Elite Mathematics
Edexcel International A Level

IAL PURE 2 · WMA12

WMA12/01 January 2020 Worked Solutions

January 2020 paper rebuilt with the worked solution after each question.

10	2019–2026	Single-paper solutions
paper questions	19 exam sessions	checked by Dr Eslam



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PAST PAPER

WMA12/01 January 2020

January 2020

10	75	1 h 30
questions	marks	exam time

Each question is followed by its worked solution and final answer.

WMA12/01 JANUARY 2020

Question 1

7 marks | Integration

1. The table below shows corresponding values of x and y for $y = \log_2(2x)$

The values of y are given to 2 decimal places as appropriate.

x	2	5	8	11	14
y	2	3.32	4	4.46	4.81

Using the trapezium rule with all the values of y in the given table,

(a) obtain an estimate for $\int_2^{14} \log_2(2x)\,dx$, giving your answer to one decimal place. **(3)**

Using your answer to part (a) and making your method clear, estimate

(b) (i) $\int_2^{14} \frac{\log_2(4x^2)}{5}\,dx$

(ii) $\int_2^{14} \log_2\left(\frac{2}{x}\right)dx$ **(4)**

WORKED SOLUTION

Question 1 · WMA12/01 January 2020

1 Use the trapezium rule

The strip width is $h = 3$. Therefore $\int_2^{14} \log_2(2x)\,dx \approx \frac{3}{2}\{2 + 4.81 + 2(3.32 + 4 + 4.46)\}$.

2 Calculate part (a)

The estimate is $\frac{3}{2}(30.37) = 45.555$, so to one decimal place the answer is 45.6.

3 Rewrite part (b)(i)

$\log_2(4x^2) = \log_2\{(2x)^2\} = 2\log_2(2x)$, so $\int_2^{14} \frac{\log_2(4x^2)}{5}\,dx = \frac{2}{5}\int_2^{14} \log_2(2x)\,dx$.

4 Estimate part (b)(i)

Using part (a), the estimate is $\frac{2}{5}(45.555) = 18.222$, so 18.2 to one decimal place.

5 **Rewrite part (b)(ii)**

$\log_2\left(\frac{2}{x}\right) = 2 - \log_2(2x)$.

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6 **Estimate part (b)(ii)**

$\int_2^{14} \log_2\left(\frac{2}{x}\right) dx \approx \int_2^{14} 2\,dx - 45.555 = 24 - 45.555 = -21.555$, so -21.6.

✓ **FINAL ANSWER**

(a) 45.6. $(b)(i)$ 18.2. $(b)(ii)$ -21.6.

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Question 2

7 marks | Binomial Expansion

2. One of the terms in the binomial expansion of $(3 + ax)^6$, where a is a constant, is $540x^4$

(a) Find the possible values of a.

(4)

(b) Hence find the term independent of x in the expansion of

$$\left(\frac{1}{81} + \frac{1}{x^6}\right)(3 + ax)^6$$

(3)

WORKED SOLUTION

Question 2 · WMA12/01 January 2020

1 Identify the x to the fourth term

In $(3+ax)^6$, the x^4 term is $\binom{6}{4}3^2(ax)^4$.

2 Form the equation for a

$\binom{6}{4}3^2a^4x^4 = 15 \cdot 9a^4x^4 = 135a^4x^4$. Since this term is $540x^4$, $135a^4 = 540$.

3 Solve for a

$a^4 = 4$, so the real possible values are $a = \sqrt{2}$ or $a = -\sqrt{2}$.

4 Find the constant contribution from 1 over 81

The constant term from $\frac{1}{81}(3+ax)^6$ is $\frac{1}{81} \cdot 3^6 = 9$.

5 Find the constant contribution from x to the minus six

The x^6 term in $(3+ax)^6$ is a^6x^6. Since $a = \pm\sqrt{2}$, $a^6 = 8$, so $\frac{1}{x^6} \cdot a^6x^6$ contributes 8.

6 Add the constant terms

The term independent of x is $9 + 8 = 17$.

FINAL ANSWER

$(a)\ a = \pm\sqrt{2}.$ $(b)\ 17.$

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Question 3

8 marks | Polynomials

3. $f(x) = 6x^3 + 17x^2 + 4x - 12$

(a) Use the factor theorem to show that $(2x + 3)$ is a factor of $f(x)$.

(2)

(b) Hence, using algebra, write $f(x)$ as a product of three linear factors.

(4)

(c) Solve, for $\frac{\pi}{2} < \theta < \pi$, the equation

$$6\tan^3\theta + 17\tan^2\theta + 4\tan\theta - 12 = 0$$

giving your answers to 3 significant figures.

(2)

WORKED SOLUTION

Question 3 · WMA12/01 January 2020

1 Use the factor theorem

For $(2x + 3)$, use $x = -\frac{3}{2}$. Substituting gives

$$f\left(-\frac{3}{2}\right) = 6\left(-\frac{3}{2}\right)^3 + 17\left(-\frac{3}{2}\right)^2 + 4\left(-\frac{3}{2}\right) - 12 = 0.$$

2 State the factor result

Since $f\left(-\frac{3}{2}\right) = 0$, $(2x + 3)$ is a factor of $f(x)$.

3 Divide by the known factor

$6x^3 + 17x^2 + 4x - 12 = (2x + 3)(3x^2 + 4x - 4)$.

4 Factorise the quadratic

$3x^2 + 4x - 4 = (3x - 2)(x + 2)$, so $f(x) = (2x + 3)(3x - 2)(x + 2)$.

5 Use the factorisation for tan theta

Let $t = \tan\theta$. Then $(2t + 3)(3t - 2)(t + 2) = 0$, so $t = -\frac{3}{2}$, $t = \frac{2}{3}$, or $t = -2$.

6 Choose the roots in the interval

For $\frac{\pi}{2} < \theta < \pi$, $\tan\theta$ is negative, so use $\tan\theta = -\frac{3}{2}$ and $\tan\theta = -2$.

7 **Find theta**

This gives $\theta = \pi - \tan^{-1}\left(\frac{3}{2}\right) = 2.16$ and $\theta = \pi - \tan^{-1}(2) = 2.03$, to 3 significant figures.

FINAL ANSWER

(a) Proven.

(b) $f(x) = (2x+3)(3x-2)(x+2)$. (c) $\theta = 2.03,\ 2.16$ radians.

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Question 4

6 marks | Integration

4.

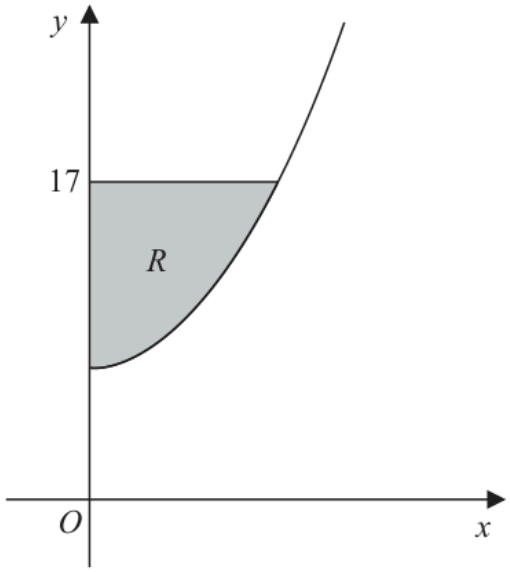


Figure 1

Figure 1 shows a sketch of the curve with equation

$$y = 2x^2 + 7 \qquad x \geqslant 0$$

The finite region R, shown shaded in Figure 1, is bounded by the curve, the y-axis and the line with equation $y = 17$

Find the exact area of R.

(6)

WORKED SOLUTION

Question 4 · WMA12/01 January 2020

1 Find the intersection with y equals 17

The curve is $y = 2x^2 + 7$. Set $2x^2 + 7 = 17$ to get $x^2 = 5$, so the right-hand limit is $x = \sqrt{5}$.

2 Set up the area

The shaded area is between $y = 17$ and $y = 2x^2 + 7$, from $x = 0$ to $x = \sqrt{5}$.

3 Integrate

$A = \int_0^{\sqrt{5}} \{17 - (2x^2 + 7)\}\, dx = \int_0^{\sqrt{5}} (10 - 2x^2)\, dx$.

4 **Evaluate the exact area**

$$A = \left[10x - \frac{2}{3}x^3\right]_0^{\sqrt{5}} = 10\sqrt{5} - \frac{2}{3}(5\sqrt{5}) = \frac{20\sqrt{5}}{3}.$$

✓ **FINAL ANSWER**

$$\frac{20\sqrt{5}}{3}.$$

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Question 5

8 marks | Modelling with Sequences & Series

5. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

A colony of bees is being studied.

The number of bees in the colony at the start of the study was 30 000

Three years after the start of the study, the number of bees in the colony is 34 000

A model predicts that the number of bees in the colony will increase by $p\%$ each year, so that the number of bees in the colony at the end of each year of study forms a geometric sequence.

Assuming the model,

(a) find the value of p, giving your answer to 2 decimal places.

(3)

According to the model, at the end of N years of study the number of bees in the colony exceeds 75 000

(b) Find, showing all steps in your working, the smallest integer value of N.

(5)

WORKED SOLUTION

Question 5 · WMA12/01 January 2020

1 **Set up the growth ratio**

Let the yearly ratio be r. After 3 years, $30000r^3 = 34000$.

2 **Find p**

$r^3 = \frac{17}{15}$, so $r = \left(\frac{17}{15}\right)^{1/3} \approx 1.04257$. Therefore $p = (r-1) \times 100 \approx 4.26$.

3 **Set up the inequality**

After N years the model gives $30000r^N$. We need $30000r^N > 75000$, so $r^N > 2.5$.

4 **Take logarithms**

$N \log r > \log(2.5)$, so $N > \frac{\log(2.5)}{\log r}$.

5 **Find the smallest integer**

Using $r \approx 1.04257$, $\dfrac{\log(2.5)}{\log r} \approx 21.98$. Hence the smallest integer value is $N = 22$.

✓ **FINAL ANSWER**

$(a)\ p = 4.26.$ $(b)\ N = 22.$

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Question 6

8 marks | Circles

6. The circle C has equation

$$x^2 + y^2 + 6x - 4y - 14 = 0$$

(a) Find

(i) the coordinates of the centre of C,

(ii) the exact radius of C.

(3)

The line with equation $y = k$, where k is a constant, is a tangent to C.

(b) Find the possible values of k.

(2)

The line with equation $y = p$, where p is a negative constant, is a chord of C.

Given that the length of this chord is 4 units,

(c) find the value of p.

(3)

WORKED SOLUTION

Question 6 · WMA12/01 January 2020

1 Complete the square

$x^2 + 6x = (x+3)^2 - 9$ and $y^2 - 4y = (y-2)^2 - 4$.

2 Find the centre and radius

The circle becomes $(x+3)^2 + (y-2)^2 = 27$. Hence the centre is $(-3, 2)$ and the radius is $\sqrt{27} = 3\sqrt{3}$.

3 Use horizontal tangents

A horizontal tangent has equation $y = k$. The possible tangent levels are one radius above and below the centre, so $k = 2 \pm 3\sqrt{3}$.

4 Use half the chord length

The chord has length 4, so half the chord is 2. If the chord is $y = p$, its perpendicular distance from the centre is $|p - 2|$.

5 **Apply Pythagoras**

$(p-2)^2+2^2=(3\sqrt{3})^2=27$, so $(p-2)^2=23$.

6 **Choose the negative value**

Since p is negative, $p=2-\sqrt{23}$.

✓ **FINAL ANSWER**

$(a)(i)\ (-3,2).\quad (a)(ii)\ 3\sqrt{3}.\quad (b)\ k=2\pm 3\sqrt{3}.\quad (c)\ p=2-\sqrt{23}.$

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Question 7

7 marks | Trigonometric Equations

7. (a) Show that the equation

$$8\tan\theta = 3\cos\theta$$

may be rewritten in the form

$$3\sin^2\theta + 8\sin\theta - 3 = 0$$

(3)

(b) Hence solve, for $0 \leqslant x \leqslant 90°$, the equation

$$8\tan 2x = 3\cos 2x$$

giving your answers to 2 decimal places.

(4)

WORKED SOLUTION

Question 7 · WMA12/01 January 2020

1 Rewrite tan theta

$8\tan\theta = 3\cos\theta$ becomes $8\dfrac{\sin\theta}{\cos\theta} = 3\cos\theta$.

2 Clear the denominator

Multiplying by $\cos\theta$ gives $8\sin\theta = 3\cos^2\theta$.

3 Use the identity

Since $\cos^2\theta = 1 - \sin^2\theta$, $8\sin\theta = 3(1 - \sin^2\theta)$, so $3\sin^2\theta + 8\sin\theta - 3 = 0$.

4 Apply the result to 2x

For $8\tan 2x = 3\cos 2x$, we get $3\sin^2(2x) + 8\sin(2x) - 3 = 0$.

5 Solve for sin 2x

$(3\sin 2x - 1)(\sin 2x + 3) = 0$. Since $\sin 2x$ cannot be -3, $\sin 2x = \dfrac{1}{3}$.

6 Find x

For $0 \leq x \leq 90°$, $0 \leq 2x \leq 180°$. Thus $2x = 19.471\ldots°$ or $160.529\ldots°$, so $x = 9.74°$ or $80.26°$.

FINAL ANSWER

(a) Proven.

(b) $x = 9.74°,\ 80.26°$.

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Question 8

7 marks | Sequences & Series

8. (i) An arithmetic series has first term a and common difference d.

Prove that the sum to n terms of this series is

$$\frac{n}{2}\{2a + (n-1)d\}$$

(3)

(ii) A sequence $u_1, u_2, u_3, \ldots$ is given by

$$u_n = 5n + 3(-1)^n$$

Find the value of

(a) u_5

(1)

(b) $\sum_{n=1}^{59} u_n$

(3)

WORKED SOLUTION

Question 8 · WMA12/01 January 2020

1 Write the series forwards

$S_n = a + (a+d) + (a+2d) + \cdots + (a+(n-1)d)$.

2 Write the series backwards

$S_n = (a+(n-1)d) + (a+(n-2)d) + \cdots + a$.

3 Add the two forms

Each pair adds to $2a + (n-1)d$, and there are n pairs. Hence $2S_n = n\{2a + (n-1)d\}$.

4 Complete the proof

Dividing by 2 gives $S_n = \frac{n}{2}\{2a + (n-1)d\}$.

5 Find u5

$u_5 = 5(5) + 3(-1)^5 = 25 - 3 = 22$.

6 **Sum the sequence**

$\sum_{n=1}^{59} u_n = 5\sum_{n=1}^{59} n + 3\sum_{n=1}^{59}(-1)^n$.

7 **Calculate the total**

$5\sum_{n=1}^{59} n = 5 \cdot \dfrac{59 \cdot 60}{2} = 8850$, and $\sum_{n=1}^{59}(-1)^n = -1$. Therefore the total is $8850 - 3 = 8847$.

✓ **FINAL ANSWER**

(i) Proven.

$(ii)(a)\ 22.$ $(ii)(b)\ 8847$.

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Question 9

7 marks | Laws of Logarithms

9. (a) Sketch the curve with equation

$$y = 3 \times 4^x$$

showing the coordinates of any points of intersection with the coordinate axes.

(2)

The curve with equation $y = 6^{1-x}$ meets the curve with equation $y = 3 \times 4^x$ at the point P.

(b) Show that the x coordinate of P is $\dfrac{\log_{10} 2}{\log_{10} 24}$

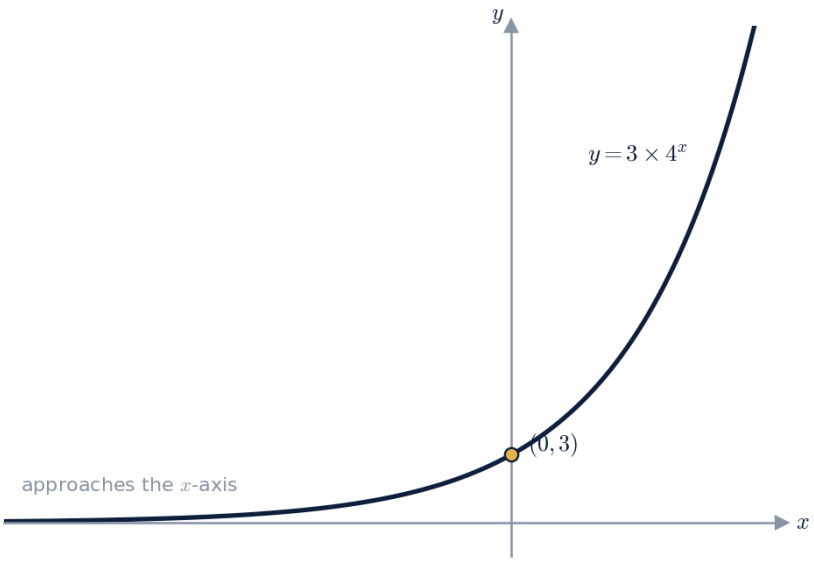
(5)

WORKED SOLUTION

Question 9 · WMA12/01 January 2020

1 Describe the sketch

$y = 3 \times 4^x$ is an increasing exponential curve. At $x = 0$, $y = 3$, so the curve crosses the y-axis at $(0, 3)$ and has no x-axis intersection.



2 Equate the two curves

At P, $6^{1-x} = 3 \times 4^x$.

3 Take logarithms

$(1 - x)\log_{10} 6 = \log_{10} 3 + x \log_{10} 4$.

4 **Collect the x terms**

$\log_{10} 6 - \log_{10} 3 = x(\log_{10} 4 + \log_{10} 6)$.

5 **Simplify the logarithms**

$\log_{10} 6 - \log_{10} 3 = \log_{10} 2$ and $\log_{10} 4 + \log_{10} 6 = \log_{10} 24$.

6 **State the x coordinate**

Therefore $x = \dfrac{\log_{10} 2}{\log_{10} 24}$.

✓ **FINAL ANSWER**

(a) Increasing exponential curve through $(0, 3)$, with no x-axis intersection.

(b) $x = \dfrac{\log_{10} 2}{\log_{10} 24}$.

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Question 10

10 marks | Applications of Differentiation

10. A curve C has equation

$$y = 4x^3 - 9x + \frac{k}{x} \qquad x > 0$$

where k is a constant.

The point P with x coordinate $\frac{1}{2}$ lies on C.

Given that P is a stationary point of C,

(a) show that $k = -\frac{3}{2}$

(4)

(b) Determine the nature of the stationary point at P, justifying your answer.

(2)

The curve C has a second stationary point.

(c) Using algebra, find the x coordinate of this second stationary point.

(4)

(Total 10 marks)

WORKED SOLUTION

Question 10 · WMA12/01 January 2020

1 Differentiate the curve

Write $y = 4x^3 - 9x + kx^{-1}$. Then $\frac{dy}{dx} = 12x^2 - 9 - kx^{-2}$.

2 Use the stationary point

At $x = \frac{1}{2}$, $\frac{dy}{dx} = 0$, so $0 = 12\left(\frac{1}{2}\right)^2 - 9 - k\left(\frac{1}{2}\right)^{-2} = 3 - 9 - 4k$.

3 Find k

$-6 - 4k = 0$, hence $k = -\frac{3}{2}$.

4 Use the second derivative

With $k = -\frac{3}{2}$, $\frac{dy}{dx} = 12x^2 - 9 + \frac{3}{2x^2}$ and $\frac{d^2y}{dx^2} = 24x - 3x^{-3}$.

5 **Determine the nature of P**

At $x = \frac{1}{2}$, $\frac{d^2y}{dx^2} = 12 - 24 = -12 < 0$, so P is a local maximum.

6 **Find the other stationary point**

Set $12x^2 - 9 + \frac{3}{2x^2} = 0$. Multiplying by $2x^2$ gives $24x^4 - 18x^2 + 3 = 0$, or $8x^4 - 6x^2 + 1 = 0$.

7 **Solve the quadratic in x squared**

$(4x^2 - 1)(2x^2 - 1) = 0$. The given point has $x = \frac{1}{2}$, so the second stationary point has $x^2 = \frac{1}{2}$.

8 **State the positive x coordinate**

Since $x > 0$, the second stationary point has $x = \frac{1}{\sqrt{2}}$.

✓ **FINAL ANSWER**

(a) $k = -\frac{3}{2}$. (b) local maximum.

(c) $x = \frac{1}{\sqrt{2}}$.