Elite Mathematics
Edexcel International A Level

IAL PURE 2 · WMA12

WMA12/01 January 2024 Worked Solutions

January 2024 paper rebuilt with the worked solution after each question.

10	2019–2026	Single-paper solutions
paper questions	19 exam sessions	checked by Dr Eslam



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PAST PAPER

WMA12/01 January 2024

January 2024

10	75	1 h 30
questions	marks	exam time

Each question is followed by its worked solution and final answer.

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Question 1

3 marks | Polynomials

1. $f(x) = ax^3 + 3x^2 - 8x + 2$ where a is a constant

Given that when $f(x)$ is divided by $(x - 2)$ the remainder is 3, find the value of a.

(3)

WORKED SOLUTION

Question 1 · WMA12/01 January 2024

1 Use the remainder theorem

Dividing by $(x - 2)$ means the remainder is $f(2)$.

2 Substitute x equals two

$f(2) = a(2)^3 + 3(2)^2 - 8(2) + 2 = 8a + 12 - 16 + 2 = 8a - 2$.

3 Use the given remainder

The remainder is 3, so $8a - 2 = 3$.

4 Solve for a

$8a = 5$, hence $a = \frac{5}{8}$.

FINAL ANSWER

$a = \frac{5}{8}$.

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Question 2

3 marks | Binomial Expansion

2. Find the coefficient of the term in x^7 of the binomial expansion of

$$\left(\frac{3}{8}+4x\right)^{12}$$

giving your answer in simplest form.

(3)

WORKED SOLUTION

Question 2 · WMA12/01 January 2024

1 Choose the x power

In $\left(\frac{3}{8}+4x\right)^{12}$, the term in x^7 comes from choosing $(4x)^7$ and $\left(\frac{3}{8}\right)^5$.

2 Write the term

The coefficient is $\binom{12}{7}\left(\frac{3}{8}\right)^5 4^7$.

3 Simplify the powers

$4^7 = 2^{14}$ and $8^5 = 2^{15}$, so $\left(\frac{3}{8}\right)^5 4^7 = \frac{3^5}{2}$.

4 Evaluate

$\binom{12}{7} = 792$, so the coefficient is $792 \cdot \frac{243}{2} = 96228$.

FINAL ANSWER

96228.

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Question 3

6 marks | Circles

3. The circle C

- has centre $A(3, 5)$
- passes through the point $B(8, -7)$

(a) Find an equation for C.

(3)

The points M and N lie on C such that MN is a chord of C.

Given that MN

- lies above the x-axis
- is parallel to the x-axis
- has length $4\sqrt{22}$

(b) find an equation for the line passing through points M and N.

(3)

WORKED SOLUTION

Question 3 · WMA12/01 January 2024

1 Find the radius squared

The centre is $A(3, 5)$ and the circle passes through $B(8, -7)$, so $r^2 = (8-3)^2 + (-7-5)^2 = 25 + 144 = 169$.

2 Write the circle equation

Hence C has equation $(x-3)^2 + (y-5)^2 = 169$.

3 Use the horizontal chord

Since MN is parallel to the x-axis, its equation has the form $y = c$.

4 Use half the chord length

The chord length is $4\sqrt{22}$, so the half-length is $2\sqrt{22}$.

5 Relate chord length to the circle

At height $y = c$, the horizontal half-length satisfies $169 - (c-5)^2 = (2\sqrt{22})^2 = 88$.

6 Solve for c

$(c-5)^2 = 81$, so $c = 14$ or $c = -4$.

7 **Use the position above the x-axis**

The chord lies above the x-axis, so $c = 14$. Therefore the line through M and N is $y = 14$.

FINAL ANSWER

$(a)\ (x-3)^2 + (y-5)^2 = 169$.

$(b)\ y = 14$.

Question 4

9 marks | Integration

4. (a) Sketch the curve with equation

$$y = a^{-x} + 4$$

where a is a constant and $a > 1$

On your sketch show

- the coordinates of the point of intersection of the curve with the y-axis
- the equation of the asymptote to the curve.

(3)

x	-4	-1.5	1	3.5	6	8.5
y	13	6.280	4.577	4.146	4.037	4.009

The table above shows corresponding values of x and y for $y = 3^{-\frac{1}{2}x} + 4$

The values of y are given to four significant figures, as appropriate.

Using the trapezium rule with all the values of y in the table,

(b) find an approximate value for

$$\int_{-4}^{8.5} \left(3^{-\frac{1}{2}x} + 4\right) \mathrm{d}x$$

giving your answer to two significant figures.

(3)

(c) **Using the answer to part (b)**, find an approximate value for

(i) $\displaystyle\int_{-4}^{8.5} \left(3^{-\frac{1}{2}x}\right) \mathrm{d}x$

(ii) $\displaystyle\int_{-4}^{8.5} \left(3^{-\frac{1}{2}x} + 4\right) \mathrm{d}x + \int_{-8.5}^{4} \left(3^{\frac{1}{2}x} + 4\right) \mathrm{d}x$

(3)

1 **Find the y-intercept**

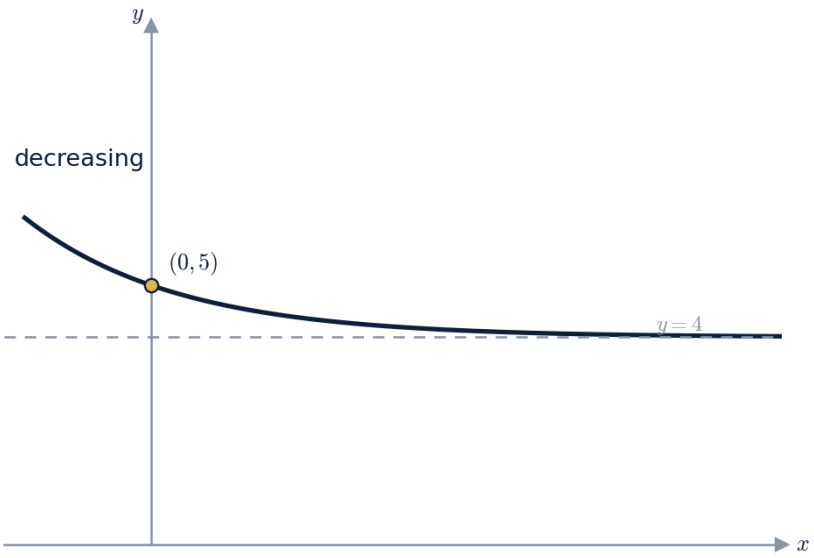
For $y = a^{-x} + 4$, set $x = 0$: $y = a^0 + 4 = 5$, so the curve meets the y-axis at $(0, 5)$.

2 **Find the asymptote**

Since $a > 1$, $a^{-x} \to 0$ as $x \to \infty$, so the horizontal asymptote is $y = 4$.

3 **Describe the sketch**

The curve is decreasing, passes through $(0, 5)$, and approaches the line $y = 4$ from above.



4 **Use the trapezium width**

The x values increase by 2.5, so $h = 2.5$.

5 **Apply the trapezium rule**

$$\int_{-4}^{8.5} \left(3^{-x/2} + 4\right) dx \approx \frac{2.5}{2}[13 + 4.009 + 2(6.280 + 4.577 + 4.146 + 4.037)].$$

6 **Round part b**

This gives 68.86125, so to two significant figures the estimate is 69.

7 **Remove the constant part**

$$\int_{-4}^{8.5} 4\,dx = 4(12.5) = 50.$$

8 **Estimate part c i**

Using the answer to part (b), $\displaystyle\int_{-4}^{8.5} 3^{-x/2} dx \approx 69 - 50 = 19$.

9 **Use symmetry by substitution**

For $\int_{-8.5}^{4}(3^{x/2}+4)dx$, let $u=-x$. This changes it to the same integral as part (b).

10 **Estimate part c ii**

Therefore the two integrals in part (c)(ii) are approximately equal, so the total is $69+69=138$.

✓ **FINAL ANSWER**

(a) y-intercept $(0,5)$ and asymptote $y=4$.

(b) 69.

$(c)(i)$ 19.

$(c)(ii)$ 138.

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Question 5

8 marks | Sequences & Series

5. (i) Find the value of

$$\sum_{r=1}^{\infty} 6 \times (0.25)^r$$

(3)

(ii) A sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = 3$$
$$u_{n+1} = \frac{u_n - 3}{u_n - 2} \qquad n \in \mathbb{N}$$

(a) Show that this sequence is periodic.

(2)

(b) State the order of this sequence.

(1)

(c) Hence find

$$\sum_{n=1}^{70} u_n$$

(2)

WORKED SOLUTION

Question 5 · WMA12/01 January 2024

1 **Identify the geometric series**

$\sum_{r=1}^{\infty} 6(0.25)^r$ has first term $6(0.25) = 1.5$ and common ratio 0.25.

2 **Sum to infinity**

$$S_\infty = \frac{1.5}{1 - 0.25} = \frac{1.5}{0.75} = 2.$$

3 **Find the first few terms**

$u_1 = 3$. Then $u_2 = \frac{3-3}{3-2} = 0$.

4 **Continue the recurrence**

$u_3 = \dfrac{0-3}{0-2} = \dfrac{3}{2}$, and $u_4 = \dfrac{\frac{3}{2}-3}{\frac{3}{2}-2} = 3$.

5 **Show periodicity**

Since $u_4 = 3 = u_1$, the terms repeat as $3, 0, \dfrac{3}{2}, 3, 0, \dfrac{3}{2}, \ldots$.

6 **State the order**

The repeating cycle has length 3, so the order is 3.

7 **Sum one cycle**

One full cycle sums to $3 + 0 + \dfrac{3}{2} = \dfrac{9}{2}$.

8 **Use 70 terms**

Since $70 = 23 \cdot 3 + 1$, the first 70 terms contain 23 complete cycles and one extra term $u_1 = 3$.

9 **Find the required sum**

$$\sum_{n=1}^{70} u_n = 23\left(\frac{9}{2}\right) + 3 = \frac{207}{2} + \frac{6}{2} = \frac{213}{2}.$$

✓ **FINAL ANSWER**

(i) 2.

$(ii)(a)$ Periodic.

$(ii)(b)$ Order 3.

$(ii)(c)$ $\dfrac{213}{2}$.

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Question 6

8 marks | Laws of Logarithms

6. (a) Given that

$$2\log_4(x+3) + \log_4 x = \log_4(4x+2) + \frac{1}{2}$$

show that

$$x^3 + 6x^2 + x - 4 = 0$$

(4)

(b) Given also that -1 is a root of the equation

$$x^3 + 6x^2 + x - 4 = 0$$

(i) use algebra to find the other two roots of the equation.

(3)

(ii) Hence solve

$$2\log_4(x+3) + \log_4 x = \log_4(4x+2) + \frac{1}{2}$$

(1)

WORKED SOLUTION

Question 6 · WMA12/01 January 2024

1 Combine the logarithms

$2\log_4(x+3) = \log_4((x+3)^2)$, so the left side is $\log_4(x(x+3)^2)$.

2 Rewrite one half

$\frac{1}{2} = \log_4 2$, so the right side is $\log_4(4x+2) + \log_4 2 = \log_4(8x+4)$.

3 Equate arguments

Therefore $x(x+3)^2 = 8x+4$.

4 Expand

$x(x^2+6x+9) = 8x+4$, so $x^3+6x^2+9x = 8x+4$.

5 **Show the cubic**

Rearranging gives $x^3 + 6x^2 + x - 4 = 0$.

6 **Use the given root**

Since -1 is a root, $(x + 1)$ is a factor of $x^3 + 6x^2 + x - 4$.

7 **Factor the cubic**

$x^3 + 6x^2 + x - 4 = (x + 1)(x^2 + 5x - 4)$.

8 **Solve the quadratic**

$x^2 + 5x - 4 = 0$ gives $x = \dfrac{-5 \pm \sqrt{25 + 16}}{2} = \dfrac{-5 \pm \sqrt{41}}{2}$.

9 **Apply the logarithm domain**

The original log equation requires $x > 0$. The only positive root is $x = \dfrac{-5 + \sqrt{41}}{2}$.

✓ **FINAL ANSWER**

(a) Proven.

$(b)(i)$ $x = \dfrac{-5 + \sqrt{41}}{2}$ and $x = \dfrac{-5 - \sqrt{41}}{2}$.

$(b)(ii)$ $x = \dfrac{-5 + \sqrt{41}}{2}$.

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Question 7

9 marks | Modelling with Sequences & Series

7. Wheat is grown on a farm.

- In year 1, the farm produced 300 tonnes of wheat.
- In year 12, the farm is predicted to produce 4000 tonnes of wheat.

Model A assumes that the amount of wheat produced on the farm will increase by the same amount each year.

(a) Using model A, find the amount of wheat produced on the farm in year 4.
Give your answer to the nearest 10 tonnes.

(3)

Model B assumes that the amount of wheat produced on the farm will increase by the same percentage each year.

(b) Using model B, find the amount of wheat produced on the farm in year 2.
Give your answer to the nearest 10 tonnes.

(3)

(c) Calculate, according to the two models, the difference between the total amounts of wheat predicted to be produced on the farm from year 1 to year 12 inclusive.
Give your answer to the nearest 10 tonnes.

(3)

WORKED SOLUTION

Question 7 · WMA12/01 January 2024

1 Model A common difference

Model A is arithmetic. From year 1 to year 12 there are 11 equal increases, so $d = \dfrac{4000 - 300}{11} = \dfrac{3700}{11}$.

2 Find year 4 in model A

Year 4 is $300 + 3d = 300 + 3\left(\dfrac{3700}{11}\right) = 1309.09\ldots$, so the amount is 1310 tonnes to the nearest 10 tonnes.

3 Model B common ratio

Model B is geometric. If the ratio is r, then $300r^{11} = 4000$.

4 **Find the ratio**

$r = \left(\frac{4000}{300}\right)^{1/11} = 1.2655\ldots$.

5 **Find year 2 in model B**

Year 2 is $300r = 379.65\ldots$, so the amount is 380 tonnes to the nearest 10 tonnes.

6 **Total for model A**

$S_A = \frac{12}{2}(300 + 4000) = 25800$ tonnes.

7 **Total for model B**

$S_B = 300\frac{r^{12} - 1}{r - 1} = 17935.20\ldots$ tonnes.

8 **Find the difference**

The difference is $25800 - 17935.20\ldots = 7864.79\ldots$, which is 7860 tonnes to the nearest 10 tonnes.

✓ **FINAL ANSWER**

(a) 1310 tonnes.

(b) 380 tonnes.

(c) 7860 tonnes.

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Question 8

6 marks | Proof

8. (i) Use a counter example to show that the following statement is **false**

"$n^2 + 3n + 1$ is prime for all $n \in \mathbb{N}$"

(2)

(ii) Use algebra to prove by exhaustion that for all $n \in \mathbb{N}$

"$n^2 - 2$ is **not** a multiple of 4"

(4)

WORKED SOLUTION

Question 8 · WMA12/01 January 2024

1 **Choose a counterexample**

Take $n = 6$.

2 **Evaluate the expression**

$n^2 + 3n + 1 = 6^2 + 3(6) + 1 = 36 + 18 + 1 = 55$.

3 **Conclude false**

55 is not prime, so the statement is false.

4 **Use residues modulo four**

Every natural number is one of $4k$, $4k + 1$, $4k + 2$ or $4k + 3$.

5 **Check n equals 4k**

If $n = 4k$, then $n^2 - 2 = 16k^2 - 2 = 4(4k^2 - 1) + 2$, not a multiple of 4.

6 **Check n equals 4k plus one**

If $n = 4k + 1$, then $n^2 - 2 = 16k^2 + 8k - 1 = 4(4k^2 + 2k - 1) + 3$, not a multiple of 4.

7 **Check n equals 4k plus two**

If $n = 4k + 2$, then $n^2 - 2 = 16k^2 + 16k + 2 = 4(4k^2 + 4k) + 2$, not a multiple of 4.

8 **Check n equals 4k plus three**

If $n = 4k + 3$, then $n^2 - 2 = 16k^2 + 24k + 7 = 4(4k^2 + 6k + 1) + 3$, not a multiple of 4.

9 **Complete the proof**

All possible cases leave remainder 2 or 3 when divided by 4, so $n^2 - 2$ is not a multiple of 4.

✓ **FINAL ANSWER**

(i) Counterexample: $n = 6$ gives 55, which is not prime.

(ii) Proven by considering $n = 4k, 4k+1, 4k+2, 4k+3$.

Question 9

14 marks | Trigonometric Equations

9. **In this question you must show detailed reasoning.**

Solutions relying entirely on calculator technology are not acceptable.

(i) Solve, for $0 \leqslant x < 360°$, the equation

$$\sin x \tan x = 5$$

giving your answers to one decimal place.

(6)

(ii)

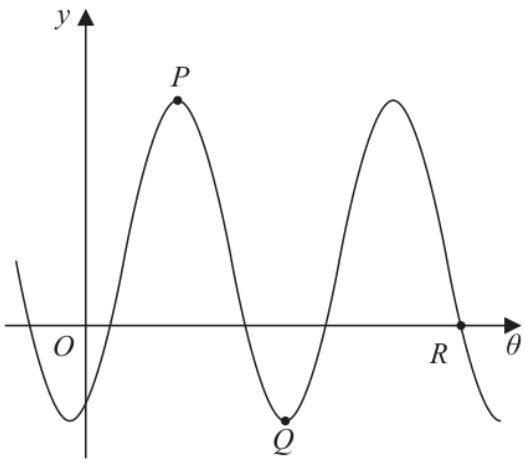


Figure 1

Figure 1 shows a sketch of part of the curve with equation

$$y = A\sin\left(2\theta - \frac{3\pi}{8}\right) + 2$$

where A is a constant and θ is measured in radians.

The points P, Q and R lie on the curve and are shown in Figure 1.

Given that the y coordinate of P is 7

(a) state the value of A,

(1)

(b) find the exact coordinates of Q,

(3)

(c) find the value of θ at R, giving your answer to 3 significant figures.

(4)

1 Rewrite the equation

$\sin x \tan x = 5$ becomes $\dfrac{\sin^2 x}{\cos x} = 5$.

2 Use sine squared

Multiplying by $\cos x$ gives $\sin^2 x = 5\cos x$. Since $\sin^2 x = 1 - \cos^2 x$, $1 - \cos^2 x = 5\cos x$.

3 Solve for cos x

$\cos^2 x + 5\cos x - 1 = 0$, so $\cos x = \dfrac{-5 \pm \sqrt{29}}{2}$.

4 Choose the possible cosine value

$\dfrac{-5 - \sqrt{29}}{2} < -1$, so the valid value is $\cos x = \dfrac{-5 + \sqrt{29}}{2}$.

5 Find x

For $0 \leq x < 360°$, this gives $x = 78.9°$ or $281.1°$ to one decimal place.

6 Find A from P

The point P is a maximum point on the sine curve. Its y coordinate is 7, so $A + 2 = 7$ and $A = 5$.

7 Locate Q

Point Q is the next minimum after P, so $\sin\left(2\theta - \dfrac{3\pi}{8}\right) = -1$.

8 Find theta at Q

For this minimum, $2\theta - \dfrac{3\pi}{8} = \dfrac{3\pi}{2}$, so $2\theta = \dfrac{15\pi}{8}$ and $\theta = \dfrac{15\pi}{16}$.

9 Find y at Q

At the minimum, $y = 2 - 5 = -3$, so $Q = \left(\dfrac{15\pi}{16}, -3\right)$.

10 Set up R

At R, $y = 0$, so $5\sin\left(2\theta - \dfrac{3\pi}{8}\right) + 2 = 0$.

11 Choose the shown crossing

Thus $\sin\left(2\theta - \dfrac{3\pi}{8}\right) = -\dfrac{2}{5}$. The crossing R shown after the second maximum has $2\theta - \dfrac{3\pi}{8} = 3\pi + \sin^{-1}\left(\dfrac{2}{5}\right)$.

12 **Find theta at R**

$\theta = \frac{1}{2}\left(3\pi + \sin^{-1}\left(\frac{2}{5}\right) + \frac{3\pi}{8}\right) = 5.507\ldots$, so $\theta = 5.51$ to 3 significant figures.

FINAL ANSWER

(i) $x = 78.9°, 281.1°$.

$(ii)(a)$ $A = 5$.

$(ii)(b)$ $Q = \left(\frac{15\pi}{16}, -3\right)$.

$(ii)(c)$ $\theta = 5.51$.

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Question 10

9 marks | Applications of Differentiation

10. **In this question you must show detailed reasoning.**

Solutions relying entirely on calculator technology are not acceptable.

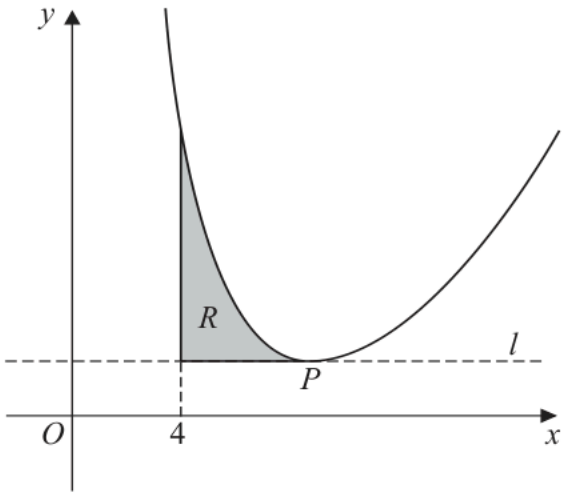


Figure 2

Figure 2 shows a sketch of the curve with equation

$$y = \frac{1}{2}x^2 + \frac{1458}{\sqrt{x^3}} - 74 \qquad x > 0$$

The point P is the only stationary point on the curve.

(a) Use calculus to show that the x coordinate of P is 9

(4)

The line l passes through the point P and is parallel to the x-axis.

The region R, shown shaded in Figure 2, is bounded by the curve, the line l and the line with equation $x = 4$

(b) Use algebraic integration to find the exact area of R.

(5)

WORKED SOLUTION

Question 10 · WMA12/01 January 2024

1 **Rewrite the curve**

$$y = \frac{1}{2}x^2 + 1458x^{-3/2} - 74.$$

2 **Differentiate**

$$\frac{dy}{dx} = x + 1458\left(-\frac{3}{2}\right)x^{-5/2} = x - 2187x^{-5/2}.$$

3 **Use the stationary condition**

At a stationary point, $x - 2187x^{-5/2} = 0$.

4 **Solve for x**

$x = 2187x^{-5/2}$, so $x^{7/2} = 2187 = 3^7$.

5 **Show x coordinate**

Raising both sides to the power $\frac{2}{7}$ gives $x = 3^2 = 9$. Since $x > 0$, this is the only stationary point.

6 **Find the horizontal line**

At $x = 9$, $y = \frac{1}{2}(9)^2 + \frac{1458}{\sqrt{9^3}} - 74 = \frac{81}{2} + 54 - 74 = \frac{41}{2}$.

7 **Set up the area**

The region lies from $x = 4$ to $x = 9$ between the curve and the horizontal line $y = \frac{41}{2}$.

8 **Integrate the difference**

$$R = \int_4^9 \left(\frac{1}{2}x^2 + 1458x^{-3/2} - 74 - \frac{41}{2}\right)dx.$$

9 **Find an antiderivative**

$$\int \left(\frac{1}{2}x^2 + 1458x^{-3/2} - \frac{189}{2}\right)dx = \frac{x^3}{6} - 2916x^{-1/2} - \frac{189}{2}x.$$

10 **Evaluate exactly**

$$R = \left[\frac{x^3}{6} - 2916x^{-1/2} - \frac{189}{2}x\right]_4^9 = \frac{373}{3}.$$

✓ **FINAL ANSWER**

(a) Proven.

(b) $\frac{373}{3}$.