Elite Mathematics
Edexcel International A Level

IAL PURE 2 · WMA12

WMA12/01 January 2023 Worked Solutions

January 2023 paper rebuilt with the worked solution after each question.

10	2019–2026	Single-paper solutions
paper questions	19 exam sessions	checked by Dr Eslam



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PAST PAPER

WMA12/01 January 2023

January 2023

10	75	1 h 30
questions	marks	exam time

Each question is followed by its worked solution and final answer.

Question 1

6 marks | Integration

1.

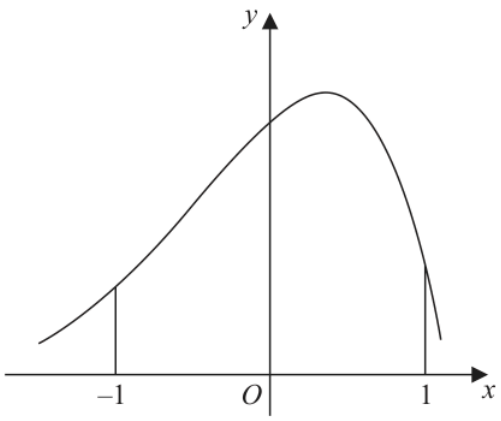


Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$

The table below shows some corresponding values of x and y for this curve.

The values of y are given to 3 decimal places.

x	–1	–0.5	0	0.5	1
y	2.287	4.470	6.719	7.291	2.834

Using the trapezium rule with all the values of y in the given table,

(a) obtain an estimate for

$$\int_{-1}^{1} \mathrm{f}(x)\,\mathrm{d}x$$

giving your answer to 2 decimal places.

(3)

(b) Use your answer to part (a) to estimate

(i) $\displaystyle\int_{-1}^{1} \left(\mathrm{f}(x) - 2\right)\mathrm{d}x$

(ii) $\displaystyle\int_{1}^{3} \mathrm{f}(x-2)\,\mathrm{d}x$

(3)

1 **Use the trapezium width**

The x values run from -1 to 1 in steps of 0.5, so $h = 0.5$.

2 **Apply the trapezium rule**

$$\int_{-1}^{1} f(x)\,dx \approx \frac{0.5}{2}[2.287 + 2.834 + 2(4.470 + 6.719 + 7.291)].$$

3 **Evaluate the estimate**

This gives 10.52025, so the estimate is 10.52 to 2 decimal places.

4 **Estimate the shifted-down area**

$$\int_{-1}^{1} (f(x) - 2)\,dx = \int_{-1}^{1} f(x)\,dx - \int_{-1}^{1} 2\,dx.$$

5 **Subtract the rectangle**

The interval has width 2, so $\int_{-1}^{1} 2\,dx = 4$. Hence the estimate is $10.52 - 4 = 6.52$.

6 **Use a substitution**

For $\int_{1}^{3} f(x-2)\,dx$, let $u = x - 2$. When $x = 1$, $u = -1$, and when $x = 3$, $u = 1$.

7 **Recognise the same integral**

So $\int_{1}^{3} f(x-2)\,dx = \int_{-1}^{1} f(u)\,du$, which has the same estimate 10.52.

FINAL ANSWER

(a) 10.52.

$(b)(i)$ 6.52.\quad $(b)(ii)$ 10.52.

Question 2

9 marks | Applications of Differentiation

2. **In this question you must show all stages of your working.**

Solutions based entirely on calculator technology are not acceptable.

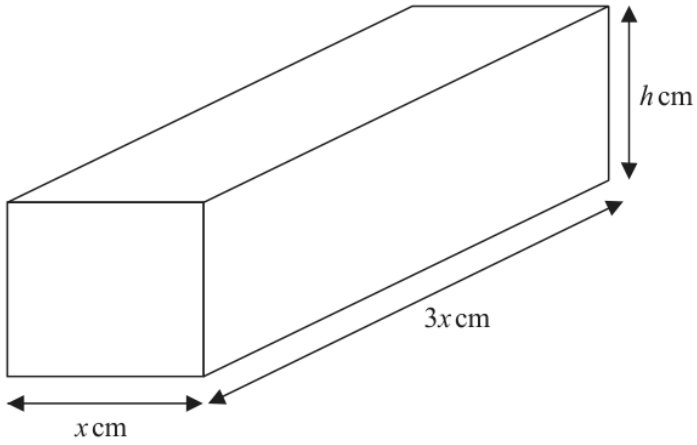


Figure 2

A brick is in the shape of a cuboid with width x cm, length $3x$ cm and height h cm, as shown in Figure 2.

The volume of the brick is $972\,\text{cm}^3$

(a) Show that the surface area of the brick, $S\,\text{cm}^2$, is given by

$$S = 6x^2 + \frac{2592}{x}$$

(3)

(b) Find $\dfrac{\mathrm{d}S}{\mathrm{d}x}$

(1)

(c) Hence find the value of x for which S is stationary.

(2)

(d) Find $\dfrac{\mathrm{d}^2S}{\mathrm{d}x^2}$ and hence show that the value of x found in part (c) gives the minimum value of S.

(2)

(e) Hence find the minimum surface area of the brick.

(1)

1 Use the volume

The cuboid has dimensions x, $3x$ and h, so its volume is $3x^2h = 972$.

2 Write h in terms of x

Hence $h = \dfrac{972}{3x^2} = \dfrac{324}{x^2}$.

3 Build the surface area

The surface area is $S = 2(3x^2 + 3xh + xh) = 6x^2 + 8xh$.

4 Substitute h

$S = 6x^2 + 8x\left(\dfrac{324}{x^2}\right) = 6x^2 + \dfrac{2592}{x}$.

5 Differentiate

$S = 6x^2 + 2592x^{-1}$, so $\dfrac{dS}{dx} = 12x - 2592x^{-2} = 12x - \dfrac{2592}{x^2}$.

6 Set the derivative to zero

For a stationary value, $12x - \dfrac{2592}{x^2} = 0$.

7 Solve for x

Multiplying by x^2 gives $12x^3 = 2592$, so $x^3 = 216$ and $x = 6$.

8 Use the second derivative

$\dfrac{d^2S}{dx^2} = 12 + \dfrac{5184}{x^3}$.

9 Show it is a minimum

At $x = 6$, $\dfrac{d^2S}{dx^2} = 12 + \dfrac{5184}{216} = 36 > 0$, so the stationary value is a minimum.

10 Find the minimum surface area

$S = 6(6)^2 + \dfrac{2592}{6} = 216 + 432 = 648$, so the minimum surface area is $648\ \text{cm}^2$.

FINAL ANSWER

(a) $S = 6x^2 + \dfrac{2592}{x}$.

(b) $\dfrac{dS}{dx} = 12x - \dfrac{2592}{x^2}$.

(c) $x = 6$.

(d) Minimum shown.

(e) $648\ \text{cm}^2$.

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Question 3

7 marks | Binomial Expansion

3. $$f(x) = \left(2 + \frac{kx}{8}\right)^7$$ where k is a non-zero constant

(a) Find the first 4 terms, in ascending powers of x, of the binomial expansion of $f(x)$. Give each term in simplest form.

(4)

Given that, in the binomial expansion of $f(x)$, the coefficients of x, x^2 and x^3 are the first 3 terms of an arithmetic progression,

(b) find, using algebra, the possible values of k.

(Solutions relying entirely on calculator technology are not acceptable.)

(3)

WORKED SOLUTION

Question 3 · WMA12/01 January 2023

1 Start the binomial expansion

$\left(2 + \frac{kx}{8}\right)^7$ begins with $2^7 + 7(2^6)\left(\frac{kx}{8}\right) + \binom{7}{2}(2^5)\left(\frac{kx}{8}\right)^2 + \binom{7}{3}(2^4)\left(\frac{kx}{8}\right)^3$.

2 Simplify the terms

This gives $128 + 56kx + \frac{21}{2}k^2x^2 + \frac{35}{32}k^3x^3$.

3 Use the arithmetic progression condition

The coefficients of x, x^2 and x^3 are $56k$, $\frac{21}{2}k^2$ and $\frac{35}{32}k^3$.

4 Set middle twice

For three terms in an arithmetic progression, twice the middle term equals the sum of the outer terms: $21k^2 = 56k + \frac{35}{32}k^3$.

5 Use k non-zero

Since $k \neq 0$, divide by k and multiply by 32: $672k = 1792 + 35k^2$.

6 Solve the quadratic

So $35k^2 - 672k + 1792 = 0$. The discriminant is $672^2 - 4(35)(1792) = 200704 = 448^2$.

7 **Find the possible values**

$k = \dfrac{672 \pm 448}{70}$, giving $k = 16$ or $k = \dfrac{16}{5}$.

FINAL ANSWER

$(a)\ 128 + 56kx + \dfrac{21}{2}k^2x^2 + \dfrac{35}{32}k^3x^3$.

$(b)\ k = \dfrac{16}{5}$ or $k = 16$.

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Question 4

6 marks | Laws of Logarithms

4. (i) Using the laws of logarithms, solve

$$\log_3(4x) + 2 = \log_3(5x + 7)$$

(3)

(ii) Given that

$$\sum_{r=1}^{2} \log_a\left(y^r\right) = \sum_{r=1}^{2} \left(\log_a y\right)^r \qquad y > 1, a > 1, y \neq a$$

find y in terms of a, giving your answer in simplest form.

(3)

WORKED SOLUTION

Question 4 · WMA12/01 January 2023

1 Rewrite the constant

Since $2 = \log_3 9$, the equation becomes $\log_3(4x) + \log_3 9 = \log_3(5x + 7)$.

2 Combine the logarithms

$\log_3(4x) + \log_3 9 = \log_3(36x)$.

3 Solve for x

Therefore $36x = 5x + 7$, so $31x = 7$ and $x = \frac{7}{31}$.

4 Introduce L

Let $L = \log_a y$. Then $\log_a(y^r) = r \log_a y = rL$.

5 Evaluate the left sum

$\sum_{r=1}^{2} \log_a(y^r) = L + 2L = 3L$.

6 Evaluate the right sum

$\sum_{r=1}^{2} (\log_a y)^r = L + L^2$.

7 **Solve for L**

So $3L = L + L^2$, which gives $L^2 - 2L = 0$ and $L(L-2) = 0$.

8 **Use the restrictions**

Since $a > 1$ and $y > 1$, $\log_a y > 0$, so $L \neq 0$. Hence $L = 2$ and $y = a^2$.

✓ **FINAL ANSWER**

$(i)\ x = \dfrac{7}{31}$.

$(ii)\ y = a^2$.

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Question 5

9 marks | Polynomials

5. $$f(x) = x^3 + (p+3)x^2 - x + q$$

where p and q are constants and $p > 0$

Given that $(x - 3)$ is a factor of $f(x)$

(a) show that

$$9p + q = -51$$

(2)

Given also that when $f(x)$ is divided by $(x + p)$ the remainder is 9

(b) show that

$$3p^2 + p + q - 9 = 0$$

(2)

(c) Hence find the value of p and the value of q.

(3)

(d) Hence find a quadratic expression $g(x)$ such that

$$f(x) = (x - 3)g(x)$$

(2)

WORKED SOLUTION

Question 5 · WMA12/01 January 2023

1 Use the factor theorem

Since $(x - 3)$ is a factor, $f(3) = 0$.

2 Substitute x equals three

$27 + 9(p + 3) - 3 + q = 0$, so $51 + 9p + q = 0$.

3 Show the first relation

Therefore $9p + q = -51$.

4 Use the remainder theorem

Dividing by $(x + p)$ means substituting $x = -p$. The remainder is $f(-p)$.

5 Substitute x equals negative p

$f(-p) = -p^3 + (p + 3)p^2 + p + q = 3p^2 + p + q$.

6 **Show the second relation**

The remainder is 9, so $3p^2+p+q=9$, hence $3p^2+p+q-9=0$.

7 **Eliminate q**

From $9p+q=-51$, $q=-51-9p$. Substitute into $3p^2+p+q-9=0$.

8 **Solve for p**

$3p^2+p-51-9p-9=0$, so $3p^2-8p-60=0$. This gives $p=6$ or $p=-\dfrac{10}{3}$.

9 **Use p greater than zero**

Since $p>0$, $p=6$. Then $q=-51-9(6)=-105$.

10 **Form f with p and q**

$f(x)=x^3+9x^2-x-105$.

11 **Divide by x minus three**

$x^3+9x^2-x-105=(x-3)(x^2+12x+35)$, so $g(x)=x^2+12x+35$.

✓ **FINAL ANSWER**

(a) Proven.

(b) Proven.

(c) $p=6,\ q=-105$.

(d) $g(x)=x^2+12x+35$.

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Question 6

10 marks | Circles

6. The circle C has equation

$$x^2 + y^2 + 8x - 4y = 0$$

(a) Find

(i) the coordinates of the centre of C,

(ii) the exact radius of C.

(3)

The point P lies on C.

Given that the tangent to C at P has equation $x + 2y + 10 = 0$

(b) find the coordinates of P

(4)

(c) Find the equation of the normal to C at P, giving your answer in the form $y = mx + c$ where m and c are integers to be found.

(3)

WORKED SOLUTION

Question 6 · WMA12/01 January 2023

1 Complete the square

$x^2 + y^2 + 8x - 4y = 0$ becomes $(x + 4)^2 - 16 + (y - 2)^2 - 4 = 0$.

2 Write the circle form

So $(x + 4)^2 + (y - 2)^2 = 20$.

3 Read the centre and radius

The centre is $(-4, 2)$ and the radius is $\sqrt{20} = 2\sqrt{5}$.

4 Use the tangent gradient

The tangent $x + 2y + 10 = 0$ has gradient $-\frac{1}{2}$.

5 Find the radius gradient

The radius to the point of contact is perpendicular to the tangent, so its gradient is 2.

6 **Write the normal through the centre**

The radius line through the centre $(-4, 2)$ is $y - 2 = 2(x + 4)$, so $y = 2x + 10$.

7 **Intersect with the tangent**

Substitute $y = 2x + 10$ into $x + 2y + 10 = 0$: $x + 2(2x + 10) + 10 = 0$.

8 **Find P**

$5x + 30 = 0$, so $x = -6$. Then $y = 2(-6) + 10 = -2$, hence $P = (-6, -2)$.

9 **State the normal**

The normal at P is the radius line already found, so its equation is $y = 2x + 10$.

✓ **FINAL ANSWER**

$(a)(i)$ $(-4, 2)$.

$(a)(ii)$ $2\sqrt{5}$.

(b) $P = (-6, -2)$.

(c) $y = 2x + 10$.

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Question 7

7 marks | Geometric Sequences

7. A geometric sequence has first term a and common ratio r, where $r > 0$

Given that

- the 3rd term is 20
- the 5th term is 12.8

(a) show that $r = 0.8$

(1)

(b) Hence find the value of a.

(2)

Given that the sum of the first n terms of this sequence is greater than 156

(c) find the smallest possible value of n.

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(4)

WORKED SOLUTION

Question 7 · WMA12/01 January 2023

1 Write the given terms

For a geometric sequence, the third term is $ar^2 = 20$ and the fifth term is $ar^4 = 12.8$.

2 Divide the equations

$\frac{ar^4}{ar^2} = \frac{12.8}{20}$, so $r^2 = 0.64$.

3 Use r positive

Since $r > 0$, $r = 0.8$.

4 Find a

Using $ar^2 = 20$, $a(0.8)^2 = 20$, so $a = \frac{20}{0.64} = 31.25$.

5 Write the sum formula

$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{31.25(1 - 0.8^n)}{0.2} = 156.25(1 - 0.8^n)$.

6 **Set up the inequality**

We need $156.25(1 - 0.8^n) > 156$.

7 **Isolate the power**

$1 - 0.8^n > \dfrac{156}{156.25} = \dfrac{624}{625}$, so $0.8^n < \dfrac{1}{625}$.

8 **Use logarithms**

$n \log(0.8) < \log\left(\dfrac{1}{625}\right)$. Since $\log(0.8) < 0$, dividing reverses the inequality:

$n > \dfrac{\log(1/625)}{\log(0.8)} = 28.85\ldots$.

9 **Choose the smallest integer**

The smallest possible integer value is therefore $n = 29$.

✓ **FINAL ANSWER**

(a) $r = 0.8$.

(b) $a = 31.25$.

(c) $n = 29$.

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Question 8

9 marks | Trigonometric Equations

8. **In this question you must show all stages of your working.**

Solutions based entirely on calculator technology are not acceptable.

(i) Solve, for $-\frac{\pi}{2} < x < \pi$, the equation

$$5\sin(3x + 0.1) + 2 = 0$$

giving your answers, **in radians**, to 2 decimal places.

(4)

(ii) Solve, for $0 < \theta < 360°$, the equation

$$2\tan\theta\sin\theta = 5 + \cos\theta$$

giving your answers, **in degrees**, to one decimal place.

(5)

WORKED SOLUTION

Question 8 · WMA12/01 January 2023

1 **Isolate the sine term**

$5\sin(3x + 0.1) + 2 = 0$ gives $\sin(3x + 0.1) = -0.4$.

2 **Find the angle interval**

Since $-\frac{\pi}{2} < x < \pi$, multiplying by 3 and adding 0.1 gives $-\frac{3\pi}{2} + 0.1 < 3x + 0.1 < 3\pi + 0.1$.

3 **Solve for the inner angle**

In this interval, $3x + 0.1 = -2.7301\ldots, -0.4115\ldots, 3.5531\ldots, 5.8717\ldots$.

4 **Find x values**

Subtract 0.1 and divide by 3: $x = -0.9433\ldots, -0.1705\ldots, 1.1510\ldots, 1.9239\ldots$.

5 **Round the radian answers**

To 2 decimal places, $x = -0.94, -0.17, 1.15, 1.92$.

6 **Rewrite the second equation**

$2\tan\theta\sin\theta = 5 + \cos\theta$. Using $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ gives $\dfrac{2\sin^2\theta}{\cos\theta} = 5 + \cos\theta$.

7 **Use sine squared**

Multiplying by $\cos\theta$ gives $2\sin^2\theta = 5\cos\theta + \cos^2\theta$. Substitute $\sin^2\theta = 1 - \cos^2\theta$.

8 **Solve for cos theta**

$2(1 - \cos^2\theta) = 5\cos\theta + \cos^2\theta$, so $3\cos^2\theta + 5\cos\theta - 2 = 0$.

9 **Factor and reject impossible value**

$(3\cos\theta - 1)(\cos\theta + 2) = 0$. Since $\cos\theta = -2$ is impossible, $\cos\theta = \dfrac{1}{3}$.

10 **Find theta**

For $0 < \theta < 360°$, $\cos\theta = \dfrac{1}{3}$ gives $\theta = 70.5°$ or $289.5°$ to one decimal place.

✓ **FINAL ANSWER**

(i) $x = -0.94, -0.17, 1.15, 1.92$.

(ii) $\theta = 70.5°, 289.5°$.

Question 9

8 marks | Integration

9. **In this question you must show all stages of your working.**

Solutions based entirely on calculator technology are not acceptable.

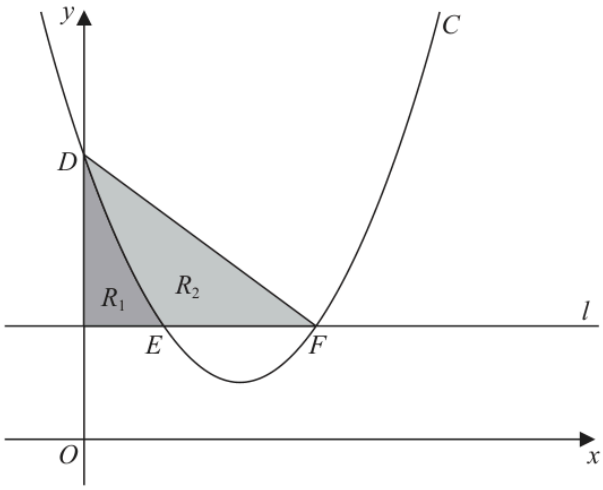


Figure 3

Figure 3 shows

- the curve C with equation $y = x^2 - 4x + 5$
- the line l with equation $y = 2$

The curve C intersects the y-axis at the point D.

(a) Write down the coordinates of D.

(1)

The curve C intersects the line l at the points E and F, as shown in Figure 3.

(b) Find the x coordinate of E and the x coordinate of F.

(2)

Shown shaded in Figure 3 is

- the region R_1 which is bounded by C, l and the y-axis
- the region R_2 which is bounded by C and the line segments EF and DF

Given that $\dfrac{\text{area of } R_1}{\text{area of } R_2} = k$, where k is a constant,

(c) use algebraic integration to find the exact value of k, giving your answer as a simplified fraction.

(5)

1 **Find D**

At the y-axis, $x = 0$. On C, $y = 0^2 - 4(0) + 5 = 5$, so $D = (0, 5)$.

2 **Find intersections with the line**

The line is $y = 2$, so solve $x^2 - 4x + 5 = 2$.

3 **Solve the quadratic**

$x^2 - 4x + 3 = 0$, so $(x - 1)(x - 3) = 0$. Hence $x_E = 1$ and $x_F = 3$.

4 **Area of R1**

R_1 lies between the curve and the line $y = 2$ from $x = 0$ to $x = 1$.

5 **Integrate R1**

$$R_1 = \int_0^1 \left(x^2 - 4x + 5 - 2\right) dx = \int_0^1 (x^2 - 4x + 3) dx.$$

6 **Evaluate R1**

$$R_1 = \left[\frac{x^3}{3} - 2x^2 + 3x\right]_0^1 = \frac{1}{3} - 2 + 3 = \frac{4}{3}.$$

7 **Find the line DF**

The line through $D = (0, 5)$ and $F = (3, 2)$ has gradient -1, so its equation is $y = 5 - x$.

8 **Split R2**

For $0 \le x \le 1$, R_2 is between $y = 5 - x$ and the curve. For $1 \le x \le 3$, it is between $y = 5 - x$ and $y = 2$.

9 **Integrate R2**

$$R_2 = \int_0^1 \left[(5 - x) - (x^2 - 4x + 5)\right] dx + \int_1^3 \left[(5 - x) - 2\right] dx.$$

10 **Evaluate R2**

$$R_2 = \int_0^1 (-x^2 + 3x) dx + \int_1^3 (3 - x) dx = \frac{7}{6} + 2 = \frac{19}{6}.$$

11 **Find the ratio**

$$k = \frac{\text{area of } R_1}{\text{area of } R_2} = \frac{4/3}{19/6} = \frac{8}{19}.$$

FINAL ANSWER

$(a)\ D = (0, 5).$

$(b)\ x_E = 1,\ x_F = 3.$

$(c)\ k = \dfrac{8}{19}.$

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Question 10

4 marks | Proof

10. A student was asked to prove by exhaustion that

if n is an integer then $2n^2 + n + 1$ is **not** divisible by 3

The start of the student's proof is shown in the box below.

> Consider the case when $n = 3k$
>
> $$2n^2 + n + 1 = 18k^2 + 3k + 1 = 3(6k^2 + k) + 1$$
>
> which is not divisible by 3

Complete this proof.

(4)

WORKED SOLUTION

Question 10 · WMA12/01 January 2023

1 Use the three possible forms

Every integer is exactly one of the forms $3k$, $3k + 1$ or $3k + 2$, where k is an integer.

2 First case already given

For $n = 3k$, the expression becomes $3(6k^2 + k) + 1$, so it leaves remainder 1 when divided by 3.

3 Try n equals 3k plus 1

If $n = 3k + 1$, then $2n^2 + n + 1 = 2(3k + 1)^2 + (3k + 1) + 1$.

4 Simplify the second case

$2(9k^2 + 6k + 1) + 3k + 2 = 18k^2 + 15k + 4 = 3(6k^2 + 5k + 1) + 1$, which is not divisible by 3.

5 Try n equals 3k plus 2

If $n = 3k + 2$, then $2n^2 + n + 1 = 2(3k + 2)^2 + (3k + 2) + 1$.

6 Simplify the third case

$2(9k^2 + 12k + 4) + 3k + 3 = 18k^2 + 27k + 11 = 3(6k^2 + 9k + 3) + 2$, which is not divisible by 3.

7 **Complete the exhaustion**

All possible integer forms have been checked, and none gives a multiple of 3. Therefore $2n^2 + n + 1$ is not divisible by 3.

✓ **FINAL ANSWER**

For $n = 3k$, $n = 3k + 1$ and $n = 3k + 2$, the expression has remainders 1, 1 and 2 respectively when divided by 3. Therefore $2n^2 + n + 1$ is never divisible by 3.