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Elite Mathematics
Edexcel International A Level

IAL PURE 2 · WMA12

WMA12/01 October 2022 Worked Solutions

October 2022 paper rebuilt with the worked solution after each question.

10	2019–2026	Single-paper solutions
paper questions	19 exam sessions	checked by Dr Eslam



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PAST PAPER

WMA12/01 October 2022

October 2022

10	75	1 h 30
questions	marks	exam time

Each question is followed by its worked solution and final answer.

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Question 1

3 marks | Proof

1. Given that a, b and c are integers greater than 0 such that

- $c = b + 2$
- $a + b + c = 10$

Prove, by exhaustion, that the product of a, b and c is always even.

You may use the table below to illustrate your answer.

(3)

You may not need to use all rows of this table.

a	b	c
	1	
	2	

WORKED SOLUTION

Question 1 · WMA12/01 October 2022

1 Use the two conditions

Since $c = b + 2$ and $a + b + c = 10$, substitute for c: $a + b + (b + 2) = 10$.

2 Write a in terms of b

This gives $a + 2b + 2 = 10$, so $a = 8 - 2b$.

3 List the possible b values

The integers a, b, c are all positive. Since $b > 0$ and $a = 8 - 2b > 0$, the only possible values are $b = 1, 2, 3$.

4 **Exhaust the cases**

If $b = 1$, then $a = 6$ and $c = 3$, so $abc = 18$. If $b = 2$, then $a = 4$ and $c = 4$, so $abc = 32$. If $b = 3$, then $a = 2$ and $c = 5$, so $abc = 30$.

5 **Conclude the proof**

The list contains every possible case, and in every case the product is even. Therefore abc is always even.

✓ **FINAL ANSWER**

The possible triples are $(6, 1, 3)$, $(4, 2, 4)$ and $(2, 3, 5)$, giving products 18, 32 and 30. Each product is even, so abc is always even.

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Question 2

7 marks | Binomial Expansion

2. A curve C has equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = (2 - kx)^5$$

and k is a constant.

Given that when $\mathrm{f}(x)$ is divided by $(4x - 5)$ the remainder is $\frac{243}{32}$

(a) show that $k = \frac{2}{5}$

(2)

(b) Find the first three terms, in ascending powers of x, of the binomial expansion of

$$\left(2 - \frac{2}{5}x\right)^5$$

giving each term in simplest form.

(3)

Using the solution to part (b) and making your method clear,

(c) find the gradient of C at the point where $x = 0$

(2)

WORKED SOLUTION

Question 2 · WMA12/01 October 2022

1 Use the remainder theorem

Dividing by $4x - 5$ means the remainder is $f\left(\frac{5}{4}\right)$.

2 Substitute x equals five quarters

$f\left(\frac{5}{4}\right) = \left(2 - \frac{5k}{4}\right)^5 = \frac{243}{32}$.

3 Take the fifth root

Since $\frac{243}{32} = \left(\frac{3}{2}\right)^5$, we have $2 - \frac{5k}{4} = \frac{3}{2}$.

4 Find k

$\frac{5k}{4} = \frac{1}{2}$, so $k = \frac{2}{5}$.

5 **Expand the binomial**

With $k = \frac{2}{5}$, $f(x) = \left(2 - \frac{2}{5}x\right)^5$.

6 **Find the first three terms**

$2^5 + 5(2^4)\left(-\frac{2}{5}x\right) + 10(2^3)\left(-\frac{2}{5}x\right)^2 = 32 - 32x + \frac{64}{5}x^2$.

7 **Use the coefficient of x**

For a polynomial, the gradient at $x = 0$ is the coefficient of x in the expansion. Therefore $f'(0) = -32$.

✓ **FINAL ANSWER**

$(a)\ k = \frac{2}{5}$.

$(b)\ 32 - 32x + \frac{64}{5}x^2$.

$(c)\ -32$.

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Question 3

7 marks | Sequences & Series

3. A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_n = \cos^2\left(\frac{n\pi}{3}\right)$$

Find the exact values of

(a) (i) a_1

(ii) a_2

(iii) a_3

(3)

(b) Hence find the exact value of

$$\sum_{n=1}^{50}\left\{n + \cos^2\left(\frac{n\pi}{3}\right)\right\}$$

You must make your method clear.

(4)

WORKED SOLUTION

Question 3 · WMA12/01 October 2022

1 Find the first term

$$a_1 = \cos^2\left(\frac{\pi}{3}\right) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

2 Find the second term

$$a_2 = \cos^2\left(\frac{2\pi}{3}\right) = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}.$$

3 Find the third term

$$a_3 = \cos^2(\pi) = (-1)^2 = 1.$$

4 Recognise the repeating pattern

The values of $\cos^2\left(\frac{n\pi}{3}\right)$ repeat every 3 terms as $\frac{1}{4}, \frac{1}{4}, 1$.

5 **Sum the n part**

$\sum_{n=1}^{50} n = \dfrac{50 \cdot 51}{2} = 1275.$

6 **Sum the cosine part**

There are 16 complete groups of 3 terms in the first 48 terms, then the first 2 terms repeat. Hence the cosine-square sum is $16\left(\dfrac{1}{4}+\dfrac{1}{4}+1\right)+\dfrac{1}{4}+\dfrac{1}{4}=24+\dfrac{1}{2}=\dfrac{49}{2}$.

7 **Add the two sums**

$\sum_{n=1}^{50}\left(n+\cos^2\left(\dfrac{n\pi}{3}\right)\right)=1275+\dfrac{49}{2}=\dfrac{2599}{2}$.

✓ **FINAL ANSWER**

$(a)\ a_1=\dfrac{1}{4},\ a_2=\dfrac{1}{4},\ a_3=1.$\quad $(b)\ \dfrac{2599}{2}$.

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Question 4

8 marks | Laws of Logarithms

4. The weight of a baby mammal is monitored over a 16-month period.

The weight of the mammal, w kg, is given by

$$w = \log_a(t+5) - \log_a 4 \qquad 2 \leqslant t \leqslant 18$$

where t is the age of the mammal in months and a is a constant.

Given that the weight of the mammal was 10 kg when $t = 3$

(a) show that $a = 1.072$ correct to 3 decimal places.

(3)

Using $a = 1.072$

(b) find an equation for t in terms of w

(3)

(c) find the value of t when $w = 15$, giving your answer to 3 significant figures.

(2)

WORKED SOLUTION

Question 4 · WMA12/01 October 2022

1 Substitute the given information

When $t = 3$, $w = 10$. So $10 = \log_a(3+5) - \log_a 4 = \log_a 8 - \log_a 4$.

2 Combine the logarithms

$\log_a 8 - \log_a 4 = \log_a\left(\frac{8}{4}\right) = \log_a 2$, so $10 = \log_a 2$.

3 Find a

$10 = \log_a 2$ means $a^{10} = 2$, hence $a = 2^{1/10} = 1.071773\ldots = 1.072$ to 3 decimal places.

4 Start from the model

Using $a = 1.072$, $w = \log_{1.072}(t+5) - \log_{1.072} 4$.

5 Combine the logs again

$w = \log_{1.072}\left(\frac{t+5}{4}\right)$.

6 **Make t the subject**

Changing from logarithmic form to exponential form gives $(1.072)^w = \dfrac{t+5}{4}$, so $t = 4(1.072)^w - 5$.

7 **Substitute w equals fifteen**

When $w = 15$, $t = 4(1.072)^{15} - 5 = 6.3496\ldots$, so $t = 6.35$ months to 3 significant figures.

✓ **FINAL ANSWER**

(a) $a = 1.072$ to 3 d.p.

(b) $t = 4(1.072)^w - 5$.

(c) $t = 6.35$ months to 3 s.f.

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Question 5

8 marks | Trigonometric Equations

5. **In this question you must show detailed reasoning.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that the equation

$$(3\cos\theta - \tan\theta)\cos\theta = 2$$

can be written as

$$3\sin^2\theta + \sin\theta - 1 = 0$$

(3)

(b) Hence solve for $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

$$(3\cos 2x - \tan 2x)\cos 2x = 2$$

(5)

WORKED SOLUTION

Question 5 · WMA12/01 October 2022

1 **Rewrite the left side**

$(3\cos\theta - \tan\theta)\cos\theta = 3\cos^2\theta - \sin\theta$ because $\tan\theta\cos\theta = \sin\theta$.

2 **Move everything to one side**

The equation becomes $3\cos^2\theta - \sin\theta = 2$.

3 **Use cos squared**

Since $\cos^2\theta = 1 - \sin^2\theta$, $3(1 - \sin^2\theta) - \sin\theta = 2$.

4 **Show the required form**

Expanding and rearranging gives $3\sin^2\theta + \sin\theta - 1 = 0$.

5 **Apply the result to 2x**

For $(3\cos 2x - \tan 2x)\cos 2x = 2$, take $\theta = 2x$. Then $3\sin^2(2x) + \sin(2x) - 1 = 0$.

6 **Solve the quadratic**

Let $u = \sin(2x)$. Then $3u^2 + u - 1 = 0$, so $u = \dfrac{-1 \pm \sqrt{13}}{6}$.

7 **Use the interval**

Since $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, we have $-\pi \leq 2x \leq \pi$. Solving $\sin(2x) = \frac{-1+\sqrt{13}}{6}$ and $\sin(2x) = \frac{-1-\sqrt{13}}{6}$ in this interval gives $2x = -2.2665\ldots, -0.8751\ldots, 0.4492\ldots, 2.6924\ldots$.

8 **Divide by two**

Therefore $x = -1.133\ldots, -0.4375\ldots, 0.2246\ldots, 1.346\ldots$, giving $x = -1.13, -0.438, 0.225, 1.35$ radians.

FINAL ANSWER

(a) Proven.

(b) $x = -1.13,\ -0.438,\ 0.225,\ 1.35$ radians.

Question 6

7 marks | Integration

6. The curve C_1 has equation $y = f(x)$.

A table of values of x and y for $y = f(x)$ is shown below, with the y values rounded to 4 decimal places where appropriate.

x	0	0.5	1	1.5	2
y	3	2.6833	2.4	2.1466	1.92

(a) Use the trapezium rule with all the values of y in the table to find an approximation for

$$\int_0^2 f(x)\,dx$$

giving your answer to 3 decimal places.

(3)

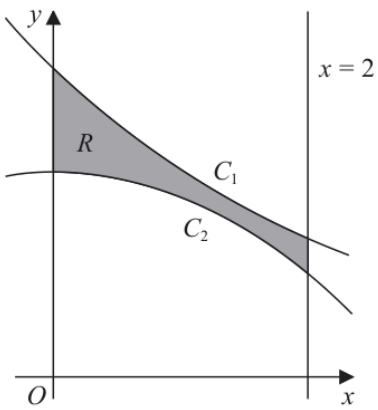


Figure 1

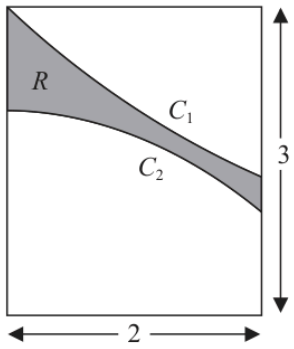


Figure 2

The region R, shown shaded in Figure 1, is bounded by

- the curve C_1
- the curve C_2 with equation $y = 2 - \frac{1}{4}x^2$
- the line with equation $x = 2$
- the y-axis

The region R forms part of the design for a logo shown in Figure 2.

The design consists of the shaded region R inside a rectangle of width 2 and height 3

Using calculus and the answer to part (a),

(b) calculate an estimate for the percentage of the logo which is shaded.

(4)

1 Identify the strip width

The x values increase by 0.5, so the trapezium-rule width is $h = 0.5$.

2 Apply the trapezium rule

$$\int_0^2 f(x)\,dx \approx \frac{0.5}{2}[3 + 1.92 + 2(2.6833 + 2.4 + 2.1466)].$$

3 Evaluate the estimate

This gives 4.84495, so the approximation is 4.845 to 3 decimal places.

4 Find the area under C2

$$\int_0^2 \left(2 - \frac{1}{4}x^2\right)dx = \left[2x - \frac{x^3}{12}\right]_0^2 = 4 - \frac{2}{3} = \frac{10}{3}.$$

5 Estimate the shaded area

The shaded region lies between C_1 and C_2, so its estimated area is $4.845 - \dfrac{10}{3} = 1.5116\ldots$.

6 Compare with the rectangle

The rectangle has area $2 \times 3 = 6$.

7 Convert to a percentage

Percentage shaded $= \dfrac{1.5116\ldots}{6} \times 100 = 25.19\ldots\%$, so the estimate is 25.2%.

FINAL ANSWER

(a) 4.845.

(b) 25.2%.

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Question 7

9 marks | Applications of Differentiation

7. The curve C has equation

$$y = \frac{12x^3(x-7) + 14x(13x-15)}{21\sqrt{x}} \qquad x > 0$$

(a) Write the equation of C in the form

$$y = ax^{\frac{7}{2}} + bx^{\frac{5}{2}} + cx^{\frac{3}{2}} + dx^{\frac{1}{2}}$$

where a, b, c and d are fully simplified constants.

(3)

The curve C has three turning points.

Using calculus,

(b) show that the x coordinates of the three turning points satisfy the equation

$$2x^3 - 10x^2 + 13x - 5 = 0$$

(3)

Given that the x coordinate of one of the turning points is 1

(c) find, using algebra, the exact x coordinates of the other two turning points.

(*Solutions based entirely on calculator technology are not acceptable.*)

(3)

WORKED SOLUTION

Question 7 · WMA12/01 October 2022

1 **Expand the numerator**

$12x^3(x-7) + 14x(13x-15) = 12x^4 - 84x^3 + 182x^2 - 210x$.

2 **Divide by 21 root x**

Since $\sqrt{x} = x^{1/2}$, divide each term by $21x^{1/2}$.

3 **Write in the required form**

$y = \frac{12}{21}x^{7/2} - \frac{84}{21}x^{5/2} + \frac{182}{21}x^{3/2} - \frac{210}{21}x^{1/2} = \frac{4}{7}x^{7/2} - 4x^{5/2} + \frac{26}{3}x^{3/2} - 10x^{1/2}$.

4 **Differentiate**

$\frac{dy}{dx} = 2x^{5/2} - 10x^{3/2} + 13x^{1/2} - 5x^{-1/2}$.

5 **Factor the derivative**

$\frac{dy}{dx} = x^{-1/2}(2x^3 - 10x^2 + 13x - 5)$.

6 **Use the turning point condition**

At a turning point, $\frac{dy}{dx} = 0$. Since $x > 0$, $x^{-1/2} \neq 0$, so the x coordinates satisfy $2x^3 - 10x^2 + 13x - 5 = 0$.

7 **Use the known root**

One root is $x = 1$, so divide $2x^3 - 10x^2 + 13x - 5$ by $(x - 1)$.

8 **Factor the cubic**

$2x^3 - 10x^2 + 13x - 5 = (x - 1)(2x^2 - 8x + 5)$.

9 **Solve the quadratic**

$2x^2 - 8x + 5 = 0$ gives $x = \frac{8 \pm \sqrt{64 - 40}}{4} = \frac{8 \pm 2\sqrt{6}}{4} = 2 \pm \frac{\sqrt{6}}{2}$.

✓ **FINAL ANSWER**

(a) $y = \frac{4}{7}x^{7/2} - 4x^{5/2} + \frac{26}{3}x^{3/2} - 10x^{1/2}$.

(b) Proven.

(c) $x = 2 - \frac{\sqrt{6}}{2}$ and $x = 2 + \frac{\sqrt{6}}{2}$.

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Question 8

7 marks | Geometric Sequences

8. A geometric sequence has first term a and common ratio r

Given that $S_\infty = 3a$

(a) show that $r = \frac{2}{3}$

(2)

Given also that

$$u_2 - u_4 = 16$$

where u_k is the k^{th} term of this sequence,

(b) find the value of S_{10} giving your answer to one decimal place.

(5)

WORKED SOLUTION

Question 8 · WMA12/01 October 2022

1 Use the sum to infinity formula

For a geometric sequence, $S_\infty = \frac{a}{1-r}$. The question gives $S_\infty = 3a$.

2 Find the common ratio

$\frac{a}{1-r} = 3a$. Since $u_2 - u_4 = 16$, $a \neq 0$, so $\frac{1}{1-r} = 3$ and $r = \frac{2}{3}$.

3 Use u2 minus u4

$u_2 - u_4 = ar - ar^3 = ar(1 - r^2) = 16$.

4 Substitute r

$a\left(\frac{2}{3}\right)\left(1 - \frac{4}{9}\right) = 16$.

5 Solve for a

$a \cdot \frac{2}{3} \cdot \frac{5}{9} = 16$, so $\frac{10a}{27} = 16$ and $a = \frac{216}{5} = 43.2$.

6 Find S10

$S_{10} = \frac{a(1 - r^{10})}{1 - r} = 3a\left(1 - \left(\frac{2}{3}\right)^{10}\right)$.

7 **Evaluate**

$S_{10} = 3\left(\frac{216}{5}\right)\left(1-\left(\frac{2}{3}\right)^{10}\right) = \frac{92840}{729} = 127.352\ldots$, so $S_{10} = 127$ to 3 significant figures.

FINAL ANSWER

(a) $a = 43.2$.

(b) $S_{10} = 127.353\ldots$, so 127 to 3 s.f.

Question 9

12 marks | Circles

9. **In this question you must show detailed reasoning.**

Solutions relying entirely on calculator technology are not acceptable.

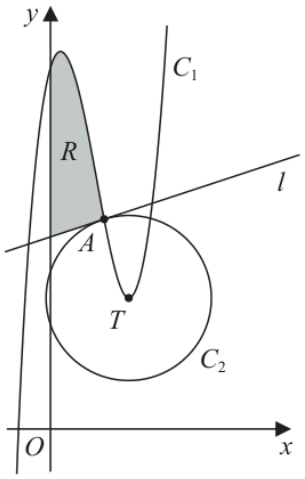


Figure 3

Figure 3 shows

- the curve C_1 with equation $y = x^3 - 5x^2 + 3x + 14$
- the circle C_2 with centre T

The point T is the minimum turning point of C_1

Using Figure 3 and calculus,

(a) find the coordinates of T

(3)

The curve C_1 intersects the circle C_2 at the point A with x coordinate 2

(b) Find an equation of the circle C_2

(3)

The line l shown in Figure 3, is the tangent to circle C_2 at A

(c) Show that an equation of l is

$$y = \frac{1}{3}x + \frac{22}{3}$$

(3)

The region R, shown shaded in Figure 3, is bounded by C_1, l and the y-axis.

(d) Find the exact area of R.

(3)

1 **Differentiate the cubic**

For $y = x^3 - 5x^2 + 3x + 14$, $\dfrac{dy}{dx} = 3x^2 - 10x + 3$.

2 **Find stationary x values**

$3x^2 - 10x + 3 = 0$ factors as $(3x - 1)(x - 3) = 0$, so $x = \dfrac{1}{3}$ or $x = 3$.

3 **Identify the minimum**

$\dfrac{d^2y}{dx^2} = 6x - 10$. At $x = 3$, this is positive, so the minimum turning point has $x = 3$.

4 **Find T**

Substitute $x = 3$: $y = 27 - 45 + 9 + 14 = 5$. Hence $T = (3, 5)$.

5 **Find A**

At $x = 2$ on C_1, $y = 8 - 20 + 6 + 14 = 8$, so $A = (2, 8)$.

6 **Find the circle radius**

The centre of C_2 is $T = (3, 5)$ and it passes through $A = (2, 8)$. Thus $r^2 = (2 - 3)^2 + (8 - 5)^2 = 10$.

7 **Write the circle equation**

Therefore C_2 has equation $(x - 3)^2 + (y - 5)^2 = 10$.

8 **Find the tangent gradient**

The radius TA has gradient $\dfrac{8 - 5}{2 - 3} = -3$, so the tangent at A has gradient $\dfrac{1}{3}$.

9 **Write the tangent equation**

Using $A = (2, 8)$, $y - 8 = \dfrac{1}{3}(x - 2)$, so $y = \dfrac{1}{3}x + \dfrac{22}{3}$.

10 **Set up the area**

From $x = 0$ to $x = 2$, the cubic is above the line. The required area is $\displaystyle\int_0^2 \left[x^3 - 5x^2 + 3x + 14 - \left(\frac{1}{3}x + \frac{22}{3}\right)\right] dx$.

11 **Simplify the integrand**

The integrand is $x^3 - 5x^2 + \dfrac{8}{3}x + \dfrac{20}{3}$.

12 **Integrate**

$$\left[\frac{x^4}{4}-\frac{5x^3}{3}+\frac{4x^2}{3}+\frac{20x}{3}\right]_0^2 = 4-\frac{40}{3}+\frac{16}{3}+\frac{40}{3}=\frac{28}{3}.$$

✓ **FINAL ANSWER**

$(a)\ T=(3,5).$

$(b)\ (x-3)^2+(y-5)^2=10.$

$(c)\ y=\frac{1}{3}x+\frac{22}{3}.$

$(d)\ \frac{28}{3}.$

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Question 10

7 marks | Laws of Logarithms

10. Given $a = \log_2 3$

(i) write, in simplest form, in terms of a,

(a) $\log_2 9$

(b) $\log_2\left(\frac{\sqrt{3}}{16}\right)$

(3)

(ii) Solve

$$3^x \times 2^{x+4} = 6$$

giving your answer, in simplest form, in terms of a.

(4)

WORKED SOLUTION

Question 10 · WMA12/01 October 2022

1 Use the power rule for log 2 of 9

$\log_2 9 = \log_2(3^2) = 2\log_2 3 = 2a$.

2 Split the second logarithm

$\log_2\left(\frac{\sqrt{3}}{16}\right) = \log_2(3^{1/2}) - \log_2(16)$.

3 Simplify in terms of a

$\log_2(3^{1/2}) = \frac{1}{2}\log_2 3 = \frac{a}{2}$ and $\log_2(16) = 4$, so the expression is $\frac{a}{2} - 4$.

4 Take log base 2 of both sides

For $3^x \times 2^{x+4} = 6$, take $\log_2$ of both sides: $\log_2(3^x) + \log_2(2^{x+4}) = \log_2 6$.

5 Rewrite each term

$\log_2(3^x) = xa$, $\log_2(2^{x+4}) = x + 4$, and $\log_2 6 = \log_2(2 \cdot 3) = 1 + a$.

6 Solve for x

So $xa + x + 4 = a + 1$. Hence $x(a+1) = a - 3$, giving $x = \frac{a-3}{a+1}$.

FINAL ANSWER

$(i)(a)\ 2a.$\quad $(i)(b)\ \dfrac{a}{2}-4.$\quad $(ii)\ x=\dfrac{a-3}{a+1}.$