Elite Mathematics
Edexcel International A Level

IAL PURE 2 · WMA12

WMA12/01 October 2023 Worked Solutions

October 2023 paper rebuilt with the worked solution after each question.

10	2019–2026	Single-paper solutions
paper questions	19 exam sessions	checked by Dr Eslam



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PAST PAPER

WMA12/01 October 2023

October 2023

10	75	1 h 30
questions	marks	exam time

Each question is followed by its worked solution and final answer.

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Question 1

3 marks | Proof

1. Given that a, b and c are integers greater than 0 such that

- $c = 3a + 1$
- $a + b + c = 15$

prove, by exhaustion, that the product abc is always a multiple of 4

You may use the table below to illustrate your answer.

(3)

You may not need to use all rows of this table.

a	b	c	abc

WORKED SOLUTION

Question 1 · WMA12/01 October 2023

1 Use the conditions

Since $c = 3a + 1$ and $a + b + c = 15$, substitute for c to get $a + b + 3a + 1 = 15$.

2 Write b in terms of a

This gives $b = 14 - 4a$.

3 List possible a values

The integers a, b, c are positive. Since $b = 14 - 4a > 0$, we need $a < 3.5$. Therefore $a = 1, 2, 3$.

4 Check each case

If $a = 1$, then $b = 10$ and $c = 4$, so $abc = 40$. If $a = 2$, then $b = 6$ and $c = 7$, so $abc = 84$. If $a = 3$, then $b = 2$ and $c = 10$, so $abc = 60$.

5

Conclude the exhaustion

The products 40, 84 and 60 are all multiples of 4. All possible cases have been checked, so abc is always a multiple of 4.

✓ **FINAL ANSWER**

The only possible triples are $(1, 10, 4)$, $(2, 6, 7)$ and $(3, 2, 10)$, giving products 40, 84 and 60. Each is a multiple of 4, so abc is always a multiple of 4.

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Question 2

5 marks | Sequences & Series

2. A sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = 3$$

$$u_{n+1} = 2 - \frac{4}{u_n}$$

(a) Find the value of u_2, the value of u_3 and the value of u_4

(3)

(b) Find the value of

$$\sum_{r=1}^{100} u_r$$

(2)

WORKED SOLUTION

Question 2 · WMA12/01 October 2023

1 Find u2

$u_2 = 2 - \frac{4}{u_1} = 2 - \frac{4}{3} = \frac{2}{3}$.

2 Find u3

$u_3 = 2 - \frac{4}{u_2} = 2 - \frac{4}{2/3} = 2 - 6 = -4$.

3 Find u4

$u_4 = 2 - \frac{4}{u_3} = 2 - \frac{4}{-4} = 3$.

4 Notice the cycle

The sequence starts $3, \frac{2}{3}, -4, 3, \frac{2}{3}, -4, \ldots$, so it repeats every 3 terms.

5 Sum one cycle

$3 + \frac{2}{3} - 4 = -\frac{1}{3}$.

6 **Use 100 terms**

There are 33 complete cycles in the first 99 terms, then one extra term $u_{100} = 3$.

7 **Find the sum**

$$\sum_{r=1}^{100} u_r = 33\left(-\frac{1}{3}\right) + 3 = -11 + 3 = -8.$$

✓ **FINAL ANSWER**

$(a)\ u_2 = \frac{2}{3},\ u_3 = -4,\ u_4 = 3.$

$(b)\ -8.$

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Question 3

7 marks | Trigonometric Equations

3. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Solve, for $0 < \theta \leqslant 360°$ the equation

$$2\tan\theta + 3\sin\theta = 0$$

giving your answers, as appropriate, to one decimal place.

(5)

(b) Hence, or otherwise, find the smallest positive solution of

$$2\tan(2x + 40°) + 3\sin(2x + 40°) = 0$$

giving your answer to one decimal place.

(2)

WORKED SOLUTION

Question 3 · WMA12/01 October 2023

1 **Factor the equation**

$2\tan\theta + 3\sin\theta = 0$ becomes $\dfrac{2\sin\theta}{\cos\theta} + 3\sin\theta = 0$.

2 **Multiply by cos theta**

Since $\tan\theta$ is defined, $\cos\theta \neq 0$. Multiplying by $\cos\theta$ gives $2\sin\theta + 3\sin\theta\cos\theta = 0$.

3 **Factor**

$\sin\theta(2 + 3\cos\theta) = 0$.

4 **Solve the sine case**

For $0 < \theta \leq 360°$, $\sin\theta = 0$ gives $\theta = 180°$ or $360°$.

5 **Solve the cosine case**

$2 + 3\cos\theta = 0$ gives $\cos\theta = -\dfrac{2}{3}$, so $\theta = 131.8°$ or $228.2°$ to one decimal place.

6 **Apply to the new angle**

For $2\tan(2x + 40°) + 3\sin(2x + 40°) = 0$, the angle $2x + 40°$ must be one of the solution angles, modulo $360°$.

7 **Find the smallest positive x**

The smallest positive value comes from $2x + 40° = 131.8...°$, so $x = \dfrac{131.8... - 40}{2} = 45.9°$ to one decimal place.

✓ **FINAL ANSWER**

$(a)\ \theta = 131.8°, 180°, 228.2°, 360°$.

$(b)\ x = 45.9°$.

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Question 4

9 marks | Polynomials

4. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

$$f(x) = 4x^3 + ax^2 - 29x + b$$

where a and b are constants.

Given that $(2x + 1)$ is a factor of $f(x)$,

(a) show that

$$a + 4b = -56$$

(2)

Given also that when $f(x)$ is divided by $(x - 2)$ the remainder is -25

(b) find a second simplified equation linking a and b.

(2)

(c) Hence, using algebra and showing your working,

(i) find the value of a and the value of b,

(ii) fully factorise $f(x)$.

(5)

WORKED SOLUTION

Question 4 · WMA12/01 October 2023

1 **Use the factor theorem**

Since $(2x+1)$ is a factor, $x = -\frac{1}{2}$ is a root, so $f\left(-\frac{1}{2}\right) = 0$.

2 **Substitute x equals negative half**

$4\left(-\frac{1}{2}\right)^3 + a\left(-\frac{1}{2}\right)^2 - 29\left(-\frac{1}{2}\right) + b = 0$.

3 **Show the first relation**

$-\frac{1}{2} + \frac{a}{4} + \frac{29}{2} + b = 0$. Multiplying by 4 gives $a + 4b + 56 = 0$, so $a + 4b = -56$.

4 **Use the remainder theorem**

Dividing by $(x-2)$ gives remainder $f(2)$.

5 **Substitute x equals two**

$f(2) = 4(8) + 4a - 58 + b = 4a + b - 26$.

6 **Find the second relation**

The remainder is -25, so $4a + b - 26 = -25$, hence $4a + b = 1$.

7 **Solve the simultaneous equations**

Solve $a + 4b = -56$ and $4a + b = 1$. Multiplying the second equation by 4 gives $16a + 4b = 4$.

8 **Find a and b**

Subtracting $a + 4b = -56$ from $16a + 4b = 4$ gives $15a = 60$, so $a = 4$. Then $16 + b = 1$, so $b = -15$.

9 **Write f(x)**

Therefore $f(x) = 4x^3 + 4x^2 - 29x - 15$.

10 **Factorise**

Using the known factors $(2x + 1)$ and $(x - 2)$, division gives $f(x) = (2x + 1)(x - 2)(2x + 15)$.

✓ **FINAL ANSWER**

(a) Proven.

(b) $4a + b = 1$.

$(c)(i)$ $a = 4,\ b = -15$.

$(c)(ii)$ $f(x) = (2x + 1)(x - 2)(2x + 15)$.

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Question 5

6 marks | Laws of Logarithms

5. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(i) Solve

$$3^a = 70$$

giving the answer to 3 decimal places.

(2)

(ii) Find the exact value of b such that

$$4 + 3\log_3 b = \log_3 5b$$

(4)

WORKED SOLUTION

Question 5 · WMA12/01 October 2023

1 **Solve the exponential equation**

$3^a = 70$ gives $a = \log_3 70 = \dfrac{\log 70}{\log 3}$.

2 **Round a**

$a = 3.867147\ldots$, so $a = 3.867$ to 3 decimal places.

3 **Rewrite four as a logarithm**

$4 = \log_3(3^4) = \log_3 81$.

4 **Combine the left side**

$4 + 3\log_3 b = \log_3 81 + \log_3(b^3) = \log_3(81b^3)$.

5 **Equate arguments**

The equation becomes $\log_3(81b^3) = \log_3(5b)$, so $81b^3 = 5b$.

6 **Use the log domain**

Since $\log_3 b$ is defined, $b > 0$. Divide by b to get $81b^2 = 5$.

7 **Find b**

$b^2 = \dfrac{5}{81}$, and $b > 0$, so $b = \dfrac{\sqrt{5}}{9}$.

FINAL ANSWER

$(i)\ a = 3.867.$

$(ii)\ b = \dfrac{\sqrt{5}}{9}.$

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Question 6

6 marks | Integration

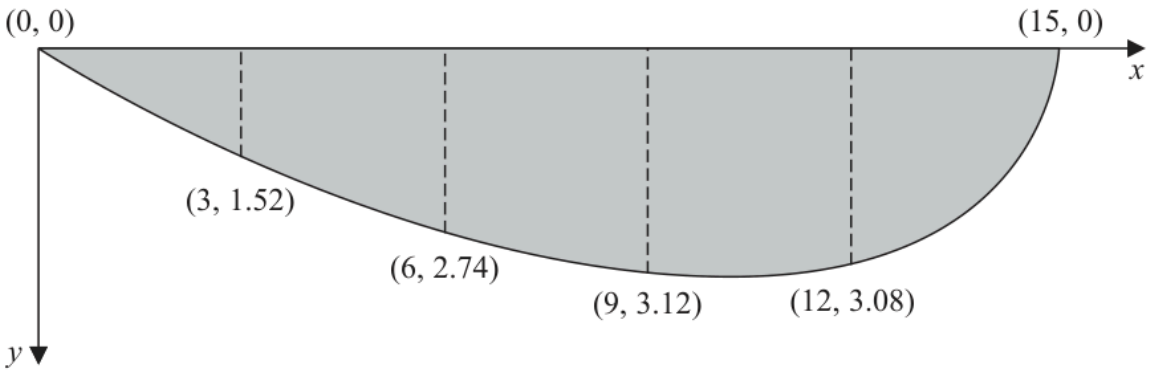


Figure 1

A river is being studied.

At one particular place, the river is 15 m wide.

The depth, y metres, of the river is measured at a point x metres from one side of the river.

Figure 1 shows a plot of the cross-section of the river and the coordinate values (x, y)

(a) Use the trapezium rule with all the y values given in Figure 1 to estimate the cross-sectional area of the river.

(3)

The water in the river is modelled as flowing at a constant speed of $1.5\,\text{m}\,\text{s}^{-1}$ across the whole of the cross-section.

(b) Use the model and the answer to part (a) to estimate the volume of water flowing through this section of the river each minute, giving your answer in m^3 to 2 significant figures.

(2)

Assuming the model,

(c) state, giving a reason for your answer, whether your answer for part (b) is an overestimate or an underestimate of the true volume of water flowing through this section of the river each minute.

(1)

WORKED SOLUTION

Question 6 · WMA12/01 October 2023

1 **Read the strip width**

The x values are $0, 3, 6, 9, 12, 15$, so the trapezium width is $h = 3$.

2 **Apply the trapezium rule**

$\text{Area} \approx \frac{3}{2}[0 + 0 + 2(1.52 + 2.74 + 3.12 + 3.08)]$.

3 **Find the cross-sectional area**

This gives $31.38\ \text{m}^2$.

4 **Convert flow speed to one minute**

The speed is $1.5\ \text{m s}^{-1}$, so in one minute the water travels $1.5 \times 60 = 90\ \text{m}$.

5 **Estimate volume**

Volume per minute $\approx 31.38 \times 90 = 2824.2\ \text{m}^3$.

6 **Round the volume**

To 2 significant figures, this is $2.8 \times 10^3\ \text{m}^3$.

7 **State over or under**

It is an underestimate, because the curve between the plotted points lies below the straight-line trapezium chords in the diagram, so the true cross-sectional area is larger.

✓ **FINAL ANSWER**

(a) $31.38\ \text{m}^2$.

(b) $2.8 \times 10^3\ \text{m}^3$.

(c) Underestimate.

Question 7

8 marks | Circles

7.

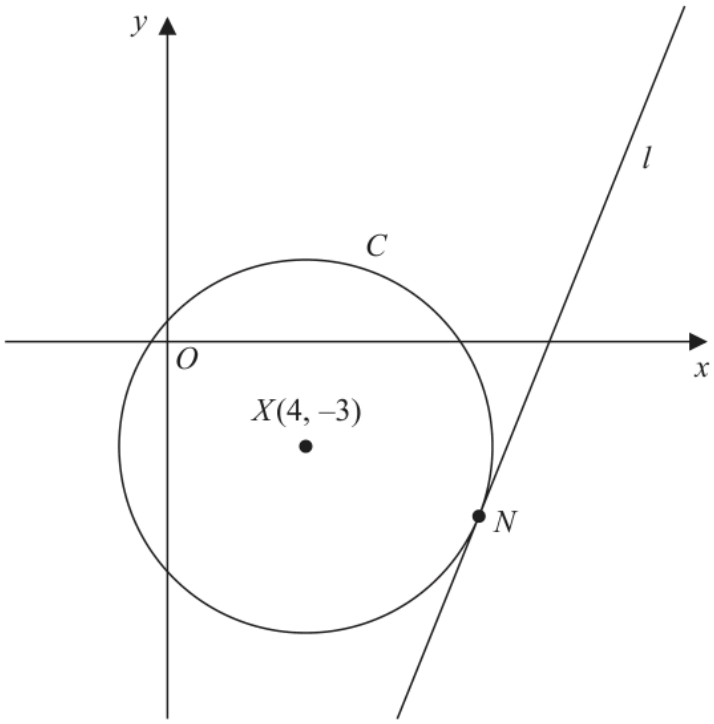


Figure 2

Figure 2 shows a sketch of

- the circle C with centre $X(4, -3)$
- the line l with equation $y = \frac{5}{2}x - \frac{55}{2}$

Given that l is the tangent to C at the point N,

(a) show that an equation for the straight line passing through X and N is

$$2x + 5y + 7 = 0$$

(3)

(b) Hence find

(i) the coordinates of N,

(ii) an equation for C.

(5)

1 Use perpendicular gradients

The tangent l has gradient $\frac{5}{2}$, so the radius XN is perpendicular to it and has gradient $-\frac{2}{5}$.

2 Use the centre

The line through $X(4,-3)$ with gradient $-\frac{2}{5}$ is $y+3=-\frac{2}{5}(x-4)$.

3 Show the required line

Multiplying by 5 gives $5y+15=-2x+8$, so $2x+5y+7=0$.

4 Intersect the radius and tangent

Use $y=\frac{5}{2}x-\frac{55}{2}$ in $2x+5y+7=0$.

5 Find x coordinate

$2x+5\left(\frac{5}{2}x-\frac{55}{2}\right)+7=0$. Multiplying by 2 gives $4x+25x-275+14=0$, so $29x=261$ and $x=9$.

6 Find y coordinate

$y=\frac{5}{2}(9)-\frac{55}{2}=\frac{45-55}{2}=-5$, so $N=(9,-5)$.

7 Find the radius squared

$XN^2=(9-4)^2+(-5+3)^2=5^2+(-2)^2=29$.

8 Write the circle equation

With centre $(4,-3)$ and radius squared 29, the circle is $(x-4)^2+(y+3)^2=29$.

FINAL ANSWER

(a) Proven.

$(b)(i)$ $N=(9,-5)$.

$(b)(ii)$ $(x-4)^2+(y+3)^2=29$.

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Question 8

7 marks | Arithmetic Sequences

8. In a large theatre there are n rows of seats, where n is a constant.

The number of seats in the first row is a, where a is a constant.

In each subsequent row there are 4 more seats than in the previous row so that

- in the 2nd row there are $(a + 4)$ seats
- in the 3rd row there are $(a + 8)$ seats
- the number of seats in each row form an **arithmetic** sequence

Given that the **total** number of seats in the first 10 rows is 360

(a) find the value of a.

(2)

Given also that the total number of seats in the n rows is 2146

(b) show that

$$n^2 + 8n - 1073 = 0$$

(2)

(c) Hence

(i) state the number of rows of seats in the theatre,

(ii) find the maximum number of seats in any one row.

(3)

WORKED SOLUTION

Question 8 · WMA12/01 October 2023

1 Use the arithmetic sum formula

The first term is a and the common difference is 4.

2 Use the first 10 rows

$S_{10} = \frac{10}{2}(2a + 9 \cdot 4) = 360$.

3 Find a

$5(2a + 36) = 360$, so $10a + 180 = 360$ and $a = 18$.

4 Write S n

$S_n = \frac{n}{2}(2(18) + (n-1)4) = \frac{n}{2}(4n + 32) = 2n^2 + 16n$.

5 **Use the total number of seats**

Since the total is 2146, $2n^2 + 16n = 2146$.

6 **Show the equation**

Dividing by 2 and rearranging gives $n^2 + 8n - 1073 = 0$.

7 **Solve for n**

$n = \dfrac{-8 \pm \sqrt{8^2 + 4(1073)}}{2} = \dfrac{-8 \pm 66}{2}$, so $n = 29$ or $n = -37$.

8 **Choose the row count**

The number of rows must be positive, so there are 29 rows.

9 **Find the maximum row size**

The largest row is the last row: $18 + (29 - 1)4 = 18 + 112 = 130$ seats.

✓ **FINAL ANSWER**

(a) $a = 18$.

(b) Proven.

$(c)(i)$ 29 rows.

$(c)(ii)$ 130 seats.

Question 9

12 marks | Applications of Differentiation

9.

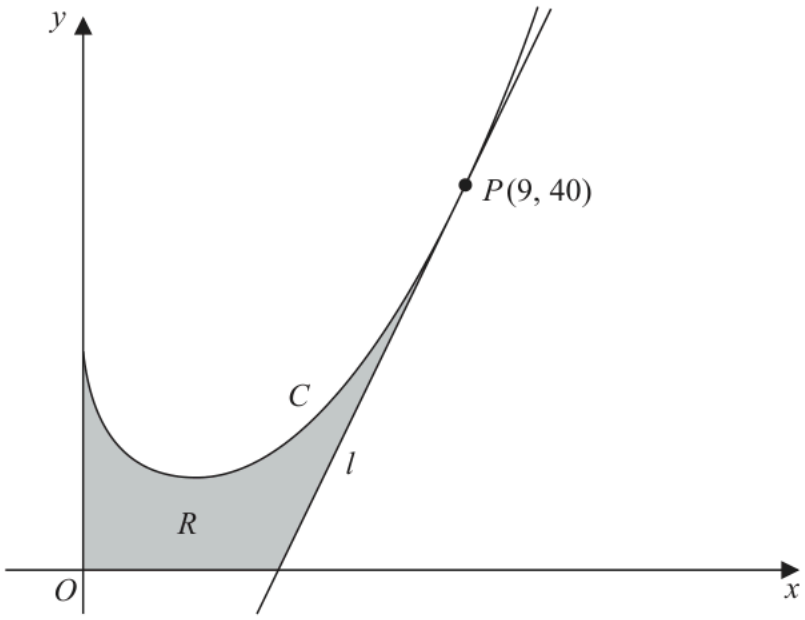


Figure 3

In this question you must show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

Figure 3 shows a sketch of part of the curve C with equation

$$y = \frac{2}{3}x^2 - 9\sqrt{x} + 13 \qquad x \geqslant 0$$

(a) Find, using calculus, the range of values of x for which y is increasing.

(4)

The point P lies on C and has coordinates (9, 40).

The line l is the tangent to C at the point P.

The finite region R, shown shaded in Figure 3, is bounded by the curve C, the line l, the x-axis and the y-axis.

(b) Find, using calculus, the exact area of R.

(8)

1 **Differentiate**

For $y = \frac{2}{3}x^2 - 9\sqrt{x} + 13$, $\frac{dy}{dx} = \frac{4}{3}x - \frac{9}{2\sqrt{x}}$.

2 **Set up increasing condition**

y is increasing when $\frac{dy}{dx} > 0$.

3 **Multiply by a positive quantity**

For $x > 0$, multiply by $6\sqrt{x}$: $8x\sqrt{x} - 27 > 0$.

4 **Solve the inequality**

$x^{3/2} > \frac{27}{8} = \left(\frac{3}{2}\right)^3$, so $\sqrt{x} > \frac{3}{2}$ and $x > \frac{9}{4}$.

5 **Find the tangent gradient**

At $P(9, 40)$, $\frac{dy}{dx} = \frac{4}{3}(9) - \frac{9}{2 \cdot 3} = 12 - \frac{3}{2} = \frac{21}{2}$.

6 **Write the tangent line**

$y - 40 = \frac{21}{2}(x - 9)$, so $y = \frac{21}{2}x - \frac{109}{2}$.

7 **Find where the tangent meets the x-axis**

Set $y = 0$: $\frac{21}{2}x - \frac{109}{2} = 0$, so $x = \frac{109}{21}$.

8 **Split the area**

From $x = 0$ to $x = \frac{109}{21}$, the lower boundary is the x-axis. From $x = \frac{109}{21}$ to $x = 9$, the lower boundary is the tangent.

9 **Set up the integral**

$$R = \int_0^{109/21} \left(\frac{2}{3}x^2 - 9\sqrt{x} + 13\right) dx + \int_{109/21}^{9} \left(\frac{2}{3}x^2 - 9\sqrt{x} + 13 - \left(\frac{21}{2}x - \frac{109}{2}\right)\right) dx.$$

10 **Use antiderivatives**

$\int \left(\frac{2}{3}x^2 - 9\sqrt{x} + 13\right) dx = \frac{2}{9}x^3 - 6x^{3/2} + 13x$, and $\int \left(\frac{21}{2}x - \frac{109}{2}\right) dx = \frac{21}{4}x^2 - \frac{109}{2}x$.

11 **Evaluate**

Substituting the limits and simplifying gives $R = \frac{857}{21}$.

FINAL ANSWER

(a) y is increasing for $x > \frac{9}{4}$.

(b) Area $= \frac{857}{21}$.

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Question 10

12 marks | Binomial Expansion

10. (i) (a) Find, in ascending powers of x, the 2nd, 3rd and 5th terms of the binomial expansion of

$$(3 + 2x)^6$$

(3)

For a particular value of x, these three terms form consecutive terms in a geometric series.

(b) Find this value of x.

(3)

(ii) In a **different** geometric series,

- the first term is $\sin^2\theta$
- the common ratio is $2\cos\theta$
- the sum to infinity is $\frac{8}{5}$

(a) Show that

$$5\cos^2\theta - 16\cos\theta + 3 = 0$$

(3)

(b) Hence find the exact value of the 2nd term in the series.

(3)

WORKED SOLUTION

Question 10 · WMA12/01 October 2023

1 Find the second term

The 2nd term of $(3+2x)^6$ is $\binom{6}{1}3^5(2x) = 6 \cdot 243 \cdot 2x = 2916x$.

2 Find the third term

The 3rd term is $\binom{6}{2}3^4(2x)^2 = 15 \cdot 81 \cdot 4x^2 = 4860x^2$.

3 Find the fifth term

The 5th term is $\binom{6}{4}3^2(2x)^4 = 15 \cdot 9 \cdot 16x^4 = 2160x^4$.

4 Use the geometric condition

For three consecutive terms in a geometric series, the middle term squared equals the product of the two outer terms.

5 **Set up the equation**

$(4860x^2)^2 = (2916x)(2160x^4)$.

6 **Solve for x**

For the non-zero geometric case, divide by x^4: $4860^2 = 2916 \cdot 2160\, x$, so $x = \dfrac{4860^2}{2916 \cdot 2160} = \dfrac{15}{4}$.

7 **Use the sum to infinity**

In the different geometric series, first term $= \sin^2\theta$, common ratio $= 2\cos\theta$, and $S_\infty = \dfrac{8}{5}$.

8 **Form the equation**

$\dfrac{\sin^2\theta}{1 - 2\cos\theta} = \dfrac{8}{5}$.

9 **Use sine squared**

$5\sin^2\theta = 8(1 - 2\cos\theta)$. With $\sin^2\theta = 1 - \cos^2\theta$, this becomes $5(1 - \cos^2\theta) = 8 - 16\cos\theta$.

10 **Show the required equation**

Rearranging gives $5\cos^2\theta - 16\cos\theta + 3 = 0$.

11 **Solve for cosine**

$5\cos^2\theta - 16\cos\theta + 3 = 0$ gives $\cos\theta = 3$ or $\cos\theta = \dfrac{1}{5}$. The possible value is $\cos\theta = \dfrac{1}{5}$.

12 **Find the second term**

The 2nd term is first term times common ratio: $\sin^2\theta(2\cos\theta) = \left(1 - \dfrac{1}{25}\right) \cdot \dfrac{2}{5} = \dfrac{24}{25} \cdot \dfrac{2}{5} = \dfrac{48}{125}$.

FINAL ANSWER

$(i)(a)$ $2916x$, $4860x^2$, $2160x^4$.

$(i)(b)$ $x = \dfrac{15}{4}$.

$(ii)(a)$ Proven.

$(ii)(b)$ $\dfrac{48}{125}$.