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Elite Mathematics
Edexcel International A Level

IAL PURE 2 · WMA12

WMA12/01 October 2021 Worked Solutions

October 2021 paper rebuilt with the worked solution after each question.

10	2019–2026	Single-paper solutions
paper questions	19 exam sessions	checked by Dr Eslam



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PAST PAPER

WMA12/01 October 2021

October 2021

10	75	1 h 30
questions	marks	exam time

Each question is followed by its worked solution and final answer.

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Question 1

6 marks | Binomial Expansion

1. The first three terms, in ascending powers of x, of the binomial expansion of $(1+kx)^{16}$ are

$$1, \quad -4x \quad \text{and} \quad px^2$$

where k and p are constants.

(a) Find, in simplest form,

(i) the value of k

(ii) the value of p

(3)

$$g(x) = \left(2 + \frac{16}{x}\right)(1+kx)^{16}$$

Using the value of k found in part (a),

(b) find the term in x^2 in the expansion of $g(x)$.

(3)

WORKED SOLUTION

Question 1 · WMA12/01 October 2021

1 Compare the linear term

In $(1+kx)^{16}$, the coefficient of x is $16k$.

2 Find k

Since the coefficient of x is -4, $16k = -4$, so $k = -\frac{1}{4}$.

3 Find p

The coefficient of x^2 is $\binom{16}{2}k^2 = 120\left(\frac{1}{16}\right) = \frac{15}{2}$.

4 Find the needed binomial terms

For $k = -\frac{1}{4}$, the x^2 coefficient in $(1+kx)^{16}$ is $\frac{15}{2}$ and the x^3 coefficient is $\binom{16}{3}\left(-\frac{1}{4}\right)^3 = -\frac{35}{4}$.

5 **Build the x squared term in g**

In $g(x) = \left(2 + \frac{16}{x}\right)(1 + kx)^{16}$, the x^2 term comes from $2\left(\frac{15}{2}x^2\right)$ and $\frac{16}{x}\left(-\frac{35}{4}x^3\right)$.

6 **Simplify**

The required term is $15x^2 - 140x^2 = -125x^2$.

✓ **FINAL ANSWER**

$(a)(i)\ k = -\frac{1}{4}.$ $(a)(ii)\ p = \frac{15}{2}.$ $(b)\ -125x^2.$

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Question 2

5 marks | Sequences & Series

2. A sequence is defined by

$$u_1 = 6$$
$$u_{n+1} = ku_n + 3$$

where k is a positive constant.

(a) Find, in terms of k, an expression for u_3

(2)

Given that $\sum_{n=1}^{3} u_n = 117$

(b) find the value of k.

(3)

WORKED SOLUTION

Question 2 · WMA12/01 October 2021

1 Find u two

$u_2 = ku_1 + 3 = 6k + 3$.

2 Find u three

$u_3 = ku_2 + 3 = k(6k + 3) + 3 = 6k^2 + 3k + 3$.

3 Use the given sum

$u_1 + u_2 + u_3 = 117$, so $6 + (6k + 3) + (6k^2 + 3k + 3) = 117$.

4 Simplify the equation

$6k^2 + 9k + 12 = 117$, hence $2k^2 + 3k - 35 = 0$.

5 Solve for k

$(2k - 7)(k + 5) = 0$. Since k is positive, $k = \frac{7}{2}$.

FINAL ANSWER

$(a)\ u_3 = 6k^2 + 3k + 3$. $(b)\ k = \frac{7}{2}$.

Question 3

8 marks | Integration

3.

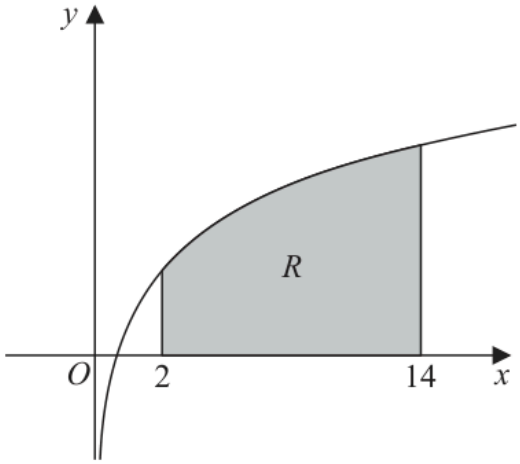


Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = \log_{10} x$

The region R, shown shaded in Figure 1, is bounded by the curve, the line with equation $x = 2$, the x-axis and the line with equation $x = 14$

Using the trapezium rule with four strips of equal width,

(a) show that the area of R is approximately 10.10

(3)

(b) Explain how the trapezium rule could be used to obtain a more accurate estimate for the area of R.

(1)

(c) Using the answer to part (a) and making your method clear, estimate the value of

(i) $\displaystyle\int_2^{14} \log_{10} \sqrt{x}\, \mathrm{d}x$

(ii) $\displaystyle\int_2^{14} \log_{10} 100x^3\, \mathrm{d}x$

(4)

1 Find the strip width

Four equal strips from $x = 2$ to $x = 14$ have width $h = \dfrac{14 - 2}{4} = 3$.

2 List the ordinates

Use $x = 2, 5, 8, 11, 14$ and $y = \log_{10} x$.

3 Apply the trapezium rule

$$A \approx \frac{3}{2}\{\log_{10} 2 + \log_{10} 14 + 2(\log_{10} 5 + \log_{10} 8 + \log_{10} 11)\} = 10.10\ldots.$$

4 Improve the estimate

Use more strips of smaller width to make the trapezium rule estimate more accurate.

5 Estimate the square-root integral

$\log_{10} \sqrt{x} = \dfrac{1}{2}\log_{10} x$, so $\displaystyle\int_2^{14} \log_{10} \sqrt{x}\, dx \approx \frac{1}{2}(10.10) = 5.05$.

6 Rewrite the second logarithm

$\log_{10}(100x^3) = \log_{10} 100 + \log_{10} x^3 = 2 + 3\log_{10} x$.

7 Estimate the second integral

$\displaystyle\int_2^{14} (2 + 3\log_{10} x)\, dx \approx 2(12) + 3(10.10) = 54.30$.

FINAL ANSWER

(a) 10.10. (b) Use more strips.

$(c)(i)$ 5.05. $(c)(ii)$ 54.30.

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Question 4

8 marks | Polynomials

4. $f(x) = (x^2 - 2)(2x - 3) - 21$

(a) State the value of the remainder when $f(x)$ is divided by $(2x - 3)$

(1)

(b) Use the factor theorem to show that $(x - 3)$ is a factor of $f(x)$

(2)

(c) Hence,

(i) factorise $f(x)$

(ii) show that the equation $f(x) = 0$ has only one real root.

(5)

WORKED SOLUTION

Question 4 · WMA12/01 October 2021

1 Use the remainder theorem

Dividing by $(2x - 3)$ means use $x = \frac{3}{2}$.

2 Find the remainder

$f\left(\frac{3}{2}\right) = \left(\frac{9}{4} - 2\right)(0) - 21 = -21.$

3 Check x equals three

$f(3) = (9 - 2)(6 - 3) - 21 = 7 \cdot 3 - 21 = 0.$

4 Use the factor theorem

Since $f(3) = 0$, $(x - 3)$ is a factor of $f(x)$.

5 Expand f

$f(x) = (x^2 - 2)(2x - 3) - 21 = 2x^3 - 3x^2 - 4x - 15.$

6 Factorise

Dividing by $(x - 3)$ gives $2x^2 + 3x + 5$, so $f(x) = (x - 3)(2x^2 + 3x + 5)$.

7 **Check the quadratic roots**

For $2x^2 + 3x + 5$, the discriminant is $3^2 - 4(2)(5) = 9 - 40 = -31 < 0$.

8 **Conclude**

The quadratic has no real roots, so $f(x) = 0$ has only one real root, $x = 3$.

✓ **FINAL ANSWER**

$(a)\ -21$. (b) Proven.

$(c)(i)\ f(x) = (x - 3)(2x^2 + 3x + 5)$. $(c)(ii)$ Only real root: $x = 3$.

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Question 5

6 marks | Arithmetic Sequences

5. A company that owned a silver mine

- extracted 480 tonnes of silver from the mine in year 1
- extracted 465 tonnes of silver from the mine in year 2
- extracted 450 tonnes of silver from the mine in year 3

and so on, forming an arithmetic sequence.

(a) Find the mass of silver extracted in year 14

(2)

After a total of 7770 tonnes of silver was extracted, the company stopped mining.

Given that this occurred at the end of year N,

(b) show that

$$N^2 - 65N + 1036 = 0$$

(3)

(c) Hence, state the value of N.

(1)

WORKED SOLUTION

Question 5 · WMA12/01 October 2021

1 Identify the arithmetic sequence

The first term is 480 and the common difference is $465 - 480 = -15$.

2 Find year fourteen

$u_{14} = 480 + 13(-15) = 285$, so 285 tonnes were extracted in year 14.

3 Set up the total

The total after N years is $S_N = \frac{N}{2}\{2(480) + (N-1)(-15)\}$.

4 Use the total extraction

$\frac{N}{2}(975 - 15N) = 7770$.

5 **Show the quadratic**

Multiplying by 2 and rearranging gives $15N^2 - 975N + 15540 = 0$, so $N^2 - 65N + 1036 = 0$.

6 **Solve and choose N**

$N = 28$ or 37. Since extraction amounts would become negative before year 37, the stopping year is $N = 28$.

✓ **FINAL ANSWER**

(a) 285 tonnes.

(b) Proven: $N^2 - 65N + 1036 = 0$.

(c) $N = 28$.

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Question 6

8 marks | Circles

6. (i) The circle C_1 has equation

$$x^2 + y^2 + 10x - 12y = k \qquad \text{where } k \text{ is a constant}$$

(a) Find the coordinates of the centre of C_1

(2)

(b) State the possible range in values for k.

(2)

(ii) The point $P(p, 0)$, the point $Q(-2, 10)$ and the point $R(8, -14)$ lie on a different circle, C_2

Given that

- p is a positive constant
- QR is a diameter of C_2

find the exact value of p.

(4)

WORKED SOLUTION

Question 6 · WMA12/01 October 2021

1 Complete the square

$x^2 + 10x = (x+5)^2 - 25$ and $y^2 - 12y = (y-6)^2 - 36$.

2 Write centre-radius form

The circle becomes $(x+5)^2 + (y-6)^2 = k + 61$.

3 Read the centre

The centre is $(-5, 6)$.

4 State the range for k

For a circle with positive radius, $k + 61 > 0$, so $k > -61$.

5 Use the diameter property

If QR is a diameter and P lies on the circle, then $\angle QPR = 90°$.

6 **Use perpendicular vectors**

So $(Q-P)\cdot(R-P)=0$. With $P=(p,0)$, $Q=(-2,10)$, and $R=(8,-14)$, this gives $(-2-p,10)\cdot(8-p,-14)=0$.

7 **Form the quadratic**

$(-2-p)(8-p)-140=0$, so $p^2-6p-156=0$.

8 **Choose the positive value**

$p=\dfrac{6\pm\sqrt{660}}{2}=3\pm\sqrt{165}$. Since p is positive, $p=3+\sqrt{165}$.

✓ **FINAL ANSWER**

$(i)(a)\ (-5,6).\quad (i)(b)\ k>-61.\quad (ii)\ p=3+\sqrt{165}.$

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Question 7

10 marks | Geometric Sequences

7. (i) A geometric sequence has first term 4 and common ratio 6
Given that the n^{th} term is greater than 10^{100}, find the minimum possible value of n.

(3)

(ii) A different geometric sequence has first term a and common ratio r.

Given that

- the second term of the sequence is -6
- the sum to infinity of the series is 25

(a) show that

$$25r^2 - 25r - 6 = 0$$

(3)

(b) Write down the solutions of

$$25r^2 - 25r - 6 = 0$$

(1)

Hence,

(c) state the value of r, giving a reason for your answer,

(1)

(d) find the sum of the first 4 terms of the series.

(2)

WORKED SOLUTION

Question 7 · WMA12/01 October 2021

1 **Write the nth term**

The nth term of the first sequence is $4 \cdot 6^{n-1}$.

2 **Use logarithms**

We need $4 \cdot 6^{n-1} > 10^{100}$, so $n - 1 > \dfrac{100 - \log_{10} 4}{\log_{10} 6} = 127.736\ldots$.

3 **Choose the smallest integer**

Thus n must be greater than $128.736\ldots$, so the minimum possible value is 129.

4 **Use the second term**

For the second sequence, $ar = -6$, so $a = -\frac{6}{r}$.

5 **Use the sum to infinity**

$\frac{a}{1-r} = 25$. Substituting $a = -\frac{6}{r}$ gives $-6 = 25r(1-r)$.

6 **Show the quadratic**

Rearranging gives $25r^2 - 25r - 6 = 0$.

7 **Solve for r**

$25r^2 - 25r - 6 = 0$ gives $r = \frac{6}{5}$ or $r = -\frac{1}{5}$.

8 **Choose the valid ratio**

A sum to infinity exists only when $|r| < 1$, so $r = -\frac{1}{5}$.

9 **Find the first term**

Since $ar = -6$, $a = 30$.

10 **Sum four terms**

$S_4 = 30\frac{1-(-1/5)^4}{1-(-1/5)} = \frac{624}{25}$.

✓ **FINAL ANSWER**

$(i)\ n = 129.$ $(ii)(a)$ Proven.

$(ii)(b)\ r = \frac{6}{5}$ or $-\frac{1}{5}.$ $(ii)(c)\ r = -\frac{1}{5}.$ $(ii)(d)\ \frac{624}{25}.$

Question 8

10 marks | Applications of Differentiation

8. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

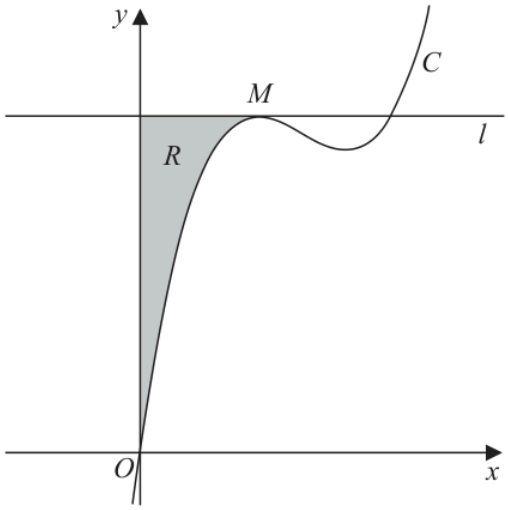


Figure 2

Figure 2 shows a sketch of part of the curve C with equation

$$y = \frac{4}{3}x^3 - 11x^2 + kx \qquad \text{where } k \text{ is a constant}$$

The point M is the maximum turning point of C and is shown in Figure 2.

Given that the x coordinate of M is 2

(a) show that $k = 28$

(3)

(b) Determine the range of values of x for which y is increasing.

(2)

The line l passes through M and is parallel to the x-axis.

The region R, shown shaded in Figure 2, is bounded by the curve C, the line l and the y-axis.

(c) Find, by algebraic integration, the exact area of R.

(5)

1 **Differentiate the curve**

For $y = \frac{4}{3}x^3 - 11x^2 + kx$, $\frac{dy}{dx} = 4x^2 - 22x + k$.

2 **Use the turning point**

At the turning point M, $x = 2$ and $\frac{dy}{dx} = 0$.

3 **Show k**

$4(2)^2 - 22(2) + k = 0$, so $16 - 44 + k = 0$ and $k = 28$.

4 **Factor the derivative**

With $k = 28$, $\frac{dy}{dx} = 4x^2 - 22x + 28 = 2(2x - 7)(x - 2)$.

5 **Find where y is increasing**

The derivative is positive outside its roots, so y is increasing for $x < 2$ or $x > \frac{7}{2}$.

6 **Find the height of the line**

At $x = 2$, $y = \frac{4}{3}(8) - 11(4) + 28(2) = \frac{68}{3}$. Hence line l has equation $y = \frac{68}{3}$.

7 **Set up the shaded area**

The region is between the line and the curve from $x = 0$ to $x = 2$, so

$A = \int_0^2 \left(\frac{68}{3} - \left(\frac{4}{3}x^3 - 11x^2 + 28x \right) \right) dx$.

8 **Integrate**

$A = \left[\frac{68}{3}x - \frac{1}{3}x^4 + \frac{11}{3}x^3 - 14x^2 \right]_0^2$.

9 **Evaluate exactly**

Substituting $x = 2$ gives $\frac{136}{3} - \frac{16}{3} + \frac{88}{3} - 56 = \frac{40}{3}$.

✓ **FINAL ANSWER**

(a) Proven: $k = 28$.

(b) $x < 2$ or $x > \frac{7}{2}$. (c) $\frac{40}{3}$.

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Question 9

4 marks | Proof

9. (a) Prove that for all positive values of x and y,

$$\frac{x+y}{2} \geqslant \sqrt{xy}$$

(3)

(b) Prove by counter-example that this inequality does not hold when x and y are both negative.

(1)

WORKED SOLUTION

Question 9 · WMA12/01 October 2021

1 Start with a square

For positive x and y, both $\sqrt{x}$ and $\sqrt{y}$ are real, so $(\sqrt{x}-\sqrt{y})^2 \geq 0$.

2 Expand

$(\sqrt{x}-\sqrt{y})^2 = x - 2\sqrt{xy} + y \geq 0$.

3 Rearrange

Thus $x+y \geq 2\sqrt{xy}$, and dividing by 2 gives $\dfrac{x+y}{2} \geq \sqrt{xy}$.

4 Give a negative counterexample

Take $x=-1$ and $y=-1$. Then $\dfrac{x+y}{2} = -1$, while $\sqrt{xy} = \sqrt{1} = 1$.

5 Conclude

Since $-1 \geq 1$ is false, the inequality does not hold for all negative x and y.

✓ **FINAL ANSWER**

(a) Proven.

(b) For example, $x=y=-1$ gives $-1 \not\geqslant 1$.

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Question 10

10 marks | Trigonometric Equations

10. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(i) Solve, for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$$\tan^2\left(2x + \frac{\pi}{4}\right) = 3$$

(5)

(ii) Solve, for $0 < \theta < 360°$

$$(2\sin\theta - \cos\theta)^2 = 1$$

giving your answers, as appropriate, to one decimal place.

(5)

(Total 10 marks)

WORKED SOLUTION

Question 10 · WMA12/01 October 2021

1 **Take the square root**

Let $u = 2x + \frac{\pi}{4}$. From $\tan^2 u = 3$, $\tan u = \pm\sqrt{3}$.

2 **Write the angle families**

Thus $u = \frac{\pi}{3} + n\pi$ or $u = -\frac{\pi}{3} + n\pi$, where n is an integer.

3 **Return to x**

So $x = \frac{\pi}{24} + \frac{n\pi}{2}$ or $x = -\frac{7\pi}{24} + \frac{n\pi}{2}$.

4 **Use the interval**

For $-\frac{\pi}{2} < x < \frac{\pi}{2}$, the solutions are $x = -\frac{11\pi}{24}, -\frac{7\pi}{24}, \frac{\pi}{24}, \frac{5\pi}{24}$.

5 **Expand the squared equation**

$(2\sin\theta - \cos\theta)^2 = 1$ gives $4\sin^2\theta - 4\sin\theta\cos\theta + \cos^2\theta = 1$.

6 **Use the identity**

Since $1 = \sin^2\theta + \cos^2\theta$, this simplifies to $3\sin^2\theta - 4\sin\theta\cos\theta = 0$.

7 **Factor**

$\sin\theta(3\sin\theta - 4\cos\theta) = 0$.

8 **Solve the two cases**

$\sin\theta = 0$ gives $\theta = 180°$ in the interval. Also $3\sin\theta = 4\cos\theta$ gives $\tan\theta = \frac{4}{3}$.

9 **State theta**

For $0 < \theta < 360°$, $\tan\theta = \frac{4}{3}$ gives $\theta = 53.1°$ and $233.1°$. Therefore $\theta = 53.1°, 180°, 233.1°$.

✓ **FINAL ANSWER**

$(i)\ x = -\frac{11\pi}{24},\ -\frac{7\pi}{24},\ \frac{\pi}{24},\ \frac{5\pi}{24}.$ $(ii)\ \theta = 53.1°,\ 180°,\ 233.1°.$