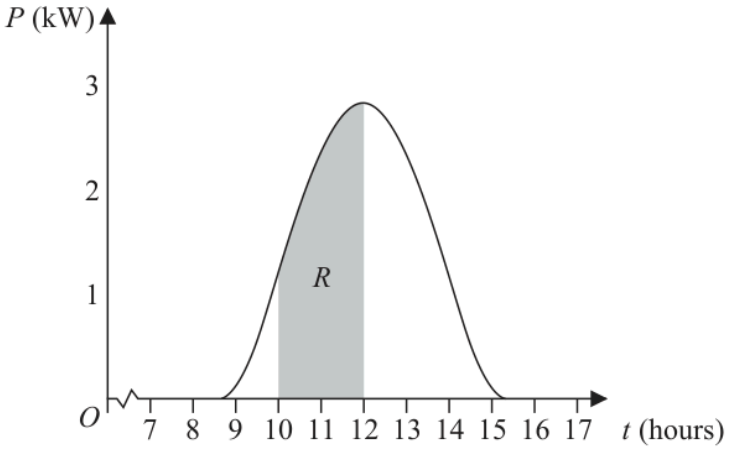


Figure 1

Solar panels are installed on the roof of a building.

The power, P, produced on a particular day, in kW, can be modelled by the equation

$$P = 0.95 + 2^{t-12} + 2^{12-t} - (t-12)^2 \qquad 8.5 \leqslant t \leqslant 15.2$$

where t is the time in hours after midnight. The graph of P against t is shown in Figure 1.

A table of values of t and P is shown below, with the values of P given to 4 significant figures where appropriate.

Time, t (hours)	10	10.5	11	11.5	12
Power, P (kW)		1.882	2.45		2.95

(a) Use the given equation to complete the table, giving the values of P to 4 significant figures where appropriate.

(2)

The amount of energy, in kWh, produced between 10:00 and 12:00 can be found by calculating the area of region R, shown shaded in Figure 1.

(b) Use the trapezium rule, with all the values of P in the completed table, to find an estimate for the amount of energy produced between 10:00 and 12:00. Give your answer to 2 decimal places.

(4)

1 **Use the model at t equals ten**

$P(10) = 0.95 + 2^{-2} + 2^{2} - (10 - 12)^{2} = 0.95 + 0.25 + 4 - 4 = 1.20$.

2 **Use the model at t equals eleven point five**

$P(11.5) = 0.95 + 2^{-0.5} + 2^{0.5} - (11.5 - 12)^{2} = 2.821\ldots$.

3 **Identify the trapezium width**

The time values increase by 0.5, so $h = 0.5$.

4 **Apply the trapezium rule**

The estimate is $\frac{0.5}{2}\{1.20 + 2.95 + 2(1.882 + 2.45 + 2.821\ldots)\}$.

5 **Calculate the energy**

This gives $4.614\ldots$, so the estimated energy is $4.61\ \text{kWh}$ to 2 decimal places.

✓ **FINAL ANSWER**

$(a)\ P(10) = 1.20,\ P(11.5) = 2.821.\quad (b)\ 4.61\ \text{kWh}$.

WMA12/01 MAY/JUNE 2023

Question 6

9 marks | Modelling with Sequences & Series | Main topic: Geometric Sequences

6. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

A software developer released an app to download.

The numbers of downloads of the app each month, in thousands, for the first three months after the app was released were

$$2k - 15 \qquad k \qquad k + 4$$

where k is a constant.

Given that the numbers of downloads each month are modelled as a geometric series,

(a) show that $k^2 - 7k - 60 = 0$

(2)

(b) predict the number of downloads in the 4th month.

(4)

The **total** number of all downloads of the app is predicted to exceed 3 million for the first time in the Nth month.

(c) Calculate the value of N according to the model.

(3)

WORKED SOLUTION

Question 6 · WMA12/01 May/June 2023

1 **Use the geometric condition**

For three consecutive terms of a geometric sequence, the middle term squared equals the product of the outer terms.

2 **Set up the equation**

So $k^2 = (2k - 15)(k + 4)$.

3 **Show the required quadratic**

Expanding gives $k^2 = 2k^2 - 7k - 60$, hence $k^2 - 7k - 60 = 0$.

4 **Find k**

$k^2 - 7k - 60 = (k - 12)(k + 5) = 0$, so $k = 12$ or $k = -5$. The download numbers must be positive, so $k = 12$.

5 **Find the common ratio**

The first three monthly values are 9, 12 and 16 thousand. The common ratio is $\frac{12}{9} = \frac{4}{3}$.

6 **Predict month four**

The 4th month is $16 \cdot \frac{4}{3} = \frac{64}{3}$ thousand downloads, which is about 21300 downloads.

7 **Write the cumulative total**

In thousands, the total after N months is $S_N = \frac{9\left(\left(\frac{4}{3}\right)^N - 1\right)}{\frac{4}{3} - 1} = 27\left(\left(\frac{4}{3}\right)^N - 1\right)$.

8 **Compare with three million**

Three million downloads is 3000 thousand, so we need $27\left(\left(\frac{4}{3}\right)^N - 1\right) > 3000$.

9 **Solve the inequality**

$\left(\frac{4}{3}\right)^N > \frac{1009}{9}$. Taking logarithms gives $N > \frac{\log(1009/9)}{\log(4/3)} = 16.405\ldots$.

10 **Choose the first month**

The first integer month for which the total exceeds 3 million is $N = 17$.

✓ **FINAL ANSWER**

(a) Proven.

(b) $\frac{64}{3}$ thousand downloads, about 21300 downloads.

(c) $N = 17$.

WMA12/01 OCTOBER 2023

Question 6

6 marks | Modelling with Sequences & Series | Main topic: Integration

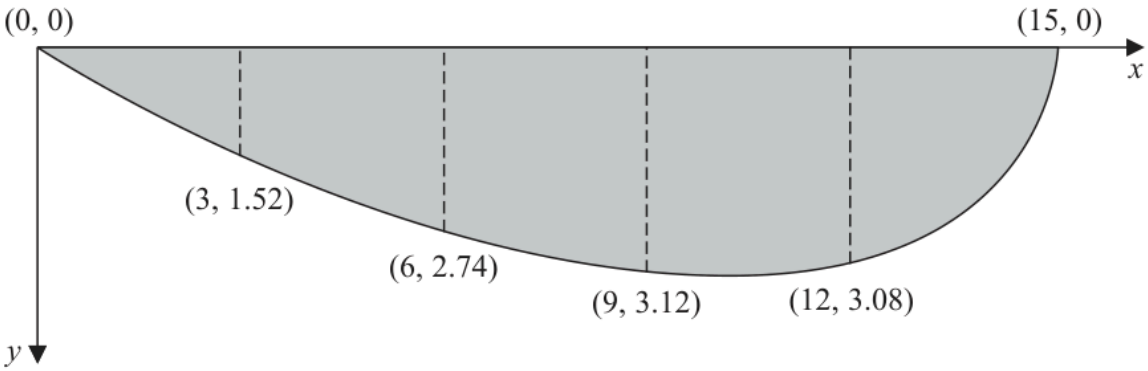


Figure 1

A river is being studied.

At one particular place, the river is 15 m wide.

The depth, y metres, of the river is measured at a point x metres from one side of the river.

Figure 1 shows a plot of the cross-section of the river and the coordinate values (x, y)

(a) Use the trapezium rule with all the y values given in Figure 1 to estimate the cross-sectional area of the river.

(3)

The water in the river is modelled as flowing at a constant speed of $1.5\,\text{m}\,\text{s}^{-1}$ across the whole of the cross-section.

(b) Use the model and the answer to part (a) to estimate the volume of water flowing through this section of the river each minute, giving your answer in m^3 to 2 significant figures.

(2)

Assuming the model,

(c) state, giving a reason for your answer, whether your answer for part (b) is an overestimate or an underestimate of the true volume of water flowing through this section of the river each minute.

(1)

WORKED SOLUTION

Question 6 · WMA12/01 October 2023

1 **Read the strip width**

The x values are $0, 3, 6, 9, 12, 15$, so the trapezium width is $h = 3$.

2 **Apply the trapezium rule**

$\text{Area} \approx \frac{3}{2}[0 + 0 + 2(1.52 + 2.74 + 3.12 + 3.08)]$.

3 **Find the cross-sectional area**

This gives $31.38\ \text{m}^2$.

4 **Convert flow speed to one minute**

The speed is $1.5\ \text{m s}^{-1}$, so in one minute the water travels $1.5 \times 60 = 90\ \text{m}$.

5 **Estimate volume**

Volume per minute $\approx 31.38 \times 90 = 2824.2\ \text{m}^3$.

6 **Round the volume**

To 2 significant figures, this is $2.8 \times 10^3\ \text{m}^3$.

7 **State over or under**

It is an underestimate, because the curve between the plotted points lies below the straight-line trapezium chords in the diagram, so the true cross-sectional area is larger.

✓ **FINAL ANSWER**

(a) $31.38\ \text{m}^2$.

(b) $2.8 \times 10^3\ \text{m}^3$.

(c) Underestimate.

WMA12/01 MAY/JUNE 2024

Question 8

12 marks | Modelling with Sequences & Series | Main topic: Trigonometric Equations

8. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(i) Solve, for $0 < x \leqslant \pi$, the equation

$$5\sin x \tan x + 13 = \cos x$$

giving your answer in radians to 3 significant figures.

(5)

(ii) The temperature inside a greenhouse is monitored on one particular day.

The temperature, $H°C$, inside the greenhouse, t hours after midnight, is modelled by the equation

$$H = 10 + 12\sin(kt + 18)° \qquad 0 \leqslant t < 24$$

where k is a constant.

Use the equation of the model to answer parts (a) to (c).

Given that

- the temperature inside the greenhouse was 20 °C at 6 am
- $0 < k < 20$

(a) find all possible values for k, giving each answer to 2 decimal places.

(4)

Given further that $0 < k < 10$

(b) find the maximum temperature inside the greenhouse,

(1)

(c) find the time of day at which this maximum temperature occurs.

Give your answer to the nearest minute.

(2)

WORKED SOLUTION

Question 8 · WMA12/01 May/June 2024

1 **Rewrite the trig equation**

$5\sin x \tan x + 13 = \cos x$ becomes $\dfrac{5\sin^2 x}{\cos x} + 13 = \cos x$.

2 **Multiply by cos x**

$5\sin^2 x + 13\cos x = \cos^2 x$.

3 **Use sine squared**

$5(1 - \cos^2 x) + 13\cos x = \cos^2 x$, so $6\cos^2 x - 13\cos x - 5 = 0$.

4 **Solve for cos x**

$(2\cos x - 5)(3\cos x + 1) = 0$, so $\cos x = \frac{5}{2}$ or $\cos x = -\frac{1}{3}$.

5 **Choose the valid root**

$\cos x = \frac{5}{2}$ is impossible, so $\cos x = -\frac{1}{3}$. For $0 < x \leq \pi$, $x = 1.9106\ldots$, so $x = 1.91$ radians to 3 significant figures.

6 **Use the temperature at 6 am**

At $t = 6$, $20 = 10 + 12\sin(6k + 18)°$, so $\sin(6k + 18)° = \frac{5}{6}$.

7 **Solve for k**

$\sin^{-1}\left(\frac{5}{6}\right) = 56.442\ldots°$. Since $0 < k < 20$, $18 < 6k + 18 < 138$, so $6k + 18 = 56.442\ldots$ or $123.557\ldots$.

8 **Find possible k values**

This gives $k = 6.407\ldots$ or $k = 17.592\ldots$, so $k = 6.41$ or 17.59 to 2 decimal places.

9 **Use the tighter condition**

Given further that $0 < k < 10$, use $k = 6.407\ldots$.

10 **Find the maximum temperature**

The maximum value of $\sin$ is 1, so the maximum temperature is $10 + 12 = 22°\text{C}$.

11 **Find the time of maximum**

The maximum occurs when $kt + 18 = 90$, so $t = \frac{72}{6.407\ldots} = 11.2375\ldots$ hours after midnight.

12 **Convert to time of day**

$0.2375\ldots$ hours is $14.25\ldots$ minutes, so the maximum occurs at about $11{:}14$ am.

FINAL ANSWER

(i) $x = 1.91$ radians.

$(ii)(a)$ $k = 6.41$ or 17.59.

$(ii)(b)$ $22°\text{C}$.

$(ii)(c)$ $11{:}14$ am.

Question 9

11 marks | Modelling with Sequences & Series | Main topic: Applications of Differentiation

9:

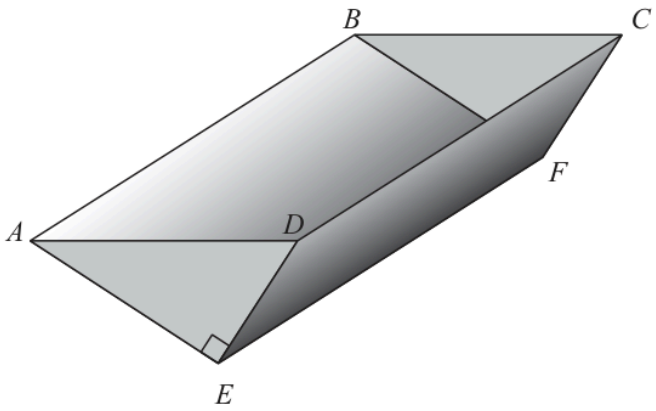


Figure 2

Figure 2 shows a sketch of an open container.

The sides $CDEF$ and $ABFE$ are rectangles.

The ends ADE and BCF are congruent (identical) right-angled triangles.

The container is made from metal of negligible thickness.

Given that

- $AE = BF = 3x$ metres
- $DE = CF = 2x$ metres
- $AB = DC = EF = L$ metres

and the capacity of the container is $12\,\text{m}^3$

(a) show that the area of metal used to make the container, $S\,\text{m}^2$, is given by

$$S = Px^2 + \frac{Q}{x}$$

where P and Q are positive integers to be found.

(4)

Given that x can vary,

(b) use algebraic calculus to find the minimum value of S, giving your answer to one decimal place.

(5)

(c) Justify that the value of S found in part (b) is a minimum.

(2)

1 **Find the triangular end area**

Each end is a right-angled triangle with perpendicular sides $3x$ and $2x$, so its area is $\frac{1}{2}(3x)(2x) = 3x^2$.

2 **Use the capacity**

The volume is cross-section area times length, so $3x^2L = 12$ and $L = \frac{4}{x^2}$.

3 **Write the metal area**

There are two triangular ends with total area $6x^2$, and two rectangular sides with total area $3xL + 2xL = 5xL$.

4 **Substitute for L**

$S = 6x^2 + 5x\left(\frac{4}{x^2}\right) = 6x^2 + \frac{20}{x}$. Hence $P = 6$ and $Q = 20$.

5 **Differentiate**

$\frac{dS}{dx} = 12x - 20x^{-2} = 12x - \frac{20}{x^2}$.

6 **Find the stationary value of x**

Set $\frac{dS}{dx} = 0$: $12x = \frac{20}{x^2}$, so $x^3 = \frac{5}{3}$.

7 **Evaluate S**

At $x = \sqrt[3]{\frac{5}{3}}$, $S = 6x^2 + \frac{20}{x} = 25.3029\ldots$, so the minimum value is 25.3 to one decimal place.

8 **Differentiate again**

$\frac{d^2S}{dx^2} = 12 + 40x^{-3} = 12 + \frac{40}{x^3}$.

9 **Justify the minimum**

Since $x > 0$, $12 + \frac{40}{x^3} > 0$, so the stationary value found is a minimum.

✓ **FINAL ANSWER**

(a) $S = 6x^2 + \frac{20}{x}$, so $P = 6$ and $Q = 20$.

(b) 25.3.

(c) Minimum justified.

WMA12/01 OCTOBER 2025

Question 8

8 marks | Modelling with Sequences & Series | Main topic: Geometric Sequences

8. (i) A geometric series has first term a and common ratio r.

Prove that the sum of the first n terms of this series S_n is given by

$$S_n = \frac{a(1-r^n)}{1-r}$$

(3)

(ii) A liquid is to be stored in a barrel.

Due to evaporation, the volume of the liquid in the barrel at the end of each year is 8% less than the volume of the liquid in the barrel at the start of the year.

At the start of the first year, the barrel is filled with 150 litres of the liquid.

(a) Show that the amount of the liquid in the barrel at the end of 6 years is approximately 91 litres.

(2)

At the start of each year a new barrel is filled with 150 litres of the liquid so that, at the end of 40 years, there are 40 barrels containing the liquid.

(b) Calculate the total amount of the liquid, to the nearest litre, in the 40 barrels at the end of 40 years.

(3)

WORKED SOLUTION

Question 8 · WMA12/01 October 2025

1 Write S n

$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$.

2 Multiply by r

$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n$.

3 Subtract

$S_n - rS_n = a - ar^n = a(1 - r^n)$.

4 Rearrange

$(1 - r)S_n = a(1 - r^n)$, so $S_n = \dfrac{a(1 - r^n)}{1 - r}$.

5 **Use the yearly multiplier**

An 8% decrease each year means the multiplier is 0.92.

6 **Show the amount after six years**

After 6 years, the amount is $150(0.92)^6 = 90.953\ldots$, which is approximately 91 litres.

7 **Set up the forty barrel total**

At the end of 40 years, the amounts are $150(0.92), 150(0.92)^2, \ldots, 150(0.92)^{40}$.

8 **Sum the geometric sequence**

The total is $150(0.92)\dfrac{1-0.92^{40}}{1-0.92} = 1663.581\ldots$.

9 **Round the total**

To the nearest litre, the total amount is 1664 litres.

✓ **FINAL ANSWER**

(i) Proven.

$(ii)(a)$ Approximately 91 litres.

$(ii)(b)$ 1664 litres.

TOPIC

Laws of Logarithms

Every past-paper question on this topic, October 2019 to January 2026.

WMA12/01 OCTOBER 2019

Question 7

7 marks | Laws of Logarithms

7. Given $\log_a b = k$, find, in simplest form in terms of k,

(i) $\log_a\left(\frac{\sqrt{a}}{b}\right)$ **(2)**

(ii) $\frac{\log_a a^2 b}{\log_a b^3}$ **(2)**

(iii) $\sum_{n=1}^{50}(k + \log_a b^n)$ **(3)**

WORKED SOLUTION

Question 7 · WMA12/01 October 2019

1 Use the given logarithm

Since $\log_a b = k$, we can replace every $\log_a b$ by k.

2 Simplify part (i)

$\log_a\left(\frac{\sqrt{a}}{b}\right) = \log_a(a^{1/2}) - \log_a b = \frac{1}{2} - k.$

3 Simplify the numerator and denominator

$\log_a(a^2 b) = \log_a a^2 + \log_a b = 2 + k$, and $\log_a(b^3) = 3\log_a b = 3k.$

4 Write part (ii)

Therefore $\frac{\log_a(a^2 b)}{\log_a(b^3)} = \frac{k+2}{3k}$.

5 Simplify the summand

$\log_a(b^n) = n\log_a b = nk$, so $k + \log_a(b^n) = k + nk = (n+1)k.$

6 Sum the series

$\sum_{n=1}^{50}(n+1)k = k\left(\sum_{n=1}^{50} n + 50\right) = k(1275 + 50) = 1325k.$

FINAL ANSWER

$(i)\ \frac{1}{2} - k.\quad (ii)\ \frac{k+2}{3k}.\quad (iii)\ 1325k.$

WMA12/01 JANUARY 2020

Question 9

7 marks | Laws of Logarithms

9. (a) Sketch the curve with equation

$$y = 3 \times 4^x$$

showing the coordinates of any points of intersection with the coordinate axes.

(2)

The curve with equation $y = 6^{1-x}$ meets the curve with equation $y = 3 \times 4^x$ at the point P.

(b) Show that the x coordinate of P is $\dfrac{\log_{10} 2}{\log_{10} 24}$

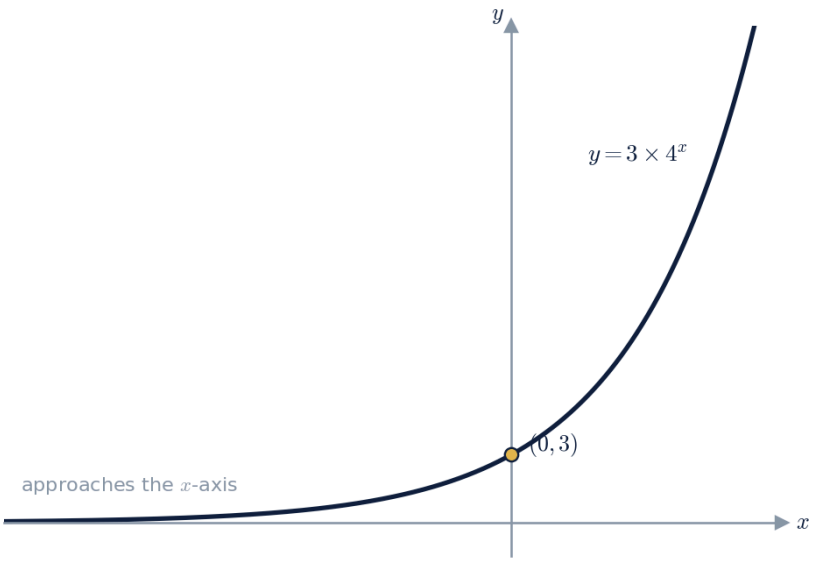
(5)

WORKED SOLUTION

Question 9 · WMA12/01 January 2020

1 Describe the sketch

$y = 3 \times 4^x$ is an increasing exponential curve. At $x = 0$, $y = 3$, so the curve crosses the y-axis at $(0, 3)$ and has no x-axis intersection.



2 Equate the two curves

At P, $6^{1-x} = 3 \times 4^x$.

3 Take logarithms

$(1 - x)\log_{10} 6 = \log_{10} 3 + x \log_{10} 4$.

4 **Collect the x terms**

$\log_{10} 6 - \log_{10} 3 = x(\log_{10} 4 + \log_{10} 6)$.

5 **Simplify the logarithms**

$\log_{10} 6 - \log_{10} 3 = \log_{10} 2$ and $\log_{10} 4 + \log_{10} 6 = \log_{10} 24$.

6 **State the x coordinate**

Therefore $x = \dfrac{\log_{10} 2}{\log_{10} 24}$.

✓ **FINAL ANSWER**

(a) Increasing exponential curve through $(0, 3)$, with no x-axis intersection.

(b) $x = \dfrac{\log_{10} 2}{\log_{10} 24}$.

WMA12/01 MAY/JUNE 2021

Question 7

10 marks | Laws of Logarithms

7. (a) Given that

$$3\log_3(2x-1) = 2 + \log_3(14x-25)$$

show that

$$2x^3 - 3x^2 - 30x + 56 = 0$$

(4)

(b) Show that -4 is a root of this cubic equation.

(2)

(c) Hence, using algebra and showing each step of your working, solve

$$3\log_3(2x-1) = 2 + \log_3(14x-25)$$

(4)

WORKED SOLUTION

Question 7 · WMA12/01 May/June 2021

1 Use the power law

$3\log_3(2x-1) = \log_3(2x-1)^3$.

2 Rewrite two as a logarithm

Since $2 = \log_3 9$, the right side is $\log_3 9 + \log_3(14x-25) = \log_3\{9(14x-25)\}$.

3 Equate the arguments

So $(2x-1)^3 = 9(14x-25)$.

4 Expand and simplify

$8x^3 - 12x^2 + 6x - 1 = 126x - 225$, which simplifies to $2x^3 - 3x^2 - 30x + 56 = 0$.

5 Check the given root

Substituting $x = -4$ gives $2(-64) - 3(16) - 30(-4) + 56 = 0$, so -4 is a root.

6 Factorise the cubic

Dividing by $(x+4)$ gives $2x^2 - 11x + 14 = (2x-7)(x-2)$.

7 **Apply the logarithm domain**

The possible roots are -4, 2, and $\frac{7}{2}$. The original logarithms require $2x-1>0$ and $14x-25>0$, so $x>\frac{25}{14}$.

8 **State the valid solutions**

Therefore the solutions are $x=2$ and $x=\frac{7}{2}$.

✓ **FINAL ANSWER**

(a) Proven.

(b) -4 is a root of the cubic.

(c) $x=2,\ \frac{7}{2}$.

WMA12/01 OCTOBER 2022

Question 10

7 marks | Laws of Logarithms

10. Given $a = \log_2 3$

(i) write, in simplest form, in terms of a,

(a) $\log_2 9$

(b) $\log_2\left(\frac{\sqrt{3}}{16}\right)$

(3)

(ii) Solve

$$3^x \times 2^{x+4} = 6$$

giving your answer, in simplest form, in terms of a.

(4)

WORKED SOLUTION

Question 10 · WMA12/01 October 2022

1 Use the power rule for log 2 of 9

$\log_2 9 = \log_2(3^2) = 2\log_2 3 = 2a$.

2 Split the second logarithm

$\log_2\left(\frac{\sqrt{3}}{16}\right) = \log_2(3^{1/2}) - \log_2(16)$.

3 Simplify in terms of a

$\log_2(3^{1/2}) = \frac{1}{2}\log_2 3 = \frac{a}{2}$ and $\log_2(16) = 4$, so the expression is $\frac{a}{2} - 4$.

4 Take log base 2 of both sides

For $3^x \times 2^{x+4} = 6$, take $\log_2$ of both sides: $\log_2(3^x) + \log_2(2^{x+4}) = \log_2 6$.

5 Rewrite each term

$\log_2(3^x) = xa$, $\log_2(2^{x+4}) = x + 4$, and $\log_2 6 = \log_2(2 \cdot 3) = 1 + a$.

6 Solve for x

So $xa + x + 4 = a + 1$. Hence $x(a+1) = a - 3$, giving $x = \frac{a-3}{a+1}$.

FINAL ANSWER

$(i)(a)\ 2a.$\quad $(i)(b)\ \dfrac{a}{2}-4.$\quad $(ii)\ x=\dfrac{a-3}{a+1}.$

WMA12/01 JANUARY 2024

Question 6

8 marks | Laws of Logarithms

6. (a) Given that

$$2\log_4(x+3) + \log_4 x = \log_4(4x+2) + \frac{1}{2}$$

show that

$$x^3 + 6x^2 + x - 4 = 0$$

(4)

(b) Given also that -1 is a root of the equation

$$x^3 + 6x^2 + x - 4 = 0$$

(i) use algebra to find the other two roots of the equation.

(3)

(ii) Hence solve

$$2\log_4(x+3) + \log_4 x = \log_4(4x+2) + \frac{1}{2}$$

(1)

WORKED SOLUTION

Question 6 · WMA12/01 January 2024

1 Combine the logarithms

$2\log_4(x+3) = \log_4((x+3)^2)$, so the left side is $\log_4(x(x+3)^2)$.

2 Rewrite one half

$\frac{1}{2} = \log_4 2$, so the right side is $\log_4(4x+2) + \log_4 2 = \log_4(8x+4)$.

3 Equate arguments

Therefore $x(x+3)^2 = 8x+4$.

4 Expand

$x(x^2+6x+9) = 8x+4$, so $x^3+6x^2+9x = 8x+4$.

5 **Show the cubic**

Rearranging gives $x^3 + 6x^2 + x - 4 = 0$.

6 **Use the given root**

Since -1 is a root, $(x + 1)$ is a factor of $x^3 + 6x^2 + x - 4$.

7 **Factor the cubic**

$x^3 + 6x^2 + x - 4 = (x + 1)(x^2 + 5x - 4)$.

8 **Solve the quadratic**

$x^2 + 5x - 4 = 0$ gives $x = \dfrac{-5 \pm \sqrt{25 + 16}}{2} = \dfrac{-5 \pm \sqrt{41}}{2}$.

9 **Apply the logarithm domain**

The original log equation requires $x > 0$. The only positive root is $x = \dfrac{-5 + \sqrt{41}}{2}$.

✓ **FINAL ANSWER**

(a) Proven.

$(b)(i)$ $x = \dfrac{-5 + \sqrt{41}}{2}$ and $x = \dfrac{-5 - \sqrt{41}}{2}$.

$(b)(ii)$ $x = \dfrac{-5 + \sqrt{41}}{2}$.

WMA12/01 MAY/JUNE R 2024

Question 9

9 marks | Laws of Logarithms

9. Given that

$$3\log_2(t+4) - 2\log_2(t-2) = 7$$

(a) verify that $t = 4$ is a solution of the above equation,

(2)

(b) show that

$$t^3 - 116t^2 + 560t - 448 = 0$$

(3)

(c) Hence, using algebra and showing your working, solve

$$3\log_2(t+4) - 2\log_2(t-2) = 7$$

giving each answer in simplest form.

(*Solutions based entirely on calculator technology are not acceptable.*)

(4)

WORKED SOLUTION

Question 9 · WMA12/01 May/June R 2024

1 Verify t equals four

Substitute $t = 4$: $3\log_2(8) - 2\log_2(2) = 3(3) - 2(1) = 7$.

2 Combine the logarithms

$3\log_2(t+4) - 2\log_2(t-2) = \log_2\left(\dfrac{(t+4)^3}{(t-2)^2}\right)$.

3 Use seven as a logarithm

$7 = \log_2(128)$, so $\dfrac{(t+4)^3}{(t-2)^2} = 128$.

4 Clear the denominator

$(t+4)^3 = 128(t-2)^2$.

5 Expand

$t^3 + 12t^2 + 48t + 64 = 128t^2 - 512t + 512$.

6 **Show the cubic**

Rearranging gives $t^3 - 116t^2 + 560t - 448 = 0$.

7 **Use the known root**

Since $t = 4$ is a root, $(t - 4)$ is a factor.

8 **Factor the cubic**

$t^3 - 116t^2 + 560t - 448 = (t - 4)(t^2 - 112t + 112)$.

9 **Solve the quadratic**

$t^2 - 112t + 112 = 0$ gives $t = \dfrac{112 \pm \sqrt{112^2 - 4(112)}}{2} = 56 \pm 12\sqrt{21}$.

10 **Use the log domain**

The original equation requires $t > 2$. The value $56 - 12\sqrt{21}$ is less than 2, so it is rejected. Hence $t = 4$ or $t = 56 + 12\sqrt{21}$.

✓ **FINAL ANSWER**

(a) Verified.

(b) Proven.

(c) $t = 4$ or $t = 56 + 12\sqrt{21}$.

WMA12/01 OCTOBER 2024

Question 6

5 marks | Laws of Logarithms

6. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

Using the laws of logarithms, solve

$$\log_4(12 - 2x) = 2 + 2\log_4(x + 1)$$

(5)

WORKED SOLUTION

Question 6 · WMA12/01 October 2024

1 State the domain

The logarithms require $12 - 2x > 0$ and $x + 1 > 0$, so $-1 < x < 6$.

2 Combine the right hand side

$2 = \log_4(16)$, so $2 + 2\log_4(x + 1) = \log_4\left(16(x + 1)^2\right)$.

3 Equate the arguments

$\log_4(12 - 2x) = \log_4\left(16(x + 1)^2\right)$ gives $12 - 2x = 16(x + 1)^2$.

4 Form the quadratic

$12 - 2x = 16x^2 + 32x + 16$, so $8x^2 + 17x + 2 = 0$.

5 Factor and solve

$8x^2 + 17x + 2 = (8x + 1)(x + 2) = 0$, so $x = -\frac{1}{8}$ or $x = -2$.

6 Check the domain

The value $x = -2$ is outside $-1 < x < 6$, so the solution is $x = -\frac{1}{8}$.

✓ **FINAL ANSWER**

$x = -\frac{1}{8}$.

WMA12/01 JANUARY 2025

Question 7

8 marks | Laws of Logarithms

7. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(i) The table below shows values of x and y, where $y = \log_{10}(x + 5)$, for x values between -1 and 4

x	-1	0	1	2	3	4
$y = \log_{10}(x + 5)$	$\log_{10} 4$	$\log_{10} 5$	$\log_{10} 6$	$\log_{10} 7$	$\log_{10} 8$	$\log_{10} 9$

Using the trapezium rule with all the y values in the given table, show that

$$\int_{-1}^{4} \log_{10}(x + 5)\,\mathrm{d}x \approx \log_{10} k$$

where k is an integer to be found.

(3)

(ii) Find the value of a such that

$$2\log_5(5 - a) - \log_5(a + 25) = 1$$

(5)

WORKED SOLUTION

Question 7 · WMA12/01 January 2025

1 **Use the trapezium rule**

The width is 1, so

$$\int_{-1}^{4} \log_{10}(x + 5)\,dx \approx \frac{1}{2}\{\log_{10} 4 + \log_{10} 9 + 2(\log_{10} 5 + \log_{10} 6 + \log_{10} 7 + \log_{10} 8)\}.$$

2 **Combine the logarithms**

This is $\log_{10}\left(\sqrt{4 \times 9} \times 5 \times 6 \times 7 \times 8\right)$.

3 **Find k**

$\sqrt{4 \times 9} \times 5 \times 6 \times 7 \times 8 = 6 \times 1680 = 10080$, so $k = 10080$.

4 **State the domain**

The logarithms require $5 - a > 0$ and $a + 25 > 0$, so $-25 < a < 5$.

5 **Combine the logarithms**

$2\log_5(5-a) - \log_5(a+25) = \log_5\left(\dfrac{(5-a)^2}{a+25}\right)$.

6 **Use one as a logarithm**

$1 = \log_5 5$, so $\dfrac{(5-a)^2}{a+25} = 5$.

7 **Form the quadratic**

$(5-a)^2 = 5(a+25)$, so $a^2 - 10a + 25 = 5a + 125$. Hence $a^2 - 15a - 100 = 0$.

8 **Solve and check**

$(a-20)(a+5) = 0$, so $a = 20$ or $a = -5$. Only $a = -5$ is in the domain, so $a = -5$.

✓ **FINAL ANSWER**

(i) $k = 10080$.

(ii) $a = -5$.

WMA12/01 MAY/JUNE 2025

Question 6

10 marks | Laws of Logarithms

6: **In this question you must show all stages of your working.**
Solutions relying on calculator technology are not acceptable.

(i) Solve

$$2\log_2(4-x) = 3 + \log_2\left(\frac{x+11}{2}\right)$$

(5)

(ii) The curves C_1 and C_2 with equations

$$y = 3^{2x+1} \qquad \text{and} \qquad y = 6 \times 3^x$$

meet at the point P.

Find the exact coordinates of P, writing your answer in the form $(\log_3 a, b)$ where a and b are integers.

(5)

WORKED SOLUTION

Question 6 · WMA12/01 May/June 2025

1 **State the domain**

The logarithms require $4 - x > 0$ and $\frac{x+11}{2} > 0$, so $-11 < x < 4$.

2 **Use log laws**

$2\log_2(4-x) = \log_2((4-x)^2)$ and $3 + \log_2\left(\frac{x+11}{2}\right) = \log_2\left(8 \cdot \frac{x+11}{2}\right)$.

3 **Equate the arguments**

Thus $(4-x)^2 = 4(x+11)$.

4 **Solve the quadratic**

$x^2 - 8x + 16 = 4x + 44$, so $x^2 - 12x - 28 = 0$. Hence $(x-14)(x+2) = 0$.

5 **Check the domain**

The root 14 is outside the domain, so $x = -2$.

6 **Equate the curves**

At P, $3^{2x+1} = 6 \times 3^x$.

7 **Let u equal 3 to the x**

Let $u = 3^x$. Then $3^{2x+1} = 3u^2$, so $3u^2 = 6u$.

8 **Find u**

Since $u > 0$, divide by $3u$ to get $u = 2$. Hence $3^x = 2$ and $x = \log_3 2$.

9 **Find y**

$y = 6 \times 3^x = 6 \times 2 = 12$, so $P = (\log_3 2, 12)$.

✓ **FINAL ANSWER**

$(i)\ x = -2$.

$(ii)\ P = (\log_3 2, 12)$.

WMA12/01 OCTOBER 2025

Question 9

10 marks | Laws of Logarithms

9. **In this question you must show detailed reasoning.**
Solutions relying on calculator technology are not acceptable.

(i) Solve

$$2\log_3(4x+5) - \log_3(x+3) = 2$$

(5)

(ii) Given that $a > 0$, $b > 0$ and

$$\log_{10} a + \log_{10} b = \log_{10}(a+b)$$

(a) prove that $a = \dfrac{b}{b-1}$

(3)

(b) Hence write down the full restriction on the value of b, giving a reason for your answer.

(2)

WORKED SOLUTION

Question 9 · WMA12/01 October 2025

1 **State the domain**

The logarithms require $4x+5>0$ and $x+3>0$, so $x > -\dfrac{5}{4}$.

2 **Combine the logarithms**

$2\log_3(4x+5) - \log_3(x+3) = \log_3\left(\dfrac{(4x+5)^2}{x+3}\right)$.

3 **Use two as a logarithm**

$2 = \log_3 9$, so $\dfrac{(4x+5)^2}{x+3} = 9$.

4 **Form the quadratic**

$(4x+5)^2 = 9(x+3)$, so $16x^2+40x+25 = 9x+27$. Hence $16x^2+31x-2=0$.

5 **Solve the quadratic**

$16x^2+31x-2 = (16x-1)(x+2) = 0$, so $x = \dfrac{1}{16}$ or $x=-2$.

6 **Check the domain**

$x = -2$ is outside the domain, so $x = \frac{1}{16}$.

7 **Use the logarithm equation**

$\log_{10} a + \log_{10} b = \log_{10}(a + b)$ gives $\log_{10}(ab) = \log_{10}(a + b)$.

8 **Equate the arguments**

Since the logarithm function is one-to-one, $ab = a + b$.

9 **Rearrange for a**

$ab - a = b$, so $a(b - 1) = b$ and therefore $a = \frac{b}{b - 1}$.

10 **Restrict b**

Given $a > 0$ and $b > 0$, the fraction $\frac{b}{b - 1}$ must be positive. Since the numerator is positive, the denominator must also be positive, so $b > 1$.

✓ **FINAL ANSWER**

$(i)\ x = \frac{1}{16}$.

$(ii)(a)$ Proven.

$(ii)(b)\ b > 1$.

WMA12/01 JANUARY 2026

Question 7

9 marks | Laws of Logarithms

7. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(i) Find the exact solution of the equation

$$8^{2x-5} = 20$$

giving the answer in the form $a + b\log_2 5$ where a and b are rational constants.

(4)

(ii) Using the laws of logarithms, solve the equation

$$\log_3(13 + 2y) + 3 = 2\log_3(4 - y)$$

(5)

WORKED SOLUTION

Question 7 · WMA12/01 January 2026

1 Take logarithms base 2

From $8^{2x-5} = 20$, take $\log_2$ of both sides to get $(2x - 5)\log_2 8 = \log_2 20$.

2 Simplify the logarithms

Since $\log_2 8 = 3$ and $\log_2 20 = \log_2(4 \cdot 5) = 2 + \log_2 5$, we get $3(2x - 5) = 2 + \log_2 5$.

3 Solve for x

$6x - 15 = 2 + \log_2 5$, so $6x = 17 + \log_2 5$ and $x = \frac{17}{6} + \frac{1}{6}\log_2 5$.

4 Use log laws

$3 = \log_3 27$ and $2\log_3(4 - y) = \log_3(4 - y)^2$.

5 Remove the logarithms

$\log_3(13 + 2y) + \log_3 27 = \log_3(4 - y)^2$, so $27(13 + 2y) = (4 - y)^2$.

6 Solve the quadratic

$351 + 54y = y^2 - 8y + 16$, hence $y^2 - 62y - 335 = 0 = (y - 67)(y + 5)$.

7 Check the logarithm domains

We need $13 + 2y > 0$ and $4 - y > 0$. Therefore $y = 67$ is not valid, and the solution is $y = -5$.

FINAL ANSWER

$(i)\ x = \dfrac{17}{6} + \dfrac{1}{6}\log_2 5. \quad (ii)\ y = -5.$

Question 9

9 marks | Laws of Logarithms | Main topic: Modelling with Sequences & Series

9. A scientist is using carbon-14 dating to determine the age of some wooden items.

The equation for carbon-14 dating an item is given by

$$N = k\lambda^t$$

where

- N grams is the amount of carbon-14 **currently** present in the item
- k grams was the **initial** amount of carbon-14 present in the item
- t is the number of years since the item was made
- λ is a constant, with $0 < \lambda < 1$

(a) Sketch the graph of N against t for $k = 1$

(2)

Given that it takes 5700 years for the amount of carbon-14 to reduce to half its initial value,

(b) show that the value of the constant λ is 0.999878 to 6 decimal places.

(2)

Given that Item A

- is known to have had 15 grams of carbon-14 present initially
- is thought to be 3250 years old

(c) calculate, to 3 significant figures, how much carbon-14 the equation predicts is currently in Item A.

(2)

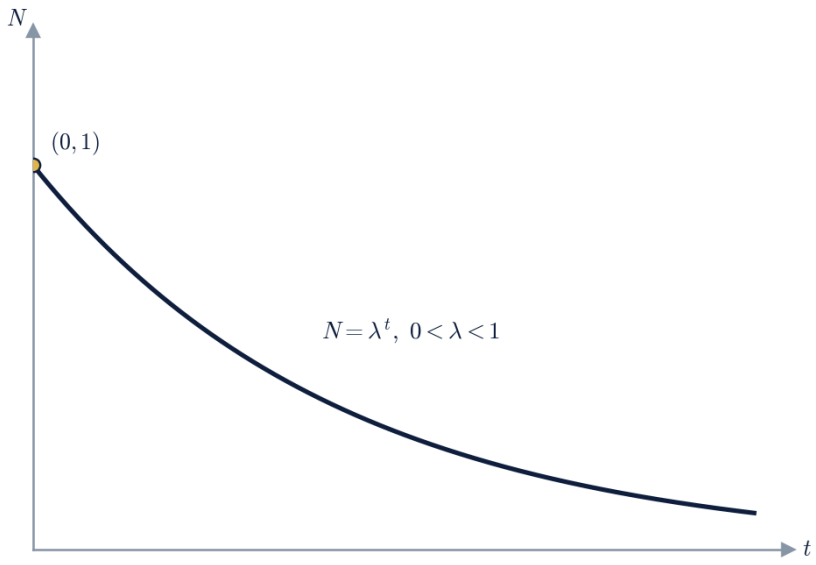
Item B is known to have initially had 25 grams of carbon-14 present, but only 18 grams now remain.

(d) Use algebra to calculate the age of Item B to the nearest 100 years.

(3)

1 **Sketch the model for k equals one**

For $k = 1$, $N = \lambda^t$ with $0 < \lambda < 1$. The graph starts at $(0, 1)$, decreases as t increases, and tends towards 0.



2 **Use the half-life information**

After 5700 years, $N = \dfrac{k}{2}$, so $k\lambda^{5700} = \dfrac{k}{2}$.

3 **Find lambda**

$\lambda^{5700} = \dfrac{1}{2}$, hence $\lambda = \left(\dfrac{1}{2}\right)^{1/5700} = 0.999878\ldots$, so $\lambda = 0.999878$ to 6 decimal places.

4 **Model item A**

For item A, $k = 15$ and $t = 3250$, so $N = 15\lambda^{3250}$.

5 **Calculate item A**

$N = 15(0.9998784026\ldots)^{3250} = 10.103\ldots$, so the current amount is 10.1 g to 3 significant figures.

6 **Set up item B**

For item B, $18 = 25\lambda^t$, so $\lambda^t = \dfrac{18}{25}$.

7 **Take logarithms**

$t \log \lambda = \log\left(\dfrac{18}{25}\right)$, so $t = \dfrac{\log(18/25)}{\log \lambda}$.

8 **Round the age**

Using $\lambda = (1/2)^{1/5700}$ gives $t = 2701\ldots$, so the age is 2700 years to the nearest 100 years.

FINAL ANSWER

(a) Decreasing exponential through $(0, 1)$.

(b) $\lambda = 0.999878$.

(c) $10.1\ \text{g}$. (d) 2700 years.

WMA12/01 MAY/JUNE 2024

Question 10

8 marks | Laws of Logarithms | Main topic: Modelling with Sequences & Series

10. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The number of dormice and the number of voles on an island are being monitored.

Initially there are 2000 dormice on the island.

A model predicts that the number of dormice will increase by 3% each year, so that the numbers of dormice on the island at the end of each year form a geometric sequence.

(a) Find, according to the model, the number of dormice on the island 6 years after monitoring began. Give your answer to 3 significant figures.

(2)

The number of voles on the island is being monitored over the same period of time.

Given that
- 4 years after monitoring began there were 3690 voles on the island
- 7 years after monitoring began there were 3470 voles on the island
- the number of voles on the island at the end of each year is modelled as a geometric sequence

(b) find the equation of this model in the form

$$N = ab^t$$

where N is the number of voles, t years after monitoring began and a and b are constants. Give the value of a and the value of b to 2 significant figures.

(3)

When $t = T$, the number of dormice on the island is equal to the number of voles on the island.

(c) Find, according to the models, the value of T, giving your answer to one decimal place.

(3)

WORKED SOLUTION

Question 10 · WMA12/01 May/June 2024

1 **Model the dormice**

The dormice model is $D = 2000(1.03)^t$.

2 **Find the dormice after six years**

$D = 2000(1.03)^6 = 2388.10\ldots$, so there are $\mathbf{2390}$ dormice to 3 significant figures.

3 **Set up the vole model**

Let the vole model be $N = ab^t$.

4 **Use the two data points**

At $t = 4$, $ab^4 = 3690$. At $t = 7$, $ab^7 = 3470$.

5 **Find b**

Dividing gives $b^3 = \dfrac{3470}{3690}$, so $b = 0.9797\ldots$, which is 0.98 to 2 significant figures.

6 **Find a**

$a = \dfrac{3690}{b^4} = 4005.18\ldots$, which is 4000 to 2 significant figures.

7 **Write the vole model**

So the model is approximately $N = 4000(0.98)^t$.

8 **Set the two models equal**

Use $2000(1.03)^T = ab^T$ with the unrounded constants from the vole model.

9 **Solve using logarithms**

$\left(\dfrac{1.03}{b}\right)^T = \dfrac{a}{2000}$, so $T = \dfrac{\log(a/2000)}{\log(1.03/b)}$.

10 **Calculate T**

Substituting $a = 4005.18\ldots$ and $b = 0.9797\ldots$ gives $T = 13.875\ldots$, so $T = 13.9$ to one decimal place.

✓ **FINAL ANSWER**

(a) 2390 dormice.

(b) $N = 4000(0.98)^t$ to 2 s.f. constants.

(c) $T = 13.9$.

TOPIC

Trigonometric Equations

Every past-paper question on this topic, October 2019 to January 2026.

WMA12/01 OCTOBER 2019

Question 9

12 marks | Trigonometric Equations

9. Solutions based entirely on graphical or numerical methods are not acceptable in this question.

(i) Solve, for $0 \leqslant \theta < 180°$, the equation

$$3\sin(2\theta - 10°) = 1$$

giving your answers to one decimal place.

(4)

(ii) The first three terms of an arithmetic sequence are

$$\sin\alpha, \quad \frac{1}{\tan\alpha} \quad \text{and} \quad 2\sin\alpha$$

where α is a constant.

(a) Show that $2\cos\alpha = 3\sin^2\alpha$

(3)

Given that $\pi < \alpha < 2\pi$,

(b) find, showing all working, the value of α to 3 decimal places.

(5)

WORKED SOLUTION

Question 9 · WMA12/01 October 2019

1 Solve the sine equation

Let $u = 2\theta - 10°$. Then $\sin u = \frac{1}{3}$.

2 Find the valid angle values

Since $0 \leq \theta < 180°$, $-10° \leq u < 350°$. In this interval, $u = 19.471\ldots°$ or $u = 160.529\ldots°$.

3 Return to theta

$\theta = \frac{u + 10°}{2}$, giving $\theta = 14.7°$ and $85.3°$ to one decimal place.

4 **Use the arithmetic sequence condition**

For three consecutive terms of an arithmetic sequence, second minus first equals third minus second. Hence $\frac{1}{\tan\alpha} - \sin\alpha = 2\sin\alpha - \frac{1}{\tan\alpha}$.

5 **Prove the required identity**

This gives $\frac{2}{\tan\alpha} = 3\sin\alpha$. Since $\frac{1}{\tan\alpha} = \frac{\cos\alpha}{\sin\alpha}$, we get $\frac{2\cos\alpha}{\sin\alpha} = 3\sin\alpha$, so $2\cos\alpha = 3\sin^2\alpha$.

6 **Form a quadratic in cos alpha**

Using $\sin^2\alpha = 1 - \cos^2\alpha$, $2\cos\alpha = 3(1-\cos^2\alpha)$, so $3\cos^2\alpha + 2\cos\alpha - 3 = 0$.

7 **Solve in the given interval**

$\cos\alpha = \frac{-1+\sqrt{10}}{3}$; the other root is less than -1. Since $\pi < \alpha < 2\pi$ and cosine is positive, α is in the fourth quadrant.

8 **State alpha**

$\alpha = 2\pi - \cos^{-1}\left(\frac{-1+\sqrt{10}}{3}\right) = 5.517$ radians to 3 decimal places.

✓ **FINAL ANSWER**

$(i)\ \theta = 14.7°,\ 85.3°.$ $(ii)(a)$ Proven.

$(ii)(b)\ \alpha = 5.517$ radians.

WMA12/01 JANUARY 2020

Question 7

7 marks | Trigonometric Equations

7. (a) Show that the equation

$$8\tan\theta = 3\cos\theta$$

may be rewritten in the form

$$3\sin^2\theta + 8\sin\theta - 3 = 0$$

(3)

(b) Hence solve, for $0 \leqslant x \leqslant 90°$, the equation

$$8\tan 2x = 3\cos 2x$$

giving your answers to 2 decimal places.

(4)

WORKED SOLUTION

Question 7 · WMA12/01 January 2020

1 Rewrite tan theta

$8\tan\theta = 3\cos\theta$ becomes $8\dfrac{\sin\theta}{\cos\theta} = 3\cos\theta$.

2 Clear the denominator

Multiplying by $\cos\theta$ gives $8\sin\theta = 3\cos^2\theta$.

3 Use the identity

Since $\cos^2\theta = 1 - \sin^2\theta$, $8\sin\theta = 3(1 - \sin^2\theta)$, so $3\sin^2\theta + 8\sin\theta - 3 = 0$.

4 Apply the result to 2x

For $8\tan 2x = 3\cos 2x$, we get $3\sin^2(2x) + 8\sin(2x) - 3 = 0$.

5 Solve for sin 2x

$(3\sin 2x - 1)(\sin 2x + 3) = 0$. Since $\sin 2x$ cannot be -3, $\sin 2x = \dfrac{1}{3}$.

6 Find x

For $0 \leq x \leq 90°$, $0 \leq 2x \leq 180°$. Thus $2x = 19.471\ldots°$ or $160.529\ldots°$, so $x = 9.74°$ or $80.26°$.

FINAL ANSWER

(a) Proven.

(b) $x = 9.74°,\ 80.26°$.

WMA12/01 JANUARY 2021

Question 6

8 marks | Trigonometric Equations

6. (a) Show that the equation

$$\frac{3\sin\theta\cos\theta}{2\sin\theta - 1} = 5\tan\theta \qquad \sin\theta \neq \frac{1}{2}$$

can be written in the form

$$3\sin^3\theta + 10\sin^2\theta - 8\sin\theta = 0$$

(4)

(b) Hence solve, for $-\frac{\pi}{4} < x < \frac{\pi}{4}$

$$\frac{3\sin 2x\cos 2x}{2\sin 2x - 1} = 5\tan 2x$$

giving your answers to 3 decimal places where appropriate.

(4)

WORKED SOLUTION

Question 6 · WMA12/01 January 2021

1 Clear the fraction

Starting with $\frac{3\sin\theta\cos\theta}{2\sin\theta - 1} = 5\tan\theta$, multiply by $(2\sin\theta - 1)\cos\theta$.

2 Use tan theta

This gives $3\sin\theta\cos^2\theta = 5\sin\theta(2\sin\theta - 1)$.

3 Use cos squared

Substitute $\cos^2\theta = 1 - \sin^2\theta$: $3\sin\theta(1 - \sin^2\theta) = 10\sin^2\theta - 5\sin\theta$.

4 Collect terms

Rearranging gives $3\sin^3\theta + 10\sin^2\theta - 8\sin\theta = 0$.

5 Factor the sine equation

$3s^3 + 10s^2 - 8s = 0$ gives $s(3s - 2)(s + 4) = 0$, where $s = \sin 2x$.

6 **Use the valid sine values**

Since $-\frac{\pi}{2} < 2x < \frac{\pi}{2}$, the valid values are $\sin 2x = 0$ and $\sin 2x = \frac{2}{3}$.

7 **Find x**

$2x = 0$ gives $x = 0$. Also $2x = \sin^{-1}\left(\frac{2}{3}\right)$, so $x = 0.364863\ldots = 0.365$ to 3 decimal places.

✓ **FINAL ANSWER**

(a) Proven.

(b) $x = 0,\ 0.365$.

WMA12/01 MAY/JUNE 2021

Question 8

10 marks | Trigonometric Equations

8. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(i) Solve, for $0 < \theta < 360°$, the equation

$$3\sin(\theta + 30°) = 7\cos(\theta + 30°)$$

giving your answers to one decimal place.

(4)

(ii) (a) Show that the equation

$$3\sin^3 x = 5\sin x - 7\sin x\cos x$$

can be written in the form

$$\sin x(a\cos^2 x + b\cos x + c) = 0$$

where a, b and c are constants to be found.

(b) Hence solve for $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$ the equation

$$3\sin^3 x = 5\sin x - 7\sin x\cos x$$

(6)

WORKED SOLUTION

Question 8 · WMA12/01 May/June 2021

1 **Solve for theta plus thirty**

Let $u = \theta + 30°$. Then $3\sin u = 7\cos u$, so $\tan u = \frac{7}{3}$.

2 **Find u in the interval**

Since $0 < \theta < 360°$, $30° < u < 390°$. Thus $u = 66.801\ldots°$ or $246.801\ldots°$.

3 **Return to theta**

$\theta = u - 30°$, giving $\theta = 36.8°$ and $216.8°$ to one decimal place.

4 **Rearrange the second equation**

$3\sin^3 x = 5\sin x - 7\sin x\cos x$ gives $3\sin^3 x - 5\sin x + 7\sin x\cos x = 0$.

5 **Factor out sine**

$\sin x(3\sin^2 x - 5 + 7\cos x) = 0$.

6 **Use sin squared**

Since $\sin^2 x = 1 - \cos^2 x$, this becomes $\sin x(-3\cos^2 x + 7\cos x - 2) = 0$, or $\sin x(3\cos^2 x - 7\cos x + 2) = 0$.

7 **Solve in the interval**

For $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, $\sin x = 0$ gives $x = 0$. Also $3\cos^2 x - 7\cos x + 2 = 0$ gives $(3\cos x - 1)(\cos x - 2) = 0$.

8 **State the solutions**

$\cos x = 2$ is impossible, so $\cos x = \frac{1}{3}$. The solutions are $x = -\cos^{-1}\left(\frac{1}{3}\right)$, 0, and $\cos^{-1}\left(\frac{1}{3}\right)$.

✓ **FINAL ANSWER**

$(i)\ \theta = 36.8°,\ 216.8°.\quad (ii)(a)\ a = 3,\ b = -7,\ c = 2.\quad (ii)(b)\ x = -\cos^{-1}\left(\frac{1}{3}\right),\ 0,\ \cos^{-1}\left(\frac{1}{3}\right)$

.

WMA12/01 OCTOBER 2021

Question 10

10 marks | Trigonometric Equations

10. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(i) Solve, for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$$\tan^2\left(2x + \frac{\pi}{4}\right) = 3$$

(5)

(ii) Solve, for $0 < \theta < 360°$

$$(2\sin\theta - \cos\theta)^2 = 1$$

giving your answers, as appropriate, to one decimal place.

(5)

(Total 10 marks)

WORKED SOLUTION

Question 10 · WMA12/01 October 2021

1 **Take the square root**

Let $u = 2x + \frac{\pi}{4}$. From $\tan^2 u = 3$, $\tan u = \pm\sqrt{3}$.

2 **Write the angle families**

Thus $u = \frac{\pi}{3} + n\pi$ or $u = -\frac{\pi}{3} + n\pi$, where n is an integer.

3 **Return to x**

So $x = \frac{\pi}{24} + \frac{n\pi}{2}$ or $x = -\frac{7\pi}{24} + \frac{n\pi}{2}$.

4 **Use the interval**

For $-\frac{\pi}{2} < x < \frac{\pi}{2}$, the solutions are $x = -\frac{11\pi}{24}, -\frac{7\pi}{24}, \frac{\pi}{24}, \frac{5\pi}{24}$.

5 **Expand the squared equation**

$(2\sin\theta - \cos\theta)^2 = 1$ gives $4\sin^2\theta - 4\sin\theta\cos\theta + \cos^2\theta = 1$.

6 **Use the identity**

Since $1 = \sin^2\theta + \cos^2\theta$, this simplifies to $3\sin^2\theta - 4\sin\theta\cos\theta = 0$.

7 **Factor**

$\sin\theta(3\sin\theta - 4\cos\theta) = 0$.

8 **Solve the two cases**

$\sin\theta = 0$ gives $\theta = 180°$ in the interval. Also $3\sin\theta = 4\cos\theta$ gives $\tan\theta = \frac{4}{3}$.

9 **State theta**

For $0 < \theta < 360°$, $\tan\theta = \frac{4}{3}$ gives $\theta = 53.1°$ and $233.1°$. Therefore $\theta = 53.1°, 180°, 233.1°$.

✓ **FINAL ANSWER**

$(i)\ x = -\frac{11\pi}{24},\ -\frac{7\pi}{24},\ \frac{\pi}{24},\ \frac{5\pi}{24}.$ $(ii)\ \theta = 53.1°,\ 180°,\ 233.1°.$

WMA12/01 JANUARY 2022

Question 7

8 marks | Trigonometric Equations

7. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(i) Solve, for $-90° < x < 90°$, the equation

$$3\sin(2x - 15°) = \cos(2x - 15°)$$

giving your answers to one decimal place.

(4)

(ii) Solve, for $0 < \theta < 2\pi$, the equation

$$4\sin^2\theta + 8\cos\theta = 3$$

giving your answers to 3 significant figures.

(4)

WORKED SOLUTION

Question 7 · WMA12/01 January 2022

1 Solve the first tangent equation

Let $u = 2x - 15°$. Then $3\sin u = \cos u$, so $\tan u = \frac{1}{3}$.

2 Use the interval for u

Since $-90° < x < 90°$, $-195° < u < 165°$.

3 Find x

$u = 18.434\ldots°$ or $u = -161.565\ldots°$, so $x = 16.7°$ or $x = -73.3°$.

4 Rewrite the second equation

$4\sin^2\theta + 8\cos\theta = 3$.

5 Use sin squared

Substitute $\sin^2\theta = 1 - \cos^2\theta$ to get $4(1 - \cos^2\theta) + 8\cos\theta = 3$.

6 Solve for cos theta

This gives $4\cos^2\theta - 8\cos\theta - 1 = 0$, so $\cos\theta = 1 \pm \frac{\sqrt{5}}{2}$.

7 **Choose the valid cosine**

Only $\cos\theta = 1 - \frac{\sqrt{5}}{2}$ lies between -1 and 1.

8 **State theta**

For $0 < \theta < 2\pi$, $\theta = 1.69$ or 4.59 to 3 significant figures.

FINAL ANSWER

$(i)\ x = -73.3°,\ 16.7°.\quad (ii)\ \theta = 1.69,\ 4.59.$

WMA12/01 JANUARY 2023

Question 8

9 marks | Trigonometric Equations

8. **In this question you must show all stages of your working.**

Solutions based entirely on calculator technology are not acceptable.

(i) Solve, for $-\frac{\pi}{2} < x < \pi$, the equation

$$5\sin(3x + 0.1) + 2 = 0$$

giving your answers, **in radians**, to 2 decimal places.

(4)

(ii) Solve, for $0 < \theta < 360°$, the equation

$$2\tan\theta\sin\theta = 5 + \cos\theta$$

giving your answers, **in degrees**, to one decimal place.

(5)

WORKED SOLUTION

Question 8 · WMA12/01 January 2023

1 **Isolate the sine term**

$5\sin(3x + 0.1) + 2 = 0$ gives $\sin(3x + 0.1) = -0.4$.

2 **Find the angle interval**

Since $-\frac{\pi}{2} < x < \pi$, multiplying by 3 and adding 0.1 gives $-\frac{3\pi}{2} + 0.1 < 3x + 0.1 < 3\pi + 0.1$.

3 **Solve for the inner angle**

In this interval, $3x + 0.1 = -2.7301\ldots, -0.4115\ldots, 3.5531\ldots, 5.8717\ldots$.

4 **Find x values**

Subtract 0.1 and divide by 3: $x = -0.9433\ldots, -0.1705\ldots, 1.1510\ldots, 1.9239\ldots$.

5 **Round the radian answers**

To 2 decimal places, $x = -0.94, -0.17, 1.15, 1.92$.

6 **Rewrite the second equation**

$2\tan\theta\sin\theta = 5 + \cos\theta$. Using $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ gives $\dfrac{2\sin^2\theta}{\cos\theta} = 5 + \cos\theta$.

7 **Use sine squared**

Multiplying by $\cos\theta$ gives $2\sin^2\theta = 5\cos\theta + \cos^2\theta$. Substitute $\sin^2\theta = 1 - \cos^2\theta$.

8 **Solve for cos theta**

$2(1 - \cos^2\theta) = 5\cos\theta + \cos^2\theta$, so $3\cos^2\theta + 5\cos\theta - 2 = 0$.

9 **Factor and reject impossible value**

$(3\cos\theta - 1)(\cos\theta + 2) = 0$. Since $\cos\theta = -2$ is impossible, $\cos\theta = \dfrac{1}{3}$.

10 **Find theta**

For $0 < \theta < 360°$, $\cos\theta = \dfrac{1}{3}$ gives $\theta = 70.5°$ or $289.5°$ to one decimal place.

✓ **FINAL ANSWER**

(i) $x = -0.94, -0.17, 1.15, 1.92$.

(ii) $\theta = 70.5°, 289.5°$.

WMA12/01 MAY/JUNE 2023

Question 9

7 marks | Trigonometric Equations

9. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that

$$3\cos\theta(\tan\theta\sin\theta+3)=11-5\cos\theta$$

may be written as

$$3\cos^2\theta-14\cos\theta+8=0$$

(3)

(b) Hence solve, for $0<x<360°$

$$3\cos 2x(\tan 2x\sin 2x+3)=11-5\cos 2x$$

giving your answers to one decimal place.

(4)

WORKED SOLUTION

Question 9 · WMA12/01 May/June 2023

1 **Rewrite tan theta sin theta**

$\tan\theta\sin\theta=\dfrac{\sin^2\theta}{\cos\theta}$.

2 **Simplify the left side**

$3\cos\theta\left(\tan\theta\sin\theta+3\right)=3\sin^2\theta+9\cos\theta$.

3 **Use sine squared**

The equation becomes $3(1-\cos^2\theta)+9\cos\theta=11-5\cos\theta$.

4 **Rearrange**

$3-3\cos^2\theta+9\cos\theta=11-5\cos\theta$, so $3\cos^2\theta-14\cos\theta+8=0$.

5 **Apply the result to 2x**

For the equation in part (b), take $\theta=2x$. Then $3\cos^2(2x)-14\cos(2x)+8=0$.

6 **Solve the quadratic**

Let $u=\cos(2x)$. Then $3u^2-14u+8=0$, so $(3u-2)(u-4)=0$.

7 **Reject the impossible value**

$u = 4$ is impossible for cosine, so $\cos(2x) = \frac{2}{3}$.

8 **Use the interval**

Since $0 < x < 360°$, we have $0 < 2x < 720°$.

9 **Find x**

In this interval, $2x = 48.189\ldots°, 311.810\ldots°, 408.189\ldots°, 671.810\ldots°$. Dividing by 2 gives $x = 24.1°, 155.9°, 204.1°, 335.9°$ to one decimal place.

✓ **FINAL ANSWER**

(a) Proven.

(b) $x = 24.1°, 155.9°, 204.1°, 335.9°$.

Question 9

14 marks | Trigonometric Equations

9. **In this question you must show detailed reasoning.**

Solutions relying entirely on calculator technology are not acceptable.

(i) Solve, for $0 \leqslant x < 360°$, the equation

$$\sin x \tan x = 5$$

giving your answers to one decimal place.

(6)

(ii)

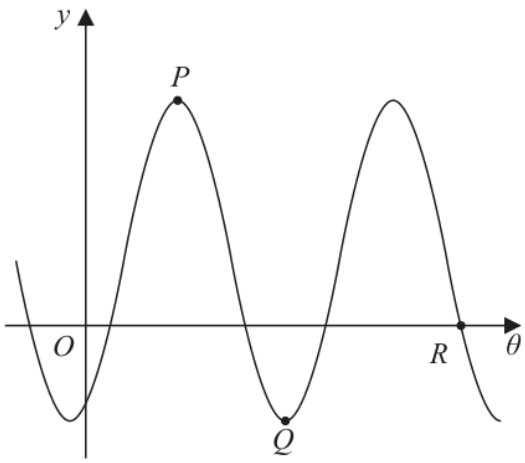


Figure 1

Figure 1 shows a sketch of part of the curve with equation

$$y = A\sin\left(2\theta - \frac{3\pi}{8}\right) + 2$$

where A is a constant and θ is measured in radians.

The points P, Q and R lie on the curve and are shown in Figure 1.

Given that the y coordinate of P is 7

(a) state the value of A,

(1)

(b) find the exact coordinates of Q,

(3)

(c) find the value of θ at R, giving your answer to 3 significant figures.

(4)

1 Rewrite the equation

$\sin x \tan x = 5$ becomes $\dfrac{\sin^2 x}{\cos x} = 5$.

2 Use sine squared

Multiplying by $\cos x$ gives $\sin^2 x = 5\cos x$. Since $\sin^2 x = 1 - \cos^2 x$, $1 - \cos^2 x = 5\cos x$.

3 Solve for cos x

$\cos^2 x + 5\cos x - 1 = 0$, so $\cos x = \dfrac{-5 \pm \sqrt{29}}{2}$.

4 Choose the possible cosine value

$\dfrac{-5 - \sqrt{29}}{2} < -1$, so the valid value is $\cos x = \dfrac{-5 + \sqrt{29}}{2}$.

5 Find x

For $0 \leq x < 360°$, this gives $x = 78.9°$ or $281.1°$ to one decimal place.

6 Find A from P

The point P is a maximum point on the sine curve. Its y coordinate is 7, so $A + 2 = 7$ and $A = 5$.

7 Locate Q

Point Q is the next minimum after P, so $\sin\left(2\theta - \dfrac{3\pi}{8}\right) = -1$.

8 Find theta at Q

For this minimum, $2\theta - \dfrac{3\pi}{8} = \dfrac{3\pi}{2}$, so $2\theta = \dfrac{15\pi}{8}$ and $\theta = \dfrac{15\pi}{16}$.

9 Find y at Q

At the minimum, $y = 2 - 5 = -3$, so $Q = \left(\dfrac{15\pi}{16}, -3\right)$.

10 Set up R

At R, $y = 0$, so $5\sin\left(2\theta - \dfrac{3\pi}{8}\right) + 2 = 0$.

11 Choose the shown crossing

Thus $\sin\left(2\theta - \dfrac{3\pi}{8}\right) = -\dfrac{2}{5}$. The crossing R shown after the second maximum has $2\theta - \dfrac{3\pi}{8} = 3\pi + \sin^{-1}\left(\dfrac{2}{5}\right)$.

12 **Find theta at R**

$\theta = \frac{1}{2}\left(3\pi + \sin^{-1}\left(\frac{2}{5}\right) + \frac{3\pi}{8}\right) = 5.507\ldots$, so $\theta = 5.51$ to 3 significant figures.

FINAL ANSWER

(i) $x = 78.9°, 281.1°$.

$(ii)(a)$ $A = 5$.

$(ii)(b)$ $Q = \left(\frac{15\pi}{16}, -3\right)$.

$(ii)(c)$ $\theta = 5.51$.

WMA12/01 MAY/JUNE 2024

Question 8

12 marks | Trigonometric Equations

8. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(i) Solve, for $0 < x \leqslant \pi$, the equation

$$5\sin x \tan x + 13 = \cos x$$

giving your answer in radians to 3 significant figures.

(5)

(ii) The temperature inside a greenhouse is monitored on one particular day.

The temperature, $H°C$, inside the greenhouse, t hours after midnight, is modelled by the equation

$$H = 10 + 12\sin(kt + 18)° \qquad 0 \leqslant t < 24$$

where k is a constant.

Use the equation of the model to answer parts (a) to (c).

Given that

- the temperature inside the greenhouse was 20 °C at 6 am
- $0 < k < 20$

(a) find all possible values for k, giving each answer to 2 decimal places.

(4)

Given further that $0 < k < 10$

(b) find the maximum temperature inside the greenhouse,

(1)

(c) find the time of day at which this maximum temperature occurs.

Give your answer to the nearest minute.

(2)

WORKED SOLUTION

Question 8 · WMA12/01 May/June 2024

1 **Rewrite the trig equation**

$5\sin x \tan x + 13 = \cos x$ becomes $\dfrac{5\sin^2 x}{\cos x} + 13 = \cos x$.

2 **Multiply by cos x**

$5\sin^2 x + 13\cos x = \cos^2 x$.

3 **Use sine squared**

$5(1 - \cos^2 x) + 13\cos x = \cos^2 x$, so $6\cos^2 x - 13\cos x - 5 = 0$.

4 **Solve for cos x**

$(2\cos x - 5)(3\cos x + 1) = 0$, so $\cos x = \frac{5}{2}$ or $\cos x = -\frac{1}{3}$.

5 **Choose the valid root**

$\cos x = \frac{5}{2}$ is impossible, so $\cos x = -\frac{1}{3}$. For $0 < x \leq \pi$, $x = 1.9106\ldots$, so $x = 1.91$ radians to 3 significant figures.

6 **Use the temperature at 6 am**

At $t = 6$, $20 = 10 + 12\sin(6k + 18)°$, so $\sin(6k + 18)° = \frac{5}{6}$.

7 **Solve for k**

$\sin^{-1}\left(\frac{5}{6}\right) = 56.442\ldots°$. Since $0 < k < 20$, $18 < 6k + 18 < 138$, so $6k + 18 = 56.442\ldots$ or $123.557\ldots$.

8 **Find possible k values**

This gives $k = 6.407\ldots$ or $k = 17.592\ldots$, so $k = 6.41$ or 17.59 to 2 decimal places.

9 **Use the tighter condition**

Given further that $0 < k < 10$, use $k = 6.407\ldots$.

10 **Find the maximum temperature**

The maximum value of $\sin$ is 1, so the maximum temperature is $10 + 12 = 22°\text{C}$.

11 **Find the time of maximum**

The maximum occurs when $kt + 18 = 90$, so $t = \frac{72}{6.407\ldots} = 11.2375\ldots$ hours after midnight.

12 **Convert to time of day**

$0.2375\ldots$ hours is $14.25\ldots$ minutes, so the maximum occurs at about $11{:}14$ am.

FINAL ANSWER

(i) $x = 1.91$ radians.

$(ii)(a)$ $k = 6.41$ or 17.59.

$(ii)(b)$ $22°\text{C}$.

$(ii)(c)$ $11{:}14$ am.

WMA12/01 MAY/JUNE R 2024

Question 7

7 marks | Trigonometric Equations

7. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(i) Solve, for $0 \leqslant x < 2\pi$, the equation

$$3\sin x \tan x = 11 + \cos x$$

giving the answers in radians to 3 decimal places.

(5)

(ii) Given that

- $0 < \theta < 90°$
- $\cos\theta = \frac{1}{3}$

find, in simplest form, the exact value of $\tan\theta$

(2)

WORKED SOLUTION

Question 7 · WMA12/01 May/June R 2024

1 Rewrite tan x

$3\sin x \tan x = 11 + \cos x$ becomes $\dfrac{3\sin^2 x}{\cos x} = 11 + \cos x$.

2 Multiply by cos x

$3\sin^2 x = 11\cos x + \cos^2 x$.

3 Use sine squared

$3(1 - \cos^2 x) = 11\cos x + \cos^2 x$, so $4\cos^2 x + 11\cos x - 3 = 0$.

4 Solve for cosine

$(4\cos x - 1)(\cos x + 3) = 0$, so $\cos x = \dfrac{1}{4}$ or $\cos x = -3$.

5 Reject impossible value

$\cos x = -3$ is impossible, so $\cos x = \dfrac{1}{4}$.

6 Find x values

For $0 \leq x < 2\pi$, this gives $x = 1.318\ldots$ or $x = 4.965\ldots$, so $x = 1.318, 4.965$ radians.

7 **Use the right triangle identity**

Given $0 < \theta < 90°$ and $\cos\theta = \frac{1}{3}$, $\sin\theta = \sqrt{1 - \frac{1}{9}} = \frac{2\sqrt{2}}{3}$.

8 **Find tan theta**

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2\sqrt{2}/3}{1/3} = 2\sqrt{2}$.

✓ **FINAL ANSWER**

$(i)\ x = 1.318,\ 4.965$.

$(ii)\ 2\sqrt{2}$.

WMA12/01 OCTOBER 2024

Question 9

7 marks | Trigonometric Equations

9. **In this question you must show detailed reasoning.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that the equation

$$2\tan\theta = 3\cos\theta$$

can be written as

$$3\sin^2\theta + 2\sin\theta - 3 = 0$$

(3)

(b) Hence solve, for $-\pi < x < \pi$, the equation

$$2\tan\left(2x + \frac{\pi}{3}\right) = 3\cos\left(2x + \frac{\pi}{3}\right)$$

giving your answers to 3 significant figures.

(4)

WORKED SOLUTION

Question 9 · WMA12/01 October 2024

1 **Write tan in terms of sin and cos**

$2\tan\theta = 3\cos\theta$ gives $2\dfrac{\sin\theta}{\cos\theta} = 3\cos\theta$.

2 **Multiply by cos theta**

Since $\tan\theta$ is defined, $\cos\theta \neq 0$. Hence $2\sin\theta = 3\cos^2\theta$.

3 **Use the identity**

$2\sin\theta = 3(1 - \sin^2\theta)$, so $3\sin^2\theta + 2\sin\theta - 3 = 0$.

4 **Substitute the new angle**

Let $\theta = 2x + \dfrac{\pi}{3}$. Then $3\sin^2\theta + 2\sin\theta - 3 = 0$.

5 **Solve for sin theta**

$\sin\theta = \dfrac{-2 \pm \sqrt{40}}{6} = \dfrac{-1 \pm \sqrt{10}}{3}$. The root $\dfrac{-1 - \sqrt{10}}{3}$ is less than -1, so reject it.

6 **Find the angle range**

Since $-\pi < x < \pi$, $-\dfrac{5\pi}{3} < \theta < \dfrac{7\pi}{3}$.

7 **Find the theta values**

For $\sin\theta = \dfrac{-1+\sqrt{10}}{3}$ in this range, $\theta \approx -3.94649$, 0.804897, 2.33670, and 7.08808.

8 **Convert back to x**

Using $x = \dfrac{\theta - \pi/3}{2}$ gives $x \approx -2.49684$, -0.121150, 0.644749, and 3.02044.

9 **Round the answers**

To 3 significant figures, $x = -2.50$, -0.121, 0.645, and 3.02.

✓ **FINAL ANSWER**

(a) Proven.

(b) $x = -2.50$, -0.121, 0.645, 3.02.

WMA12/01 JANUARY 2025

Question 10

8 marks | Trigonometric Equations

10. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(a) Show that

$$\cos\theta\left(3\tan\theta+\frac{2}{\tan\theta}\right)\equiv\sin\theta+\frac{2}{\sin\theta}\qquad\theta\neq\frac{n\pi}{2}$$

(4)

(b) Hence solve, for $0<x<2\pi$, the equation

$$\cos x\left(3\tan x+\frac{2}{\tan x}\right)=4\sin x-5$$

giving your answers to 3 significant figures.

(4)

WORKED SOLUTION

Question 10 · WMA12/01 January 2025

1 **Expand the left side**

$\cos\theta\left(3\tan\theta+\dfrac{2}{\tan\theta}\right)=3\sin\theta+\dfrac{2\cos\theta}{\tan\theta}$.

2 **Rewrite the second term**

Since $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$, $\dfrac{2\cos\theta}{\tan\theta}=\dfrac{2\cos^2\theta}{\sin\theta}$.

3 **Use the identity**

$3\sin\theta+\dfrac{2\cos^2\theta}{\sin\theta}=3\sin\theta+\dfrac{2(1-\sin^2\theta)}{\sin\theta}$.

4 **Simplify**

$3\sin\theta+\dfrac{2}{\sin\theta}-2\sin\theta=\sin\theta+\dfrac{2}{\sin\theta}$, as required.

5 **Use part a**

The equation becomes $\sin x+\dfrac{2}{\sin x}=4\sin x-5$.

6 **Form a quadratic in sin x**

Multiplying by $\sin x$ gives $\sin^2 x + 2 = 4\sin^2 x - 5\sin x$, so $3\sin^2 x - 5\sin x - 2 = 0$.

7 **Solve for sin x**

$(3\sin x + 1)(\sin x - 2) = 0$. Since $\sin x = 2$ is impossible, $\sin x = -\frac{1}{3}$.

8 **Find x in the interval**

For $0 < x < 2\pi$, $\sin x = -\frac{1}{3}$ gives $x = \pi + \sin^{-1}\left(\frac{1}{3}\right)$ or $x = 2\pi - \sin^{-1}\left(\frac{1}{3}\right)$.

9 **Round the answers**

Thus $x = 3.481\ldots$ or $x = 5.943\ldots$, so to 3 significant figures $x = 3.48$ or $x = 5.94$.

✓ **FINAL ANSWER**

(a) Proven.

(b) $x = 3.48, 5.94$.

WMA12/01 MAY/JUNE 2025

Question 8

8 marks | Trigonometric Equations

8: **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(i) Solve, for $0 \leqslant x < \pi$, the equation

$$3\tan\left(2x + \frac{\pi}{5}\right) = \sqrt{3}$$

giving the answers in radians in the form $k\pi$, where k is a rational constant to be found.

(3)

(ii) Solve, for $0 \leqslant \theta < 360°$, the equation

$$5\sin\theta\tan\theta = \cos\theta + 4$$

giving your answers, in degrees, to one decimal place.

(5)

WORKED SOLUTION

Question 8 · WMA12/01 May/June 2025

1 **Solve the tangent equation**

$3\tan\left(2x + \frac{\pi}{5}\right) = \sqrt{3}$ gives $\tan\left(2x + \frac{\pi}{5}\right) = \frac{1}{\sqrt{3}}$.

2 **Write the general angle**

Therefore $2x + \frac{\pi}{5} = \frac{\pi}{6} + n\pi$.

3 **Find x**

$2x = -\frac{\pi}{30} + n\pi$, so $x = -\frac{\pi}{60} + \frac{n\pi}{2}$.

4 **Use the interval**

For $0 \leq x < \pi$, the possible values are $x = \frac{29\pi}{60}$ and $x = \frac{59\pi}{60}$.

5 **Rewrite the second equation**

Using $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $5\sin\theta\tan\theta = \cos\theta + 4$ becomes $\frac{5\sin^2\theta}{\cos\theta} = \cos\theta + 4$.

6 **Form an equation in cos theta**

Multiplying by $\cos\theta$ gives $5\sin^2\theta = \cos^2\theta + 4\cos\theta$. Using $\sin^2\theta = 1 - \cos^2\theta$ gives $6\cos^2\theta + 4\cos\theta - 5 = 0$.

7 **Solve for cos theta**

$\cos\theta = \dfrac{-2 \pm \sqrt{34}}{6}$. The negative option is less than -1, so reject it.

8 **Find theta**

$\cos\theta = \dfrac{-2 + \sqrt{34}}{6}$ gives $\theta = 50.320\ldots°$ or $309.679\ldots°$.

9 **Round the answers**

To one decimal place, $\theta = 50.3°$ or $309.7°$.

✓ **FINAL ANSWER**

$(i)\ x = \dfrac{29\pi}{60}, \dfrac{59\pi}{60}$.

$(ii)\ \theta = 50.3°, 309.7°$.

WMA12/01 JANUARY 2026

Question 8

10 marks | Trigonometric Equations

8. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(i) Solve, for $0 \leqslant \theta < 180$

$$4\tan^2(2\theta - 30)° + 1 = 49$$

giving the answers to one decimal place.

(5)

(ii) Solve, for $0 \leqslant x < 2\pi$

$$2\tan x \sin x + 3 = 0$$

giving the answers, in radians, in the form $k\pi$ where k is a rational constant.

(5)

WORKED SOLUTION

Question 8 · WMA12/01 January 2026

1 **Make the tangent expression the subject**

$4\tan^2(2\theta - 30°) + 1 = 49$ gives $\tan^2(2\theta - 30°) = 12$.

2 **Find the possible angle values**

Let $\alpha = 2\theta - 30°$. Since $0 \leq \theta < 180°$, $-30° \leq \alpha < 330°$. The solutions of $\tan\alpha = \pm\sqrt{12}$ in this range are approximately $73.9°, 106.1°, 253.9°, 286.1°$.

3 **Return to theta**

$\theta = \dfrac{\alpha + 30°}{2}$, giving $\theta = 51.9°, 68.1°, 141.9°, 158.1°$ to one decimal place.

4 **Rewrite tan x**

$2\tan x \sin x + 3 = 0$ becomes $2\left(\dfrac{\sin x}{\cos x}\right)\sin x + 3 = 0$.

5 **Form a quadratic in cos x**

Multiplying by $\cos x$ gives $2\sin^2 x + 3\cos x = 0$. Using $\sin^2 x = 1 - \cos^2 x$, $2(1 - \cos^2 x) + 3\cos x = 0$, so $2\cos^2 x - 3\cos x - 2 = 0$.

6 **Solve for x**

$(2\cos x + 1)(\cos x - 2) = 0$. Since $\cos x = 2$ is impossible, $\cos x = -\frac{1}{2}$. In $0 \leq x < 2\pi$, $x = \frac{2\pi}{3}$ or $x = \frac{4\pi}{3}$.

✓ **FINAL ANSWER**

$(i)\ \theta = 51.9°,\ 68.1°,\ 141.9°,\ 158.1°.$ $(ii)\ x = \frac{2\pi}{3},\ \frac{4\pi}{3}.$

WMA12/01 OCTOBER 2023

Question 10

12 marks | Trigonometric Equations | Main topic: Binomial Expansion

10. (i) (a) Find, in ascending powers of x, the 2nd, 3rd and 5th terms of the binomial expansion of

$$(3 + 2x)^6$$

(3)

For a particular value of x, these three terms form consecutive terms in a geometric series.

(b) Find this value of x.

(3)

(ii) In a **different** geometric series,

- the first term is $\sin^2\theta$
- the common ratio is $2\cos\theta$
- the sum to infinity is $\frac{8}{5}$

(a) Show that

$$5\cos^2\theta - 16\cos\theta + 3 = 0$$

(3)

(b) Hence find the exact value of the 2nd term in the series.

(3)

WORKED SOLUTION

Question 10 · WMA12/01 October 2023

1 Find the second term

The 2nd term of $(3+2x)^6$ is $\binom{6}{1}3^5(2x) = 6 \cdot 243 \cdot 2x = 2916x$.

2 Find the third term

The 3rd term is $\binom{6}{2}3^4(2x)^2 = 15 \cdot 81 \cdot 4x^2 = 4860x^2$.

3 Find the fifth term

The 5th term is $\binom{6}{4}3^2(2x)^4 = 15 \cdot 9 \cdot 16x^4 = 2160x^4$.

4 Use the geometric condition

For three consecutive terms in a geometric series, the middle term squared equals the product of the two outer terms.

5 **Set up the equation**

$(4860x^2)^2 = (2916x)(2160x^4)$.

6 **Solve for x**

For the non-zero geometric case, divide by x^4: $4860^2 = 2916 \cdot 2160\,x$, so $x = \dfrac{4860^2}{2916 \cdot 2160} = \dfrac{15}{4}$.

7 **Use the sum to infinity**

In the different geometric series, first term $= \sin^2\theta$, common ratio $= 2\cos\theta$, and $S_\infty = \dfrac{8}{5}$.

8 **Form the equation**

$\dfrac{\sin^2\theta}{1 - 2\cos\theta} = \dfrac{8}{5}$.

9 **Use sine squared**

$5\sin^2\theta = 8(1 - 2\cos\theta)$. With $\sin^2\theta = 1 - \cos^2\theta$, this becomes $5(1 - \cos^2\theta) = 8 - 16\cos\theta$.

10 **Show the required equation**

Rearranging gives $5\cos^2\theta - 16\cos\theta + 3 = 0$.

11 **Solve for cosine**

$5\cos^2\theta - 16\cos\theta + 3 = 0$ gives $\cos\theta = 3$ or $\cos\theta = \dfrac{1}{5}$. The possible value is $\cos\theta = \dfrac{1}{5}$.

12 **Find the second term**

The 2nd term is first term times common ratio: $\sin^2\theta(2\cos\theta) = \left(1 - \dfrac{1}{25}\right) \cdot \dfrac{2}{5} = \dfrac{24}{25} \cdot \dfrac{2}{5} = \dfrac{48}{125}$.

FINAL ANSWER

$(i)(a)\ 2916x,\ 4860x^2,\ 2160x^4$.

$(i)(b)\ x = \dfrac{15}{4}$.

$(ii)(a)$ Proven.

$(ii)(b)\ \dfrac{48}{125}$.

TOPIC

Applications of Differentiation

Every past-paper question on this topic, October 2019 to January 2026.

WMA12/01 OCTOBER 2019

Question 10

7 marks | Applications of Differentiation

10. The curve C has equation

$$y = ax^3 - 3x^2 + 3x + b$$

where a and b are constants.

Given that

- the point $(2, 5)$ lies on C
- the gradient of the curve at $(2, 5)$ is 7

(a) find the value of a and the value of b.

(4)

(b) Prove that C has no turning points.

(3)

WORKED SOLUTION

Question 10 · WMA12/01 October 2019

1 Use the point on the curve

Substitute $(2, 5)$ into $y = ax^3 - 3x^2 + 3x + b$: $5 = 8a - 12 + 6 + b$, so $8a + b = 11$.

2 Differentiate

$\frac{dy}{dx} = 3ax^2 - 6x + 3$.

3 Use the given gradient

At $x = 2$, the gradient is 7, so $12a - 12 + 3 = 7$. Hence $12a = 16$ and $a = \frac{4}{3}$.

4 Find b

Using $8a + b = 11$, $b = 11 - 8\left(\frac{4}{3}\right) = \frac{1}{3}$.

5 Check for turning points

With $a = \frac{4}{3}$, $\frac{dy}{dx} = 4x^2 - 6x + 3$.

6 **Use the discriminant**

For $4x^2 - 6x + 3 = 0$, the discriminant is $(-6)^2 - 4(4)(3) = 36 - 48 = -12 < 0$.

7 **Conclude**

The derivative is never zero, so the curve has no turning points.

✓ **FINAL ANSWER**

(a) $a = \frac{4}{3}$, $b = \frac{1}{3}$. (b) Proven: C has no turning points.

WMA12/01 JANUARY 2020

Question 10

10 marks | Applications of Differentiation

10. A curve C has equation

$$y = 4x^3 - 9x + \frac{k}{x} \qquad x > 0$$

where k is a constant.

The point P with x coordinate $\frac{1}{2}$ lies on C.

Given that P is a stationary point of C,

(a) show that $k = -\frac{3}{2}$

(4)

(b) Determine the nature of the stationary point at P, justifying your answer.

(2)

The curve C has a second stationary point.

(c) Using algebra, find the x coordinate of this second stationary point.

(4)

(Total 10 marks)

WORKED SOLUTION

Question 10 · WMA12/01 January 2020

1 Differentiate the curve

Write $y = 4x^3 - 9x + kx^{-1}$. Then $\frac{dy}{dx} = 12x^2 - 9 - kx^{-2}$.

2 Use the stationary point

At $x = \frac{1}{2}$, $\frac{dy}{dx} = 0$, so $0 = 12\left(\frac{1}{2}\right)^2 - 9 - k\left(\frac{1}{2}\right)^{-2} = 3 - 9 - 4k$.

3 Find k

$-6 - 4k = 0$, hence $k = -\frac{3}{2}$.

4 Use the second derivative

With $k = -\frac{3}{2}$, $\frac{dy}{dx} = 12x^2 - 9 + \frac{3}{2x^2}$ and $\frac{d^2y}{dx^2} = 24x - 3x^{-3}$.

5 **Determine the nature of P**

At $x = \frac{1}{2}$, $\frac{d^2y}{dx^2} = 12 - 24 = -12 < 0$, so P is a local maximum.

6 **Find the other stationary point**

Set $12x^2 - 9 + \frac{3}{2x^2} = 0$. Multiplying by $2x^2$ gives $24x^4 - 18x^2 + 3 = 0$, or $8x^4 - 6x^2 + 1 = 0$.

7 **Solve the quadratic in x squared**

$(4x^2 - 1)(2x^2 - 1) = 0$. The given point has $x = \frac{1}{2}$, so the second stationary point has $x^2 = \frac{1}{2}$.

8 **State the positive x coordinate**

Since $x > 0$, the second stationary point has $x = \frac{1}{\sqrt{2}}$.

✓ **FINAL ANSWER**

(a) $k = -\frac{3}{2}$. (b) local maximum.

(c) $x = \frac{1}{\sqrt{2}}$.

WMA12/01 MAY/JUNE 2021

Question 9

10 marks | Applications of Differentiation

9.

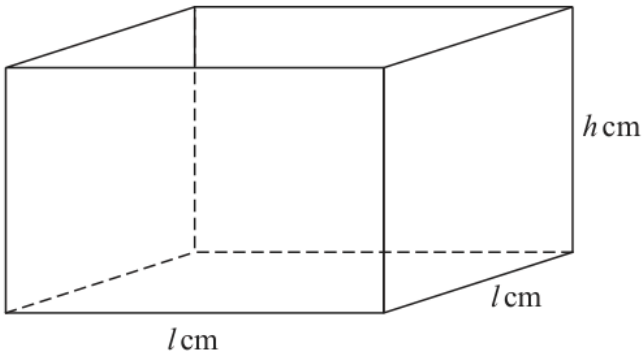


Figure 3

Figure 3 shows a sketch of a square based, open top box.

The height of the box is h cm, and the base edges each have length l cm.

Given that the volume of the box is 250 000 cm^3

(a) show that the external surface area, S cm^2, of the box is given by

$$S = \frac{250\,000}{h} + 2000\sqrt{h}$$

(3)

(b) Use algebraic differentiation to show that S has a stationary point when $h = 250^k$ where k is a rational constant to be found.

(5)

(c) Justify by further differentiation that this value of h gives the minimum external surface area of the box.

(2)

(Total 10 marks)

TOTAL FOR PAPER IS 75 MARKS

WORKED SOLUTION

Question 9 · WMA12/01 May/June 2021

1 Use the volume

The volume is $l^2h = 250000$, so $l^2 = \dfrac{250000}{h}$.

2 Write the surface area

The open box has one square base and four side faces, so $S = l^2 + 4lh$.

3 **Substitute l**

Since $l = \sqrt{\dfrac{250000}{h}} = \dfrac{500}{\sqrt{h}}$, $4lh = 4 \cdot \dfrac{500}{\sqrt{h}} \cdot h = 2000\sqrt{h}$.

4 **Show S**

Therefore $S = \dfrac{250000}{h} + 2000\sqrt{h}$.

5 **Differentiate S**

$S = 250000h^{-1} + 2000h^{1/2}$, so $\dfrac{dS}{dh} = -250000h^{-2} + 1000h^{-1/2}$.

6 **Set the derivative to zero**

$-250000h^{-2} + 1000h^{-1/2} = 0$, so $1000h^{-1/2} = 250000h^{-2}$.

7 **Solve for h**

Multiplying by h^2 gives $1000h^{3/2} = 250000$, so $h^{3/2} = 250$ and $h = 250^{2/3}$.

8 **Find k**

Since $h = 250^k$, we have $k = \dfrac{2}{3}$.

9 **Use the second derivative**

$\dfrac{d^2S}{dh^2} = 500000h^{-3} - 500h^{-3/2}$.

10 **Check the sign**

At $h = 250^{2/3}$, $h^{3/2} = 250$, so the second derivative is $\dfrac{500000}{250^2} - \dfrac{500}{250} = 8 - 2 = 6 > 0$.

11 **Conclude**

The stationary point is therefore a minimum.

✓ **FINAL ANSWER**

(a) Proven: $S = \dfrac{250000}{h} + 2000\sqrt{h}$.

(b) $h = 250^{2/3}$, so $k = \dfrac{2}{3}$.

(c) Minimum.

Question 8

10 marks | Applications of Differentiation

8. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

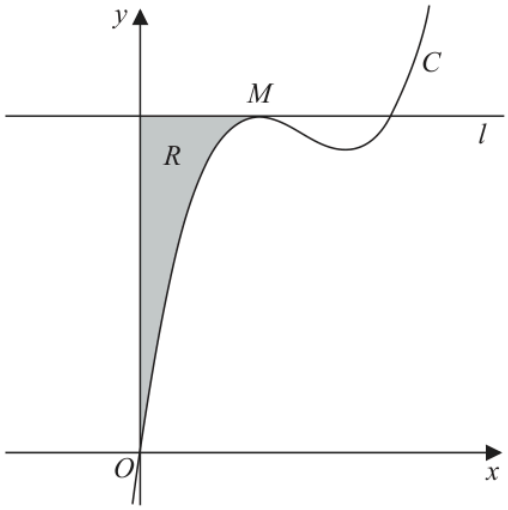


Figure 2

Figure 2 shows a sketch of part of the curve C with equation

$$y = \frac{4}{3}x^3 - 11x^2 + kx \qquad \text{where } k \text{ is a constant}$$

The point M is the maximum turning point of C and is shown in Figure 2.

Given that the x coordinate of M is 2

(a) show that $k = 28$

(3)

(b) Determine the range of values of x for which y is increasing.

(2)

The line l passes through M and is parallel to the x-axis.

The region R, shown shaded in Figure 2, is bounded by the curve C, the line l and the y-axis.

(c) Find, by algebraic integration, the exact area of R.

(5)

1 **Differentiate the curve**

For $y = \frac{4}{3}x^3 - 11x^2 + kx$, $\frac{dy}{dx} = 4x^2 - 22x + k$.

2 **Use the turning point**

At the turning point M, $x = 2$ and $\frac{dy}{dx} = 0$.

3 **Show k**

$4(2)^2 - 22(2) + k = 0$, so $16 - 44 + k = 0$ and $k = 28$.

4 **Factor the derivative**

With $k = 28$, $\frac{dy}{dx} = 4x^2 - 22x + 28 = 2(2x - 7)(x - 2)$.

5 **Find where y is increasing**

The derivative is positive outside its roots, so y is increasing for $x < 2$ or $x > \frac{7}{2}$.

6 **Find the height of the line**

At $x = 2$, $y = \frac{4}{3}(8) - 11(4) + 28(2) = \frac{68}{3}$. Hence line l has equation $y = \frac{68}{3}$.

7 **Set up the shaded area**

The region is between the line and the curve from $x = 0$ to $x = 2$, so

$A = \int_0^2 \left(\frac{68}{3} - \left(\frac{4}{3}x^3 - 11x^2 + 28x \right) \right) dx$.

8 **Integrate**

$A = \left[\frac{68}{3}x - \frac{1}{3}x^4 + \frac{11}{3}x^3 - 14x^2 \right]_0^2$.

9 **Evaluate exactly**

Substituting $x = 2$ gives $\frac{136}{3} - \frac{16}{3} + \frac{88}{3} - 56 = \frac{40}{3}$.

✓ **FINAL ANSWER**

(a) Proven: $k = 28$.

(b) $x < 2$ or $x > \frac{7}{2}$. (c) $\frac{40}{3}$.

WMA12/01 MAY/JUNE 2022

Question 8

8 marks | Applications of Differentiation

8. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

A curve has equation

$$y = 256x^4 - 304x - 35 + \frac{27}{x^2} \qquad x \neq 0$$

(a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$

(3)

(b) Hence find the coordinates of the stationary points of the curve.

(5)

WORKED SOLUTION

Question 8 · WMA12/01 May/June 2022

1 **Rewrite with powers**

$y = 256x^4 - 304x - 35 + 27x^{-2}$.

2 **Differentiate**

$\dfrac{dy}{dx} = 1024x^3 - 304 - 54x^{-3} = 1024x^3 - 304 - \dfrac{54}{x^3}$.

3 **Set the derivative to zero**

$1024x^3 - 304 - \dfrac{54}{x^3} = 0$.

4 **Use a substitution**

Multiplying by x^3 gives $1024x^6 - 304x^3 - 54 = 0$. Let $u = x^3$.

5 **Solve for u**

$1024u^2 - 304u - 54 = 0$, or $512u^2 - 152u - 27 = 0 = (8u + 1)(64u - 27)$.

6 **Find x**

Thus $x^3 = -\dfrac{1}{8}$ or $x^3 = \dfrac{27}{64}$, giving $x = -\dfrac{1}{2}$ or $x = \dfrac{3}{4}$.

7 **Find the y values**

Substitution into $y = 256x^4 - 304x - 35 + \dfrac{27}{x^2}$ gives $y = 241$ at $x = -\dfrac{1}{2}$ and $y = -134$ at $x = \dfrac{3}{4}$.

8 **State the stationary points**

The stationary points are $\left(-\dfrac{1}{2}, 241\right)$ and $\left(\dfrac{3}{4}, -134\right)$.

✓ **FINAL ANSWER**

$(a)\ \dfrac{dy}{dx} = 1024x^3 - 304 - \dfrac{54}{x^3}.$ $(b)\ \left(-\dfrac{1}{2}, 241\right),\ \left(\dfrac{3}{4}, -134\right).$

WMA12/01 OCTOBER 2022

Question 7

9 marks | Applications of Differentiation

7. The curve C has equation

$$y = \frac{12x^3(x-7) + 14x(13x-15)}{21\sqrt{x}} \qquad x > 0$$

(a) Write the equation of C in the form

$$y = ax^{\frac{7}{2}} + bx^{\frac{5}{2}} + cx^{\frac{3}{2}} + dx^{\frac{1}{2}}$$

where a, b, c and d are fully simplified constants.

(3)

The curve C has three turning points.

Using calculus,

(b) show that the x coordinates of the three turning points satisfy the equation

$$2x^3 - 10x^2 + 13x - 5 = 0$$

(3)

Given that the x coordinate of one of the turning points is 1

(c) find, using algebra, the exact x coordinates of the other two turning points.

(*Solutions based entirely on calculator technology are not acceptable.*)

(3)

WORKED SOLUTION

Question 7 · WMA12/01 October 2022

1 **Expand the numerator**

$12x^3(x-7) + 14x(13x-15) = 12x^4 - 84x^3 + 182x^2 - 210x$.

2 **Divide by 21 root x**

Since $\sqrt{x} = x^{1/2}$, divide each term by $21x^{1/2}$.

3 **Write in the required form**

$y = \frac{12}{21}x^{7/2} - \frac{84}{21}x^{5/2} + \frac{182}{21}x^{3/2} - \frac{210}{21}x^{1/2} = \frac{4}{7}x^{7/2} - 4x^{5/2} + \frac{26}{3}x^{3/2} - 10x^{1/2}$.

4 **Differentiate**

$\frac{dy}{dx} = 2x^{5/2} - 10x^{3/2} + 13x^{1/2} - 5x^{-1/2}$.

5 **Factor the derivative**

$\frac{dy}{dx} = x^{-1/2}(2x^3 - 10x^2 + 13x - 5)$.

6 **Use the turning point condition**

At a turning point, $\frac{dy}{dx} = 0$. Since $x > 0$, $x^{-1/2} \neq 0$, so the x coordinates satisfy $2x^3 - 10x^2 + 13x - 5 = 0$.

7 **Use the known root**

One root is $x = 1$, so divide $2x^3 - 10x^2 + 13x - 5$ by $(x - 1)$.

8 **Factor the cubic**

$2x^3 - 10x^2 + 13x - 5 = (x - 1)(2x^2 - 8x + 5)$.

9 **Solve the quadratic**

$2x^2 - 8x + 5 = 0$ gives $x = \frac{8 \pm \sqrt{64 - 40}}{4} = \frac{8 \pm 2\sqrt{6}}{4} = 2 \pm \frac{\sqrt{6}}{2}$.

✓ **FINAL ANSWER**

(a) $y = \frac{4}{7}x^{7/2} - 4x^{5/2} + \frac{26}{3}x^{3/2} - 10x^{1/2}$.

(b) Proven.

(c) $x = 2 - \frac{\sqrt{6}}{2}$ and $x = 2 + \frac{\sqrt{6}}{2}$.

WMA12/01 MAY/JUNE 2023

Question 7

8 marks | Applications of Differentiation

7. The height of a river above a fixed point on the riverbed was monitored over a 7-day period.

The height of the river, H metres, t days after monitoring began, was given by

$$H = \frac{\sqrt{t}}{20}(20 + 6t - t^2) + 17 \qquad 0 \leqslant t \leqslant 7$$

Given that H has a stationary value at $t = \alpha$

(a) use calculus to show that α satisfies the equation

$$5\alpha^2 - 18\alpha - 20 = 0$$

(5)

(b) Hence find the value of α, giving your answer to 3 decimal places.

(1)

(c) Use further calculus to prove that H is a maximum at this value of α.

(2)

WORKED SOLUTION

Question 7 · WMA12/01 May/June 2023

1 Rewrite H with powers

$H = \frac{1}{20}t^{1/2}(20 + 6t - t^2) + 17.$

2 Expand before differentiating

$H = \frac{1}{20}(20t^{1/2} + 6t^{3/2} - t^{5/2}) + 17.$

3 Differentiate

$\frac{dH}{dt} = \frac{1}{20}\left(10t^{-1/2} + 9t^{1/2} - \frac{5}{2}t^{3/2}\right).$

4 Put over a common factor

$\frac{dH}{dt} = \frac{-5t^2 + 18t + 20}{40\sqrt{t}}.$

5 Use the stationary condition

At $t = \alpha$, $\frac{dH}{dt} = 0$. Since $\alpha > 0$, the denominator is not zero, so $-5\alpha^2 + 18\alpha + 20 = 0$.

6 **Show the required equation**

Multiplying by -1 gives $5\alpha^2 - 18\alpha - 20 = 0$.

7 **Find alpha**

$$\alpha = \frac{18 \pm \sqrt{18^2 + 4(5)(20)}}{10} = \frac{9 \pm \sqrt{181}}{5}.$$

8 **Use the time interval**

Only the positive value lies in $0 \le t \le 7$, so $\alpha = \dfrac{9 + \sqrt{181}}{5} = 4.491$ to 3 decimal places.

9 **Differentiate again**

$$\frac{d^2H}{dt^2} = \frac{-15t^2 + 18t - 20}{80t^{3/2}}.$$

10 **Use the stationary equation**

At $t = \alpha$, $5\alpha^2 = 18\alpha + 20$. Therefore the numerator of $\dfrac{d^2H}{dt^2}$ is $-15\alpha^2 + 18\alpha - 20 = -3(18\alpha + 20) + 18\alpha - 20 = -36\alpha - 80$.

11 **Show it is negative**

Since $\alpha > 0$, $-36\alpha - 80 < 0$, and the denominator $80\alpha^{3/2}$ is positive. Thus $\dfrac{d^2H}{dt^2} < 0$, so H is a maximum.

✓ **FINAL ANSWER**

(a) Proven.

(b) $\alpha = 4.491$.

(c) H is a maximum at this value.

Question 9

12 marks | Applications of Differentiation

9.

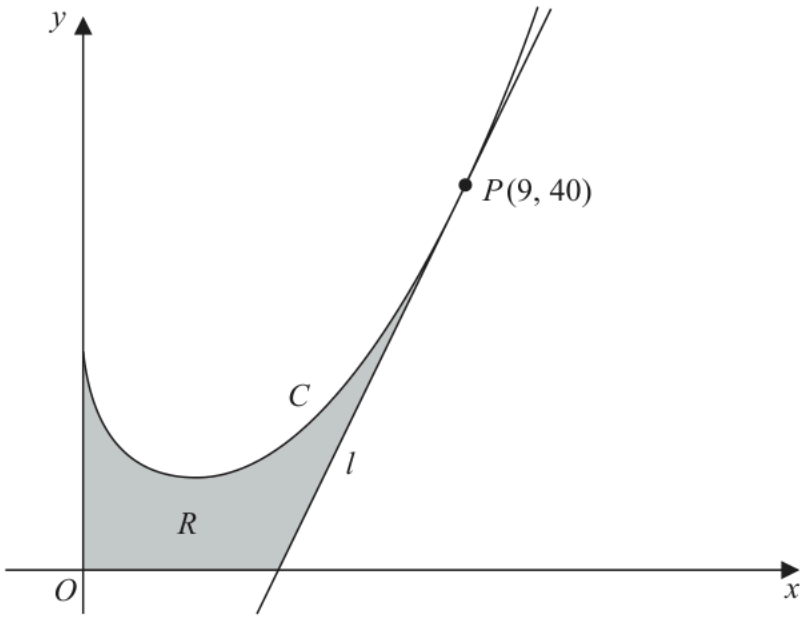


Figure 3

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 3 shows a sketch of part of the curve C with equation

$$y = \frac{2}{3}x^2 - 9\sqrt{x} + 13 \qquad x \geqslant 0$$

(a) Find, using calculus, the range of values of x for which y is increasing.

(4)

The point P lies on C and has coordinates (9, 40).

The line l is the tangent to C at the point P.

The finite region R, shown shaded in Figure 3, is bounded by the curve C, the line l, the x-axis and the y-axis.

(b) Find, using calculus, the exact area of R.

(8)

1 **Differentiate**

For $y = \frac{2}{3}x^2 - 9\sqrt{x} + 13$, $\frac{dy}{dx} = \frac{4}{3}x - \frac{9}{2\sqrt{x}}$.

2 **Set up increasing condition**

y is increasing when $\frac{dy}{dx} > 0$.

3 **Multiply by a positive quantity**

For $x > 0$, multiply by $6\sqrt{x}$: $8x\sqrt{x} - 27 > 0$.

4 **Solve the inequality**

$x^{3/2} > \frac{27}{8} = \left(\frac{3}{2}\right)^3$, so $\sqrt{x} > \frac{3}{2}$ and $x > \frac{9}{4}$.

5 **Find the tangent gradient**

At $P(9, 40)$, $\frac{dy}{dx} = \frac{4}{3}(9) - \frac{9}{2 \cdot 3} = 12 - \frac{3}{2} = \frac{21}{2}$.

6 **Write the tangent line**

$y - 40 = \frac{21}{2}(x - 9)$, so $y = \frac{21}{2}x - \frac{109}{2}$.

7 **Find where the tangent meets the x-axis**

Set $y = 0$: $\frac{21}{2}x - \frac{109}{2} = 0$, so $x = \frac{109}{21}$.

8 **Split the area**

From $x = 0$ to $x = \frac{109}{21}$, the lower boundary is the x-axis. From $x = \frac{109}{21}$ to $x = 9$, the lower boundary is the tangent.

9 **Set up the integral**

$$R = \int_0^{109/21} \left(\frac{2}{3}x^2 - 9\sqrt{x} + 13\right) dx + \int_{109/21}^{9} \left(\frac{2}{3}x^2 - 9\sqrt{x} + 13 - \left(\frac{21}{2}x - \frac{109}{2}\right)\right) dx.$$

10 **Use antiderivatives**

$\int \left(\frac{2}{3}x^2 - 9\sqrt{x} + 13\right) dx = \frac{2}{9}x^3 - 6x^{3/2} + 13x$, and $\int \left(\frac{21}{2}x - \frac{109}{2}\right) dx = \frac{21}{4}x^2 - \frac{109}{2}x$.

11 **Evaluate**

Substituting the limits and simplifying gives $R = \frac{857}{21}$.

FINAL ANSWER

(a) y is increasing for $x > \frac{9}{4}$.

(b) Area $= \frac{857}{21}$.

WMA12/01 JANUARY 2024

Question 10

9 marks | Applications of Differentiation

10. **In this question you must show detailed reasoning.**

Solutions relying entirely on calculator technology are not acceptable.

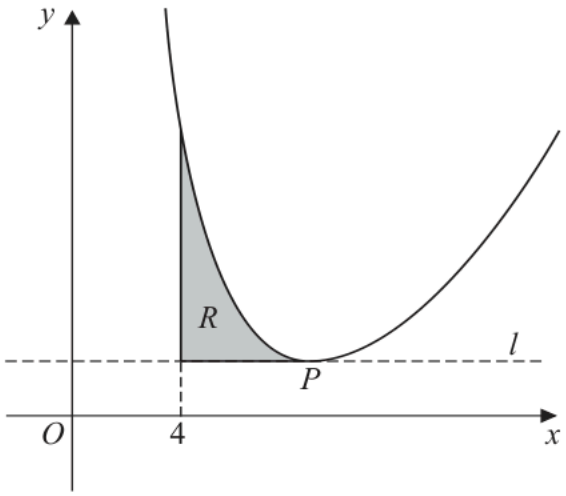


Figure 2

Figure 2 shows a sketch of the curve with equation

$$y = \frac{1}{2}x^2 + \frac{1458}{\sqrt{x^3}} - 74 \qquad x > 0$$

The point P is the only stationary point on the curve.

(a) Use calculus to show that the x coordinate of P is 9

(4)

The line l passes through the point P and is parallel to the x-axis.

The region R, shown shaded in Figure 2, is bounded by the curve, the line l and the line with equation $x = 4$

(b) Use algebraic integration to find the exact area of R.

(5)

WORKED SOLUTION

Question 10 · WMA12/01 January 2024

1 **Rewrite the curve**

$$y = \frac{1}{2}x^2 + 1458x^{-3/2} - 74.$$

2 **Differentiate**

$$\frac{dy}{dx} = x + 1458\left(-\frac{3}{2}\right)x^{-5/2} = x - 2187x^{-5/2}.$$

3 **Use the stationary condition**

At a stationary point, $x - 2187x^{-5/2} = 0$.

4 **Solve for x**

$x = 2187x^{-5/2}$, so $x^{7/2} = 2187 = 3^7$.

5 **Show x coordinate**

Raising both sides to the power $\frac{2}{7}$ gives $x = 3^2 = 9$. Since $x > 0$, this is the only stationary point.

6 **Find the horizontal line**

At $x = 9$, $y = \frac{1}{2}(9)^2 + \frac{1458}{\sqrt{9^3}} - 74 = \frac{81}{2} + 54 - 74 = \frac{41}{2}$.

7 **Set up the area**

The region lies from $x = 4$ to $x = 9$ between the curve and the horizontal line $y = \frac{41}{2}$.

8 **Integrate the difference**

$$R = \int_4^9 \left(\frac{1}{2}x^2 + 1458x^{-3/2} - 74 - \frac{41}{2}\right)dx.$$

9 **Find an antiderivative**

$$\int \left(\frac{1}{2}x^2 + 1458x^{-3/2} - \frac{189}{2}\right)dx = \frac{x^3}{6} - 2916x^{-1/2} - \frac{189}{2}x.$$

10 **Evaluate exactly**

$$R = \left[\frac{x^3}{6} - 2916x^{-1/2} - \frac{189}{2}x\right]_4^9 = \frac{373}{3}.$$

✓ **FINAL ANSWER**

(a) Proven.

(b) $\frac{373}{3}$.

Question 9

8 marks | Applications of Differentiation

9.

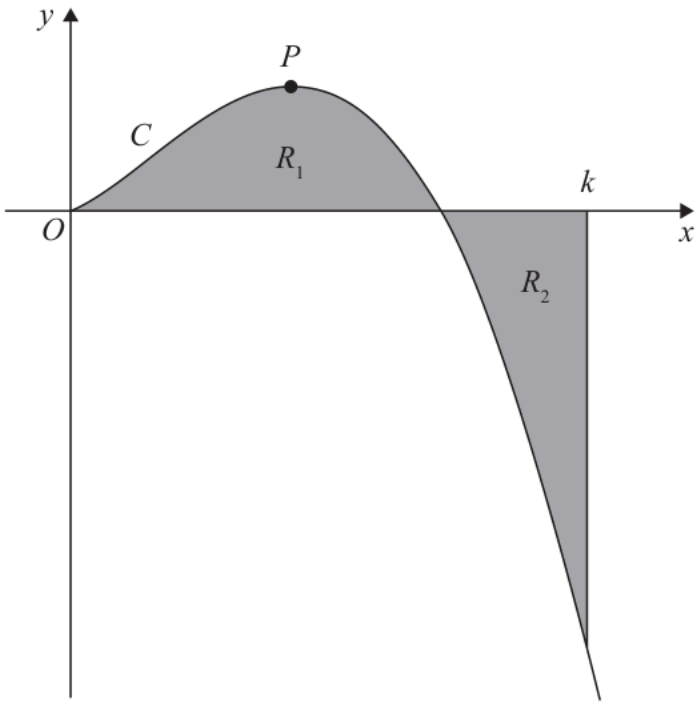


Figure 1

Figure 1 is a sketch of the curve C with equation

$$y = 2x^{\frac{3}{2}}(4 - x) \qquad x \geqslant 0$$

The point P is the stationary point of C.

(a) Find, using calculus, the x coordinate of P.

(4)

The region R_1, shown shaded in Figure 1, is bounded by C and the x-axis.

The region R_2, also shown shaded in Figure 1, is bounded by C, the x-axis and the line with equation $x = k$, where k is a constant.

Given that the area of R_1 is equal to the area of R_2

(b) find, using calculus, the exact value of k.

(4)

1 **Rewrite the curve**

$y = 2x^{3/2}(4 - x) = 8x^{3/2} - 2x^{5/2}$.

2 **Differentiate**

$\dfrac{dy}{dx} = 12x^{1/2} - 5x^{3/2} = \sqrt{x}(12 - 5x)$.

3 **Find the stationary point**

For the interior stationary point P, $12 - 5x = 0$, so $x = \dfrac{12}{5}$.

4 **Set up the area condition**

R_1 is the area from $x = 0$ to $x = 4$ above the x-axis. R_2 is the area from $x = 4$ to $x = k$ below the x-axis.

5 **Use equal areas**

Equal areas mean $\displaystyle\int_0^4 y\,dx = -\int_4^k y\,dx$, so $\displaystyle\int_0^k y\,dx = 0$.

6 **Integrate y**

$$\int 2x^{3/2}(4 - x)dx = \int (8x^{3/2} - 2x^{5/2})dx = \frac{16}{5}x^{5/2} - \frac{4}{7}x^{7/2}.$$

7 **Solve for k**

$\dfrac{16}{5}k^{5/2} - \dfrac{4}{7}k^{7/2} = 0$.

8 **Factor**

$4k^{5/2}\left(\dfrac{4}{5} - \dfrac{k}{7}\right) = 0$.

9 **Choose the value beyond four**

Since $k > 4$, $\dfrac{4}{5} - \dfrac{k}{7} = 0$, so $k = \dfrac{28}{5}$.

✓ **FINAL ANSWER**

(a) $x = \dfrac{12}{5}$.

(b) $k = \dfrac{28}{5}$.

WMA12/01 MAY/JUNE R 2024

Question 6

9 marks | Applications of Differentiation

6. $f(x) = 4x^3 + px^2 + 8x + q$

where p and q are constants.

Given that

- $(2x + 3)$ is a factor of $f(x)$
- $f(x)$ has a remainder of -5 when divided by $(x + 2)$

(a) (i) show that $p = 10$

(ii) find the value of q.

(5)

(b) Hence find the range of values of x for which $f(x)$ is decreasing.

(*Solutions based entirely on calculator technology are not acceptable.*)

(4)

WORKED SOLUTION

Question 6 · WMA12/01 May/June R 2024

1 Use the factor theorem

Since $(2x+3)$ is a factor, $x = -\frac{3}{2}$ is a root.

2 Substitute x equals negative three halves

$4\left(-\frac{3}{2}\right)^3 + p\left(-\frac{3}{2}\right)^2 + 8\left(-\frac{3}{2}\right) + q = 0.$

3 Simplify the first condition

$-\frac{27}{2} + \frac{9}{4}p - 12 + q = 0$, so $\frac{9}{4}p + q = \frac{51}{2}$.

4 Use the remainder theorem

The remainder on division by $(x+2)$ is $f(-2)$.

5 Substitute x equals negative two

$f(-2) = 4(-8) + 4p - 16 + q = 4p + q - 48.$

6 **Use the given remainder**

$4p + q - 48 = -5$, so $4p + q = 43$.

7 **Solve for p and q**

Subtract $\frac{9}{4}p + q = \frac{51}{2}$ from $4p + q = 43$: $\frac{7}{4}p = \frac{35}{2}$, so $p = 10$. Then $4(10) + q = 43$, so $q = 3$.

8 **Differentiate**

With $p = 10$, $f(x) = 4x^3 + 10x^2 + 8x + 3$, so $f'(x) = 12x^2 + 20x + 8$.

9 **Factor the derivative**

$f'(x) = 4(3x + 2)(x + 1)$.

10 **Find where decreasing**

The quadratic derivative is negative between its roots, so $f(x)$ is decreasing for $-1 < x < -\frac{2}{3}$.

✓ **FINAL ANSWER**

$(a)(i)\ p = 10$.

$(a)(ii)\ q = 3$.

$(b)\ -1 < x < -\frac{2}{3}$.

WMA12/01 MAY/JUNE R 2024

Question 10

9 marks | Applications of Differentiation

10.

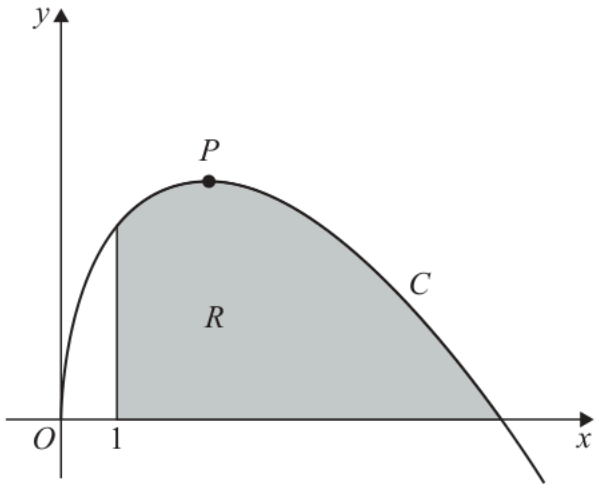


Figure 1

In this question you must show all stages of your working.
Solutions relying entirely on calculator technology are not acceptable.

Figure 1 shows a sketch of part of the curve C with equation

$$y = \frac{9x - x^2}{2\sqrt{x}} \qquad x > 0$$

The point P is the stationary point on C.

(a) Find, using calculus, the x coordinate of P.

(4)

The finite region R, shown shaded in Figure 1, is bounded by the curve C, the x-axis and the line with equation $x = 1$

(b) Using calculus, calculate the exact area of R.

(5)

WORKED SOLUTION

Question 10 · WMA12/01 May/June R 2024

1 **Rewrite the curve**

$$y = \frac{9x - x^2}{2\sqrt{x}} = \frac{1}{2}(9x^{1/2} - x^{3/2}).$$

2 **Differentiate**

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{9}{2}x^{-1/2} - \frac{3}{2}x^{1/2}\right) = \frac{3(3-x)}{4\sqrt{x}}.$$

3 **Use the stationary condition**

For $x > 0$, the denominator is not zero. So $\frac{dy}{dx} = 0$ when $3 - x = 0$.

4 **Find P**

Therefore the x coordinate of P is 3.

5 **Find where the curve meets the x-axis**

$y = 0$ when $9x - x^2 = 0$, so $x = 9$ for the positive intercept.

6 **Set up the area**

The region is under the curve from $x = 1$ to $x = 9$, so $R = \int_1^9 \frac{9x - x^2}{2\sqrt{x}}dx$.

7 **Integrate**

$$R = \int_1^9 \frac{1}{2}(9x^{1/2} - x^{3/2})dx = \left[3x^{3/2} - \frac{1}{5}x^{5/2}\right]_1^9.$$

8 **Evaluate**

$$R = \left(3(27) - \frac{1}{5}(243)\right) - \left(3 - \frac{1}{5}\right) = \frac{148}{5}.$$

✓ **FINAL ANSWER**

(a) $x = 3$.

(b) $\frac{148}{5}$.

Question 9

11 marks | Applications of Differentiation

9:

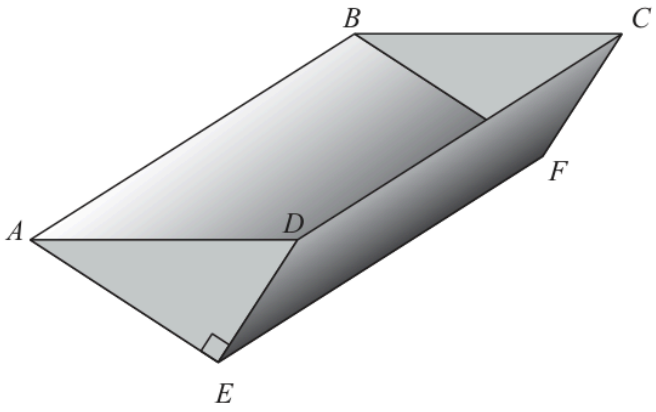


Figure 2

Figure 2 shows a sketch of an open container.

The sides $CDEF$ and $ABFE$ are rectangles.

The ends ADE and BCF are congruent (identical) right-angled triangles.

The container is made from metal of negligible thickness.

Given that

- $AE = BF = 3x$ metres
- $DE = CF = 2x$ metres
- $AB = DC = EF = L$ metres

and the capacity of the container is $12\,\text{m}^3$

(a) show that the area of metal used to make the container, $S\,\text{m}^2$, is given by

$$S = Px^2 + \frac{Q}{x}$$

where P and Q are positive integers to be found.

(4)

Given that x can vary,

(b) use algebraic calculus to find the minimum value of S, giving your answer to one decimal place.

(5)

(c) Justify that the value of S found in part (b) is a minimum.

(2)

1 **Find the triangular end area**

Each end is a right-angled triangle with perpendicular sides $3x$ and $2x$, so its area is $\frac{1}{2}(3x)(2x) = 3x^2$.

2 **Use the capacity**

The volume is cross-section area times length, so $3x^2L = 12$ and $L = \frac{4}{x^2}$.

3 **Write the metal area**

There are two triangular ends with total area $6x^2$, and two rectangular sides with total area $3xL + 2xL = 5xL$.

4 **Substitute for L**

$S = 6x^2 + 5x\left(\frac{4}{x^2}\right) = 6x^2 + \frac{20}{x}$. Hence $P = 6$ and $Q = 20$.

5 **Differentiate**

$\frac{dS}{dx} = 12x - 20x^{-2} = 12x - \frac{20}{x^2}$.

6 **Find the stationary value of x**

Set $\frac{dS}{dx} = 0$: $12x = \frac{20}{x^2}$, so $x^3 = \frac{5}{3}$.

7 **Evaluate S**

At $x = \sqrt[3]{\frac{5}{3}}$, $S = 6x^2 + \frac{20}{x} = 25.3029\ldots$, so the minimum value is 25.3 to one decimal place.

8 **Differentiate again**

$\frac{d^2S}{dx^2} = 12 + 40x^{-3} = 12 + \frac{40}{x^3}$.

9 **Justify the minimum**

Since $x > 0$, $12 + \frac{40}{x^3} > 0$, so the stationary value found is a minimum.

✓ **FINAL ANSWER**

(a) $S = 6x^2 + \frac{20}{x}$, so $P = 6$ and $Q = 20$.

(b) 25.3.

(c) Minimum justified.

Question 10

8 marks | Applications of Differentiation

10.

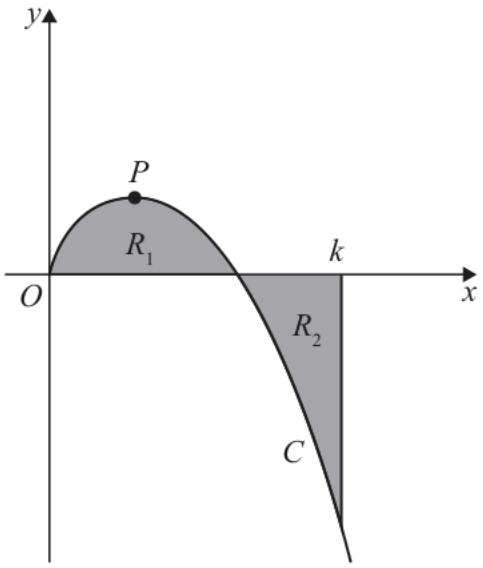


Figure 2

In this question you must show all stages of your working.
Solutions relying entirely on calculator technology are not acceptable.

Figure 2 shows a sketch of the curve C with equation

$$y = \frac{\sqrt{x}\left(100 - x^2\right)}{40} \qquad x \geqslant 0$$

The point P is a stationary point on C.

(a) Use algebraic differentiation to find the exact x coordinate of P.

(4)

The region R_1, shown shaded in Figure 2, is bounded by C and the x-axis.

The region R_2, also shown shaded in Figure 2, is bounded by C, the x-axis and the line with equation $x = k$, where k is a constant.

Given that the area of R_1 is equal to the area of R_2

(b) use algebraic integration to find the exact value of k.

(4)

1 **Rewrite the curve**

$y = \dfrac{\sqrt{x}(100 - x^2)}{40} = \dfrac{5}{2}x^{1/2} - \dfrac{1}{40}x^{5/2}$.

2 **Differentiate**

$\dfrac{dy}{dx} = \dfrac{5}{4}x^{-1/2} - \dfrac{1}{16}x^{3/2}$.

3 **Use the stationary point**

At P, $\dfrac{dy}{dx} = 0$. Thus $\dfrac{5}{4}x^{-1/2} = \dfrac{1}{16}x^{3/2}$. Multiplying by $16x^{1/2}$ gives $20 = x^2$.

4 **Find the x-coordinate**

Since $x \geq 0$, $x = \sqrt{20} = 2\sqrt{5}$.

5 **Integrate the curve**

$\displaystyle\int y\,dx = \int \left(\frac{5}{2}x^{1/2} - \frac{1}{40}x^{5/2}\right)dx = \frac{5}{3}x^{3/2} - \frac{1}{140}x^{7/2}$.

6 **Use equal areas**

The signed area from 0 to k is zero because the positive area R_1 equals the negative area R_2 in magnitude.

7 **Solve for k**

$\dfrac{5}{3}k^{3/2} - \dfrac{1}{140}k^{7/2} = 0$. Since $k > 0$, divide by $k^{3/2}$ to get $\dfrac{5}{3} - \dfrac{k^2}{140} = 0$, so $k^2 = \dfrac{700}{3}$.

8 **Write the exact value**

$k = \sqrt{\dfrac{700}{3}} = \dfrac{10\sqrt{21}}{3}$.

✓ **FINAL ANSWER**

$(a)\ x = 2\sqrt{5}$. $\quad (b)\ k = \dfrac{10\sqrt{21}}{3}$.

WMA12/01 JANUARY 2020

Question 6

8 marks | Applications of Differentiation | Main topic: Circles

6. The circle C has equation

$$x^2 + y^2 + 6x - 4y - 14 = 0$$

(a) Find

(i) the coordinates of the centre of C,

(ii) the exact radius of C.

(3)

The line with equation $y = k$, where k is a constant, is a tangent to C.

(b) Find the possible values of k.

(2)

The line with equation $y = p$, where p is a negative constant, is a chord of C.

Given that the length of this chord is 4 units,

(c) find the value of p.

(3)

WORKED SOLUTION

Question 6 · WMA12/01 January 2020

1 Complete the square

$x^2 + 6x = (x+3)^2 - 9$ and $y^2 - 4y = (y-2)^2 - 4$.

2 Find the centre and radius

The circle becomes $(x+3)^2 + (y-2)^2 = 27$. Hence the centre is $(-3, 2)$ and the radius is $\sqrt{27} = 3\sqrt{3}$.

3 Use horizontal tangents

A horizontal tangent has equation $y = k$. The possible tangent levels are one radius above and below the centre, so $k = 2 \pm 3\sqrt{3}$.

4 Use half the chord length

The chord has length 4, so half the chord is 2. If the chord is $y = p$, its perpendicular distance from the centre is $|p - 2|$.

5 **Apply Pythagoras**

$(p-2)^2+2^2=(3\sqrt{3})^2=27$, so $(p-2)^2=23$.

6 **Choose the negative value**

Since p is negative, $p=2-\sqrt{23}$.

✓ **FINAL ANSWER**

$(a)(i)\ (-3,2).\quad (a)(ii)\ 3\sqrt{3}.\quad (b)\ k=2\pm 3\sqrt{3}.\quad (c)\ p=2-\sqrt{23}.$

WMA12/01 JANUARY 2021

Question 9

10 marks | Applications of Differentiation | Main topic: Circles

9. A circle C has equation

$$(x-k)^2+(y-2k)^2=k+7$$

where k is a positive constant.

(a) Write down, in terms of k,

(i) the coordinates of the centre of C,

(ii) the radius of C.

(2)

Given that the point $P(2,3)$ lies on C

(b) (i) show that $5k^2-17k+6=0$

(ii) hence find the possible values of k.

(3)

The tangent to the circle at P intersects the x-axis at point T.

Given that $k<2$

(c) calculate the exact area of triangle OPT.

(5)

WORKED SOLUTION

Question 9 · WMA12/01 January 2021

1 Read the centre and radius

From $(x-k)^2+(y-2k)^2=k+7$, the centre is $(k,2k)$ and the radius is $\sqrt{k+7}$.

2 Substitute point P

Since $P(2,3)$ lies on the circle, $(2-k)^2+(3-2k)^2=k+7$.

3 Show the quadratic

Expanding gives $k^2-4k+4+4k^2-12k+9=k+7$, hence $5k^2-17k+6=0$.

4 **Solve for k**

$5k^2 - 17k + 6 = (5k - 2)(k - 3)$, so $k = \dfrac{2}{5}$ or $k = 3$.

5 **Use the condition k less than two**

For the tangent calculation, $k < 2$, so the centre is $\left(\dfrac{2}{5}, \dfrac{4}{5}\right)$.

6 **Find the tangent gradient**

The radius from the centre to P has gradient $\dfrac{3 - 4/5}{2 - 2/5} = \dfrac{11}{8}$, so the tangent gradient is $-\dfrac{8}{11}$.

7 **Find the x intercept**

The tangent is $y - 3 = -\dfrac{8}{11}(x - 2)$. Setting $y = 0$ gives $x = \dfrac{49}{8}$, so $T = \left(\dfrac{49}{8}, 0\right)$.

8 **Calculate the area**

Triangle OPT has base $\dfrac{49}{8}$ on the x-axis and height 3, so its area is $\dfrac{1}{2} \cdot \dfrac{49}{8} \cdot 3 = \dfrac{147}{16}$.

✓ **FINAL ANSWER**

$(a)(i)\ (k, 2k).$ $(a)(ii)\ \sqrt{k + 7}.$ $(b)(ii)\ k = 3 \text{ or } \dfrac{2}{5}.$ $(c)\ \dfrac{147}{16}.$

WMA12/01 MAY/JUNE 2021

Question 6

7 marks | Applications of Differentiation | Main topic: Circles

6. A circle has equation

$$x^2 - 6x + y^2 + 8y + k = 0$$

where k is a positive constant.

Given that the x-axis is a tangent to this circle,

(a) find the value of k.

(3)

The circle meets the coordinate axes at the points R, S and T.

(b) Find the exact area of the triangle RST.

(4)

WORKED SOLUTION

Question 6 · WMA12/01 May/June 2021

1 Complete the square

$x^2 - 6x = (x-3)^2 - 9$ and $y^2 + 8y = (y+4)^2 - 16$.

2 Write centre-radius form

The circle becomes $(x-3)^2 + (y+4)^2 = 25 - k$.

3 Use the tangent condition

The centre is $(3, -4)$. Since the x-axis is tangent, the radius is the distance from the centre to the x-axis, which is 4.

4 Find k

Thus $25 - k = 4^2 = 16$, so $k = 9$.

5 Find the intercepts

With $k = 9$, the circle is $(x-3)^2 + (y+4)^2 = 16$. On the x-axis, $y = 0$, giving $x = 3$.

6 Find the y-axis points

On the y-axis, $x = 0$, so $9 + (y+4)^2 = 16$. Hence $y = -4 \pm \sqrt{7}$.

7 Calculate the triangle area

The vertical base on the y-axis has length $2\sqrt{7}$, and its perpendicular distance to $(3,0)$ is 3. The area is $\frac{1}{2}(2\sqrt{7})(3) = 3\sqrt{7}$.

FINAL ANSWER

$(a)\ k = 9.\quad (b)\ 3\sqrt{7}.$

WMA12/01 JANUARY 2022

Question 6

8 marks | Applications of Differentiation | Main topic: Circles

6.

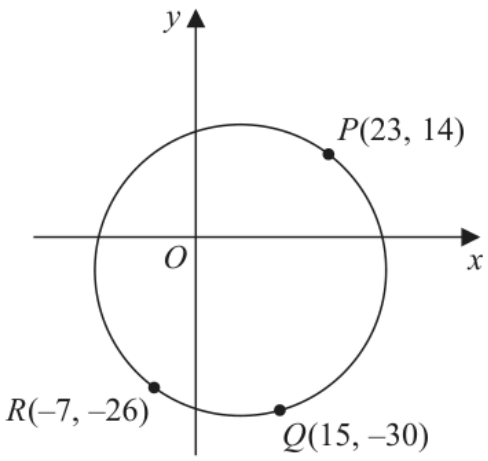


Figure 1

The points $P(23, 14)$, $Q(15, -30)$ and $R(-7, -26)$ lie on the circle C, as shown in Figure 1.

(a) Show that angle $PQR = 90°$

(2)

(b) Hence, or otherwise, find

(i) the centre of C,

(ii) the radius of C.

(3)

Given that the point S lies on C such that the distance QS is greatest,

(c) find an equation of the tangent to C at S, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers to be found.

(3)

WORKED SOLUTION

Question 6 · WMA12/01 January 2022

1 Find the gradient QP

The gradient of QP is $\dfrac{14-(-30)}{23-15} = \dfrac{44}{8} = \dfrac{11}{2}$.

2 Find the gradient QR

The gradient of QR is $\dfrac{-26-(-30)}{-7-15} = \dfrac{4}{-22} = -\dfrac{2}{11}$.

3 **Show the right angle**

The product of the two gradients is -1, so QP is perpendicular to QR and $\angle PQR = 90°$.

4 **Use the diameter**

Since the angle at Q is a right angle, PR is a diameter of the circle.

5 **Find the centre**

The centre is the midpoint of $P(23, 14)$ and $R(-7, -26)$, so it is $(8, -6)$.

6 **Find the radius**

$PR = \sqrt{30^2 + 40^2} = 50$, so the radius is 25.

7 **Find the farthest point from Q**

The greatest distance from Q occurs at the point opposite Q on the diameter through the centre. Thus $S = 2(8, -6) - (15, -30) = (1, 18)$.

8 **Find the tangent gradient**

The radius from the centre to S has gradient $\dfrac{18 - (-6)}{1 - 8} = -\dfrac{24}{7}$, so the tangent gradient is $\dfrac{7}{24}$.

9 **Write the tangent equation**

$y - 18 = \dfrac{7}{24}(x - 1)$, which rearranges to $7x - 24y + 425 = 0$.

✓ **FINAL ANSWER**

(a) Proven: $\angle PQR = 90°$.

$(b)(i)$ $(8, -6)$. $(b)(ii)$ 25. (c) $7x - 24y + 425 = 0$.

Question 9

10 marks | Applications of Differentiation | Main topic: Integration

9.

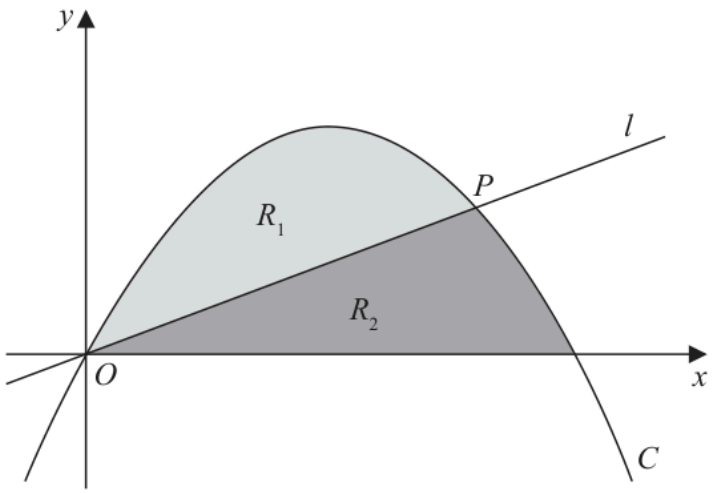


Figure 2

Figure 2 shows

- the curve C with equation $y = x - x^2$
- the line l with equation $y = mx$, where m is a constant and $0 < m < 1$

The line and the curve intersect at the origin O and at the point P.

(a) Find, in terms of m, the coordinates of P.

(2)

The region R_1, shown shaded in Figure 2, is bounded by C and l.

(b) Show that the area of R_1 is

$$\frac{(1-m)^3}{6}$$

(5)

The region R_2, also shown shaded in Figure 2, is bounded by C, the x-axis and l.

Given that the area of R_1 is equal to the area of R_2

(c) find the exact value of m.

(3)

1 Find the second intersection

At an intersection, $x - x^2 = mx$, so $x(1 - m - x) = 0$.

2 State P

The non-zero intersection has $x = 1 - m$, and then $y = m(1 - m)$, so $P = (1 - m, m(1 - m))$.

3 Set up area R one

Region R_1 is between the curve and the line from $x = 0$ to $x = 1 - m$.

4 Integrate the difference

$A_{R_1} = \int_0^{1-m} \{(x - x^2) - mx\}\,dx = \int_0^{1-m} \{(1 - m)x - x^2\}\,dx$.

5 Evaluate

$$A_{R_1} = \left[\frac{1 - m}{2}x^2 - \frac{1}{3}x^3\right]_0^{1-m} = \frac{(1 - m)^3}{2} - \frac{(1 - m)^3}{3} = \frac{(1 - m)^3}{6}.$$

6 Use the total curve area

The total area under $y = x - x^2$ from $x = 0$ to $x = 1$ is $\int_0^1 (x - x^2)\,dx = \frac{1}{6}$.

7 Use equal regions

Since R_1 and R_2 are equal and together make this total area, each has area $\frac{1}{12}$.

8 Solve for m

$\frac{(1 - m)^3}{6} = \frac{1}{12}$, so $(1 - m)^3 = \frac{1}{2}$ and $m = 1 - \frac{1}{\sqrt[3]{2}}$.

FINAL ANSWER

(a) $P = (1 - m,\ m(1 - m))$. (b) Proven: $\frac{(1 - m)^3}{6}$.

(c) $m = 1 - \frac{1}{\sqrt[3]{2}}$.

Question 6

7 marks | Applications of Differentiation | Main topic: Integration

6. The curve C_1 has equation $y = f(x)$.

A table of values of x and y for $y = f(x)$ is shown below, with the y values rounded to 4 decimal places where appropriate.

x	0	0.5	1	1.5	2
y	3	2.6833	2.4	2.1466	1.92

(a) Use the trapezium rule with all the values of y in the table to find an approximation for

$$\int_0^2 f(x)\,dx$$

giving your answer to 3 decimal places.

(3)

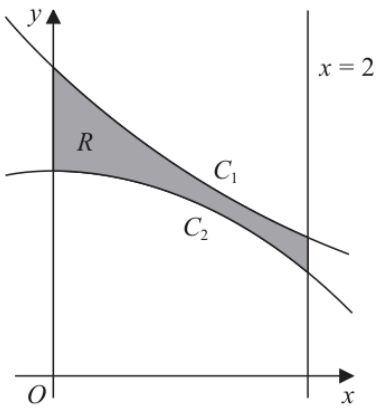


Figure 1

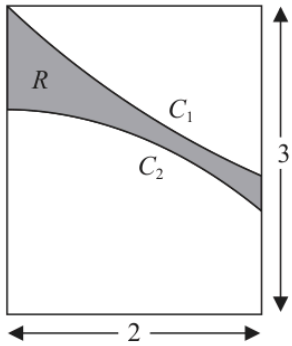


Figure 2

The region R, shown shaded in Figure 1, is bounded by

- the curve C_1
- the curve C_2 with equation $y = 2 - \frac{1}{4}x^2$
- the line with equation $x = 2$
- the y-axis

The region R forms part of the design for a logo shown in Figure 2.

The design consists of the shaded region R inside a rectangle of width 2 and height 3

Using calculus and the answer to part (a),

(b) calculate an estimate for the percentage of the logo which is shaded.

(4)

1 Identify the strip width

The x values increase by 0.5, so the trapezium-rule width is $h = 0.5$.

2 Apply the trapezium rule

$$\int_0^2 f(x)\,dx \approx \frac{0.5}{2}[3 + 1.92 + 2(2.6833 + 2.4 + 2.1466)].$$

3 Evaluate the estimate

This gives 4.84495, so the approximation is 4.845 to 3 decimal places.

4 Find the area under C2

$$\int_0^2 \left(2 - \frac{1}{4}x^2\right)dx = \left[2x - \frac{x^3}{12}\right]_0^2 = 4 - \frac{2}{3} = \frac{10}{3}.$$

5 Estimate the shaded area

The shaded region lies between C_1 and C_2, so its estimated area is $4.845 - \dfrac{10}{3} = 1.5116\ldots$.

6 Compare with the rectangle

The rectangle has area $2 \times 3 = 6$.

7 Convert to a percentage

Percentage shaded $= \dfrac{1.5116\ldots}{6} \times 100 = 25.19\ldots\%$, so the estimate is 25.2%.

FINAL ANSWER

(a) 4.845.

(b) 25.2%.

Question 9

12 marks | Applications of Differentiation | Main topic: Circles

9. **In this question you must show detailed reasoning.**

Solutions relying entirely on calculator technology are not acceptable.

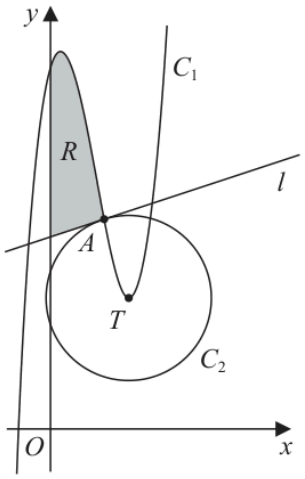


Figure 3

Figure 3 shows

- the curve C_1 with equation $y = x^3 - 5x^2 + 3x + 14$
- the circle C_2 with centre T

The point T is the minimum turning point of C_1

Using Figure 3 and calculus,

(a) find the coordinates of T

(3)

The curve C_1 intersects the circle C_2 at the point A with x coordinate 2

(b) Find an equation of the circle C_2

(3)

The line l shown in Figure 3, is the tangent to circle C_2 at A

(c) Show that an equation of l is

$$y = \frac{1}{3}x + \frac{22}{3}$$

(3)

The region R, shown shaded in Figure 3, is bounded by C_1, l and the y-axis.

(d) Find the exact area of R.

(3)

1 **Differentiate the cubic**

For $y = x^3 - 5x^2 + 3x + 14$, $\dfrac{dy}{dx} = 3x^2 - 10x + 3$.

2 **Find stationary x values**

$3x^2 - 10x + 3 = 0$ factors as $(3x - 1)(x - 3) = 0$, so $x = \dfrac{1}{3}$ or $x = 3$.

3 **Identify the minimum**

$\dfrac{d^2y}{dx^2} = 6x - 10$. At $x = 3$, this is positive, so the minimum turning point has $x = 3$.

4 **Find T**

Substitute $x = 3$: $y = 27 - 45 + 9 + 14 = 5$. Hence $T = (3, 5)$.

5 **Find A**

At $x = 2$ on C_1, $y = 8 - 20 + 6 + 14 = 8$, so $A = (2, 8)$.

6 **Find the circle radius**

The centre of C_2 is $T = (3, 5)$ and it passes through $A = (2, 8)$. Thus $r^2 = (2 - 3)^2 + (8 - 5)^2 = 10$.

7 **Write the circle equation**

Therefore C_2 has equation $(x - 3)^2 + (y - 5)^2 = 10$.

8 **Find the tangent gradient**

The radius TA has gradient $\dfrac{8 - 5}{2 - 3} = -3$, so the tangent at A has gradient $\dfrac{1}{3}$.

9 **Write the tangent equation**

Using $A = (2, 8)$, $y - 8 = \dfrac{1}{3}(x - 2)$, so $y = \dfrac{1}{3}x + \dfrac{22}{3}$.

10 **Set up the area**

From $x = 0$ to $x = 2$, the cubic is above the line. The required area is $\displaystyle\int_0^2 \left[x^3 - 5x^2 + 3x + 14 - \left(\frac{1}{3}x + \frac{22}{3}\right)\right] dx$.

11 **Simplify the integrand**

The integrand is $x^3 - 5x^2 + \dfrac{8}{3}x + \dfrac{20}{3}$.

12 **Integrate**

$$\left[\frac{x^4}{4}-\frac{5x^3}{3}+\frac{4x^2}{3}+\frac{20x}{3}\right]_0^2=4-\frac{40}{3}+\frac{16}{3}+\frac{40}{3}=\frac{28}{3}.$$

✓ **FINAL ANSWER**

$(a)\ T=(3,5).$

$(b)\ (x-3)^2+(y-5)^2=10.$

$(c)\ y=\frac{1}{3}x+\frac{22}{3}.$

$(d)\ \frac{28}{3}.$

WMA12/01 JANUARY 2023

Question 6

10 marks | Applications of Differentiation | Main topic: Circles

6. The circle C has equation

$$x^2 + y^2 + 8x - 4y = 0$$

(a) Find

(i) the coordinates of the centre of C,

(ii) the exact radius of C.

(3)

The point P lies on C.

Given that the tangent to C at P has equation $x + 2y + 10 = 0$

(b) find the coordinates of P

(4)

(c) Find the equation of the normal to C at P, giving your answer in the form $y = mx + c$ where m and c are integers to be found.

(3)

WORKED SOLUTION

Question 6 · WMA12/01 January 2023

1 Complete the square

$x^2 + y^2 + 8x - 4y = 0$ becomes $(x + 4)^2 - 16 + (y - 2)^2 - 4 = 0$.

2 Write the circle form

So $(x + 4)^2 + (y - 2)^2 = 20$.

3 Read the centre and radius

The centre is $(-4, 2)$ and the radius is $\sqrt{20} = 2\sqrt{5}$.

4 Use the tangent gradient

The tangent $x + 2y + 10 = 0$ has gradient $-\frac{1}{2}$.

5 Find the radius gradient

The radius to the point of contact is perpendicular to the tangent, so its gradient is 2.

6 **Write the normal through the centre**

The radius line through the centre $(-4, 2)$ is $y - 2 = 2(x + 4)$, so $y = 2x + 10$.

7 **Intersect with the tangent**

Substitute $y = 2x + 10$ into $x + 2y + 10 = 0$: $x + 2(2x + 10) + 10 = 0$.

8 **Find P**

$5x + 30 = 0$, so $x = -6$. Then $y = 2(-6) + 10 = -2$, hence $P = (-6, -2)$.

9 **State the normal**

The normal at P is the radius line already found, so its equation is $y = 2x + 10$.

✓ **FINAL ANSWER**

$(a)(i)$ $(-4, 2)$.

$(a)(ii)$ $2\sqrt{5}$.

(b) $P = (-6, -2)$.

(c) $y = 2x + 10$.

Question 10

10 marks | Applications of Differentiation | Main topic: Integration

10. The curve C has equation

$$y = \frac{(x-k)^2}{\sqrt{x}} \qquad x > 0$$

where k is a **positive** constant.

(a) Show that

$$\int_1^{16} \frac{(x-k)^2}{\sqrt{x}}\,\mathrm{d}x = ak^2 + bk + \frac{2046}{5}$$

where a and b are integers to be found.

(5)

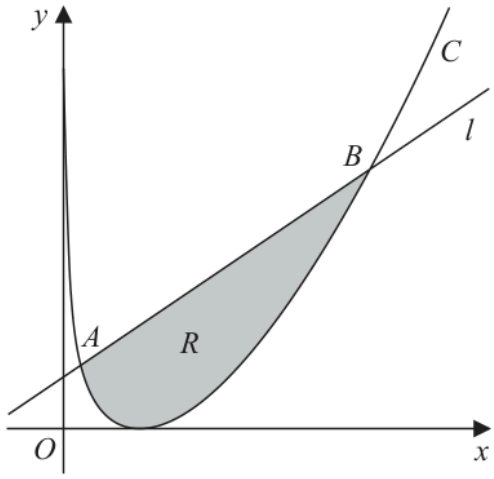


Figure 1

Figure 1 shows a sketch of the curve C and the line l.

Given that l intersects C at the point $A(1, 9)$ and at the point $B(16, q)$ where q is a constant,

(b) show that $k = 4$

(2)

The region R, shown shaded in Figure 1, is bounded by C and l

Using the answers to parts (a) and (b),

(c) find the area of region R

(3)

1 **Expand the integrand**

$$\frac{(x-k)^2}{\sqrt{x}} = \frac{x^2 - 2kx + k^2}{x^{1/2}} = x^{3/2} - 2kx^{1/2} + k^2x^{-1/2}.$$

2 **Integrate**

$$\int \left(x^{3/2} - 2kx^{1/2} + k^2x^{-1/2}\right)dx = \frac{2}{5}x^{5/2} - \frac{4k}{3}x^{3/2} + 2k^2x^{1/2}.$$

3 **Substitute the limits**

From 1 to 16, this gives $\left(\frac{2048}{5} - \frac{256k}{3} + 8k^2\right) - \left(\frac{2}{5} - \frac{4k}{3} + 2k^2\right)$.

4 **Simplify the integral**

The result is $6k^2 - 84k + \frac{2046}{5}$, so $a = 6$ and $b = -84$.

5 **Use point A**

Since $A(1, 9)$ lies on C, $9 = \frac{(1-k)^2}{\sqrt{1}} = (1-k)^2$.

6 **Find k**

Thus $1 - k = \pm 3$, so $k = 4$ or $k = -2$. Since k is positive, $k = 4$.

7 **Find point B**

When $k = 4$ and $x = 16$, $q = \frac{(16-4)^2}{\sqrt{16}} = \frac{144}{4} = 36$, so $B = (16, 36)$.

8 **Find the line**

The line through $A(1, 9)$ and $B(16, 36)$ has gradient $\frac{36-9}{16-1} = \frac{9}{5}$, so l has equation $y = \frac{9}{5}x + \frac{36}{5}$.

9 **Area under the line**

$$\int_1^{16} \left(\frac{9}{5}x + \frac{36}{5}\right)dx = \left[\frac{9}{10}x^2 + \frac{36}{5}x\right]_1^{16} = \frac{675}{2}.$$

10 **Area under the curve**

From part (a), with $k = 4$, $\int_1^{16} \frac{(x-4)^2}{\sqrt{x}}dx = 6(4)^2 - 84(4) + \frac{2046}{5} = \frac{846}{5}$.

11 **Subtract curve from line**

The shaded area is $\frac{675}{2} - \frac{846}{5} = \frac{1683}{10}$.

FINAL ANSWER

(a) $a = 6,\ b = -84$.

(b) $k = 4$.

(c) Area $= \dfrac{1683}{10}$.

Question 7

8 marks | Applications of Differentiation | Main topic: Circles

7.

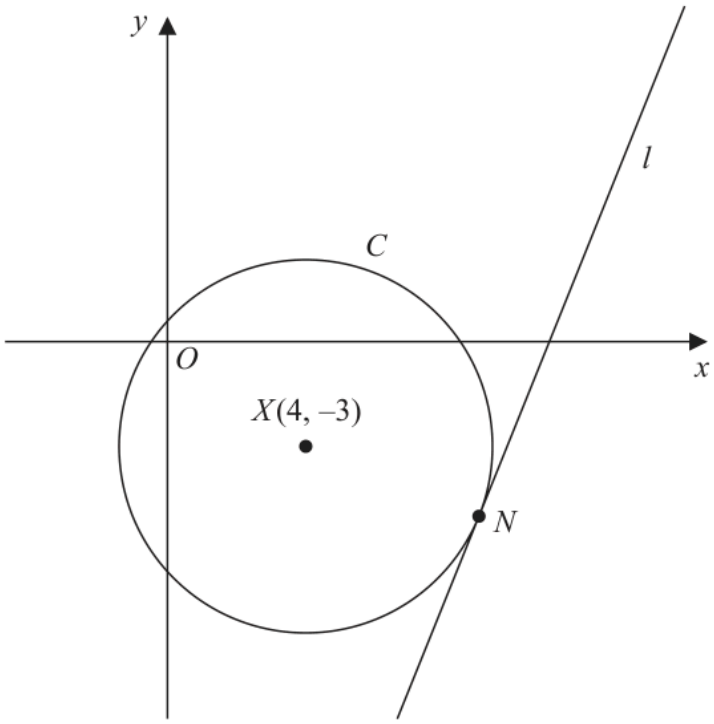


Figure 2

Figure 2 shows a sketch of

- the circle C with centre $X(4, -3)$
- the line l with equation $y = \frac{5}{2}x - \frac{55}{2}$

Given that l is the tangent to C at the point N,

(a) show that an equation for the straight line passing through X and N is

$$2x + 5y + 7 = 0$$

(3)

(b) Hence find

(i) the coordinates of N,

(ii) an equation for C.

(5)

1 Use perpendicular gradients

The tangent l has gradient $\frac{5}{2}$, so the radius XN is perpendicular to it and has gradient $-\frac{2}{5}$.

2 Use the centre

The line through $X(4,-3)$ with gradient $-\frac{2}{5}$ is $y+3=-\frac{2}{5}(x-4)$.

3 Show the required line

Multiplying by 5 gives $5y+15=-2x+8$, so $2x+5y+7=0$.

4 Intersect the radius and tangent

Use $y=\frac{5}{2}x-\frac{55}{2}$ in $2x+5y+7=0$.

5 Find x coordinate

$2x+5\left(\frac{5}{2}x-\frac{55}{2}\right)+7=0$. Multiplying by 2 gives $4x+25x-275+14=0$, so $29x=261$ and $x=9$.

6 Find y coordinate

$y=\frac{5}{2}(9)-\frac{55}{2}=\frac{45-55}{2}=-5$, so $N=(9,-5)$.

7 Find the radius squared

$XN^2=(9-4)^2+(-5+3)^2=5^2+(-2)^2=29$.

8 Write the circle equation

With centre $(4,-3)$ and radius squared 29, the circle is $(x-4)^2+(y+3)^2=29$.

FINAL ANSWER

(a) Proven.

$(b)(i)$ $N=(9,-5)$.

$(b)(ii)$ $(x-4)^2+(y+3)^2=29$.

WMA12/01 OCTOBER 2024

Question 10

10 marks | Applications of Differentiation | Main topic: Circles

10. The circle C has equation

$$x^2 + y^2 + 4x - 30y + 209 = 0$$

(a) Find

(i) the coordinates of the centre of C,

(ii) the exact value of the radius of C.

(3)

The line L has equation $y = mx + 1$, where m is a constant.

Given that L is the tangent to C at the point P,

(b) show that

$$2m^2 - 7m - 22 = 0$$

(3)

(c) Hence find the possible pairs of coordinates of P.

(4)

WORKED SOLUTION

Question 10 · WMA12/01 October 2024

1 Complete the square

$x^2 + y^2 + 4x - 30y + 209 = 0$ becomes $(x+2)^2 + (y-15)^2 = 20$.

2 Read the centre and radius

The centre is $(-2, 15)$ and the radius is $\sqrt{20} = 2\sqrt{5}$.

3 Use the tangent distance

The line $y = mx + 1$ can be written as $mx - y + 1 = 0$. Its distance from $(-2, 15)$ must be $2\sqrt{5}$.

4 Set up the equation for m

$\dfrac{|m(-2) - 15 + 1|}{\sqrt{m^2+1}} = 2\sqrt{5}$, so $\dfrac{|-2m - 14|}{\sqrt{m^2+1}} = 2\sqrt{5}$.

5 Show the quadratic

Squaring gives $4(m+7)^2 = 20(m^2+1)$. Hence $(m+7)^2 = 5(m^2+1)$, so $2m^2 - 7m - 22 = 0$.

6 **Find the possible gradients**

$2m^2 - 7m - 22 = 0$ gives $m = -2$ or $m = \frac{11}{2}$.

7 **Use perpendicular radii**

A tangent is perpendicular to the radius at the point of contact.

8 **Find the first point**

For $m = -2$, the radius gradient is $\frac{1}{2}$. The radius through the centre is $y - 15 = \frac{1}{2}(x + 2)$. Solving with $y = -2x + 1$ gives $x = -6$ and $y = 13$.

9 **Find the second point**

For $m = \frac{11}{2}$, the radius gradient is $-\frac{2}{11}$. The radius through the centre is $y - 15 = -\frac{2}{11}(x + 2)$. Solving with $y = \frac{11}{2}x + 1$ gives $x = \frac{12}{5}$ and $y = \frac{71}{5}$.

✓ **FINAL ANSWER**

$(a)(i)$ $(-2, 15)$.

$(a)(ii)$ $2\sqrt{5}$.

(b) Proven.

(c) $(-6, 13)$ and $\left(\frac{12}{5}, \frac{71}{5}\right)$.

Question 9

12 marks | Applications of Differentiation | Main topic: Integration

9.

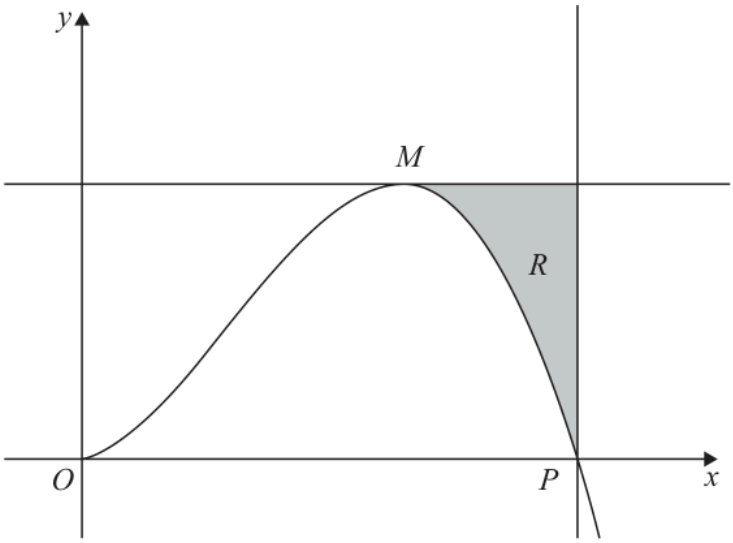


Figure 3

In this question you must show all stages of your working.
Solutions relying entirely on calculator technology are not acceptable.

Figure 3 shows a sketch of part of the curve with equation

$$y = \frac{9x^2\left(5 - \sqrt{x}\right)}{5} \qquad x \geqslant 0$$

The curve has a turning point at the point M, as shown in Figure 3.

(a) Using calculus, find the coordinates of M.

(5)

The curve crosses the x-axis at the point P, as shown in Figure 3.

(b) Use algebra to find the x coordinate of P.

(2)

The finite region R, shown shaded in Figure 3, is bounded by the curve, the line through M parallel to the x-axis and the line through P parallel to the y-axis.

(c) Use algebraic integration to find the area of R, giving your answer to one decimal place.

(5)

1 Rewrite the curve

$y = \dfrac{9x^2(5-\sqrt{x})}{5} = 9x^2 - \dfrac{9}{5}x^{5/2}$.

2 Differentiate

$\dfrac{dy}{dx} = 18x - \dfrac{9}{2}x^{3/2} = \dfrac{9}{2}x(4-\sqrt{x})$.

3 Find the turning point x coordinate

For the turning point shown away from the origin, set $4-\sqrt{x}=0$. Hence $x=16$.

4 Find the y coordinate

At $x=16$, $y = \dfrac{9(16)^2(5-4)}{5} = \dfrac{2304}{5}$.

5 State M

Therefore $M\left(16, \dfrac{2304}{5}\right)$.

6 Find the positive x intercept

For the positive intercept, $\dfrac{9x^2(5-\sqrt{x})}{5} = 0$ gives $5-\sqrt{x}=0$, so $x=25$.

7 Set up the area

The top boundary is $y = \dfrac{2304}{5}$ and the curve is below it from $x=16$ to $x=25$. So

$R = \displaystyle\int_{16}^{25}\left(\frac{2304}{5} - 9x^2 + \frac{9}{5}x^{5/2}\right)dx$.

8 Integrate

$R = \left[\dfrac{2304}{5}x - 3x^3 + \dfrac{18}{35}x^{7/2}\right]_{16}^{25}$.

9 Evaluate

$R = \dfrac{9189}{7} = 1312.714\ldots$, so the area is 1312.7 to one decimal place.

FINAL ANSWER

(a) $M\left(16, \dfrac{2304}{5}\right)$.

(b) $x_P = 25$.

(c) 1312.7.

Question 7

10 marks | Applications of Differentiation | Main topic: Integration

7.

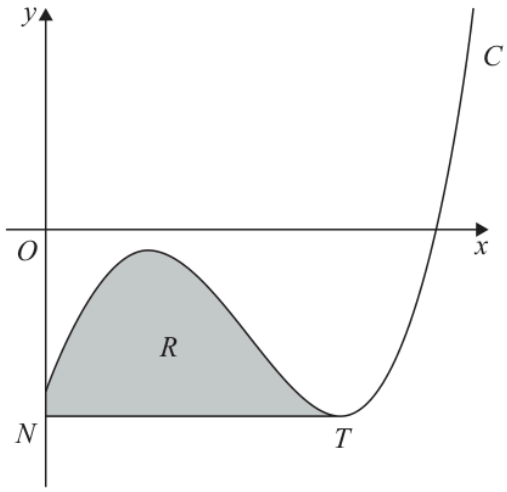


Figure 3

In this question you must show all stages of your working.
Solutions relying entirely on calculator technology are not acceptable.

Figure 3 shows a sketch of part of the curve C with equation

$$y = x^3 - 4x^{\frac{5}{2}} - kx^{\frac{1}{2}} + 28x - 44 \qquad x \geqslant 0$$

where k is a positive constant.

(a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in simplest form.

(2)

The point T, shown in Figure 3, is a minimum stationary point on C.

Given that the x coordinate of T is 9

(b) show that $k = 6$

(2)

The line through T parallel to the x-axis meets the y-axis at the point N.

The finite region R, shown shaded in Figure 3, is bounded by C, the y-axis and the line segment NT.

(c) Use algebraic integration to find the area of R, giving the answer to 3 significant figures.

(6)

1 **Differentiate**

For $y = x^3 - 4x^{5/2} - kx^{1/2} + 28x - 44$, $\dfrac{dy}{dx} = 3x^2 - 10x^{3/2} - \dfrac{k}{2}x^{-1/2} + 28$.

2 **Write in simplest form**

$\dfrac{dy}{dx} = 3x^2 - 10x^{3/2} - \dfrac{k}{2\sqrt{x}} + 28$.

3 **Use the stationary point**

At T, $x = 9$ and $\dfrac{dy}{dx} = 0$.

4 **Substitute x equals nine**

$0 = 3(9)^2 - 10(9)^{3/2} - \dfrac{k}{2\sqrt{9}} + 28 = 243 - 270 - \dfrac{k}{6} + 28$.

5 **Find k**

This simplifies to $0 = 1 - \dfrac{k}{6}$, so $k = 6$.

6 **Find the horizontal line**

With $k = 6$, the y coordinate of T is $9^3 - 4(9)^{5/2} - 6\sqrt{9} + 28(9) - 44 = -53$.

7 **Set up the area**

The line NT is $y = -53$. The region is between the curve and this line from $x = 0$ to $x = 9$.

8 **Write the integral**

$R = \displaystyle\int_0^9 \left(x^3 - 4x^{5/2} - 6x^{1/2} + 28x - 44 + 53\right)dx$.

9 **Integrate**

$R = \left[\dfrac{x^4}{4} - \dfrac{8}{7}x^{7/2} - 4x^{3/2} + 14x^2 + 9x\right]_0^9$.

10 **Evaluate**

$R = \dfrac{6939}{28} = 247.821\ldots$, so the area is 248 to 3 significant figures.

FINAL ANSWER

(a) $3x^2 - 10x^{3/2} - \dfrac{k}{2\sqrt{x}} + 28$.

(b) Proven.

(c) 248.

WMA12/01 JANUARY 2026

Question 6

8 marks | Applications of Differentiation | Main topic: Proof

6. (i) Given that p and q are consecutive odd numbers, where $p > q > 0$, prove that

$$p^2 - q^2$$

is a multiple of 8

(4)

(ii) The curve C has equation

$$y = x^3 + 12x^2 + 49x + 2$$

Prove that C has no stationary points.

(4)

WORKED SOLUTION

Question 6 · WMA12/01 January 2026

1 Represent the consecutive odd numbers

Let $q = 2n - 1$ and $p = 2n + 1$, where n is a positive integer. Then $p > q > 0$ and the numbers are consecutive odd numbers.

2 Subtract the squares

$p^2 - q^2 = (2n + 1)^2 - (2n - 1)^2$.

3 Simplify the expression

$p^2 - q^2 = (4n^2 + 4n + 1) - (4n^2 - 4n + 1) = 8n$.

4 Conclude the proof

Since n is an integer, $8n$ is a multiple of 8, so $p^2 - q^2$ is a multiple of 8.

5 Differentiate the curve

For $y = x^3 + 12x^2 + 49x + 2$, $\dfrac{dy}{dx} = 3x^2 + 24x + 49$.

6 Show the gradient is never zero

$3x^2 + 24x + 49 = 3(x + 4)^2 + 1$. This is always positive for real x.

7 State the conclusion

Since $\dfrac{dy}{dx}$ is never 0, the curve has no stationary points.

FINAL ANSWER

(i) Proven.

(ii) C has no stationary points.

TOPIC

Integration

Every past-paper question on this topic, October 2019 to January 2026.

WMA12/01 OCTOBER 2019

Question 8

9 marks | Integration

8. Solutions relying on calculator technology are not acceptable in this question.

(i)

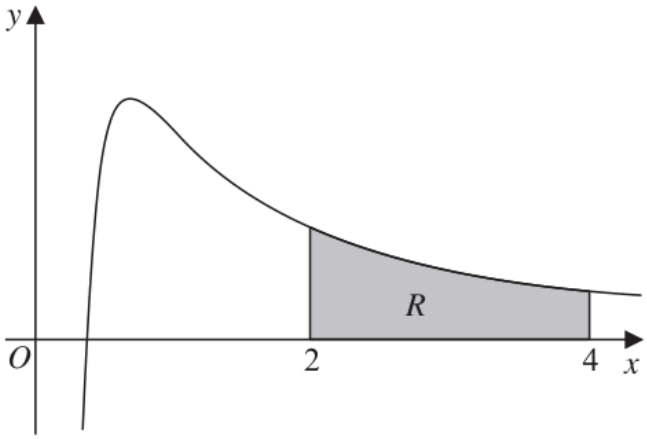


Figure 2

Figure 2 shows a sketch of part of a curve with equation

$$y = \frac{8\sqrt{x} - 5}{2x^2} \qquad x > 0$$

The region R, shown shaded in Figure 2, is bounded by the curve, the line with equation $x = 2$, the x-axis and the line with equation $x = 4$

Find the exact area of R.

(5)

(ii) Find the value of the constant k such that

$$\int_{-3}^{6} \left(\frac{1}{2}x^2 + k \right) \mathrm{d}x = 55$$

(4)

WORKED SOLUTION

Question 8 · WMA12/01 October 2019

1 **Rewrite the curve**

$$y = \frac{8\sqrt{x} - 5}{2x^2} = 4x^{-3/2} - \frac{5}{2}x^{-2}.$$

2 **Integrate the curve**

$$\int \left(4x^{-3/2} - \frac{5}{2}x^{-2}\right)dx = -8x^{-1/2} + \frac{5}{2x}.$$

3 **Apply the limits for the area**

$$\left[-8x^{-1/2} + \frac{5}{2x}\right]_2^4 = \left(-4 + \frac{5}{8}\right) - \left(-4\sqrt{2} + \frac{5}{4}\right).$$

4 **Simplify the exact area**

This gives $4\sqrt{2} - \frac{37}{8}$.

5 **Integrate the second expression**

$$\int \left(\frac{1}{2}x^2 + k\right)dx = \frac{x^3}{6} + kx.$$

6 **Use the limits**

$$\left[\frac{x^3}{6} + kx\right]_{-3}^{6} = (36 + 6k) - \left(-\frac{9}{2} - 3k\right) = \frac{81}{2} + 9k.$$

7 **Solve for k**

$\frac{81}{2} + 9k = 55$, so $9k = \frac{29}{2}$ and $k = \frac{29}{18}$.

✓ **FINAL ANSWER**

$(i)\ 4\sqrt{2} - \frac{37}{8}$. $\quad (ii)\ k = \frac{29}{18}$.

Question 7

6 marks | Integration

7.

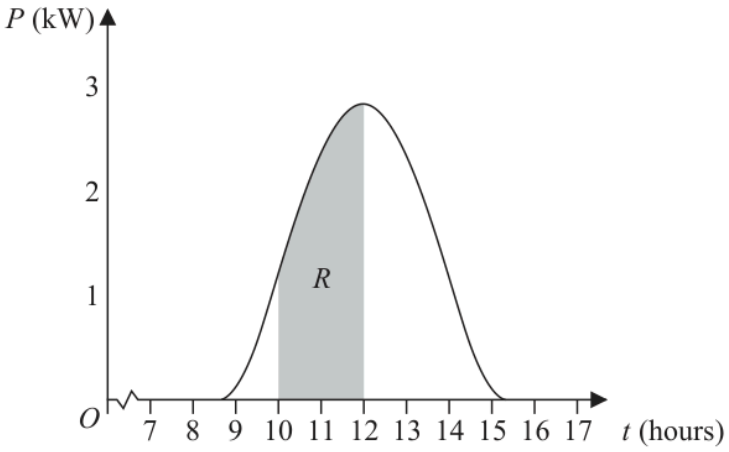


Figure 1

Solar panels are installed on the roof of a building.

The power, P, produced on a particular day, in kW, can be modelled by the equation

$$P = 0.95 + 2^{t-12} + 2^{12-t} - (t-12)^2 \qquad 8.5 \leqslant t \leqslant 15.2$$

where t is the time in hours after midnight. The graph of P against t is shown in Figure 1.

A table of values of t and P is shown below, with the values of P given to 4 significant figures where appropriate.

Time, t (hours)	10	10.5	11	11.5	12
Power, P (kW)		1.882	2.45		2.95

(a) Use the given equation to complete the table, giving the values of P to 4 significant figures where appropriate.

(2)

The amount of energy, in kWh, produced between 10:00 and 12:00 can be found by calculating the area of region R, shown shaded in Figure 1.

(b) Use the trapezium rule, with all the values of P in the completed table, to find an estimate for the amount of energy produced between 10:00 and 12:00. Give your answer to 2 decimal places.

(4)

1 **Use the model at t equals ten**

$P(10) = 0.95 + 2^{-2} + 2^{2} - (10 - 12)^{2} = 0.95 + 0.25 + 4 - 4 = 1.20$.

2 **Use the model at t equals eleven point five**

$P(11.5) = 0.95 + 2^{-0.5} + 2^{0.5} - (11.5 - 12)^{2} = 2.821\ldots$.

3 **Identify the trapezium width**

The time values increase by 0.5, so $h = 0.5$.

4 **Apply the trapezium rule**

The estimate is $\frac{0.5}{2}\{1.20 + 2.95 + 2(1.882 + 2.45 + 2.821\ldots)\}$.

5 **Calculate the energy**

This gives $4.614\ldots$, so the estimated energy is $4.61\ \mathrm{kWh}$ to 2 decimal places.

✓ **FINAL ANSWER**

$(a)\ P(10) = 1.20,\ P(11.5) = 2.821.\quad (b)\ 4.61\ \mathrm{kWh}$.

Question 9

10 marks | Integration

9.

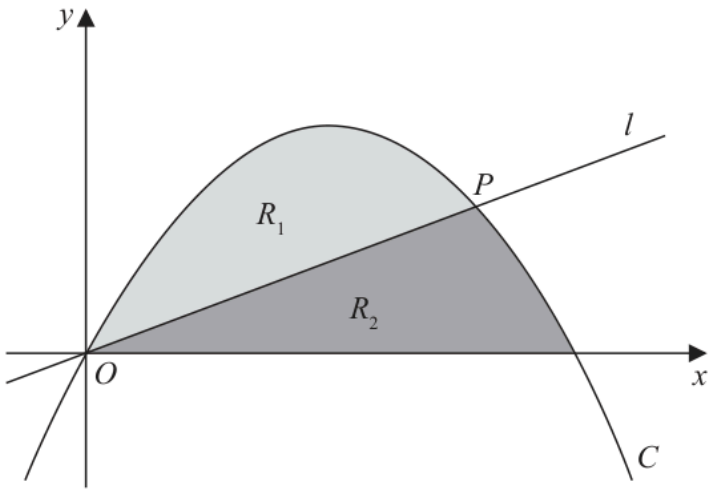


Figure 2

Figure 2 shows

- the curve C with equation $y = x - x^2$
- the line l with equation $y = mx$, where m is a constant and $0 < m < 1$

The line and the curve intersect at the origin O and at the point P.

(a) Find, in terms of m, the coordinates of P.

(2)

The region R_1, shown shaded in Figure 2, is bounded by C and l.

(b) Show that the area of R_1 is

$$\frac{(1-m)^3}{6}$$

(5)

The region R_2, also shown shaded in Figure 2, is bounded by C, the x-axis and l.

Given that the area of R_1 is equal to the area of R_2

(c) find the exact value of m.

(3)

1 Find the second intersection

At an intersection, $x - x^2 = mx$, so $x(1 - m - x) = 0$.

2 State P

The non-zero intersection has $x = 1 - m$, and then $y = m(1 - m)$, so $P = (1 - m, m(1 - m))$.

3 Set up area R one

Region R_1 is between the curve and the line from $x = 0$ to $x = 1 - m$.

4 Integrate the difference

$A_{R_1} = \int_0^{1-m} \{(x - x^2) - mx\}\,dx = \int_0^{1-m} \{(1 - m)x - x^2\}\,dx$.

5 Evaluate

$$A_{R_1} = \left[\frac{1 - m}{2}x^2 - \frac{1}{3}x^3\right]_0^{1-m} = \frac{(1 - m)^3}{2} - \frac{(1 - m)^3}{3} = \frac{(1 - m)^3}{6}.$$

6 Use the total curve area

The total area under $y = x - x^2$ from $x = 0$ to $x = 1$ is $\int_0^1 (x - x^2)\,dx = \frac{1}{6}$.

7 Use equal regions

Since R_1 and R_2 are equal and together make this total area, each has area $\frac{1}{12}$.

8 Solve for m

$\frac{(1 - m)^3}{6} = \frac{1}{12}$, so $(1 - m)^3 = \frac{1}{2}$ and $m = 1 - \frac{1}{\sqrt[3]{2}}$.

FINAL ANSWER

(a) $P = (1 - m,\ m(1 - m))$. (b) Proven: $\frac{(1 - m)^3}{6}$.

(c) $m = 1 - \frac{1}{\sqrt[3]{2}}$.

WMA12/01 MAY/JUNE 2022

Question 7

7 marks | Integration

7. $\mathrm{f}(x) = Ax^3 + 6x^2 - 4x + B$

where A and B are constants.

Given that

- $(x + 2)$ is a factor of $\mathrm{f}(x)$
- $\int_3^5 \mathrm{f}(x)\,\mathrm{d}x = 176$

find the value of A and the value of B.

(7)

WORKED SOLUTION

Question 7 · WMA12/01 May/June 2022

1 Use the factor condition

Since $(x + 2)$ is a factor, $f(-2) = 0$.

2 Form the first equation

$A(-2)^3 + 6(-2)^2 - 4(-2) + B = 0$, so $-8A + 32 + B = 0$ and $B = 8A - 32$.

3 Set up the integral

$\int_3^5 (Ax^3 + 6x^2 - 4x + B)\,dx = 176$.

4 Integrate

$$\left[\frac{A}{4}x^4 + 2x^3 - 2x^2 + Bx\right]_3^5 = 176.$$

5 Simplify the second equation

This gives $136A + 164 + 2B = 176$, so $68A + B = 6$.

6 Solve simultaneously

Using $B = 8A - 32$ in $68A + B = 6$ gives $76A = 38$, so $A = \frac{1}{2}$.

7 **Find B**

$$B = 8\left(\frac{1}{2}\right) - 32 = -28.$$

✓ **FINAL ANSWER**

$$A = \frac{1}{2}, \quad B = -28.$$

Question 6

7 marks | Integration

6. The curve C_1 has equation $y = f(x)$.

A table of values of x and y for $y = f(x)$ is shown below, with the y values rounded to 4 decimal places where appropriate.

x	0	0.5	1	1.5	2
y	3	2.6833	2.4	2.1466	1.92

(a) Use the trapezium rule with all the values of y in the table to find an approximation for

$$\int_0^2 f(x)\,dx$$

giving your answer to 3 decimal places.

(3)

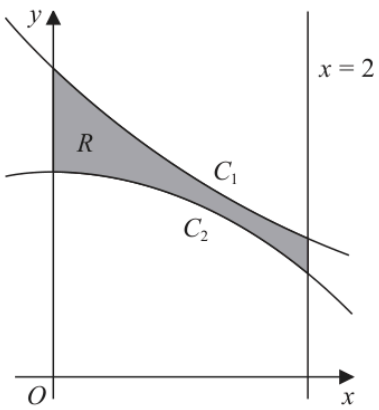


Figure 1

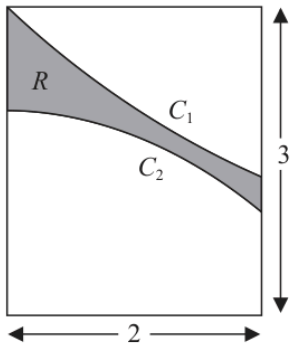


Figure 2

The region R, shown shaded in Figure 1, is bounded by

- the curve C_1
- the curve C_2 with equation $y = 2 - \frac{1}{4}x^2$
- the line with equation $x = 2$
- the y-axis

The region R forms part of the design for a logo shown in Figure 2.

The design consists of the shaded region R inside a rectangle of width 2 and height 3

Using calculus and the answer to part (a),

(b) calculate an estimate for the percentage of the logo which is shaded.

(4)

1 Identify the strip width

The x values increase by 0.5, so the trapezium-rule width is $h = 0.5$.

2 Apply the trapezium rule

$$\int_0^2 f(x)\,dx \approx \frac{0.5}{2}[3 + 1.92 + 2(2.6833 + 2.4 + 2.1466)].$$

3 Evaluate the estimate

This gives 4.84495, so the approximation is 4.845 to 3 decimal places.

4 Find the area under C2

$$\int_0^2 \left(2 - \frac{1}{4}x^2\right)dx = \left[2x - \frac{x^3}{12}\right]_0^2 = 4 - \frac{2}{3} = \frac{10}{3}.$$

5 Estimate the shaded area

The shaded region lies between C_1 and C_2, so its estimated area is $4.845 - \dfrac{10}{3} = 1.5116\ldots$.

6 Compare with the rectangle

The rectangle has area $2 \times 3 = 6$.

7 Convert to a percentage

Percentage shaded $= \dfrac{1.5116\ldots}{6} \times 100 = 25.19\ldots\%$, so the estimate is 25.2%.

FINAL ANSWER

(a) 4.845.

(b) 25.2%.

Question 9

8 marks | Integration

9. **In this question you must show all stages of your working.**

Solutions based entirely on calculator technology are not acceptable.

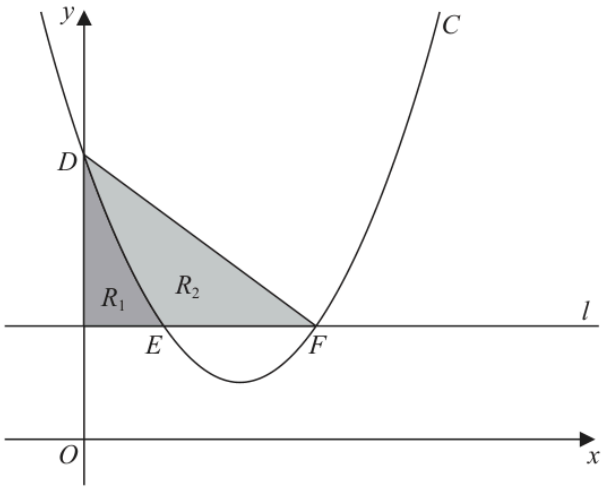


Figure 3

Figure 3 shows

- the curve C with equation $y = x^2 - 4x + 5$
- the line l with equation $y = 2$

The curve C intersects the y-axis at the point D.

(a) Write down the coordinates of D.

(1)

The curve C intersects the line l at the points E and F, as shown in Figure 3.

(b) Find the x coordinate of E and the x coordinate of F.

(2)

Shown shaded in Figure 3 is

- the region R_1 which is bounded by C, l and the y-axis
- the region R_2 which is bounded by C and the line segments EF and DF

Given that $\dfrac{\text{area of } R_1}{\text{area of } R_2} = k$, where k is a constant,

(c) use algebraic integration to find the exact value of k, giving your answer as a simplified fraction.

(5)

1 Find D

At the y-axis, $x = 0$. On C, $y = 0^2 - 4(0) + 5 = 5$, so $D = (0, 5)$.

2 Find intersections with the line

The line is $y = 2$, so solve $x^2 - 4x + 5 = 2$.

3 Solve the quadratic

$x^2 - 4x + 3 = 0$, so $(x - 1)(x - 3) = 0$. Hence $x_E = 1$ and $x_F = 3$.

4 Area of R1

R_1 lies between the curve and the line $y = 2$ from $x = 0$ to $x = 1$.

5 Integrate R1

$$R_1 = \int_0^1 \left(x^2 - 4x + 5 - 2\right) dx = \int_0^1 (x^2 - 4x + 3) dx.$$

6 Evaluate R1

$$R_1 = \left[\frac{x^3}{3} - 2x^2 + 3x\right]_0^1 = \frac{1}{3} - 2 + 3 = \frac{4}{3}.$$

7 Find the line DF

The line through $D = (0, 5)$ and $F = (3, 2)$ has gradient -1, so its equation is $y = 5 - x$.

8 Split R2

For $0 \le x \le 1$, R_2 is between $y = 5 - x$ and the curve. For $1 \le x \le 3$, it is between $y = 5 - x$ and $y = 2$.

9 Integrate R2

$$R_2 = \int_0^1 \left[(5 - x) - (x^2 - 4x + 5)\right] dx + \int_1^3 \left[(5 - x) - 2\right] dx.$$

10 Evaluate R2

$$R_2 = \int_0^1 (-x^2 + 3x) dx + \int_1^3 (3 - x) dx = \frac{7}{6} + 2 = \frac{19}{6}.$$

11 Find the ratio

$$k = \frac{\text{area of } R_1}{\text{area of } R_2} = \frac{4/3}{19/6} = \frac{8}{19}.$$

FINAL ANSWER

$(a)\ D = (0, 5).$

$(b)\ x_E = 1,\ x_F = 3.$

$(c)\ k = \dfrac{8}{19}.$

Question 10

10 marks | Integration

10. The curve C has equation

$$y = \frac{(x-k)^2}{\sqrt{x}} \qquad x > 0$$

where k is a **positive** constant.

(a) Show that

$$\int_1^{16} \frac{(x-k)^2}{\sqrt{x}}\,\mathrm{d}x = ak^2 + bk + \frac{2046}{5}$$

where a and b are integers to be found.

(5)

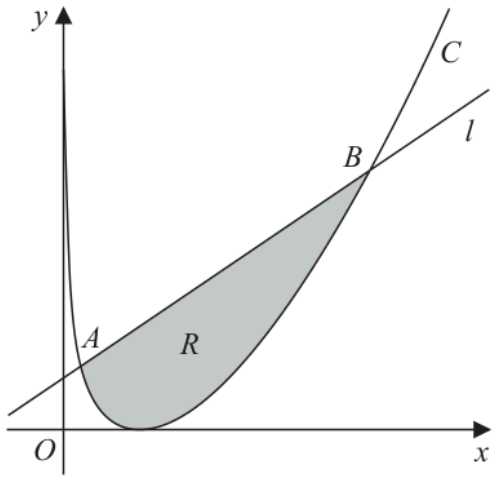


Figure 1

Figure 1 shows a sketch of the curve C and the line l.

Given that l intersects C at the point $A(1, 9)$ and at the point $B(16, q)$ where q is a constant,

(b) show that $k = 4$

(2)

The region R, shown shaded in Figure 1, is bounded by C and l

Using the answers to parts (a) and (b),

(c) find the area of region R

(3)

1 **Expand the integrand**

$\dfrac{(x-k)^2}{\sqrt{x}} = \dfrac{x^2-2kx+k^2}{x^{1/2}} = x^{3/2} - 2kx^{1/2} + k^2x^{-1/2}$.

2 **Integrate**

$\displaystyle\int \left(x^{3/2} - 2kx^{1/2} + k^2x^{-1/2}\right)dx = \frac{2}{5}x^{5/2} - \frac{4k}{3}x^{3/2} + 2k^2x^{1/2}$.

3 **Substitute the limits**

From 1 to 16, this gives $\left(\dfrac{2048}{5} - \dfrac{256k}{3} + 8k^2\right) - \left(\dfrac{2}{5} - \dfrac{4k}{3} + 2k^2\right)$.

4 **Simplify the integral**

The result is $6k^2 - 84k + \dfrac{2046}{5}$, so $a = 6$ and $b = -84$.

5 **Use point A**

Since $A(1, 9)$ lies on C, $9 = \dfrac{(1-k)^2}{\sqrt{1}} = (1-k)^2$.

6 **Find k**

Thus $1 - k = \pm 3$, so $k = 4$ or $k = -2$. Since k is positive, $k = 4$.

7 **Find point B**

When $k = 4$ and $x = 16$, $q = \dfrac{(16-4)^2}{\sqrt{16}} = \dfrac{144}{4} = 36$, so $B = (16, 36)$.

8 **Find the line**

The line through $A(1, 9)$ and $B(16, 36)$ has gradient $\dfrac{36-9}{16-1} = \dfrac{9}{5}$, so l has equation $y = \dfrac{9}{5}x + \dfrac{36}{5}$.

9 **Area under the line**

$\displaystyle\int_1^{16} \left(\frac{9}{5}x + \frac{36}{5}\right)dx = \left[\frac{9}{10}x^2 + \frac{36}{5}x\right]_1^{16} = \frac{675}{2}$.

10 **Area under the curve**

From part (a), with $k = 4$, $\displaystyle\int_1^{16} \frac{(x-4)^2}{\sqrt{x}}dx = 6(4)^2 - 84(4) + \frac{2046}{5} = \frac{846}{5}$.

11 **Subtract curve from line**

The shaded area is $\dfrac{675}{2} - \dfrac{846}{5} = \dfrac{1683}{10}$.

FINAL ANSWER

(a) $a = 6,\ b = -84$.

(b) $k = 4$.

(c) Area $= \dfrac{1683}{10}$.

WMA12/01 OCTOBER 2023

Question 6

6 marks | Integration

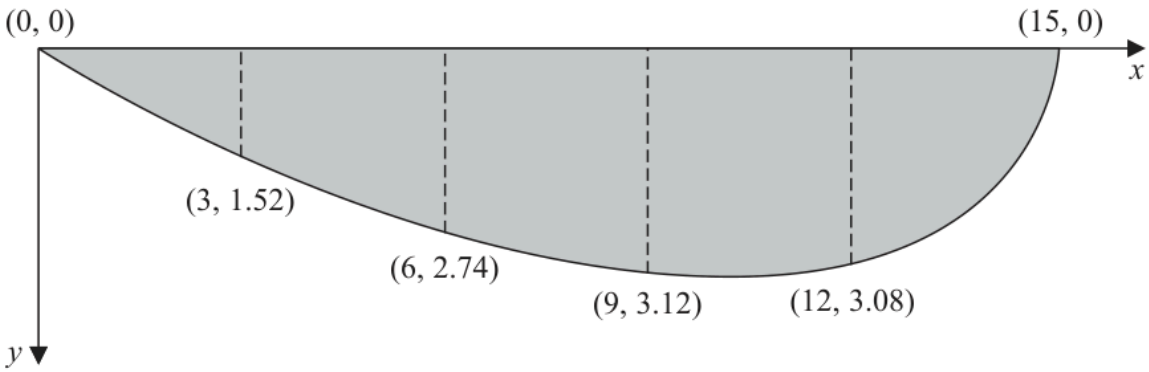


Figure 1

A river is being studied.

At one particular place, the river is 15 m wide.

The depth, y metres, of the river is measured at a point x metres from one side of the river.

Figure 1 shows a plot of the cross-section of the river and the coordinate values (x, y)

(a) Use the trapezium rule with all the y values given in Figure 1 to estimate the cross-sectional area of the river.

(3)

The water in the river is modelled as flowing at a constant speed of $1.5\,\text{m}\,\text{s}^{-1}$ across the whole of the cross-section.

(b) Use the model and the answer to part (a) to estimate the volume of water flowing through this section of the river each minute, giving your answer in m^3 to 2 significant figures.

(2)

Assuming the model,

(c) state, giving a reason for your answer, whether your answer for part (b) is an overestimate or an underestimate of the true volume of water flowing through this section of the river each minute.

(1)

WORKED SOLUTION

Question 6 · WMA12/01 October 2023

1 **Read the strip width**

The x values are $0, 3, 6, 9, 12, 15$, so the trapezium width is $h = 3$.

2 **Apply the trapezium rule**

$\text{Area} \approx \frac{3}{2}[0 + 0 + 2(1.52 + 2.74 + 3.12 + 3.08)]$.

3 **Find the cross-sectional area**

This gives $31.38\ \text{m}^2$.

4 **Convert flow speed to one minute**

The speed is $1.5\ \text{m s}^{-1}$, so in one minute the water travels $1.5 \times 60 = 90\ \text{m}$.

5 **Estimate volume**

Volume per minute $\approx 31.38 \times 90 = 2824.2\ \text{m}^3$.

6 **Round the volume**

To 2 significant figures, this is $2.8 \times 10^3\ \text{m}^3$.

7 **State over or under**

It is an underestimate, because the curve between the plotted points lies below the straight-line trapezium chords in the diagram, so the true cross-sectional area is larger.

✓ **FINAL ANSWER**

(a) $31.38\ \text{m}^2$.

(b) $2.8 \times 10^3\ \text{m}^3$.

(c) Underestimate.

Question 6

9 marks | Integration

6. (a) Sketch the curve with equation

$$y = a^x + 4$$

where a is a positive constant greater than 1

On your sketch, show

- the coordinates of the point of intersection of the curve with the y-axis
- the equation of the asymptote of the curve

(3)

x	2	2.3	2.6	2.9	3.2	3.5
y	0	0.3246	0.8629	1.6643	2.7896	4.3137

The table shows corresponding values of x and y for

$$y = 2^x - 2x$$

with the values of y given to 4 decimal places as appropriate.

Using the trapezium rule with all the values of y in the given table,

(b) obtain an estimate for $\int_2^{3.5} \left(2^x - 2x\right) \mathrm{d}x$, giving your answer to 2 decimal places.

(3)

(c) Using your answer to part (b) and making your method clear, estimate

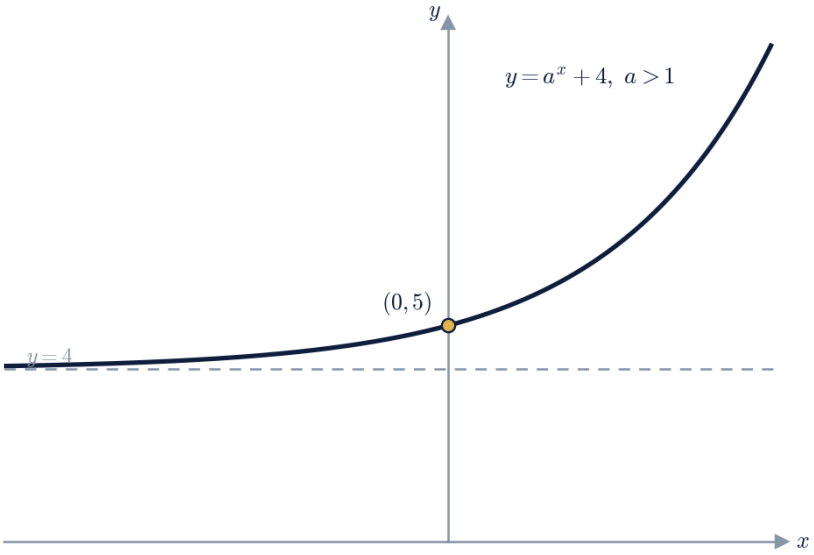
(i) $\int_2^{3.5} \left(2^x + 2x\right) \mathrm{d}x$

(ii) $\int_2^{3.5} \left(2^{x+1} - 4x\right) \mathrm{d}x$

(3)

1 **Sketch key features**

For $y = a^x + 4$ with $a > 1$, the curve is increasing.



2 **Find the y-intercept**

At $x = 0$, $y = a^0 + 4 = 5$, so the y-intercept is $(0, 5)$.

3 **Find the asymptote**

As $x \to -\infty$, $a^x \to 0$, so the horizontal asymptote is $y = 4$.

4 **Use the trapezium width**

The x values increase by 0.3, so $h = 0.3$.

5 **Apply the trapezium rule**

$$\int_2^{3.5} (2^x - 2x)dx \approx \frac{0.3}{2}[0 + 4.3137 + 2(0.3246 + 0.8629 + 1.6643 + 2.7896)].$$

6 **Round part b**

This gives 2.339475, so the estimate is 2.34 to 2 decimal places.

7 **Estimate part c i**

$2^x + 2x = (2^x - 2x) + 4x$, so $\int_2^{3.5} (2^x + 2x)dx \approx 2.34 + \int_2^{3.5} 4x\,dx$.

8 **Add the extra integral**

$\int_2^{3.5} 4x\,dx = [2x^2]_2^{3.5} = 16.5$, so the estimate is $2.34 + 16.5 = 18.84$.

9 **Estimate part c ii**

$2^{x+1} - 4x = 2(2^x - 2x)$, so the estimate is $2(2.34) = 4.68$.

✓ **FINAL ANSWER**

(a) y-intercept $(0, 5)$ and asymptote $y = 4$.

(b) 2.34.

$(c)(i)$ 18.84.

$(c)(ii)$ 4.68.

Question 8

8 marks | Integration

8. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

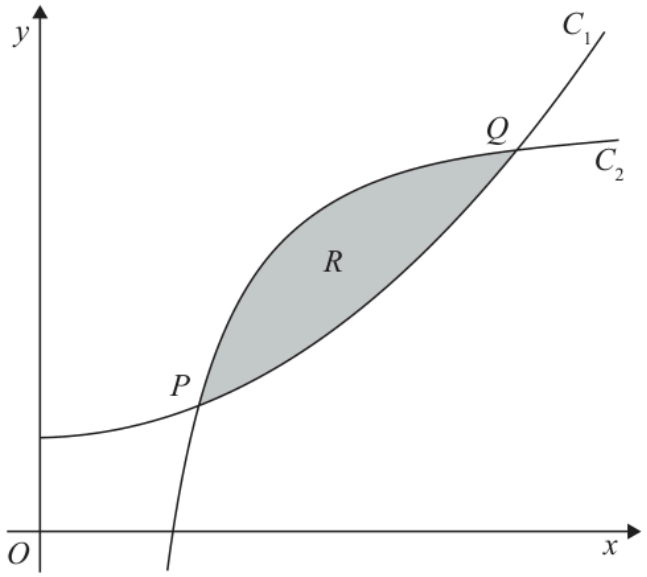


Figure 1

Figure 1 shows a sketch of part of the curve C_1 with equation

$$y = x^2 + 3 \qquad x > 0$$

and part of the curve C_2 with equation

$$y = 13 - \frac{9}{x^2} \qquad x > 0$$

The curves C_1 and C_2 intersect at the points P and Q as shown in Figure 1.

(a) Use algebra to find the x coordinate of P and the x coordinate of Q.

(4)

The finite region R, shown shaded in Figure 1, is bounded by C_1 and C_2

(b) Use algebraic integration to find the exact area of R.

(4)

1 **Equate the curves**

$x^2 + 3 = 13 - \dfrac{9}{x^2}$.

2 **Clear the denominator**

Multiplying by x^2 gives $x^4 + 3x^2 = 13x^2 - 9$, so $x^4 - 10x^2 + 9 = 0$.

3 **Factor**

Let $u = x^2$. Then $u^2 - 10u + 9 = (u - 1)(u - 9) = 0$, so $x^2 = 1$ or $x^2 = 9$.

4 **Use x greater than zero**

Since $x > 0$, the intersections have $x = 1$ and $x = 3$. Thus $x_P = 1$ and $x_Q = 3$.

5 **Set up the area**

Between $x = 1$ and $x = 3$, curve C_2 is above curve C_1, so $R = \displaystyle\int_1^3 \left(13 - \frac{9}{x^2} - (x^2 + 3)\right) dx$.

6 **Simplify the integrand**

$R = \displaystyle\int_1^3 (10 - x^2 - 9x^{-2})\, dx$.

7 **Integrate**

$R = \left[10x - \dfrac{x^3}{3} + \dfrac{9}{x}\right]_1^3$.

8 **Evaluate**

$R = (30 - 9 + 3) - \left(10 - \dfrac{1}{3} + 9\right) = 24 - \dfrac{56}{3} = \dfrac{16}{3}$.

✓ **FINAL ANSWER**

(a) $x_P = 1$, $x_Q = 3$.

(b) $\dfrac{16}{3}$.

Question 9

12 marks | Integration

9.

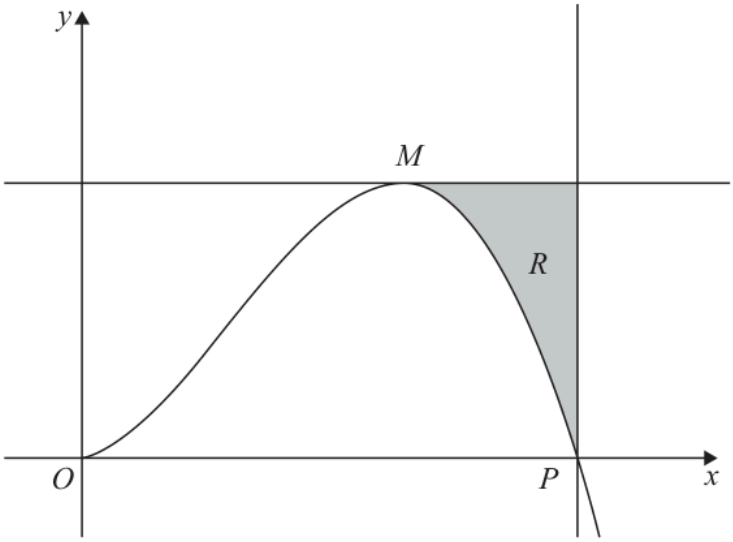


Figure 3

In this question you must show all stages of your working.
Solutions relying entirely on calculator technology are not acceptable.

Figure 3 shows a sketch of part of the curve with equation

$$y = \frac{9x^2\left(5 - \sqrt{x}\right)}{5} \qquad x \geqslant 0$$

The curve has a turning point at the point M, as shown in Figure 3.

(a) Using calculus, find the coordinates of M.

(5)

The curve crosses the x-axis at the point P, as shown in Figure 3.

(b) Use algebra to find the x coordinate of P.

(2)

The finite region R, shown shaded in Figure 3, is bounded by the curve, the line through M parallel to the x-axis and the line through P parallel to the y-axis.

(c) Use algebraic integration to find the area of R, giving your answer to one decimal place.

(5)

1 Rewrite the curve

$y = \dfrac{9x^2(5-\sqrt{x})}{5} = 9x^2 - \dfrac{9}{5}x^{5/2}$.

2 Differentiate

$\dfrac{dy}{dx} = 18x - \dfrac{9}{2}x^{3/2} = \dfrac{9}{2}x(4-\sqrt{x})$.

3 Find the turning point x coordinate

For the turning point shown away from the origin, set $4-\sqrt{x}=0$. Hence $x=16$.

4 Find the y coordinate

At $x=16$, $y = \dfrac{9(16)^2(5-4)}{5} = \dfrac{2304}{5}$.

5 State M

Therefore $M\left(16, \dfrac{2304}{5}\right)$.

6 Find the positive x intercept

For the positive intercept, $\dfrac{9x^2(5-\sqrt{x})}{5} = 0$ gives $5-\sqrt{x}=0$, so $x=25$.

7 Set up the area

The top boundary is $y = \dfrac{2304}{5}$ and the curve is below it from $x=16$ to $x=25$. So

$R = \displaystyle\int_{16}^{25}\left(\frac{2304}{5} - 9x^2 + \frac{9}{5}x^{5/2}\right)dx$.

8 Integrate

$R = \left[\dfrac{2304}{5}x - 3x^3 + \dfrac{18}{35}x^{7/2}\right]_{16}^{25}$.

9 Evaluate

$R = \dfrac{9189}{7} = 1312.714\ldots$, so the area is 1312.7 to one decimal place.

✓ FINAL ANSWER

(a) $M\left(16, \dfrac{2304}{5}\right)$.

(b) $x_P = 25$.

(c) 1312.7.

Question 7

10 marks | Integration

7.

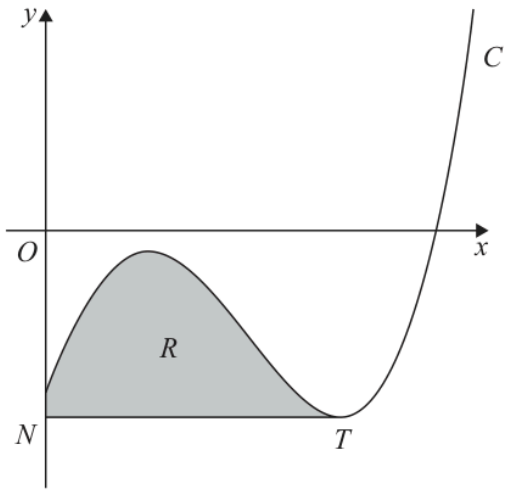


Figure 3

In this question you must show all stages of your working.
Solutions relying entirely on calculator technology are not acceptable.

Figure 3 shows a sketch of part of the curve C with equation

$$y = x^3 - 4x^{\frac{5}{2}} - kx^{\frac{1}{2}} + 28x - 44 \qquad x \geqslant 0$$

where k is a positive constant.

(a) Find $\dfrac{dy}{dx}$ in simplest form.

(2)

The point T, shown in Figure 3, is a minimum stationary point on C.

Given that the x coordinate of T is 9

(b) show that $k = 6$

(2)

The line through T parallel to the x-axis meets the y-axis at the point N.

The finite region R, shown shaded in Figure 3, is bounded by C, the y-axis and the line segment NT.

(c) Use algebraic integration to find the area of R, giving the answer to 3 significant figures.

(6)

1 **Differentiate**

For $y = x^3 - 4x^{5/2} - kx^{1/2} + 28x - 44$, $\dfrac{dy}{dx} = 3x^2 - 10x^{3/2} - \dfrac{k}{2}x^{-1/2} + 28$.

2 **Write in simplest form**

$\dfrac{dy}{dx} = 3x^2 - 10x^{3/2} - \dfrac{k}{2\sqrt{x}} + 28$.

3 **Use the stationary point**

At T, $x = 9$ and $\dfrac{dy}{dx} = 0$.

4 **Substitute x equals nine**

$0 = 3(9)^2 - 10(9)^{3/2} - \dfrac{k}{2\sqrt{9}} + 28 = 243 - 270 - \dfrac{k}{6} + 28$.

5 **Find k**

This simplifies to $0 = 1 - \dfrac{k}{6}$, so $k = 6$.

6 **Find the horizontal line**

With $k = 6$, the y coordinate of T is $9^3 - 4(9)^{5/2} - 6\sqrt{9} + 28(9) - 44 = -53$.

7 **Set up the area**

The line NT is $y = -53$. The region is between the curve and this line from $x = 0$ to $x = 9$.

8 **Write the integral**

$R = \displaystyle\int_0^9 \left(x^3 - 4x^{5/2} - 6x^{1/2} + 28x - 44 + 53\right)dx$.

9 **Integrate**

$R = \left[\dfrac{x^4}{4} - \dfrac{8}{7}x^{7/2} - 4x^{3/2} + 14x^2 + 9x\right]_0^9$.

10 **Evaluate**

$R = \dfrac{6939}{28} = 247.821\ldots$, so the area is 248 to 3 significant figures.

FINAL ANSWER

(a) $3x^2 - 10x^{3/2} - \dfrac{k}{2\sqrt{x}} + 28$.

(b) Proven.

(c) 248.

WMA12/01 JANUARY 2021

Question 9

10 marks | Integration | Main topic: Circles

9. A circle C has equation

$$(x-k)^2+(y-2k)^2=k+7$$

where k is a positive constant.

(a) Write down, in terms of k,

(i) the coordinates of the centre of C,

(ii) the radius of C.

(2)

Given that the point $P(2,3)$ lies on C

(b) (i) show that $5k^2-17k+6=0$

(ii) hence find the possible values of k.

(3)

The tangent to the circle at P intersects the x-axis at point T.

Given that $k<2$

(c) calculate the exact area of triangle OPT.

(5)

WORKED SOLUTION

Question 9 · WMA12/01 January 2021

1 Read the centre and radius

From $(x-k)^2+(y-2k)^2=k+7$, the centre is $(k,2k)$ and the radius is $\sqrt{k+7}$.

2 Substitute point P

Since $P(2,3)$ lies on the circle, $(2-k)^2+(3-2k)^2=k+7$.

3 Show the quadratic

Expanding gives $k^2-4k+4+4k^2-12k+9=k+7$, hence $5k^2-17k+6=0$.

4 **Solve for k**

$5k^2 - 17k + 6 = (5k - 2)(k - 3)$, so $k = \dfrac{2}{5}$ or $k = 3$.

5 **Use the condition k less than two**

For the tangent calculation, $k < 2$, so the centre is $\left(\dfrac{2}{5}, \dfrac{4}{5}\right)$.

6 **Find the tangent gradient**

The radius from the centre to P has gradient $\dfrac{3 - 4/5}{2 - 2/5} = \dfrac{11}{8}$, so the tangent gradient is $-\dfrac{8}{11}$.

7 **Find the x intercept**

The tangent is $y - 3 = -\dfrac{8}{11}(x - 2)$. Setting $y = 0$ gives $x = \dfrac{49}{8}$, so $T = \left(\dfrac{49}{8}, 0\right)$.

8 **Calculate the area**

Triangle OPT has base $\dfrac{49}{8}$ on the x-axis and height 3, so its area is $\dfrac{1}{2} \cdot \dfrac{49}{8} \cdot 3 = \dfrac{147}{16}$.

✓ **FINAL ANSWER**

$(a)(i)\ (k, 2k).$ $(a)(ii)\ \sqrt{k + 7}.$ $(b)(ii)\ k = 3 \text{ or } \dfrac{2}{5}.$ $(c)\ \dfrac{147}{16}.$

WMA12/01 MAY/JUNE 2021

Question 6

7 marks | Integration | Main topic: Circles

6. A circle has equation

$$x^2 - 6x + y^2 + 8y + k = 0$$

where k is a positive constant.

Given that the x-axis is a tangent to this circle,

(a) find the value of k.

(3)

The circle meets the coordinate axes at the points R, S and T.

(b) Find the exact area of the triangle RST.

(4)

WORKED SOLUTION

Question 6 · WMA12/01 May/June 2021

1 Complete the square

$x^2 - 6x = (x-3)^2 - 9$ and $y^2 + 8y = (y+4)^2 - 16$.

2 Write centre-radius form

The circle becomes $(x-3)^2 + (y+4)^2 = 25 - k$.

3 Use the tangent condition

The centre is $(3, -4)$. Since the x-axis is tangent, the radius is the distance from the centre to the x-axis, which is 4.

4 Find k

Thus $25 - k = 4^2 = 16$, so $k = 9$.

5 Find the intercepts

With $k = 9$, the circle is $(x-3)^2 + (y+4)^2 = 16$. On the x-axis, $y = 0$, giving $x = 3$.

6 Find the y-axis points

On the y-axis, $x = 0$, so $9 + (y+4)^2 = 16$. Hence $y = -4 \pm \sqrt{7}$.

7 **Calculate the triangle area**

The vertical base on the y-axis has length $2\sqrt{7}$, and its perpendicular distance to $(3, 0)$ is 3. The area is $\frac{1}{2}(2\sqrt{7})(3) = 3\sqrt{7}$.

✓ **FINAL ANSWER**

$(a)\ k = 9.\quad (b)\ 3\sqrt{7}.$

WMA12/01 MAY/JUNE 2021

Question 9

10 marks | Integration

Main topic: Applications of Differentiation

9.

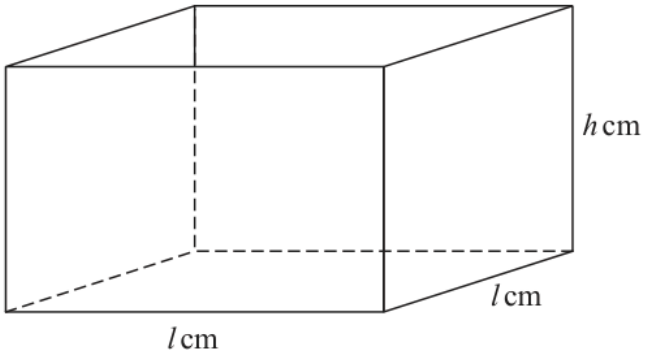


Figure 3

Figure 3 shows a sketch of a square based, open top box.

The height of the box is h cm, and the base edges each have length l cm.

Given that the volume of the box is 250 000 cm^3

(a) show that the external surface area, S cm^2, of the box is given by

$$S = \frac{250\,000}{h} + 2000\sqrt{h}$$

(3)

(b) Use algebraic differentiation to show that S has a stationary point when $h = 250^k$ where k is a rational constant to be found.

(5)

(c) Justify by further differentiation that this value of h gives the minimum external surface area of the box.

(2)

(Total 10 marks)

WORKED SOLUTION

Question 9 · WMA12/01 May/June 2021

1 Use the volume

The volume is $l^2h = 250000$, so $l^2 = \dfrac{250000}{h}$.

2 Write the surface area

The open box has one square base and four side faces, so $S = l^2 + 4lh$.

3 **Substitute l**

Since $l = \sqrt{\frac{250000}{h}} = \frac{500}{\sqrt{h}}$, $4lh = 4 \cdot \frac{500}{\sqrt{h}} \cdot h = 2000\sqrt{h}$.

4 **Show S**

Therefore $S = \frac{250000}{h} + 2000\sqrt{h}$.

5 **Differentiate S**

$S = 250000h^{-1} + 2000h^{1/2}$, so $\frac{dS}{dh} = -250000h^{-2} + 1000h^{-1/2}$.

6 **Set the derivative to zero**

$-250000h^{-2} + 1000h^{-1/2} = 0$, so $1000h^{-1/2} = 250000h^{-2}$.

7 **Solve for h**

Multiplying by h^2 gives $1000h^{3/2} = 250000$, so $h^{3/2} = 250$ and $h = 250^{2/3}$.

8 **Find k**

Since $h = 250^k$, we have $k = \frac{2}{3}$.

9 **Use the second derivative**

$\frac{d^2S}{dh^2} = 500000h^{-3} - 500h^{-3/2}$.

10 **Check the sign**

At $h = 250^{2/3}$, $h^{3/2} = 250$, so the second derivative is $\frac{500000}{250^2} - \frac{500}{250} = 8 - 2 = 6 > 0$.

11 **Conclude**

The stationary point is therefore a minimum.

✓ **FINAL ANSWER**

(a) Proven: $S = \frac{250000}{h} + 2000\sqrt{h}$.

(b) $h = 250^{2/3}$, so $k = \frac{2}{3}$.

(c) Minimum.

Question 8

10 marks | Integration

Main topic: Applications of Differentiation

8. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

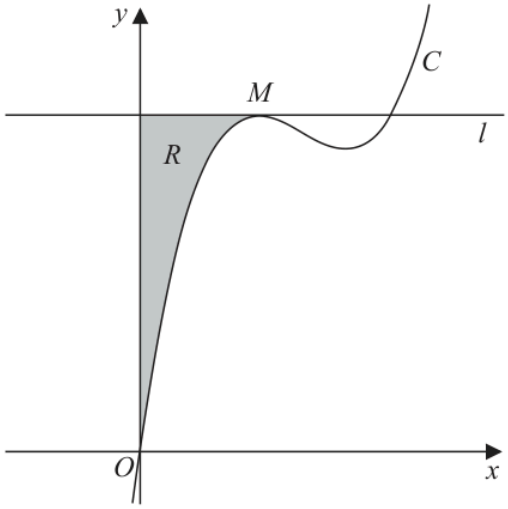


Figure 2

Figure 2 shows a sketch of part of the curve C with equation

$$y = \frac{4}{3}x^3 - 11x^2 + kx \qquad \text{where } k \text{ is a constant}$$

The point M is the maximum turning point of C and is shown in Figure 2.

Given that the x coordinate of M is 2

(a) show that $k = 28$

(3)

(b) Determine the range of values of x for which y is increasing.

(2)

The line l passes through M and is parallel to the x-axis.

The region R, shown shaded in Figure 2, is bounded by the curve C, the line l and the y-axis.

(c) Find, by algebraic integration, the exact area of R.

(5)

1 Differentiate the curve

For $y = \frac{4}{3}x^3 - 11x^2 + kx$, $\frac{dy}{dx} = 4x^2 - 22x + k$.

2 Use the turning point

At the turning point M, $x = 2$ and $\frac{dy}{dx} = 0$.

3 Show k

$4(2)^2 - 22(2) + k = 0$, so $16 - 44 + k = 0$ and $k = 28$.

4 Factor the derivative

With $k = 28$, $\frac{dy}{dx} = 4x^2 - 22x + 28 = 2(2x - 7)(x - 2)$.

5 Find where y is increasing

The derivative is positive outside its roots, so y is increasing for $x < 2$ or $x > \frac{7}{2}$.

6 Find the height of the line

At $x = 2$, $y = \frac{4}{3}(8) - 11(4) + 28(2) = \frac{68}{3}$. Hence line l has equation $y = \frac{68}{3}$.

7 Set up the shaded area

The region is between the line and the curve from $x = 0$ to $x = 2$, so

$A = \int_0^2 \left(\frac{68}{3} - \left(\frac{4}{3}x^3 - 11x^2 + 28x \right) \right) dx$.

8 Integrate

$A = \left[\frac{68}{3}x - \frac{1}{3}x^4 + \frac{11}{3}x^3 - 14x^2 \right]_0^2$.

9 Evaluate exactly

Substituting $x = 2$ gives $\frac{136}{3} - \frac{16}{3} + \frac{88}{3} - 56 = \frac{40}{3}$.

✓ FINAL ANSWER

(a) Proven: $k = 28$.

(b) $x < 2$ or $x > \frac{7}{2}$. (c) $\frac{40}{3}$.

Question 9

12 marks | Integration | Main topic: Circles

9. **In this question you must show detailed reasoning.**

Solutions relying entirely on calculator technology are not acceptable.

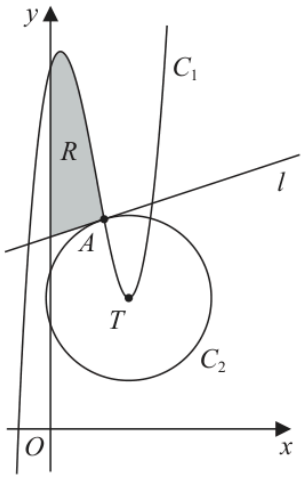


Figure 3

Figure 3 shows

- the curve C_1 with equation $y = x^3 - 5x^2 + 3x + 14$
- the circle C_2 with centre T

The point T is the minimum turning point of C_1

Using Figure 3 and calculus,

(a) find the coordinates of T

(3)

The curve C_1 intersects the circle C_2 at the point A with x coordinate 2

(b) Find an equation of the circle C_2

(3)

The line l shown in Figure 3, is the tangent to circle C_2 at A

(c) Show that an equation of l is

$$y = \frac{1}{3}x + \frac{22}{3}$$

(3)

The region R, shown shaded in Figure 3, is bounded by C_1, l and the y-axis.

(d) Find the exact area of R.

(3)

1 **Differentiate the cubic**

For $y = x^3 - 5x^2 + 3x + 14$, $\dfrac{dy}{dx} = 3x^2 - 10x + 3$.

2 **Find stationary x values**

$3x^2 - 10x + 3 = 0$ factors as $(3x - 1)(x - 3) = 0$, so $x = \dfrac{1}{3}$ or $x = 3$.

3 **Identify the minimum**

$\dfrac{d^2y}{dx^2} = 6x - 10$. At $x = 3$, this is positive, so the minimum turning point has $x = 3$.

4 **Find T**

Substitute $x = 3$: $y = 27 - 45 + 9 + 14 = 5$. Hence $T = (3, 5)$.

5 **Find A**

At $x = 2$ on C_1, $y = 8 - 20 + 6 + 14 = 8$, so $A = (2, 8)$.

6 **Find the circle radius**

The centre of C_2 is $T = (3, 5)$ and it passes through $A = (2, 8)$. Thus $r^2 = (2 - 3)^2 + (8 - 5)^2 = 10$.

7 **Write the circle equation**

Therefore C_2 has equation $(x - 3)^2 + (y - 5)^2 = 10$.

8 **Find the tangent gradient**

The radius TA has gradient $\dfrac{8 - 5}{2 - 3} = -3$, so the tangent at A has gradient $\dfrac{1}{3}$.

9 **Write the tangent equation**

Using $A = (2, 8)$, $y - 8 = \dfrac{1}{3}(x - 2)$, so $y = \dfrac{1}{3}x + \dfrac{22}{3}$.

10 **Set up the area**

From $x = 0$ to $x = 2$, the cubic is above the line. The required area is $\displaystyle\int_0^2 \left[x^3 - 5x^2 + 3x + 14 - \left(\frac{1}{3}x + \frac{22}{3}\right)\right] dx$.

11 **Simplify the integrand**

The integrand is $x^3 - 5x^2 + \dfrac{8}{3}x + \dfrac{20}{3}$.

12 **Integrate**

$$\left[\frac{x^4}{4}-\frac{5x^3}{3}+\frac{4x^2}{3}+\frac{20x}{3}\right]_0^2=4-\frac{40}{3}+\frac{16}{3}+\frac{40}{3}=\frac{28}{3}.$$

✓ **FINAL ANSWER**

$(a)\ T=(3,5).$

$(b)\ (x-3)^2+(y-5)^2=10.$

$(c)\ y=\frac{1}{3}x+\frac{22}{3}.$

$(d)\ \frac{28}{3}.$

Question 9

12 marks | Integration

Main topic: Applications of Differentiation

9.

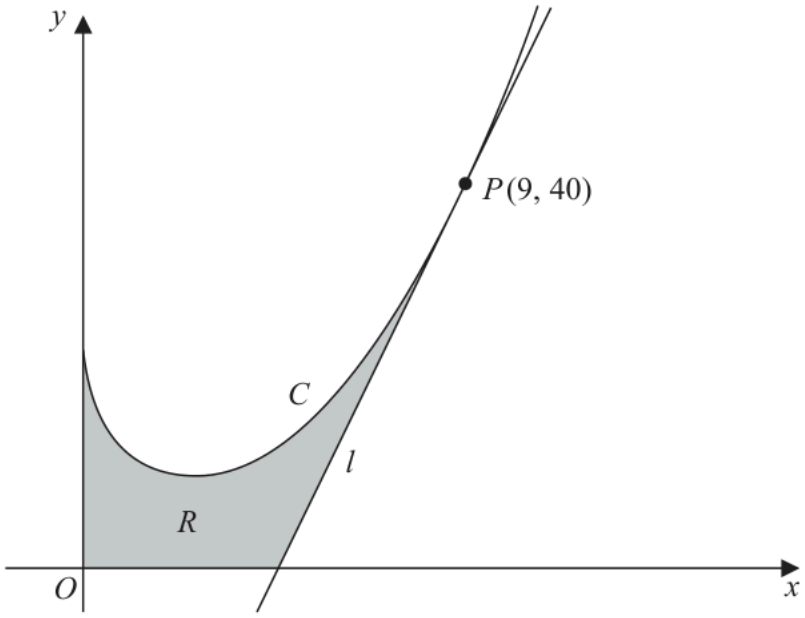


Figure 3

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 3 shows a sketch of part of the curve C with equation

$$y = \frac{2}{3}x^2 - 9\sqrt{x} + 13 \qquad x \geqslant 0$$

(a) Find, using calculus, the range of values of x for which y is increasing.

(4)

The point P lies on C and has coordinates (9, 40).

The line l is the tangent to C at the point P.

The finite region R, shown shaded in Figure 3, is bounded by the curve C, the line l, the x-axis and the y-axis.

(b) Find, using calculus, the exact area of R.

(8)

1 **Differentiate**

For $y = \frac{2}{3}x^2 - 9\sqrt{x} + 13$, $\frac{dy}{dx} = \frac{4}{3}x - \frac{9}{2\sqrt{x}}$.

2 **Set up increasing condition**

y is increasing when $\frac{dy}{dx} > 0$.

3 **Multiply by a positive quantity**

For $x > 0$, multiply by $6\sqrt{x}$: $8x\sqrt{x} - 27 > 0$.

4 **Solve the inequality**

$x^{3/2} > \frac{27}{8} = \left(\frac{3}{2}\right)^3$, so $\sqrt{x} > \frac{3}{2}$ and $x > \frac{9}{4}$.

5 **Find the tangent gradient**

At $P(9, 40)$, $\frac{dy}{dx} = \frac{4}{3}(9) - \frac{9}{2 \cdot 3} = 12 - \frac{3}{2} = \frac{21}{2}$.

6 **Write the tangent line**

$y - 40 = \frac{21}{2}(x - 9)$, so $y = \frac{21}{2}x - \frac{109}{2}$.

7 **Find where the tangent meets the x-axis**

Set $y = 0$: $\frac{21}{2}x - \frac{109}{2} = 0$, so $x = \frac{109}{21}$.

8 **Split the area**

From $x = 0$ to $x = \frac{109}{21}$, the lower boundary is the x-axis. From $x = \frac{109}{21}$ to $x = 9$, the lower boundary is the tangent.

9 **Set up the integral**

$$R = \int_0^{109/21} \left(\frac{2}{3}x^2 - 9\sqrt{x} + 13\right) dx + \int_{109/21}^{9} \left(\frac{2}{3}x^2 - 9\sqrt{x} + 13 - \left(\frac{21}{2}x - \frac{109}{2}\right)\right) dx.$$

10 **Use antiderivatives**

$\int \left(\frac{2}{3}x^2 - 9\sqrt{x} + 13\right) dx = \frac{2}{9}x^3 - 6x^{3/2} + 13x$, and $\int \left(\frac{21}{2}x - \frac{109}{2}\right) dx = \frac{21}{4}x^2 - \frac{109}{2}x$.

11 **Evaluate**

Substituting the limits and simplifying gives $R = \frac{857}{21}$.

FINAL ANSWER

(a) y is increasing for $x > \frac{9}{4}$.

(b) Area $= \frac{857}{21}$.

WMA12/01 JANUARY 2024

Question 10

9 marks | Integration

Main topic: Applications of Differentiation

10. **In this question you must show detailed reasoning.**

Solutions relying entirely on calculator technology are not acceptable.

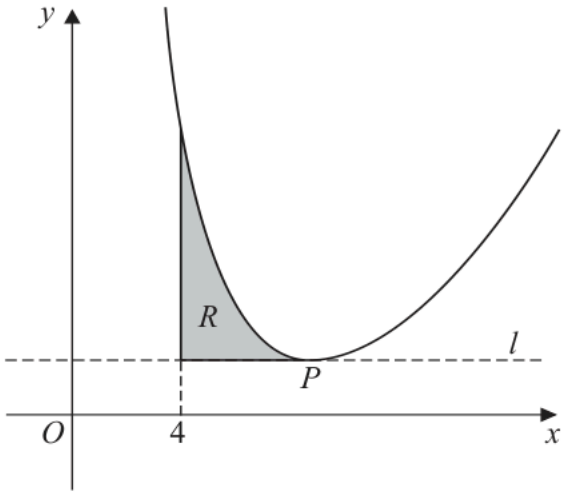


Figure 2

Figure 2 shows a sketch of the curve with equation

$$y = \frac{1}{2}x^2 + \frac{1458}{\sqrt{x^3}} - 74 \qquad x > 0$$

The point P is the only stationary point on the curve.

(a) Use calculus to show that the x coordinate of P is 9

(4)

The line l passes through the point P and is parallel to the x-axis.

The region R, shown shaded in Figure 2, is bounded by the curve, the line l and the line with equation $x = 4$

(b) Use algebraic integration to find the exact area of R.

(5)

WORKED SOLUTION

Question 10 · WMA12/01 January 2024

1 **Rewrite the curve**

$$y = \frac{1}{2}x^2 + 1458x^{-3/2} - 74.$$

2 **Differentiate**

$$\frac{dy}{dx} = x + 1458\left(-\frac{3}{2}\right)x^{-5/2} = x - 2187x^{-5/2}.$$

3 **Use the stationary condition**

At a stationary point, $x - 2187x^{-5/2} = 0$.

4 **Solve for x**

$x = 2187x^{-5/2}$, so $x^{7/2} = 2187 = 3^7$.

5 **Show x coordinate**

Raising both sides to the power $\frac{2}{7}$ gives $x = 3^2 = 9$. Since $x > 0$, this is the only stationary point.

6 **Find the horizontal line**

At $x = 9$, $y = \frac{1}{2}(9)^2 + \frac{1458}{\sqrt{9^3}} - 74 = \frac{81}{2} + 54 - 74 = \frac{41}{2}$.

7 **Set up the area**

The region lies from $x = 4$ to $x = 9$ between the curve and the horizontal line $y = \frac{41}{2}$.

8 **Integrate the difference**

$$R = \int_4^9 \left(\frac{1}{2}x^2 + 1458x^{-3/2} - 74 - \frac{41}{2}\right)dx.$$

9 **Find an antiderivative**

$$\int \left(\frac{1}{2}x^2 + 1458x^{-3/2} - \frac{189}{2}\right)dx = \frac{x^3}{6} - 2916x^{-1/2} - \frac{189}{2}x.$$

10 **Evaluate exactly**

$$R = \left[\frac{x^3}{6} - 2916x^{-1/2} - \frac{189}{2}x\right]_4^9 = \frac{373}{3}.$$

✓ **FINAL ANSWER**

(a) Proven.

(b) $\frac{373}{3}$.

Question 9

8 marks | Integration

Main topic: Applications of Differentiation

9.

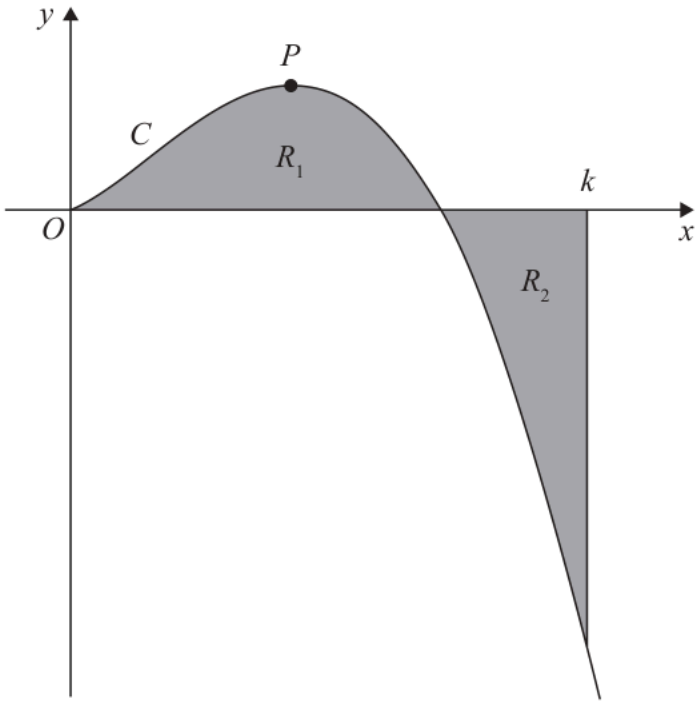


Figure 1

Figure 1 is a sketch of the curve C with equation

$$y = 2x^{\frac{3}{2}}(4 - x) \qquad x \geqslant 0$$

The point P is the stationary point of C.

(a) Find, using calculus, the x coordinate of P.

(4)

The region R_1, shown shaded in Figure 1, is bounded by C and the x-axis.

The region R_2, also shown shaded in Figure 1, is bounded by C, the x-axis and the line with equation $x = k$, where k is a constant.

Given that the area of R_1 is equal to the area of R_2

(b) find, using calculus, the exact value of k.

(4)

1 **Rewrite the curve**

$y = 2x^{3/2}(4 - x) = 8x^{3/2} - 2x^{5/2}$.

2 **Differentiate**

$\dfrac{dy}{dx} = 12x^{1/2} - 5x^{3/2} = \sqrt{x}(12 - 5x)$.

3 **Find the stationary point**

For the interior stationary point P, $12 - 5x = 0$, so $x = \dfrac{12}{5}$.

4 **Set up the area condition**

R_1 is the area from $x = 0$ to $x = 4$ above the x-axis. R_2 is the area from $x = 4$ to $x = k$ below the x-axis.

5 **Use equal areas**

Equal areas mean $\displaystyle\int_0^4 y\,dx = -\int_4^k y\,dx$, so $\displaystyle\int_0^k y\,dx = 0$.

6 **Integrate y**

$\displaystyle\int 2x^{3/2}(4 - x)dx = \int (8x^{3/2} - 2x^{5/2})dx = \frac{16}{5}x^{5/2} - \frac{4}{7}x^{7/2}$.

7 **Solve for k**

$\dfrac{16}{5}k^{5/2} - \dfrac{4}{7}k^{7/2} = 0$.

8 **Factor**

$4k^{5/2}\left(\dfrac{4}{5} - \dfrac{k}{7}\right) = 0$.

9 **Choose the value beyond four**

Since $k > 4$, $\dfrac{4}{5} - \dfrac{k}{7} = 0$, so $k = \dfrac{28}{5}$.

FINAL ANSWER

$(a)\ x = \dfrac{12}{5}$.

$(b)\ k = \dfrac{28}{5}$.

WMA12/01 MAY/JUNE R 2024

Question 10

9 marks | Integration

Main topic: Applications of Differentiation

10.

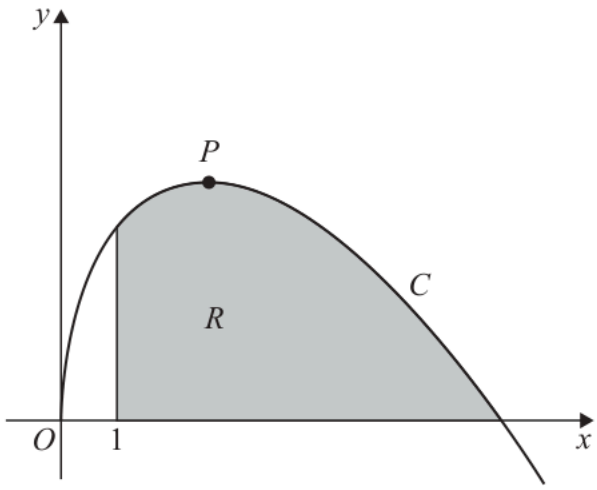


Figure 1

In this question you must show all stages of your working.
Solutions relying entirely on calculator technology are not acceptable.

Figure 1 shows a sketch of part of the curve C with equation

$$y = \frac{9x - x^2}{2\sqrt{x}} \qquad x > 0$$

The point P is the stationary point on C.

(a) Find, using calculus, the x coordinate of P.

(4)

The finite region R, shown shaded in Figure 1, is bounded by the curve C, the x-axis and the line with equation $x = 1$

(b) Using calculus, calculate the exact area of R.

(5)

WORKED SOLUTION

Question 10 · WMA12/01 May/June R 2024

1 **Rewrite the curve**

$$y = \frac{9x - x^2}{2\sqrt{x}} = \frac{1}{2}(9x^{1/2} - x^{3/2}).$$

2 **Differentiate**

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{9}{2}x^{-1/2} - \frac{3}{2}x^{1/2}\right) = \frac{3(3-x)}{4\sqrt{x}}.$$

3 **Use the stationary condition**

For $x > 0$, the denominator is not zero. So $\frac{dy}{dx} = 0$ when $3 - x = 0$.

4 **Find P**

Therefore the x coordinate of P is 3.

5 **Find where the curve meets the x-axis**

$y = 0$ when $9x - x^2 = 0$, so $x = 9$ for the positive intercept.

6 **Set up the area**

The region is under the curve from $x = 1$ to $x = 9$, so $R = \int_1^9 \frac{9x - x^2}{2\sqrt{x}}dx$.

7 **Integrate**

$$R = \int_1^9 \frac{1}{2}(9x^{1/2} - x^{3/2})dx = \left[3x^{3/2} - \frac{1}{5}x^{5/2}\right]_1^9.$$

8 **Evaluate**

$$R = \left(3(27) - \frac{1}{5}(243)\right) - \left(3 - \frac{1}{5}\right) = \frac{148}{5}.$$

✓ **FINAL ANSWER**

(a) $x = 3$.

(b) $\frac{148}{5}$.

WMA12/01 JANUARY 2025

Question 7

8 marks | Integration | Main topic: Laws of Logarithms

7. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(i) The table below shows values of x and y, where $y = \log_{10}(x + 5)$, for x values between -1 and 4

x	-1	0	1	2	3	4
$y = \log_{10}(x + 5)$	$\log_{10} 4$	$\log_{10} 5$	$\log_{10} 6$	$\log_{10} 7$	$\log_{10} 8$	$\log_{10} 9$

Using the trapezium rule with all the y values in the given table, show that

$$\int_{-1}^{4} \log_{10}(x + 5)\,\mathrm{d}x \approx \log_{10} k$$

where k is an integer to be found.

(3)

(ii) Find the value of a such that

$$2\log_5(5 - a) - \log_5(a + 25) = 1$$

(5)

WORKED SOLUTION

Question 7 · WMA12/01 January 2025

1 **Use the trapezium rule**

The width is 1, so

$$\int_{-1}^{4} \log_{10}(x + 5)\,dx \approx \frac{1}{2}\{\log_{10} 4 + \log_{10} 9 + 2(\log_{10} 5 + \log_{10} 6 + \log_{10} 7 + \log_{10} 8)\}.$$

2 **Combine the logarithms**

This is $\log_{10}\left(\sqrt{4 \times 9} \times 5 \times 6 \times 7 \times 8\right)$.

3 **Find k**

$\sqrt{4 \times 9} \times 5 \times 6 \times 7 \times 8 = 6 \times 1680 = 10080$, so $k = 10080$.

4 **State the domain**

The logarithms require $5 - a > 0$ and $a + 25 > 0$, so $-25 < a < 5$.

5 **Combine the logarithms**

$2\log_5(5-a) - \log_5(a+25) = \log_5\left(\dfrac{(5-a)^2}{a+25}\right)$.

6 **Use one as a logarithm**

$1 = \log_5 5$, so $\dfrac{(5-a)^2}{a+25} = 5$.

7 **Form the quadratic**

$(5-a)^2 = 5(a+25)$, so $a^2 - 10a + 25 = 5a + 125$. Hence $a^2 - 15a - 100 = 0$.

8 **Solve and check**

$(a-20)(a+5) = 0$, so $a = 20$ or $a = -5$. Only $a = -5$ is in the domain, so $a = -5$.

✓ **FINAL ANSWER**

(i) $k = 10080$.

(ii) $a = -5$.

Question 10

8 marks | Integration

Main topic: Applications of Differentiation

10.

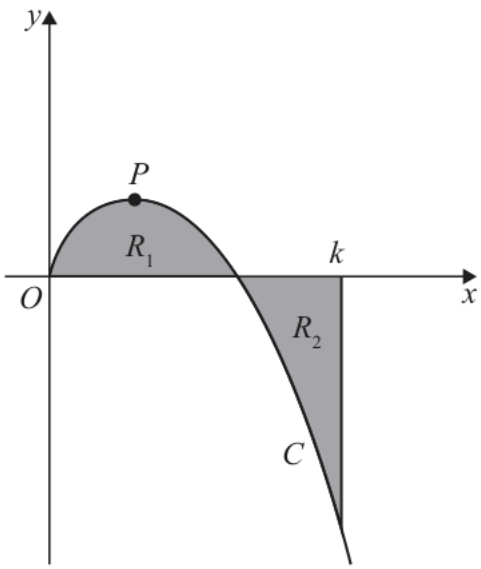


Figure 2

In this question you must show all stages of your working.
Solutions relying entirely on calculator technology are not acceptable.

Figure 2 shows a sketch of the curve C with equation

$$y = \frac{\sqrt{x}\left(100 - x^2\right)}{40} \qquad x \geqslant 0$$

The point P is a stationary point on C.

(a) Use algebraic differentiation to find the exact x coordinate of P.

(4)

The region R_1, shown shaded in Figure 2, is bounded by C and the x-axis.

The region R_2, also shown shaded in Figure 2, is bounded by C, the x-axis and the line with equation $x = k$, where k is a constant.

Given that the area of R_1 is equal to the area of R_2

(b) use algebraic integration to find the exact value of k.

(4)

1 **Rewrite the curve**

$$y = \frac{\sqrt{x}(100 - x^2)}{40} = \frac{5}{2}x^{1/2} - \frac{1}{40}x^{5/2}.$$

2 **Differentiate**

$$\frac{dy}{dx} = \frac{5}{4}x^{-1/2} - \frac{1}{16}x^{3/2}.$$

3 **Use the stationary point**

At P, $\frac{dy}{dx} = 0$. Thus $\frac{5}{4}x^{-1/2} = \frac{1}{16}x^{3/2}$. Multiplying by $16x^{1/2}$ gives $20 = x^2$.

4 **Find the x-coordinate**

Since $x \geq 0$, $x = \sqrt{20} = 2\sqrt{5}$.

5 **Integrate the curve**

$$\int y\,dx = \int \left(\frac{5}{2}x^{1/2} - \frac{1}{40}x^{5/2}\right)dx = \frac{5}{3}x^{3/2} - \frac{1}{140}x^{7/2}.$$

6 **Use equal areas**

The signed area from 0 to k is zero because the positive area R_1 equals the negative area R_2 in magnitude.

7 **Solve for k**

$\frac{5}{3}k^{3/2} - \frac{1}{140}k^{7/2} = 0$. Since $k > 0$, divide by $k^{3/2}$ to get $\frac{5}{3} - \frac{k^2}{140} = 0$, so $k^2 = \frac{700}{3}$.

8 **Write the exact value**

$$k = \sqrt{\frac{700}{3}} = \frac{10\sqrt{21}}{3}.$$

✓ **FINAL ANSWER**

$(a)\ x = 2\sqrt{5}.\quad (b)\ k = \frac{10\sqrt{21}}{3}.$