Elite Mathematics
Edexcel International A Level

IAL PURE 2 · WMA12

WMA12/01 May/June 2023 Worked Solutions

May/June 2023 paper rebuilt with the worked solution after each question.

11	2019–2026	Single-paper solutions
paper questions	19 exam sessions	checked by Dr Eslam



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PAST PAPER

WMA12/01 May/June 2023

May/June 2023

11	75	1 h 30
questions	marks	exam time

Each question is followed by its worked solution and final answer.

WMA12/01 MAY/JUNE 2023

Question 1

3 marks | Integration

1. The continuous curve C has equation $y = \mathrm{f}(x)$.

A table of values of x and y for $y = \mathrm{f}(x)$ is shown below.

x	4.0	4.2	4.4	4.6	4.8	5.0
y	9.2	8.4556	3.8512	5.0342	7.8297	8.6

Use the trapezium rule with all the values of y in the table to find an approximation for

$$\int_4^5 \mathrm{f}(x)\,\mathrm{d}x$$

giving your answer to 3 decimal places.

(3)

WORKED SOLUTION

Question 1 · WMA12/01 May/June 2023

1 Identify the interval width

The x values go from 4.0 to 5.0 in steps of 0.2, so $h = 0.2$.

2 Apply the trapezium rule

$$\int_4^5 f(x)\,dx \approx \frac{0.2}{2}[9.2 + 8.6 + 2(8.4556 + 3.8512 + 5.0342 + 7.8297)].$$

3 Evaluate

The value is 6.81414, so the approximation is 6.814 to 3 decimal places.

FINAL ANSWER

6.814.

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Question 2

6 marks | Polynomials

2. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

$$f(x) = 4x^3 - 8x^2 + 5x + a$$

where a is a constant.

Given that $(2x - 3)$ is a factor of $f(x)$,

(a) use the factor theorem to show that $a = -3$

(2)

(b) Hence show that the equation $f(x) = 0$ has only one real root.

(4)

WORKED SOLUTION

Question 2 · WMA12/01 May/June 2023

1 **Use the factor theorem**

Since $(2x - 3)$ is a factor, $x = \frac{3}{2}$ is a root, so $f\left(\frac{3}{2}\right) = 0$.

2 **Substitute the root**

$4\left(\frac{3}{2}\right)^3 - 8\left(\frac{3}{2}\right)^2 + 5\left(\frac{3}{2}\right) + a = 0$.

3 **Find a**

$13.5 - 18 + 7.5 + a = 0$, so $3 + a = 0$ and $a = -3$.

4 **Write the polynomial**

With $a = -3$, $f(x) = 4x^3 - 8x^2 + 5x - 3$.

5 **Factor using the known factor**

$4x^3 - 8x^2 + 5x - 3 = (2x - 3)(2x^2 - x + 1)$.

6 **Check the quadratic**

The discriminant of $2x^2 - x + 1$ is $(-1)^2 - 4(2)(1) = 1 - 8 = -7$.

7 **Conclude about real roots**

Since the quadratic has negative discriminant, it has no real roots. Therefore $f(x) = 0$ has only the one real root from $2x - 3 = 0$, namely $x = \dfrac{3}{2}$.

✓ **FINAL ANSWER**

(a) $a = -3$.

(b) $f(x) = 0$ has only one real root, $x = \dfrac{3}{2}$.

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Question 3

7 marks | Circles

3. A circle C has centre $(2, 5)$

Given that the point $P(8, -3)$ lies on C

(a) (i) find the radius of C

(ii) find an equation for C

(3)

(b) Find the equation of the tangent to C at P giving your answer in the form $ax + by + c = 0$ where a, b and c are integers to be found.

(4)

WORKED SOLUTION

Question 3 · WMA12/01 May/June 2023

1 Use the distance formula

The centre is $(2, 5)$ and $P = (8, -3)$. The radius is $\sqrt{(8-2)^2 + (-3-5)^2}$.

2 Find the radius

$r = \sqrt{6^2 + (-8)^2} = \sqrt{100} = 10$.

3 Write the circle equation

With centre $(2, 5)$ and radius 10, the circle is $(x-2)^2 + (y-5)^2 = 100$.

4 Find the radius gradient

The gradient of the radius from the centre to P is $\dfrac{-3-5}{8-2} = \dfrac{-8}{6} = -\dfrac{4}{3}$.

5 Find the tangent gradient

The tangent is perpendicular to the radius, so its gradient is $\dfrac{3}{4}$.

6 Use point P

The tangent through $(8, -3)$ is $y + 3 = \dfrac{3}{4}(x - 8)$.

7 Put into integer form

Multiplying by 4 gives $4y + 12 = 3x - 24$, hence $3x - 4y - 36 = 0$.

FINAL ANSWER

$(a)(i)\ 10.$

$(a)(ii)\ (x-2)^2+(y-5)^2=100.$

$(b)\ 3x-4y-36=0.$

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Question 4

7 marks | Binomial Expansion

4. The binomial expansion, in ascending powers of x, of

$$(3 + px)^5$$

where p is a constant, can be written in the form

$$A + Bx + Cx^2 + Dx^3 \ldots$$

where A, B, C and D are constants.

(a) Find the value of A

(1)

Given that

- $B = 18D$
- $p < 0$

(b) find

(i) the value of p

(ii) the value of C

(6)

WORKED SOLUTION

Question 4 · WMA12/01 May/June 2023

1 Find the constant term

In $(3 + px)^5$, the constant term is $3^5 = 243$, so $A = 243$.

2 Find B

The coefficient of x is $5(3^4)p = 405p$, so $B = 405p$.

3 Find D

The coefficient of x^3 is $\binom{5}{3}(3^2)p^3 = 10 \cdot 9p^3 = 90p^3$, so $D = 90p^3$.

4 Use B equals 18D

$405p = 18(90p^3) = 1620p^3$.

5 **Solve for p**

Since $p < 0$, $p \neq 0$. Divide by $405p$ to get $1 = 4p^2$, so $p = \pm\frac{1}{2}$. Therefore $p = -\frac{1}{2}$.

6 **Find C**

The coefficient of x^2 is $\binom{5}{2}(3^3)p^2 = 10 \cdot 27 \cdot \frac{1}{4} = \frac{135}{2}$.

FINAL ANSWER

(a) $A = 243$.

$(b)(i)$ $p = -\frac{1}{2}$.

$(b)(ii)$ $C = \frac{135}{2}$.

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Question 5

5 marks | Laws of Logarithms

5. Use the laws of logarithms to solve

$$\log_2(16x) + \log_2(x+1) = 3 + \log_2(x+6)$$

(5)

WORKED SOLUTION

Question 5 · WMA12/01 May/June 2023

1 State the domain

The logarithms require $16x > 0$, $x + 1 > 0$ and $x + 6 > 0$, so $x > 0$.

2 Combine the left side

$\log_2(16x) + \log_2(x+1) = \log_2(16x(x+1))$.

3 Rewrite the right side

$3 = \log_2 8$, so $3 + \log_2(x+6) = \log_2(8(x+6))$.

4 Equate the arguments

Therefore $16x(x+1) = 8(x+6)$.

5 Solve the quadratic

$16x^2 + 16x = 8x + 48$, so $16x^2 + 8x - 48 = 0$. Dividing by 8 gives $2x^2 + x - 6 = 0$.

6 Factor and use the domain

$(2x-3)(x+2) = 0$, so $x = \frac{3}{2}$ or $x = -2$. The domain $x > 0$ rejects -2, leaving $x = \frac{3}{2}$.

✓ **FINAL ANSWER**

$x = \frac{3}{2}$.

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Question 6

9 marks | Geometric Sequences

6. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

A software developer released an app to download.

The numbers of downloads of the app each month, in thousands, for the first three months after the app was released were

$$2k - 15 \qquad k \qquad k + 4$$

where k is a constant.

Given that the numbers of downloads each month are modelled as a geometric series,

(a) show that $k^2 - 7k - 60 = 0$

(2)

(b) predict the number of downloads in the 4th month.

(4)

The **total** number of all downloads of the app is predicted to exceed 3 million for the first time in the Nth month.

(c) Calculate the value of N according to the model.

(3)

WORKED SOLUTION

Question 6 · WMA12/01 May/June 2023

1 **Use the geometric condition**

For three consecutive terms of a geometric sequence, the middle term squared equals the product of the outer terms.

2 **Set up the equation**

So $k^2 = (2k - 15)(k + 4)$.

3 **Show the required quadratic**

Expanding gives $k^2 = 2k^2 - 7k - 60$, hence $k^2 - 7k - 60 = 0$.

4 **Find k**

$k^2 - 7k - 60 = (k - 12)(k + 5) = 0$, so $k = 12$ or $k = -5$. The download numbers must be positive, so $k = 12$.

5 **Find the common ratio**

The first three monthly values are 9, 12 and 16 thousand. The common ratio is $\frac{12}{9} = \frac{4}{3}$.

6 **Predict month four**

The 4th month is $16 \cdot \frac{4}{3} = \frac{64}{3}$ thousand downloads, which is about 21300 downloads.

7 **Write the cumulative total**

In thousands, the total after N months is $S_N = \frac{9\left(\left(\frac{4}{3}\right)^N - 1\right)}{\frac{4}{3} - 1} = 27\left(\left(\frac{4}{3}\right)^N - 1\right)$.

8 **Compare with three million**

Three million downloads is 3000 thousand, so we need $27\left(\left(\frac{4}{3}\right)^N - 1\right) > 3000$.

9 **Solve the inequality**

$\left(\frac{4}{3}\right)^N > \frac{1009}{9}$. Taking logarithms gives $N > \frac{\log(1009/9)}{\log(4/3)} = 16.405\ldots$.

10 **Choose the first month**

The first integer month for which the total exceeds 3 million is $N = 17$.

✓ **FINAL ANSWER**

(a) Proven.

(b) $\frac{64}{3}$ thousand downloads, about 21300 downloads.

(c) $N = 17$.

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Question 7

8 marks | Applications of Differentiation

7. The height of a river above a fixed point on the riverbed was monitored over a 7-day period.

The height of the river, H metres, t days after monitoring began, was given by

$$H = \frac{\sqrt{t}}{20}(20 + 6t - t^2) + 17 \qquad 0 \leqslant t \leqslant 7$$

Given that H has a stationary value at $t = \alpha$

(a) use calculus to show that α satisfies the equation

$$5\alpha^2 - 18\alpha - 20 = 0$$

(5)

(b) Hence find the value of α, giving your answer to 3 decimal places.

(1)

(c) Use further calculus to prove that H is a maximum at this value of α.

(2)

WORKED SOLUTION

Question 7 · WMA12/01 May/June 2023

1 Rewrite H with powers

$H = \frac{1}{20}t^{1/2}(20 + 6t - t^2) + 17.$

2 Expand before differentiating

$H = \frac{1}{20}(20t^{1/2} + 6t^{3/2} - t^{5/2}) + 17.$

3 Differentiate

$\frac{dH}{dt} = \frac{1}{20}\left(10t^{-1/2} + 9t^{1/2} - \frac{5}{2}t^{3/2}\right).$

4 Put over a common factor

$\frac{dH}{dt} = \frac{-5t^2 + 18t + 20}{40\sqrt{t}}.$

5 Use the stationary condition

At $t = \alpha$, $\frac{dH}{dt} = 0$. Since $\alpha > 0$, the denominator is not zero, so $-5\alpha^2 + 18\alpha + 20 = 0$.

6 **Show the required equation**

Multiplying by -1 gives $5\alpha^2 - 18\alpha - 20 = 0$.

7 **Find alpha**

$$\alpha = \frac{18 \pm \sqrt{18^2 + 4(5)(20)}}{10} = \frac{9 \pm \sqrt{181}}{5}.$$

8 **Use the time interval**

Only the positive value lies in $0 \le t \le 7$, so $\alpha = \dfrac{9 + \sqrt{181}}{5} = 4.491$ to 3 decimal places.

9 **Differentiate again**

$$\frac{d^2H}{dt^2} = \frac{-15t^2 + 18t - 20}{80t^{3/2}}.$$

10 **Use the stationary equation**

At $t = \alpha$, $5\alpha^2 = 18\alpha + 20$. Therefore the numerator of $\dfrac{d^2H}{dt^2}$ is $-15\alpha^2 + 18\alpha - 20 = -3(18\alpha + 20) + 18\alpha - 20 = -36\alpha - 80$.

11 **Show it is negative**

Since $\alpha > 0$, $-36\alpha - 80 < 0$, and the denominator $80\alpha^{3/2}$ is positive. Thus $\dfrac{d^2H}{dt^2} < 0$, so H is a maximum.

✓ **FINAL ANSWER**

(a) Proven.

(b) $\alpha = 4.491$.

(c) H is a maximum at this value.

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Question 8

5 marks | Proof

8. (i) A student writes the following statement:

"When a and b are consecutive **prime** numbers, $a^2 + b^2$ is never a multiple of 10"

Prove by counter example that this statement is **not** true.

(2)

(ii) Given that x and y are even integers greater than 0 and less than 6, prove by exhaustion, that

$$1 < x^2 - \frac{xy}{4} < 15$$

(3)

WORKED SOLUTION

Question 8 · WMA12/01 May/June 2023

1 Choose consecutive primes

7 and 11 are consecutive prime numbers.

2 Check the statement

$7^2 + 11^2 = 49 + 121 = 170$, which is a multiple of 10.

3 Conclude the counterexample

This single counterexample proves the student's statement is not true.

4 List the possible even values

The even integers greater than 0 and less than 6 are 2 and 4. Therefore the possible ordered pairs are $(2, 2), (2, 4), (4, 2), (4, 4)$.

5 Check x equals two

If $(x, y) = (2, 2)$, then $x^2 - \frac{xy}{4} = 4 - 1 = 3$. If $(x, y) = (2, 4)$, then $x^2 - \frac{xy}{4} = 4 - 2 = 2$.

6 Check x equals four

If $(x, y) = (4, 2)$, then $x^2 - \frac{xy}{4} = 16 - 2 = 14$. If $(x, y) = (4, 4)$, then $x^2 - \frac{xy}{4} = 16 - 4 = 12$.

7 **Complete the exhaustion**

The possible values are $3, 2, 14, 12$, and each lies strictly between 1 and 15. Hence $1 < x^2 - \dfrac{xy}{4} < 15$ in every possible case.

FINAL ANSWER

(i) A counterexample is $a = 7, b = 11$, since $7^2 + 11^2 = 170$.

(ii) Exhaustion proves the inequality for $(x, y) = (2, 2), (2, 4), (4, 2), (4, 4)$.

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Question 9

7 marks | Trigonometric Equations

9. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that

$$3\cos\theta(\tan\theta\sin\theta+3)=11-5\cos\theta$$

may be written as

$$3\cos^2\theta-14\cos\theta+8=0$$

(3)

(b) Hence solve, for $0<x<360°$

$$3\cos 2x(\tan 2x\sin 2x+3)=11-5\cos 2x$$

giving your answers to one decimal place.

(4)

WORKED SOLUTION

Question 9 · WMA12/01 May/June 2023

1 **Rewrite tan theta sin theta**

$\tan\theta\sin\theta=\dfrac{\sin^2\theta}{\cos\theta}$.

2 **Simplify the left side**

$3\cos\theta\left(\tan\theta\sin\theta+3\right)=3\sin^2\theta+9\cos\theta$.

3 **Use sine squared**

The equation becomes $3(1-\cos^2\theta)+9\cos\theta=11-5\cos\theta$.

4 **Rearrange**

$3-3\cos^2\theta+9\cos\theta=11-5\cos\theta$, so $3\cos^2\theta-14\cos\theta+8=0$.

5 **Apply the result to 2x**

For the equation in part (b), take $\theta=2x$. Then $3\cos^2(2x)-14\cos(2x)+8=0$.

6 **Solve the quadratic**

Let $u=\cos(2x)$. Then $3u^2-14u+8=0$, so $(3u-2)(u-4)=0$.

7 **Reject the impossible value**

$u = 4$ is impossible for cosine, so $\cos(2x) = \frac{2}{3}$.

8 **Use the interval**

Since $0 < x < 360°$, we have $0 < 2x < 720°$.

9 **Find x**

In this interval, $2x = 48.189\ldots°, 311.810\ldots°, 408.189\ldots°, 671.810\ldots°$. Dividing by 2 gives $x = 24.1°, 155.9°, 204.1°, 335.9°$ to one decimal place.

✓ **FINAL ANSWER**

(a) Proven.

(b) $x = 24.1°, 155.9°, 204.1°, 335.9°$.

Question 10

10 marks | Integration

10. The curve C has equation

$$y = \frac{(x-k)^2}{\sqrt{x}} \qquad x > 0$$

where k is a **positive** constant.

(a) Show that

$$\int_1^{16} \frac{(x-k)^2}{\sqrt{x}}\,\mathrm{d}x = ak^2 + bk + \frac{2046}{5}$$

where a and b are integers to be found.

(5)

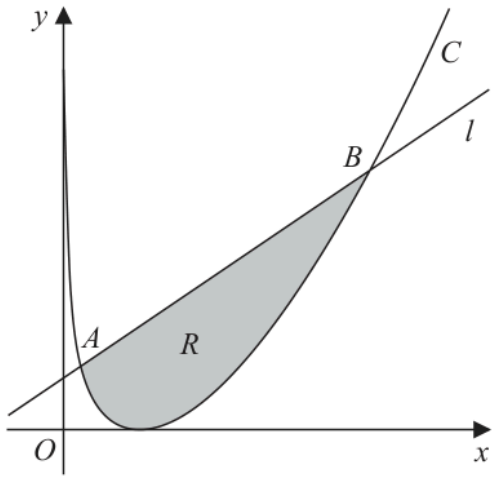


Figure 1

Figure 1 shows a sketch of the curve C and the line l.

Given that l intersects C at the point $A(1, 9)$ and at the point $B(16, q)$ where q is a constant,

(b) show that $k = 4$

(2)

The region R, shown shaded in Figure 1, is bounded by C and l

Using the answers to parts (a) and (b),

(c) find the area of region R

(3)

1 **Expand the integrand**

$\dfrac{(x-k)^2}{\sqrt{x}} = \dfrac{x^2-2kx+k^2}{x^{1/2}} = x^{3/2} - 2kx^{1/2} + k^2x^{-1/2}$.

2 **Integrate**

$\displaystyle\int \left(x^{3/2} - 2kx^{1/2} + k^2x^{-1/2}\right)dx = \frac{2}{5}x^{5/2} - \frac{4k}{3}x^{3/2} + 2k^2x^{1/2}$.

3 **Substitute the limits**

From 1 to 16, this gives $\left(\dfrac{2048}{5} - \dfrac{256k}{3} + 8k^2\right) - \left(\dfrac{2}{5} - \dfrac{4k}{3} + 2k^2\right)$.

4 **Simplify the integral**

The result is $6k^2 - 84k + \dfrac{2046}{5}$, so $a = 6$ and $b = -84$.

5 **Use point A**

Since $A(1, 9)$ lies on C, $9 = \dfrac{(1-k)^2}{\sqrt{1}} = (1-k)^2$.

6 **Find k**

Thus $1 - k = \pm 3$, so $k = 4$ or $k = -2$. Since k is positive, $k = 4$.

7 **Find point B**

When $k = 4$ and $x = 16$, $q = \dfrac{(16-4)^2}{\sqrt{16}} = \dfrac{144}{4} = 36$, so $B = (16, 36)$.

8 **Find the line**

The line through $A(1, 9)$ and $B(16, 36)$ has gradient $\dfrac{36-9}{16-1} = \dfrac{9}{5}$, so l has equation $y = \dfrac{9}{5}x + \dfrac{36}{5}$.

9 **Area under the line**

$\displaystyle\int_1^{16} \left(\frac{9}{5}x + \frac{36}{5}\right)dx = \left[\frac{9}{10}x^2 + \frac{36}{5}x\right]_1^{16} = \frac{675}{2}$.

10 **Area under the curve**

From part (a), with $k = 4$, $\displaystyle\int_1^{16} \frac{(x-4)^2}{\sqrt{x}}dx = 6(4)^2 - 84(4) + \frac{2046}{5} = \frac{846}{5}$.

11 **Subtract curve from line**

The shaded area is $\dfrac{675}{2} - \dfrac{846}{5} = \dfrac{1683}{10}$.

FINAL ANSWER

(a) $a = 6,\ b = -84$.

(b) $k = 4$.

(c) Area $= \dfrac{1683}{10}$.

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Question 11

8 marks | Sequences & Series

11. A sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_{n+1} = b - au_n$$

$$u_1 = 3$$

where a and b are constants.

(a) Find, in terms of a and b,

(i) u_2

(ii) u_3

(2)

Given

- $\sum_{n=1}^{3} u_n = 153$
- $b = a + 9$

(b) show that

$$a^2 - 5a - 66 = 0$$

(3)

(c) Hence find the larger possible value of u_2

(3)

WORKED SOLUTION

Question 11 · WMA12/01 May/June 2023

1 Find u2

Using $u_{n+1} = b - au_n$ and $u_1 = 3$, we get $u_2 = b - a(3) = b - 3a$.

2 Find u3

$u_3 = b - au_2 = b - a(b - 3a) = b - ab + 3a^2$.

3 Use the sum condition

$u_1 + u_2 + u_3 = 153$, so $3 + (b - 3a) + (b - ab + 3a^2) = 153$.

4 Substitute b equals a plus nine

Using $b = a + 9$, the left side becomes $3 + (a + 9) - 3a + (a + 9) - a(a + 9) + 3a^2$.

5 **Simplify**

This is $2a^2 - 10a + 21$, so $2a^2 - 10a + 21 = 153$.

6 **Show the required equation**

$2a^2 - 10a - 132 = 0$. Dividing by 2 gives $a^2 - 5a - 66 = 0$.

7 **Solve for a**

$a^2 - 5a - 66 = (a - 11)(a + 6) = 0$, so $a = 11$ or $a = -6$.

8 **Find the corresponding u2 values**

If $a = 11$, then $b = 20$ and $u_2 = 20 - 3(11) = -13$. If $a = -6$, then $b = 3$ and $u_2 = 3 - 3(-6) = 21$.

9 **Choose the larger value**

The larger possible value of u_2 is 21.

✓ **FINAL ANSWER**

$(a)(i)\ u_2 = b - 3a$.

$(a)(ii)\ u_3 = b - ab + 3a^2$.

(b) Proven.

$(c)\ 21$.