EA ELITE IAL MATHEMATICS

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WMA11 Pure 1 Expertise With Answers

Questions 6 and above with worked-solution pages and cross-topic placements.

109
topic placements

WMA11
Pure 1

Question bank with worked answers
standalone IAL route

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Prepared for Dr Eslam Ahmed - eliteigcse.com

P1

TOPIC

Indices & Surds

Question 6

6. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

(a) Expand and simplify

$$\left(r - \frac{1}{r}\right)^2$$

(2)

(b) Express $\dfrac{1}{3 + 2\sqrt{2}}$ in the form $p + q\sqrt{2}$ where p and q are integers.

(2)

(c) Use the results of parts (a) and (b), or otherwise, to show that

$$\sqrt{3 + 2\sqrt{2}} - \frac{1}{\sqrt{3 + 2\sqrt{2}}} = 2$$

(3)

(Total for Question 6 is 7 marks)

WMA11/01 MAY/JUNE 2023

Worked Solution - Question 6

Indices & Surds

Topic group

1. Expand the square

$$\left(r-\frac{1}{r}\right)^2=r^2-2+\frac{1}{r^2}.$$

2. Rationalise the denominator

$$\frac{1}{3+2\sqrt{2}}\times\frac{3-2\sqrt{2}}{3-2\sqrt{2}}=\frac{3-2\sqrt{2}}{9-8}=3-2\sqrt{2}.$$

3. Let r be the square root

Let $r=\sqrt{3+2\sqrt{2}}$. Then $r^2=3+2\sqrt{2}$ and $\frac{1}{r^2}=3-2\sqrt{2}$.

4. Use part (a)

$$\left(r-\frac{1}{r}\right)^2=r^2-2+\frac{1}{r^2}=(3+2\sqrt{2})-2+(3-2\sqrt{2})=4.$$

5. Take the positive square root

Since $r>1$, $r-\frac{1}{r}>0$, so $r-\frac{1}{r}=2$.

Final answer

(a) $r^2-2+\frac{1}{r^2}$.

(b) $3-2\sqrt{2}$.

(c) The expression equals 2.

TOPIC

Quadratics

Question 8

5 marks

Quadratics

8. Solve, using algebra, the equation

$$x - 6x^{\frac{1}{2}} + 4 = 0$$

Fully simplify your answers, writing them in the form $a + b\sqrt{c}$, where a, b and c are integers to be found.

(5)

(Total 5 marks)

WMA11/01 OCTOBER 2019

Worked Solution - Question 8

Quadratics

Topic group

1. Use a substitution

Let $u = \sqrt{x}$. Then $x = u^2$, so the equation becomes $u^2 - 6u + 4 = 0$.

2. Solve the quadratic

$$u = \frac{6 \pm \sqrt{36 - 16}}{2} = \frac{6 \pm 2\sqrt{5}}{2} = 3 \pm \sqrt{5}.$$

3. Return to x

Since $u = \sqrt{x}$, $x = u^2 = (3 \pm \sqrt{5})^2 = 9 + 5 \pm 6\sqrt{5} = 14 \pm 6\sqrt{5}$.

Final answer

$x = 14 + 6\sqrt{5}$ or $x = 14 - 6\sqrt{5}$.

Question 6

7 marks

Quadratics

6. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

(a) Given that

$$2xy - 3x^2 = 50$$

and

$$y - x^3 + 6x = 0$$

show that

$$2x^4 - 15x^2 - 50 = 0$$

(2)

(b) Hence solve the simultaneous equations

$$2xy - 3x^2 = 50$$
$$y - x^3 + 6x = 0$$

Give your answers in fully simplified surd form.

(5)

(Total 7 marks)

WMA11/01 MAY/JUNE 2022

Worked Solution - Question 6

Quadratics

Topic group

1. Rearrange the second equation

From $y - x^3 + 6x = 0$, we get $y = x^3 - 6x$.

2. Substitute into the first equation

$2x(x^3 - 6x) - 3x^2 = 50$, so $2x^4 - 15x^2 - 50 = 0$.

3. Solve the quadratic in x squared

Let $u = x^2$. Then $2u^2 - 15u - 50 = 0$, giving $u = 10$ or $u = -\frac{5}{2}$.

4. Reject the negative value

Since x^2 cannot be negative, $x^2 = 10$, so $x = \pm\sqrt{10}$.

5. Find y

$y = x^3 - 6x = x(x^2 - 6) = x(10 - 6) = 4x$. Therefore $y = 4\sqrt{10}$ when $x = \sqrt{10}$ and $y = -4\sqrt{10}$ when $x = -\sqrt{10}$.

Final answer

$(x, y) = (\sqrt{10}, 4\sqrt{10})$ or $(-\sqrt{10}, -4\sqrt{10})$.

Question 8

8 marks

Quadratics

8.

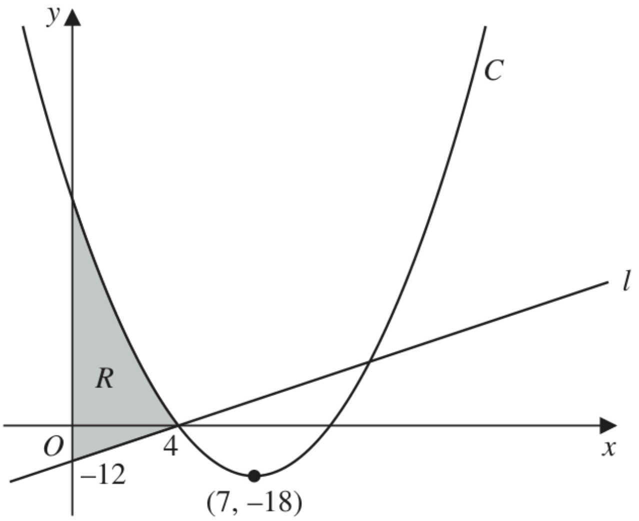

Figure 2

Figure 2 shows a sketch of the straight line l and the curve C.

Given that l cuts the y-axis at –12 and cuts the x-axis at 4, as shown in Figure 2,

(a) find an equation for l, writing your answer in the form $y = mx + c$, where m and c are constants to be found.

(2)

Given that C

- has equation $y = \mathrm{f}(x)$ where $\mathrm{f}(x)$ is a quadratic expression
- has a minimum point at (7, –18)
- cuts the x-axis at 4 and at k, where k is a constant

(b) deduce the value of k,

(1)

(c) find $\mathrm{f}(x)$.

(3)

The region R is shown shaded in Figure 2.

(d) Use inequalities to define R.

(2)

(Total for Question 8 is 8 marks)

WMA11/01 JANUARY 2023

Worked Solution - Question 8

Quadratics

Topic group

1. Find the straight line

The line cuts the y-axis at -12 and the x-axis at 4, so its gradient is $\dfrac{0-(-12)}{4-0} = 3$. Hence l is $y = 3x - 12$.

2. Use symmetry of the quadratic

The minimum point has x-coordinate 7. The roots are 4 and k, so their midpoint is 7.

3. Find k

$\dfrac{4+k}{2} = 7$, so $k = 10$.

4. Build the quadratic

Let $f(x) = a(x-4)(x-10)$. Since the minimum point is $(7, -18)$, $-18 = a(3)(-3) = -9a$, so $a = 2$.

5. Write f(x)

$f(x) = 2(x-4)(x-10) = 2x^2 - 28x + 80$.

6. Identify the shaded region

The region lies between the y-axis and $x = 4$, above the line and below the curve: $0 \leq x \leq 4$ and $3x - 12 \leq y \leq 2x^2 - 28x + 80$.

Final answer

$(a)\ y = 3x - 12.$

$(b)\ k = 10.$

$(c)\ f(x) = 2(x-4)(x-10).$

$(d)\ 0 \le x \le 4,\ 3x - 12 \le y \le 2x^2 - 28x + 80.$

WMA11/01 OCTOBER 2023

Question 6

6 marks

Quadratics

6. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

The equation

$$4(p - 2x) = \frac{12 + 15p}{x + p} \qquad x \neq -p$$

where p is a constant, has two distinct real roots.

(a) Show that

$$3p^2 - 10p - 8 > 0$$

(3)

(b) Hence, using algebra, find the range of possible values of p

(3)

(Total for Question 6 is 6 marks)

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Worked Solution - Question 6

Quadratics

Topic group

1. Clear the denominator

$4(p-2x)=\dfrac{12+15p}{x+p}$ gives $4(p-2x)(x+p)=12+15p$.

2. Form a quadratic in x

Expanding gives $4p^2-4px-8x^2=12+15p$, so
$8x^2+4px+12+15p-4p^2=0$.

3. Use the discriminant condition

For two distinct real roots, $b^2-4ac>0$.

4. Simplify the inequality

$(4p)^2-4(8)(12+15p-4p^2)>0$, so $144p^2-480p-384>0$.

5. Divide and factorise

Dividing by 48 gives $3p^2-10p-8>0$, and $3p^2-10p-8=(3p+2)(p-4)$.

6. State the range

The upward-opening quadratic is positive outside its roots, so $p<-\dfrac{2}{3}$ or $p>4$.

Final answer

$p<-\dfrac{2}{3}$ or $p>4$.

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10 marks

Question 11

Quadratics

11.

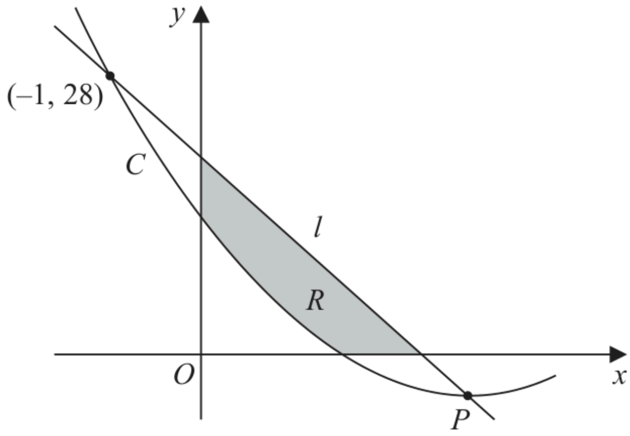


Figure 5

Figure 5 shows part of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 2x^2 - 12x + 14$$

(a) Write $2x^2 - 12x + 14$ in the form

$$a(x + b)^2 + c$$

where a, b and c are constants to be found.

(3)

Given that C has a minimum at the point P

(b) state the coordinates of P

(1)

The line l intersects C at $(-1, 28)$ and at P as shown in Figure 5.

(c) Find the equation of l giving your answer in the form $y = mx + c$ where m and c are constants to be found.

(3)

The finite region R, shown shaded in Figure 5, is bounded by the x-axis, l, the y-axis, and C.

(d) Use inequalities to define the region R.

(3)

(Total for Question 11 is 10 marks)

WMA11/01 OCTOBER 2023

Worked Solution - Question 11

Quadratics

Topic group

1. Complete the square

$2x^2 - 12x + 14 = 2(x^2 - 6x) + 14 = 2\left((x-3)^2 - 9\right) + 14 = 2(x-3)^2 - 4$

.

2. State the minimum point

The minimum of $2(x-3)^2 - 4$ is at $P = (3, -4)$.

3. Find the gradient of l

The line passes through $(-1, 28)$ and $P(3, -4)$, so

$m = \dfrac{-4-28}{3-(-1)} = \dfrac{-32}{4} = -8$.

4. Find the equation of l

Using $P(3, -4)$, $y + 4 = -8(x-3)$, so $y = -8x + 20$.

5. Use the axes as boundaries

The region is to the right of the y-axis and above the x-axis, so $x \geq 0$ and $y \geq 0$.

6. Use the line and curve as boundaries

The region is below the line and above the curve, so $y \leq -8x + 20$ and $y \geq 2x^2 - 12x + 14$.

Final answer

$(a)\ 2(x-3)^2-4.$

$(b)\ P=(3,-4).$

$(c)\ y=-8x+20.$

$(d)\ x\geq 0,\ y\geq 0,\ y\leq -8x+20,\ y\geq 2x^2-12x+14.$

Question 9

10 marks

Quadratics

9. The curve C_1 has equation $y = \mathrm{f}(x)$.

Given that

- $\mathrm{f}(x)$ is a quadratic expression
- C_1 has a maximum turning point at (2, 20)
- C_1 passes through the origin

(a) sketch a graph of C_1 showing the coordinates of any points where C_1 cuts the coordinate axes,

(2)

(b) find an expression for $\mathrm{f}(x)$.

(3)

The curve C_2 has equation $y = x(x^2 - 4)$

Curve C_1 and C_2 meet at the origin, and at the points P and Q

Given that the x coordinate of the point P is negative,

(c) using algebra and showing all stages of your working, find the coordinates of P

(5)

(Total for Question 9 is 10 marks)

WMA11/01 JANUARY 2024

Worked Solution - Question 9

Quadratics

Topic group

1. Use symmetry for the sketch

The maximum is at $(2, 20)$ and the curve passes through $(0, 0)$. Since the axis of symmetry is $x = 2$, the other x-intercept is $(4, 0)$.

2. Use completed-square form

Let $f(x) = a(x-2)^2 + 20$.

3. Use the origin

$0 = a(0-2)^2 + 20$, so $4a = -20$ and $a = -5$.

4. Write f(x)

$f(x) = -5(x-2)^2 + 20 = -5x^2 + 20x = -5x(x-4)$.

5. Equate the two curves

$-5x^2 + 20x = x(x^2-4) = x^3 - 4x$.

6. Factorise

$x^3 + 5x^2 - 24x = 0$, so $x(x^2 + 5x - 24) = 0$ and $x(x+8)(x-3) = 0$.

7. Use the negative x-coordinate

The negative x-coordinate is $x = -8$.

8. Find y

$y = x(x^2-4) = (-8)(64-4) = -480$, so $P = (-8, -480)$.

Final answer

(a) downward parabola through $(0,0)$ and $(4,0)$ with maximum $(2,20)$.

(b) $f(x) = -5x(x-4)$.

(c) $P = (-8,-480)$.

Question 10

7 marks

Quadratics

10. **In this question you must show all stages of your working.**
Solutions relying on calculator technology are not acceptable.

$$(k-1)x^6 + 4x^3 + (k-4) = 0 \qquad \text{where } k \text{ is a constant}$$

(a) Find the exact solutions to the given equation for $k = 4.5$

(3)

(b) Find the set of possible values of k for which the given equation has no real roots.

(4)

(Total for Question 10 is 7 marks)

WMA11/01 MAY/JUNE 2025

Worked Solution - Question 10

Quadratics

Topic group

1. Substitute k equals 4.5

The equation becomes $3.5x^6 + 4x^3 + 0.5 = 0$.

2. Clear decimals

Multiplying by 2 gives $7x^6 + 8x^3 + 1 = 0$.

3. Use a quadratic in x cubed

$7x^6 + 8x^3 + 1 = (x^3 + 1)(7x^3 + 1) = 0$.

4. Solve for x

$x^3 = -1$ or $x^3 = -\frac{1}{7}$, so $x = -1$ or $x = -\sqrt[3]{\frac{1}{7}}$.

5. View the equation as a quadratic

Let $u = x^3$. Then the equation is $(k-1)u^2 + 4u + (k-4) = 0$.

6. Use no real roots

Since every real value of u gives a real x, the original equation has no real roots exactly when this quadratic in u has no real roots.

7. Use the discriminant

Require $b^2 - 4ac < 0$: $16 - 4(k-1)(k-4) < 0$.

8. Simplify

$16 - 4(k^2 - 5k + 4) < 0$ gives $20k - 4k^2 < 0$, so $4k(5-k) < 0$.

9. Solve the inequality

$4k(5-k) < 0$ for $k < 0$ or $k > 5$.

Final answer

$(a)\ x = -1,\ \ -\sqrt[3]{\dfrac{1}{7}}.$

$(b)\ k < 0 \text{ or } k > 5.$

TOPIC

Simultaneous Equations

Question 8

7 marks

Simultaneous Equations

8. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

The curve C_1 has equation

$$xy = \frac{15}{2} - 5x \qquad x \neq 0$$

The curve C_2 has equation

$$y = x^3 - \frac{7}{2}x - 5$$

(a) Show that C_1 and C_2 meet when

$$2x^4 - 7x^2 - 15 = 0$$

(2)

Given that C_1 and C_2 meet at points P and Q

(b) find, using algebra, the exact distance PQ

(5)

(Total for Question 8 is 7 marks)

WMA11/01 OCTOBER 2023

Worked Solution - Question 8

Simultaneous Equations

Topic group

1. Write C1 in terms of y

$xy = \frac{15}{2} - 5x$, so $y = \frac{15}{2x} - 5$.

2. Equate the two curves

$\frac{15}{2x} - 5 = x^3 - \frac{7}{2}x - 5$.

3. Multiply by 2x

$15 = 2x^4 - 7x^2$, hence $2x^4 - 7x^2 - 15 = 0$.

4. Solve for x

Let $u = x^2$. Then $2u^2 - 7u - 15 = 0 = (2u+3)(u-5)$, so $x^2 = 5$ and $x = \pm\sqrt{5}$.

5. Find the y-coordinates

Using $y = \frac{15}{2x} - 5$, the points are $\left(\sqrt{5}, \frac{3\sqrt{5}}{2} - 5\right)$ and $\left(-\sqrt{5}, -\frac{3\sqrt{5}}{2} - 5\right)$.

6. Use the distance formula

$\Delta x = 2\sqrt{5}$ and $\Delta y = 3\sqrt{5}$, so $PQ = \sqrt{(2\sqrt{5})^2 + (3\sqrt{5})^2} = \sqrt{65}$.

Final answer

(a) $2x^4 - 7x^2 - 15 = 0$.

(b) $PQ = \sqrt{65}$.

TOPIC

Inequalities

Question 8

6 marks

Inequalities

8. The straight line l has equation $y = k(2x - 1)$, where k is a constant.

The curve C has equation $y = x^2 + 2x + 11$

Find the set of values of k for which l does not cross or touch C.

(6)

(Total 6 marks)

WMA11/01 JANUARY 2020

Worked Solution - Question 8

Inequalities

Topic group

1. Equate the line and curve

$k(2x-1)=x^2+2x+11$, so $x^2+(2-2k)x+(11+k)=0$.

2. Use the no-intersection condition

For the line not to cross or touch the curve, the quadratic in x must have no real roots, so its discriminant is less than zero.

3. Form the inequality in k

$(2-2k)^2-4(1)(11+k)<0$. Dividing by 4 gives $(1-k)^2-(11+k)<0$, so $k^2-3k-10<0$.

4. Solve the quadratic inequality

$(k-5)(k+2)<0$, so k lies between the two roots: $-2

Final answer

$-2

Question 7

6 marks

Inequalities

7.

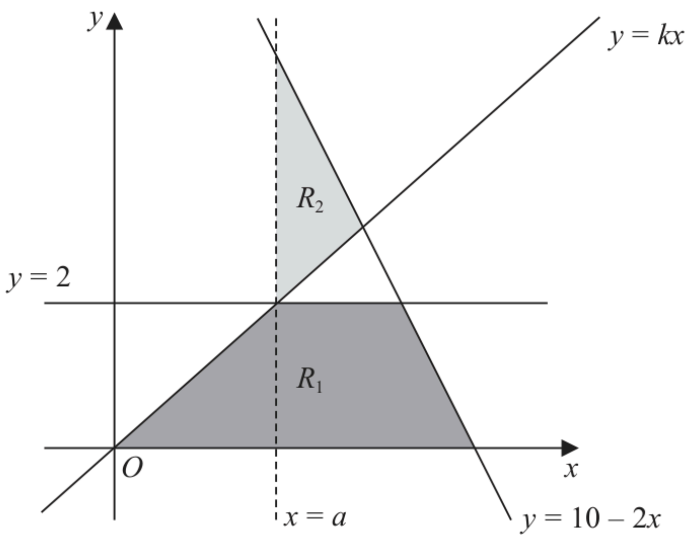


Figure 2

The region R_1, shown shaded in Figure 2, is defined by the inequalities

$$0 \leqslant y \leqslant 2 \qquad y \leqslant 10 - 2x \qquad y \leqslant kx$$

where k is a constant.

The line $x = a$, where a is a constant, passes through the intersection of the lines $y = 2$ and $y = kx$

Given that the area of R_1 is $\dfrac{27}{4}$ square units,

(a) find

(i) the value of a

(ii) the value of k

(4)

(b) Define the region R_2, also shown shaded in Figure 2, using inequalities.

(2)

(Total for Question 7 is 6 marks)

WMA11/01 MAY/JUNE 2023

Worked Solution - Question 7

Inequalities

Topic group

1. Find the area of R1

The bottom horizontal length is 5, from $x = 0$ to the intercept of $y = 10 - 2x$ at $x = 5$.

2. Use the top length

The top horizontal length is $4 - a$, because $y = 2$ meets $y = 10 - 2x$ at $x = 4$.

3. Solve for a

Area $R_1 = \frac{1}{2}(5 + 4 - a)(2) = 9 - a = \frac{27}{4}$, so $a = \frac{9}{4}$.

4. Find k

The line $x = a$ passes through the intersection of $y = 2$ and $y = kx$, so $2 = ka$. Hence $k = \frac{2}{9/4} = \frac{8}{9}$.

5. Define R2

Region R_2 is to the right of $x = a$, above $y = kx$, and below $y = 10 - 2x$.

Final answer

$(a)\ a = \frac{9}{4},\ k = \frac{8}{9}.$

$(b)\ x \geq \frac{9}{4},\ y \geq \frac{8}{9}x,\ y \leq 10 - 2x.$

Question 6

7 marks

Inequalities

6.

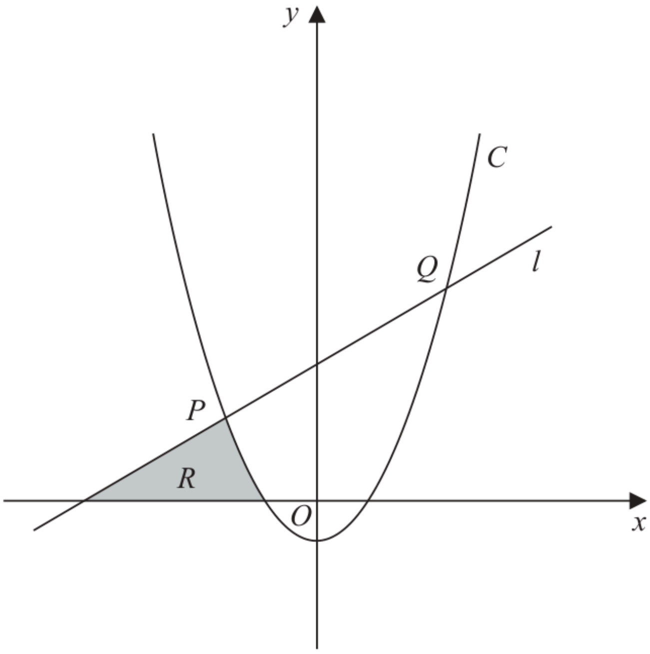


Figure 3

In this question you must show all stages of your working.

Solutions relying on calculator technology are not acceptable.

Figure 3 shows

- the line l with equation $y - 5x = 75$
- the curve C with equation $y = 2x^2 + x - 21$

The line l intersects the curve C at the points P and Q, as shown in Figure 3.

(a) Find, using algebra, the coordinates of P and the coordinates of Q.

(4)

The region R, shown shaded in Figure 3, is bounded by C, l and the x-axis.

(b) Use inequalities to define the region R.

(3)

(Total for Question 6 is 7 marks)

WMA11/01 MAY/JUNE 2024

Worked Solution - Question 6

Inequalities

Topic group

1. Write the line as y equals

$y - 5x = 75$, so $y = 5x + 75$.

2. Equate line and curve

$5x + 75 = 2x^2 + x - 21$, so $2x^2 - 4x - 96 = 0$.

3. Solve the quadratic

$x^2 - 2x - 48 = 0 = (x - 8)(x + 6)$, so $x = -6$ or $x = 8$.

4. Find the y-coordinates

Using $y = 5x + 75$, $x = -6$ gives $y = 45$ and $x = 8$ gives $y = 115$.

5. State P and Q

From the diagram, $P = (-6, 45)$ and $Q = (8, 115)$.

6. Use the x-axis and left branch

The shaded region is above the x-axis and on the left part of the diagram, so $y \geq 0$ and $x \leq 0$.

7. Use the line and curve as upper boundaries

The region is below both the line and the curve, so $y \leq 5x + 75$ and $y \leq 2x^2 + x - 21$.

Final answer

$(a)\ P=(-6,45),\ Q=(8,115).$

$(b)\ y\geq 0,\ y\leq 5x+75,\ y\leq 2x^2+x-21,\ x\leq 0.$

TOPIC

Polynomials

Question 8

9 marks

Polynomials

8. The curve C_1 has equation

$$y = 3x^2 + 6x + 9$$

(a) Write $3x^2 + 6x + 9$ in the form

$$a(x + b)^2 + c$$

where a, b and c are constants to be found.

(3)

The point P is the minimum point of C_1

(b) Deduce the coordinates of P.

(1)

A different curve C_2 has equation

$$y = Ax^3 + Bx^2 + Cx + D$$

where A, B, C and D are constants.

Given that C_2

- passes through P
- intersects the x-axis at -4, -2 and 3

(c) find, making your method clear, the values of A, B, C and D.

(5)

(Total 9 marks)

Worked Solution - Question 8

Polynomials

Topic group

1. Complete the square

$3x^2 + 6x + 9 = 3(x^2 + 2x) + 9 = 3\big((x+1)^2 - 1\big) + 9 = 3(x+1)^2 + 6.$

2. Deduce the minimum point

The minimum occurs when $(x+1)^2 = 0$, so $x = -1$ and $y = 6$. Therefore $P = (-1, 6)$.

3. Use the x-intercepts of C2

Since C_2 intersects the x-axis at $-4, -2$ and 3, it has form $y = k(x+4)(x+2)(x-3)$.

4. Use point P to find k

Substitute $P = (-1, 6)$: $6 = k(3)(1)(-4) = -12k$, so $k = -\frac{1}{2}$.

5. Expand the cubic

$(x+4)(x+2)(x-3) = (x^2 + 6x + 8)(x-3) = x^3 + 3x^2 - 10x - 24.$

6. Read off A, B, C and D

Thus $y = -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 5x + 12$, so $A = -\frac{1}{2}$, $B = -\frac{3}{2}$, $C = 5$ and $D = 12$.

Final answer

$(a)\ 3(x+1)^2 + 6.$

$(b)\ P = (-1, 6).$

$(c)\ A = -\frac{1}{2},\ B = -\frac{3}{2},\ C = 5,\ D = 12.$

WMA11/01 JANUARY 2026

Question 6

9 marks

Polynomials

6.

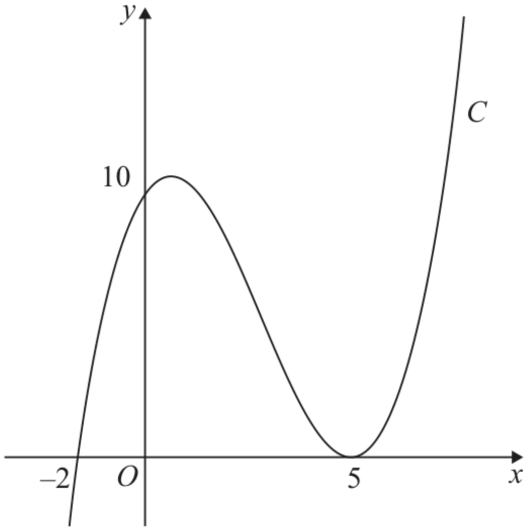


Figure 3

Figure 3 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$ where $\mathrm{f}(x)$ is a cubic function in x.

The curve C

- cuts the x-axis at $(-2, 0)$ and cuts the y-axis at $(0, 10)$
- touches the x-axis at $(5, 0)$

as shown in Figure 3.

(a) Deduce the roots of the equation

(i) $\mathrm{f}\left(\frac{1}{3}x\right) = 0$

(ii) $\mathrm{f}(x - 3) = 0$

(2)

(b) Find an expression for $\mathrm{f}(x)$. You should leave your answer in factorised form.

(3)

The curve C intersects the straight line $y = 10(x + 2)$ at exactly three points.

(c) Use algebra to find the exact x coordinates of the three points of intersection.

(*Solutions based entirely on calculator technology are not acceptable.*)

(4)

(Total for Question 6 is 9 marks)

Worked Solution - Question 6

Polynomials

Topic group

1. Roots of f one third x

The roots of $f(x) = 0$ are $x = -2$ and $x = 5$, with $x = 5$ repeated. For $f\left(\frac{1}{3}x\right) = 0$, set $\frac{1}{3}x = -2$ or 5, giving $x = -6$ or 15.

2. Roots of f of x minus 3

For $f(x-3) = 0$, set $x - 3 = -2$ or 5, giving $x = 1$ or 8.

3. Use the roots and repeated root

Since the curve cuts at $(-2, 0)$ and touches at $(5, 0)$, write $f(x) = k(x+2)(x-5)^2$.

4. Use the y-intercept

The curve passes through $(0, 10)$, so $10 = k(2)(25) = 50k$.

5. State f(x)

$k = \frac{1}{5}$, hence $f(x) = \frac{1}{5}(x+2)(x-5)^2$.

6. Set equal to the line

$\frac{1}{5}(x+2)(x-5)^2 = 10(x+2)$.

7. First solution

The factor $x + 2$ gives one solution $x = -2$.

8. Find the other solutions

For $x \neq -2$, divide by $x + 2$: $\frac{1}{5}(x-5)^2 = 10$, so $(x-5)^2 = 50$.

9. State all intersections

$x = 5 \pm \sqrt{50} = 5 \pm 5\sqrt{2}$. Therefore the three x-coordinates are -2, $5 - 5\sqrt{2}$ and $5 + 5\sqrt{2}$.

Final answer

$(a)(i)\ -6, 15.\quad (a)(ii)\ 1, 8.$

$(b)\ f(x) = \dfrac{1}{5}(x+2)(x-5)^2.$

$(c)\ x = -2,\ 5 \pm 5\sqrt{2}.$

TOPIC

Graphs of Functions

Question 6

8 marks

Graphs of Functions

6. The curve C has equation $y = \frac{4}{x} + k$, where k is a positive constant.

(a) Sketch a graph of C, stating the equation of the horizontal asymptote and the coordinates of the point of intersection with the x-axis.

(3)

The line with equation $y = 10 - 2x$ is a tangent to C.

(b) Find the possible values for k.

(5)

(Total 8 marks)

WMA11/01 OCTOBER 2019

Worked Solution - Question 6

Graphs of Functions

Topic group

1. Read the graph features

For $y = \frac{4}{x} + k$, the horizontal asymptote is $y = k$. Setting $y = 0$ gives $0 = \frac{4}{x} + k$, so the x-intercept is $x = -\frac{4}{k}$.

2. Set the tangent line equal to the curve

A tangent touches once, so solve $10 - 2x = \frac{4}{x} + k$. Multiplying by x gives $2x^2 + (k - 10)x + 4 = 0$.

3. Use the discriminant condition

For one repeated intersection, discriminant $= 0$: $(k - 10)^2 - 4(2)(4) = 0$. Hence $(k - 10)^2 = 32$.

4. Find k

$k - 10 = \pm 4\sqrt{2}$, so $k = 10 \pm 4\sqrt{2}$. Both are positive, so both satisfy the condition in the question.

Final answer

Asymptote $y = k$, x-intercept $\left(-\frac{4}{k}, 0\right)$, and $k = 10 \pm 4\sqrt{2}$.

WMA11/01 OCTOBER 2019

Question 10

10 marks

Graphs of Functions

10.

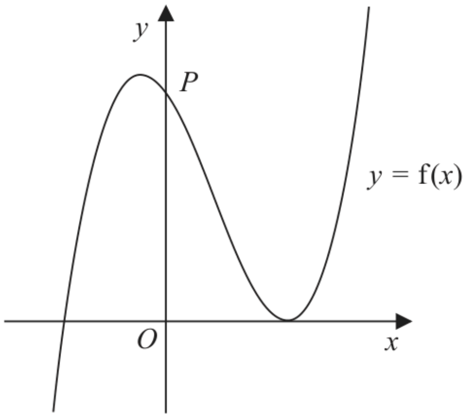


Figure 6

Figure 6 shows a sketch of part of the curve with equation $y = f(x)$, where

$$f(x) = (2x + 5)(x - 3)^2$$

(a) Deduce the values of x for which $f(x) \leqslant 0$

(2)

The curve crosses the y-axis at the point P, as shown.

(b) Expand $f(x)$ to the form

$$ax^3 + bx^2 + cx + d$$

where a, b, c and d are integers to be found.

(3)

(c) Hence, or otherwise, find

(i) the coordinates of P,

(ii) the gradient of the curve at P.

(2)

The curve with equation $y = f(x)$ is translated two units in the positive x direction to a curve with equation $y = g(x)$.

(d) (i) Find $g(x)$, giving your answer in a simplified factorised form.

(ii) Hence state the y intercept of the curve with equation $y = g(x)$.

(3)

(Total 10 marks)

Worked Solution - Question 10

Graphs of Functions

Topic group

1. Use the factorised form for the inequality

$f(x) = (2x+5)(x-3)^2$. Since $(x-3)^2 \geq 0$, the sign is controlled by $2x+5$, with an extra zero at $x = 3$. Hence $f(x) \leq 0$ for $x \leq -\frac{5}{2}$ or $x = 3$.

2. Expand the cubic

$(x-3)^2 = x^2 - 6x + 9$, so
$f(x) = (2x+5)(x^2-6x+9) = 2x^3 - 7x^2 - 12x + 45$.

3. Find P and the gradient at P

At the y-axis, $x = 0$, so $P = (0, 45)$. Also $f'(x) = 6x^2 - 14x - 12$, so the gradient at P is $f'(0) = -12$.

4. Translate the graph

A translation 2 units in the positive x direction means $g(x) = f(x-2)$. Therefore $g(x) = (2(x-2)+5)((x-2)-3)^2 = (2x+1)(x-5)^2$.

5. Find the new y-intercept

Set $x = 0$: $g(0) = (1)(25) = 25$, so the y-intercept is 25.

Final answer

$f(x) \leq 0$ for $x \leq -\frac{5}{2}$ or $x = 3$; $f(x) = 2x^3 - 7x^2 - 12x + 45$;
$P = (0, 45)$; gradient $= -12$; $g(x) = (2x+1)(x-5)^2$; y-intercept 25.

Question 10

8 marks

Graphs of Functions

10. The curve C_1 has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = (4x - 3)(x - 5)^2$$

(a) Sketch C_1 showing the coordinates of any point where the curve touches or crosses the coordinate axes.

(3)

(b) Hence or otherwise

(i) find the values of x for which $\mathrm{f}\left(\frac{1}{4}x\right) = 0$

(ii) find the value of the constant p such that the curve with equation $y = \mathrm{f}(x) + p$ passes through the origin.

(2)

A second curve C_2 has equation $y = \mathrm{g}(x)$, where $\mathrm{g}(x) = \mathrm{f}(x + 1)$

(c) (i) Find, in simplest form, $\mathrm{g}(x)$. You may leave your answer in a factorised form.

(ii) Hence, or otherwise, find the y intercept of curve C_2

(3)

(Total 8 marks)

WMA11/01 JANUARY 2020

Worked Solution - Question 10

Graphs of Functions

Topic group

1. Use the factorised form for the sketch

$f(x) = (4x-3)(x-5)^2$. The curve crosses the x-axis at $x = \frac{3}{4}$ and touches the x-axis at $x = 5$. The y-intercept is $f(0) = (-3)(25) = -75$.

2. Solve f(x/4)=0

For $f\left(\frac{1}{4}x\right) = 0$, either $4\left(\frac{x}{4}\right) - 3 = 0$ or $\frac{x}{4} - 5 = 0$. Therefore $x = 3$ or $x = 20$.

3. Find p

The curve $y = f(x) + p$ passes through the origin, so $0 = f(0) + p = -75 + p$. Hence $p = 75$.

4. Find g(x)

$g(x) = f(x+1) = (4(x+1)-3)((x+1)-5)^2 = (4x+1)(x-4)^2$.

5. Find the y-intercept of C2

At $x = 0$, $g(0) = (1)(16) = 16$. So the y-intercept is 16.

Final answer

Sketch: crosses at $\left(\frac{3}{4}, 0\right)$, touches at $(5, 0)$, crosses y-axis at $(0, -75)$. Also $x = 3, 20$, $p = 75$, $g(x) = (4x+1)(x-4)^2$, y-intercept 16.

Question 6

10 marks

Graphs of Functions

6. (a) Sketch the curve with equation

$$y = -\frac{k}{x} \qquad k > 0 \qquad x \neq 0$$

(2)

(b) On a separate diagram, sketch the curve with equation

$$y = -\frac{k}{x} + k \qquad k > 0 \qquad x \neq 0$$

stating the coordinates of the point of intersection with the x-axis and, in terms of k, the equation of the horizontal asymptote.

(3)

(c) Find the range of possible values of k for which the curve with equation

$$y = -\frac{k}{x} + k \qquad k > 0 \qquad x \neq 0$$

does not touch or intersect the line with equation $y = 3x + 4$

(5)

(Total 10 marks)

WMA11/01 JANUARY 2021

Worked Solution - Question 6

Graphs of Functions

Topic group

1. Sketch y = -k/x

Because $k > 0$, $y = -\dfrac{k}{x}$ has one branch in quadrant II and one branch in quadrant IV, with asymptotes $x = 0$ and $y = 0$.

2. Translate the graph upward

$y = -\dfrac{k}{x} + k$ is $y = -\dfrac{k}{x}$ translated up by k units. The vertical asymptote is still $x = 0$, and the horizontal asymptote is $y = k$.

3. Find the x-intercept

Set $y = 0$: $0 = -\dfrac{k}{x} + k$. Since $k > 0$, $\dfrac{k}{x} = k$, so $x = 1$. The x-intercept is $(1, 0)$.

4. Set up the intersection equation

For intersections with $y = 3x + 4$, solve $-\dfrac{k}{x} + k = 3x + 4$. Multiplying by x gives $-k + kx = 3x^2 + 4x$.

5. Use the discriminant

This rearranges to $3x^2 + (4 - k)x + k = 0$. For no touch or intersection, the discriminant must be negative: $(4 - k)^2 - 12k < 0$.

6. Solve the quadratic inequality in k

$k^2 - 20k + 16 < 0$. The roots are $k = 10 \pm 2\sqrt{21}$, so the required range is $10-2\sqrt{21}

Final answer

$(b)\ (1,0),\ y=k.$

$(c)\ 10-2\sqrt{21}

WMA11/01 JANUARY 2021

Question 8

9 marks

Graphs of Functions

8.

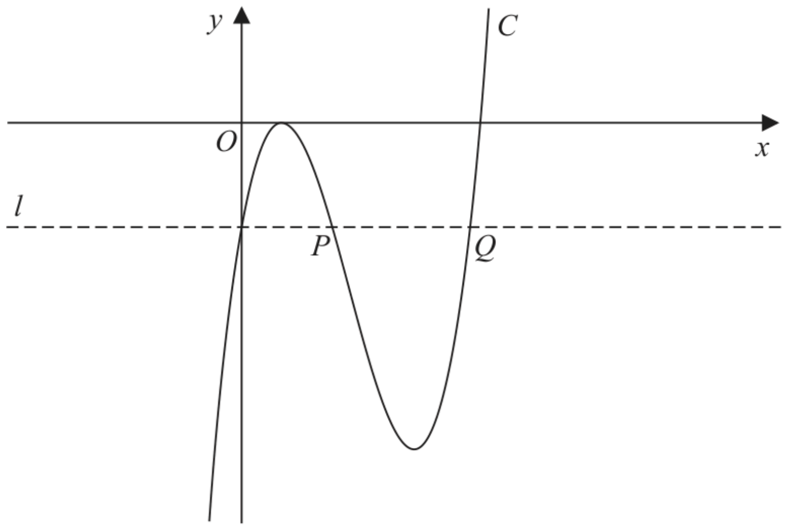


Figure 4

Figure 4 shows a sketch of part of the curve C with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = (3x - 2)^2 (x - 4)$$

(a) Deduce the values of x for which $\mathrm{f}(x) > 0$

(1)

(b) Expand $\mathrm{f}(x)$ to the form

$$ax^3 + bx^2 + cx + d$$

where a, b, c and d are integers to be found.

(3)

The line l, also shown in Figure 4, passes through the y intercept of C and is parallel to the x-axis.

The line l cuts C again at points P and Q, also shown in Figure 4.

(c) Using algebra and showing your working, find the length of line PQ. Write your answer in the form $k\sqrt{3}$, where k is a constant to be found.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(5)

(Total 9 marks)

WMA11/01 JANUARY 2021

Worked Solution - Question 8

Graphs of Functions

Topic group

1. Use the factorised form

$f(x) = (3x-2)^2(x-4)$. The squared factor is never negative, and it is zero at $x = \frac{2}{3}$. Therefore the sign of $f(x)$ away from the repeated root is controlled by $x-4$.

2. Find where f(x) is positive

For $f(x) > 0$, we need $x - 4 > 0$, so $x > 4$.

3. Expand f(x)

$(3x-2)^2 = 9x^2 - 12x + 4$. Therefore
$f(x) = (9x^2 - 12x + 4)(x-4) = 9x^3 - 48x^2 + 52x - 16$.

4. Find the line l

The y-intercept of C is $f(0) = -16$, so the horizontal line l is $y = -16$.

5. Find the other intersections with l

Solve $f(x) = -16$: $9x^3 - 48x^2 + 52x - 16 = -16$, so $x(9x^2 - 48x + 52) = 0$.
The root $x = 0$ is the y-intercept, and the two other roots give P and Q.

6. Solve the quadratic

$9x^2 - 48x + 52 = 0$. Hence

$$x = \frac{48 \pm \sqrt{48^2 - 4(9)(52)}}{18} = \frac{48 \pm 12\sqrt{3}}{18} = \frac{8 \pm 2\sqrt{3}}{3}.$$

7. Find PQ

Because P and Q lie on the same horizontal line, PQ is the difference in their x-coordinates: $PQ = \frac{8+2\sqrt{3}}{3} - \frac{8-2\sqrt{3}}{3} = \frac{4\sqrt{3}}{3}$.

Final answer

(a) $x > 4$.

(b) $f(x) = 9x^3 - 48x^2 + 52x - 16$.

(c) $PQ = \dfrac{4\sqrt{3}}{3}$.

Question 6

10 marks

Graphs of Functions

6. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

A curve C has equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 2(x+1)(x-3)^2$$

(a) Sketch a graph of C.

Show on your graph the coordinates of the points where C cuts or meets the coordinate axes.

(3)

(b) Write $\mathrm{f}(x)$ in the form $ax^3 + bx^2 + cx + d$, where a, b, c and d are constants to be found.

(3)

(c) Hence, find the equation of the tangent to C at the point where $x = \frac{1}{3}$

(4)

(Total 10 marks)

WMA11/01 OCTOBER 2021

Worked Solution - Question 6

Graphs of Functions

Topic group

1. Use the factorised form for the sketch

$f(x) = 2(x+1)(x-3)^2$. The curve cuts the x-axis at $(-1, 0)$ and meets the x-axis at $(3, 0)$ because $x = 3$ is a repeated root.

2. Find the y-intercept

$f(0) = 2(1)(9) = 18$, so the y-intercept is $(0, 18)$.

3. Describe the cubic shape

The leading term is positive, so the cubic goes down to the left and up to the right. It crosses at $x = -1$ and touches at $x = 3$.

4. Expand f(x)

$(x+1)(x-3)^2 = (x+1)(x^2 - 6x + 9) = x^3 - 5x^2 + 3x + 9$, so $f(x) = 2x^3 - 10x^2 + 6x + 18$.

5. Differentiate

$f'(x) = 6x^2 - 20x + 6$.

6. Find the gradient at x = 1/3

$f'\left(\frac{1}{3}\right) = 6\left(\frac{1}{9}\right) - 20\left(\frac{1}{3}\right) + 6 = 0$, so the tangent is horizontal.

7. Find the y-coordinate

$f\left(\frac{1}{3}\right) = 2\left(\frac{4}{3}\right)\left(-\frac{8}{3}\right)^2 = \frac{512}{27}$.

8. Write the tangent

A horizontal tangent through this point has equation $y = \frac{512}{27}$.

Final answer

(a) x-intercepts $(-1,0),(3,0)$, y-intercept $(0,18)$.

(b) $f(x) = 2x^3 - 10x^2 + 6x + 18$.

(c) $y = \dfrac{512}{27}$.

Question 10

9 marks

Graphs of Functions

10. The curve C has equation

$$y = \frac{1}{x^2} - 9$$

(a) Sketch the graph of C.

On your sketch

- show the coordinates of any points of intersection with the coordinate axes
- state clearly the equations of any asymptotes

(4)

The curve D has equation $y = kx^2$ where k is a constant.

Given that C meets D at 4 distinct points,

(b) find the range of possible values for k.

(5)

(Total 9 marks)

WMA11/01 JANUARY 2022

Worked Solution - Question 10

Graphs of Functions

Topic group

1. Find intercepts

For x-intercepts, set $\dfrac{1}{x^2} - 9 = 0$. Then $x^2 = \dfrac{1}{9}$, so $x = \pm\dfrac{1}{3}$. There is no y-intercept because $x = 0$ is not in the domain.

2. State asymptotes

The vertical asymptote is $x = 0$. As $|x|$ becomes large, $\dfrac{1}{x^2} \to 0$, so the horizontal asymptote is $y = -9$.

3. Set up intersections with D

For intersections with $y = kx^2$, solve $\dfrac{1}{x^2} - 9 = kx^2$.

4. Use u = x squared

Let $u = x^2$, where $u > 0$. Multiplying by x^2 gives $1 - 9u = ku^2$, so $ku^2 + 9u - 1 = 0$.

5. Require two positive u-values

Four distinct x-values require two distinct positive values of u, since each positive u gives $x = \pm\sqrt{u}$.

6. Find the range of k

For two positive roots, the product $-\dfrac{1}{k}$ and sum $-\dfrac{9}{k}$ must both be positive, so $k < 0$. Also the discriminant must be positive: $81 + 4k > 0$, so $k > -\dfrac{81}{4}$.

Therefore $-\dfrac{81}{4}

Final answer

(a) x-intercepts $\left(-\dfrac{1}{3},0\right),\left(\dfrac{1}{3},0\right)$; asymptotes $x=0,\ y=-9.$

$(b)\ -\dfrac{81}{4}

Question 6

6 marks

Graphs of Functions

6. (a) Given that k is a positive constant such that $0 < k < 4$ sketch, on **separate axes**, the graphs of

(i) $y = (2x - k)(x + 4)^2$

(ii) $y = \dfrac{k}{x^2}$

showing the coordinates of any points where the graphs cross or meet the coordinate axes, leaving coordinates in terms of k, where appropriate.

(5)

(b) State, with a reason, the number of roots of the equation

$$(2x - k)(x + 4)^2 = \frac{k}{x^2}$$

(1)

(Total for Question 6 is 6 marks)

WMA11/01 OCTOBER 2022

Worked Solution - Question 6

Graphs of Functions

Topic group

1. Key points for y = (2x-k)(x+4)^2

The graph has x-intercepts $(-4, 0)$ and $\left(\frac{k}{2}, 0\right)$. The root at $x = -4$ is a repeated root, so the graph touches the x-axis there.

2. Use 0 < k < 4

Since $0

3. Key features for y = k/x^2

The graph is positive on both sides of the y-axis, with asymptotes $x = 0$ and $y = 0$. It has no intercepts with the coordinate axes.

4. Compare the two graphs

The equation $(2x-k)(x+4)^2 = \frac{k}{x^2}$ represents intersections between the two graphs.

5. State the number of roots

From the sketches, the positive branch of $\frac{k}{x^2}$ meets the increasing positive branch of $(2x-k)(x+4)^2$ once only. Therefore there is one root.

Final answer

The graphs intersect once, so the equation has one root.

Question 7

7. (a) Sketch the graph of the curve C with equation

$$y = \frac{4}{x - k}$$

where k is a positive constant.

Show on your sketch

- the coordinates of any points where C cuts the coordinate axes
- the equation of the vertical asymptote to C

(4)

Given that the straight line with equation $y = 9 - x$ does not cross or touch C

(b) find the range of values of k.

(5)

(Total for Question 7 is 9 marks)

WMA11/01 JANUARY 2024

Worked Solution - Question 7

Graphs of Functions

Topic group

1. Identify the asymptote

For $y = \dfrac{4}{x-k}$, the vertical asymptote is where the denominator is zero, so $x = k$
.

2. Find axis intercepts

At $x = 0$, $y = -\dfrac{4}{k}$, so the y-intercept is $(0, -\dfrac{4}{k})$. The graph has no x-intercept because the numerator is never zero.

3. Set up intersection with the line

If the line $y = 9 - x$ meets the curve, then $9 - x = \dfrac{4}{x-k}$.

4. Form a quadratic

$(9-x)(x-k) = 4$, which simplifies to $x^2 - (9+k)x + 9k + 4 = 0$.

5. Use no real intersection

For the line not to cross or touch the curve, the discriminant must be negative.

6. Solve the inequality

$(9+k)^2 - 4(9k+4) < 0$, so $k^2 - 18k + 65 < 0$. Since
$k^2 - 18k + 65 = (k-5)(k-13)$, the range is $5

Final answer

(a) y-intercept $(0, -\dfrac{4}{k})$, no x-intercept, vertical asymptote $x = k$.

$(b)\ 5

Question 8

7 marks

Graphs of Functions

8. The curve C_1 has equation

$$y = x(4 - x^2)$$

(a) Sketch the graph of C_1 showing the coordinates of any points of intersection with the coordinate axes.

(3)

The curve C_2 has equation $y = \dfrac{A}{x}$ where A is a constant.

(b) Show that the x coordinates of the points of intersection of C_1 and C_2 satisfy the equation

$$x^4 - 4x^2 + A = 0$$

(1)

(c) Hence find the range of possible values of A for which C_1 meets C_2 at 4 distinct points.

(3)

(Total for Question 8 is 7 marks)

WMA11/01 MAY/JUNE 2024

Worked Solution - Question 8

Graphs of Functions

Topic group

1. Find the intercepts of C1

$y = x(4 - x^2)$, so $y = 0$ when $x = 0$ or $x = \pm 2$. The curve crosses the axes at $(-2, 0)$, $(0, 0)$ and $(2, 0)$.

2. Use the cubic shape

$y = 4x - x^3$ has negative leading cubic term, so the left tail rises and the right tail falls.

3. Equate C1 and C2

$x(4 - x^2) = \frac{A}{x}$.

4. Multiply by x

$x^2(4 - x^2) = A$, so $4x^2 - x^4 = A$ and therefore $x^4 - 4x^2 + A = 0$.

5. Let u equal x squared

Let $u = x^2$. Then $u^2 - 4u + A = 0$.

6. Require two positive u-values

For 4 distinct x-values, the quadratic in u must have two distinct positive roots.

7. Find the range of A

The discriminant gives $16 - 4A > 0$, so $A < 4$. The product of the two roots is A, so $A > 0$. Therefore $0

Final answer

(a) x-intercepts $(-2,0)$, $(0,0)$, $(2,0)$.

(b) $x^4-4x^2+A=0.$

$(c)\ 0

Question 6

9 marks

Graphs of Functions

6. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

(a) Sketch the curve C with equation

$$y = \frac{1}{2 - x} \qquad x \neq 2$$

State on your sketch

- the equation of the vertical asymptote
- the coordinates of the intersection of C with the y-axis

(3)

The straight line l has equation $y = kx - 4$, where k is a constant.

Given that l cuts C at least once,

(b) (i) show that

$$k^2 - 5k + 4 \geqslant 0$$

(ii) find the range of possible values for k.

(6)

(Total for Question 6 is 9 marks)

WMA11/01 OCTOBER 2024

Worked Solution - Question 6

Graphs of Functions

Topic group

1. Identify the graph features

$y = \dfrac{1}{2-x}$ has vertical asymptote $x = 2$ and horizontal asymptote $y = 0$.

2. Find the y-intercept

At $x = 0$, $y = \dfrac{1}{2}$, so the y-intercept is $(0, \dfrac{1}{2})$.

3. Set up the intersection with the line

If $y = kx - 4$ cuts C, then $kx - 4 = \dfrac{1}{2-x}$.

4. Form the quadratic

$(kx-4)(2-x) = 1$, so $kx^2 - (2k+4)x + 9 = 0$.

5. Use the discriminant

For at least one intersection, the discriminant is non-negative.

6. Derive the given inequality

$(2k+4)^2 - 4(k)(9) \geq 0$, so $4(k^2 - 5k + 4) \geq 0$, hence $k^2 - 5k + 4 \geq 0$.

7. Solve the inequality

$k^2 - 5k + 4 = (k-1)(k-4)$, so $k \leq 1$ or $k \geq 4$.

Final answer

(a) vertical asymptote $x = 2$, y-intercept $(0, \dfrac{1}{2})$.

(b) $k \leq 1$ or $k \geq 4$.

Question 7

8 marks

Graphs of Functions

7. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The curve C has equation

$$y = \frac{2}{x} - k$$

where k is a **positive** constant.

(a) Sketch the graph of C.

Show on your sketch

- the coordinates of any points of intersection of C with the coordinate axes
- the equation of the horizontal asymptote to C

stating each in terms of k.

(3)

The line l has equation $y = -kx - 6$

Given that l intersects C at 2 distinct points,

(b) find the range of possible values of k.

(5)

(Total for Question 7 is 8 marks)

WMA11/01 JANUARY 2025

Worked Solution - Question 7

Graphs of Functions

Topic group

1. Identify the horizontal asymptote

$y = \frac{2}{x} - k$ has horizontal asymptote $y = -k$.

2. Find axis intersections

Setting $y = 0$ gives $\frac{2}{x} = k$, so $x = \frac{2}{k}$. There is no y-intercept because $x = 0$ is not in the domain.

3. Set up line-curve intersections

$\frac{2}{x} - k = -kx - 6$.

4. Form a quadratic

Multiplying by x gives $2 - kx = -kx^2 - 6x$, so $kx^2 + (6-k)x + 2 = 0$.

5. Use two distinct intersections

For two distinct points, the discriminant must be positive.

6. Solve the inequality

$(6-k)^2 - 8k > 0$, so $k^2 - 20k + 36 > 0$. Since $k^2 - 20k + 36 = (k-2)(k-18)$, $k < 2$ or $k > 18$.

7. Use k positive

As $k > 0$, the range is \$018\$.

Final answer

(a) x-intercept $(\frac{2}{k}, 0)$, no y-intercept, horizontal asymptote $y = -k$.

$(b)\ 018$.

WMA11/01 MAY/JUNE 2025

Question 7

10 marks

Graphs of Functions

7.

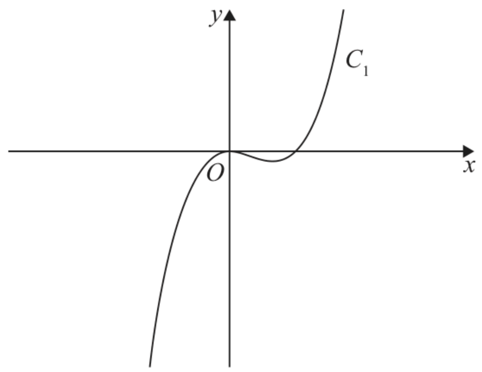


Figure 3

Figure 3 shows a sketch of part of the curve C_1

Given that C_1

- has equation $y = f(x)$ where $f(x)$ is a cubic function
- touches the x-axis at the origin and cuts the x-axis at $x = 4$
- passes through the point $(10, 120)$

(a) find $f(x)$

(3)

The curve C_2 has equation $y = 1.2x(8 - x)$

On the following page there is a copy of Figure 3 called Diagram 1.

(b) On Diagram 1 sketch a graph of the curve C_2

(2)

(c) Use algebra to find the coordinates of the points where C_1 and C_2 intersect. Show each stage of your working.

(5)

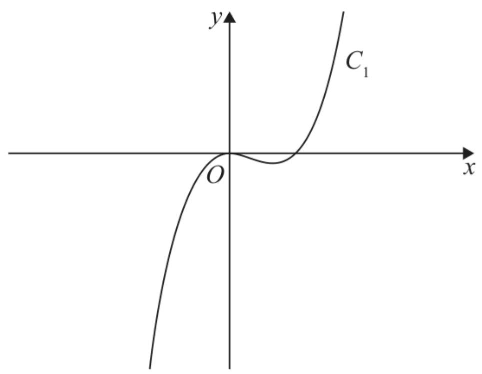


Diagram 1

(Total for Question 7 is 10 marks)

WMA11/01 MAY/JUNE 2025

Worked Solution - Question 7

Graphs of Functions

Topic group

1. Use the repeated root at the origin

Since the cubic touches the x-axis at the origin and cuts it at $x = 4$, write $f(x) = \lambda x^2(x-4)$.

2. Use the point (10,120)

$120 = \lambda(10)^2(10-4) = 600\lambda$, so $\lambda = 0.2$.

3. State f(x)

$f(x) = 0.2x^2(x-4)$.

4. Sketch C2

$C_2 : y = 1.2x(8-x)$ is a downward-opening parabola through $x = 0$ and $x = 8$, with its maximum between them.

5. Set intersections equal

$1.2x(8-x) = 0.2x^2(x-4)$.

6. Clear decimals

Divide by 0.2 to get $6x(8-x) = x^2(x-4)$.

7. Factor

$48x - 6x^2 = x^3 - 4x^2$, so $x^3 + 2x^2 - 48x = 0$ and $x(x^2 + 2x - 48) = 0$.

8. Solve for x

$x = 0$, or $x^2 + 2x - 48 = 0 = (x+8)(x-6)$, so $x = -8$ or $x = 6$.

9. Find y-values

Using $y = 1.2x(8-x)$ gives $(-8, -153.6)$, $(0, 0)$ and $(6, 14.4)$.

Final answer

(a) $f(x) = 0.2x^2(x-4)$.

(c) $(-8, -153.6),\ (0,0),\ (6, 14.4)$.

Question 6

9 marks

Graphs of Functions

6. (a) Sketch the graph of the curve C with equation

$$y = \frac{4k}{x - 2k}$$

where k is a positive constant.

On your sketch show

- the coordinates of any points where C cuts the coordinate axes
- the equation of the vertical asymptote to C

(4)

The straight line l has equation

$$y = 6 - 2x$$

Given that there is at least one point of intersection between l and C,

(b) find the range of possible values of k.

(5)

(Total for Question 6 is 9 marks)

WMA11/01 OCTOBER 2025

Worked Solution - Question 6

Graphs of Functions

Topic group

1. Find coordinate-axis intersections

$y = \dfrac{4k}{x - 2k}$ has no x-intercept because the numerator is non-zero. At $x = 0$, $y = \dfrac{4k}{-2k} = -2$, so the y-intercept is $(0, -2)$.

2. Find the vertical asymptote

The denominator is zero when $x = 2k$, so the vertical asymptote is $x = 2k$.

3. Sketch shape

The curve has one branch in quadrant I to the right of $x = 2k$, and one branch passing through $(0, -2)$ and approaching the x-axis from below as $x \to -\infty$.

4. Set line and curve equal

$\dfrac{4k}{x - 2k} = 6 - 2x$.

5. Form a quadratic in x

$4k = (6 - 2x)(x - 2k) = 6x - 12k - 2x^2 + 4kx$, so
$2x^2 - (4k + 6)x + 16k = 0$.

6. Use at least one intersection

For at least one real intersection, the discriminant must satisfy $b^2 - 4ac \geq 0$.

7. Find the discriminant

$(4k + 6)^2 - 4(2)(16k) \geq 0$, so $16k^2 - 80k + 36 \geq 0$.

8. Solve in k

$4k^2 - 20k + 9 \geq 0$, so $(2k - 1)(2k - 9) \geq 0$.

9. Use k positive

Therefore $k \le \dfrac{1}{2}$ or $k \ge \dfrac{9}{2}$. Since k is positive, $0

Final answer

(a) y-intercept $(0, -2)$, vertical asymptote $x = 2k$.

$(b)\ 0

TOPIC

Transformations

WMA11/01 OCTOBER 2021

Question 9

9 marks

Transformations

9. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

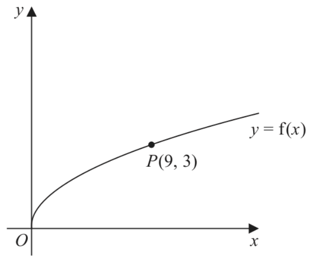


Figure 5

Figure 5 shows a sketch of the curve with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \sqrt{x} \qquad x > 0$$

The point $P(9, 3)$ lies on the curve and is shown in Figure 5.

On the next page there is a copy of Figure 5 called Diagram 1.

(a) On Diagram 1, sketch and clearly label the graphs of

$$y = \mathrm{f}(2x) \quad \text{and} \quad y = \mathrm{f}(x) + 3$$

Show on each graph the coordinates of the point to which P is transformed.

(3)

The graph of $y = \mathrm{f}(2x)$ meets the graph of $y = \mathrm{f}(x) + 3$ at the point Q.

(b) Show that the x coordinate of Q is the solution of

$$\sqrt{x} = 3\left(\sqrt{2} + 1\right)$$

(3)

(c) Hence find, in simplest form, the coordinates of Q.

(3)

Question 9 continued

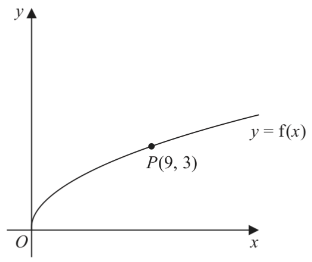
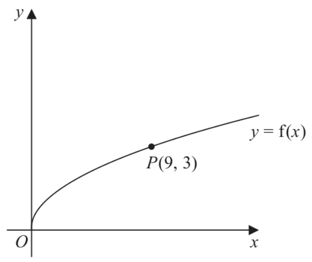


Diagram 1

Turn over for a copy of Diagram 1 if you need to redraw your graphs.

Only use this copy if you need to redraw your graphs.



Copy of Diagram 1

(Total 9 marks)

WMA11/01 OCTOBER 2021

Worked Solution - Question 9

Transformations

Topic group

1. Transform P on y = f(2x)

$y = f(2x)$ is a horizontal scale factor of $\frac{1}{2}$, so $P(9,3)$ maps to $\left(\frac{9}{2}, 3\right)$.

2. Transform P on y = f(x) + 3

$y = f(x) + 3$ is a translation 3 units upward, so $P(9,3)$ maps to $(9,6)$.

3. Set up the intersection equation

At Q, $f(2x) = f(x) + 3$. Since $f(x) = \sqrt{x}$, this gives $\sqrt{2x} = \sqrt{x} + 3$.

4. Show the required equation

Let $s = \sqrt{x}$. Then $\sqrt{2x} = \sqrt{2}\,s$, so $\sqrt{2}s = s + 3$. Hence $(\sqrt{2} - 1)s = 3$ and $s = \dfrac{3}{\sqrt{2} - 1} = 3(\sqrt{2} + 1)$.

5. Find the x-coordinate

$\sqrt{x} = 3(\sqrt{2} + 1)$, so $x = 9(\sqrt{2} + 1)^2 = 9(3 + 2\sqrt{2}) = 27 + 18\sqrt{2}$.

6. Find the y-coordinate

$y = \sqrt{x} + 3 = 3(\sqrt{2} + 1) + 3 = 6 + 3\sqrt{2}$. Therefore $Q = (27 + 18\sqrt{2},\ 6 + 3\sqrt{2})$.

Final answer

$(a)\ P \mapsto \left(\frac{9}{2}, 3\right)$ on $y = f(2x)$, $P \mapsto (9,6)$ on $y = f(x) + 3$.

$(c)\ Q = (27 + 18\sqrt{2},\ 6 + 3\sqrt{2})$.

Question 7

8 marks

Transformations

7.

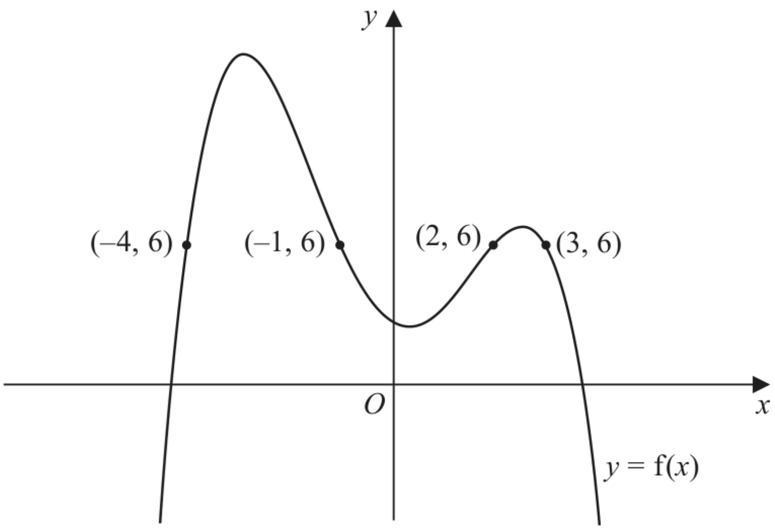


Figure 1

Figure 1 shows the curve with equation $y = \mathrm{f}(x)$.

The points $P(-4, 6)$, $Q(-1, 6)$, $R(2, 6)$ and $S(3, 6)$ lie on the curve.

(a) Using Figure 1, find the range of values of x for which

$$\mathrm{f}(x) < 6$$

(3)

(b) State the largest solution of the equation

$$\mathrm{f}(2x) = 6$$

(1)

(c) (i) Sketch the curve with equation $y = \mathrm{f}(-x)$.

On your sketch, state the coordinates of the points to which P, Q, R and S are transformed.

(ii) Hence find the set of values of x for which

$$\mathrm{f}(-x) \geqslant 6 \quad \text{and} \quad x < 0$$

(4)

(Total for Question 7 is 8 marks)

WMA11/01 OCTOBER 2022

Worked Solution - Question 7

Transformations

Topic group

1. Read f(x) < 6 from the graph

The curve is below the line $y = 6$ for $x < -4$, for -13.

2. Solve f(2x) = 6

Since $f(u) = 6$ at $u = -4, -1, 2, 3$, we set $2x = -4, -1, 2, 3$. The solutions are $x = -2, -\frac{1}{2}, 1, \frac{3}{2}$.

3. State the largest solution

The largest solution is $x = \frac{3}{2}$.

4. Reflect the points in the y-axis

For $y = f(-x)$, each point (x, y) becomes $(-x, y)$.

5. List the transformed points

$P(-4, 6) \to (4, 6)$, $Q(-1, 6) \to (1, 6)$, $R(2, 6) \to (-2, 6)$, and $S(3, 6) \to (-3, 6)$.

6. Use the reflected graph

For $x < 0$, the part of the reflected graph on or above $y = 6$ is from $x = -3$ to $x = -2$.

Final answer

$(a)\ x<-4,\ -13.$

$(b)\ \frac{3}{2}.$

$(c)(ii)\ -3 \le x \le -2.$

WMA11/01 JANUARY 2023

Question 7

10 marks

Transformations

7. (a) On Diagram 1, sketch a graph of the curve C with equation

$$y = \frac{6}{x} \qquad x \neq 0$$

(2)

The curve C is transformed onto the curve with equation $y = \frac{6}{x-2} \qquad x \neq 2$

(b) Fully describe this transformation.

(2)

The curve with equation

$$y = \frac{6}{x-2} \qquad x \neq 2$$

and the line with equation

$$y = kx + 7 \qquad \text{where } k \text{ is a constant}$$

intersect at exactly two points, P and Q.

Given that the x coordinate of point P is -4

(c) find the value of k,

(2)

(d) find, using algebra, the coordinates of point Q.

(*Solutions relying entirely on calculator technology are not acceptable.*)

(4)

Question 7 continued

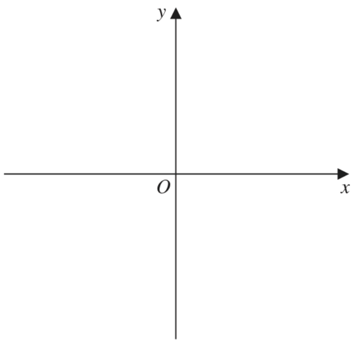
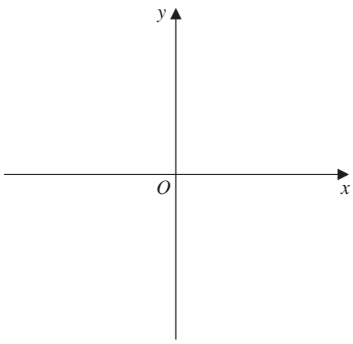


Diagram 1

Only use this copy of Diagram 1 if you need to redraw your graph.



Copy of Diagram 1

(Total for Question 7 is 10 marks)

WMA11/01 JANUARY 2023

Worked Solution - Question 7

Transformations

Topic group

1. Sketch y = 6/x

The graph is a rectangular hyperbola with asymptotes $x = 0$ and $y = 0$, with branches in quadrants I and III.

2. Describe the transformation

$y = \dfrac{6}{x-2}$ is obtained from $y = \dfrac{6}{x}$ by a translation 2 units in the positive x-direction.

3. Use P to find k

At $x = -4$ on $y = \dfrac{6}{x-2}$, $y = \dfrac{6}{-6} = -1$.

4. Substitute into the line

Since P lies on $y = kx + 7$, $-1 = -4k + 7$, so $k = 2$.

5. Find the second intersection

Solve $\dfrac{6}{x-2} = 2x + 7$. Multiplying by $x - 2$ gives $6 = (2x+7)(x-2)$.

6. Factor and choose Q

$6 = 2x^2 + 3x - 14$, so $2x^2 + 3x - 20 = 0 = (2x-5)(x+4)$. The root $x = -4$ is P, so Q has $x = \dfrac{5}{2}$.

7. Find y for Q

$y = 2\left(\dfrac{5}{2}\right) + 7 = 12$, so $Q = \left(\dfrac{5}{2}, 12\right)$.

Final answer

(b) Translation 2 units right.

(c) $k = 2.$

(d) $Q = \left(\frac{5}{2}, 12\right).$

Question 9

9.

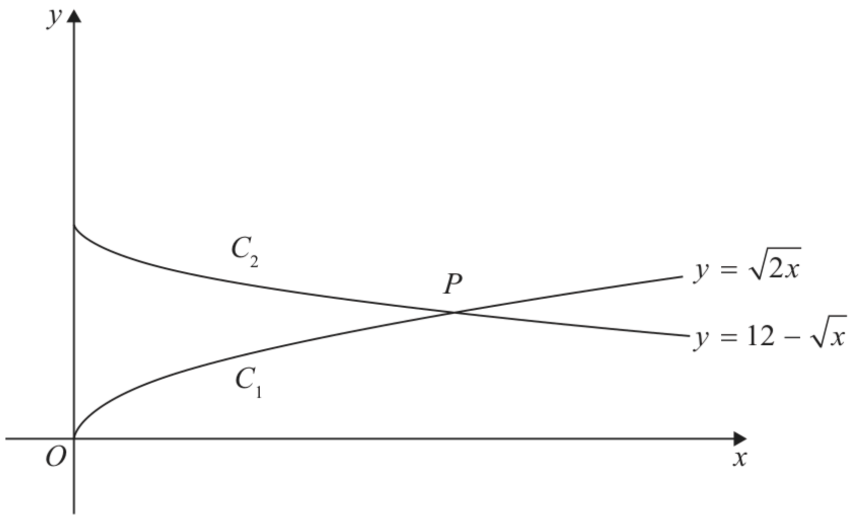


Figure 4

In this question you must show all stages of your working.
Solutions relying on calculator technology are not acceptable.

Figure 4 shows a sketch of

- the graph C_1 with equation $y = \sqrt{2x}$
- the graph C_2 with equation $y = 12 - \sqrt{x}$

(a) Describe fully the single transformation that would transform

(i) the graph with equation $y = \sqrt{x}$ onto C_1

(ii) the graph with equation $y = -\sqrt{x}$ onto C_2

(4)

The graphs C_1 and C_2 meet at the point P, as shown in Figure 4.

(b) (i) Show that the x coordinate of P is a solution of

$$\sqrt{x} = 12\left(\sqrt{2} - 1\right)$$

(ii) Hence find, in simplest form, the exact coordinates of P.

(6)

(Total for Question 9 is 10 marks)

WMA11/01 MAY/JUNE 2025

Worked Solution - Question 9

Transformations

Topic group

1. Transform y equals root x to C1

C_1 has equation $y = \sqrt{2x}$. This is obtained from $y = \sqrt{x}$ by a stretch parallel to the x-axis with scale factor $\dfrac{1}{2}$.

2. Equivalent description

Equivalently, $y = \sqrt{2x} = \sqrt{2}\sqrt{x}$ is a stretch parallel to the y-axis with scale factor $\sqrt{2}$. The mark scheme also accepts the horizontal stretch description.

3. Transform y equals minus root x to C2

C_2 has equation $y = 12 - \sqrt{x} = -\sqrt{x} + 12$, so it is a translation of $y = -\sqrt{x}$ by $\begin{pmatrix} 0 \\ 12 \end{pmatrix}$.

4. Set the curves equal

At P, $\sqrt{2x} = 12 - \sqrt{x}$.

5. Factor out root x

$\sqrt{2}\sqrt{x} + \sqrt{x} = 12$, so $(\sqrt{2}+1)\sqrt{x} = 12$.

6. Rationalise

$$\sqrt{x} = \frac{12}{\sqrt{2}+1} = \frac{12(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)} = 12(\sqrt{2}-1).$$

7. Find x

$x = [12(\sqrt{2}-1)]^2 = 144(3 - 2\sqrt{2})$.

8. Find y

$y = \sqrt{2x} = \sqrt{2}\sqrt{x} = \sqrt{2} \cdot 12(\sqrt{2}-1) = 24 - 12\sqrt{2}$.

Final answer

$(a)(i)$ stretch parallel to the x-axis scale factor $\dfrac{1}{2}$ (equivalently vertical stretch scale factor $\sqrt{2}$).

$(a)(ii)$ translation by $\begin{pmatrix} 0 \\ 12 \end{pmatrix}$.

(b) $P = (144(3 - 2\sqrt{2}),\ 24 - 12\sqrt{2})$.

Question 9

6 marks

Transformations

9.

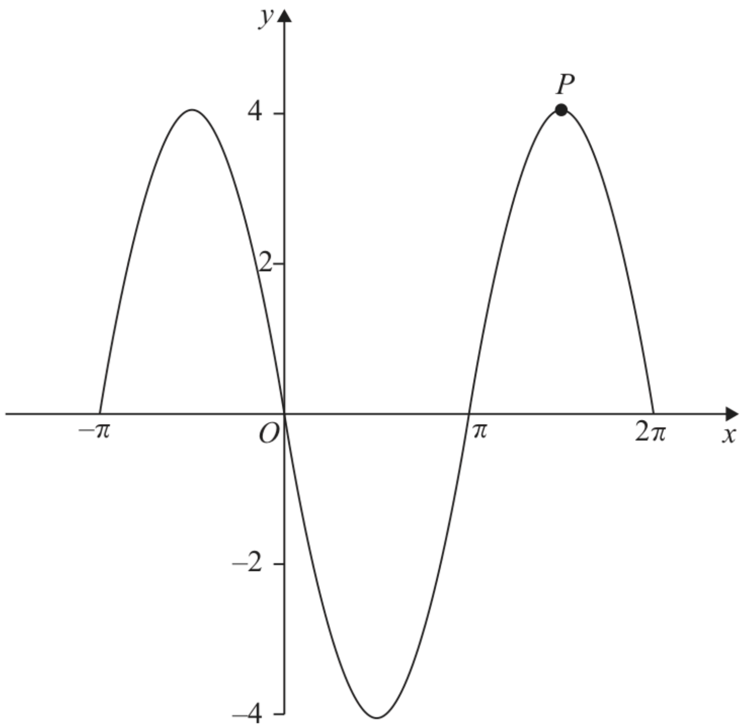


Figure 3

Figure 3 shows a sketch of part of the graph of the trigonometric function with equation $y = \mathrm{f}(x)$

(a) Write down an expression for $\mathrm{f}(x)$

(2)

The point P lies on $y = \mathrm{f}(x)$ and is shown in Figure 3.

(b) State the coordinates of the point to which P is transformed when the graph of $y = \mathrm{f}(x)$ is transformed to the graph with equation

(i) $y = \mathrm{f}\left(x - \dfrac{\pi}{6}\right)$

(2)

(ii) $y = -\dfrac{1}{2}\mathrm{f}(x)$

(2)

(Total for Question 9 is 6 marks)

WMA11/01 OCTOBER 2025

Worked Solution - Question 9

Transformations

Topic group

1. Identify amplitude and direction

The graph has amplitude 4, crosses the origin, and initially moves downwards, so $f(x) = -4\sin x$.

2. Identify P

The labelled maximum is at $P = \left(\frac{3\pi}{2}, 4\right)$.

3. Transform y equals f of x minus pi over 6

$y = f\left(x - \frac{\pi}{6}\right)$ translates the graph right by $\frac{\pi}{6}$.

4. New coordinates for part b(i)

P becomes $\left(\frac{3\pi}{2} + \frac{\pi}{6}, 4\right) = \left(\frac{5\pi}{3}, 4\right)$.

5. Transform y equals negative half f(x)

$y = -\frac{1}{2}f(x)$ leaves x-coordinates unchanged and multiplies y-coordinates by $-\frac{1}{2}$.

6. New coordinates for part b(ii)

P becomes $\left(\frac{3\pi}{2}, -2\right)$.

Final answer

$(a)\ f(x) = -4\sin x.$

$(b)(i)\ \left(\frac{5\pi}{3}, 4\right).$

$(b)(ii)\ \left(\frac{3\pi}{2}, -2\right).$

TOPIC

Straight Line

Question 6

6. The line l_1 has equation $3x - 4y + 20 = 0$

The line l_2 cuts the x-axis at $R(8,0)$ and is parallel to l_1

(a) Find the equation of l_2, writing your answer in the form $ax + by + c = 0$, where a, b and c are integers to be found.

(3)

The line l_1 cuts the x-axis at P and the y-axis at Q.

Given that $PQRS$ is a parallelogram, find

(b) the area of $PQRS$,

(3)

(c) the coordinates of S.

(2)

(Total 8 marks)

WMA11/01 JANUARY 2020

Worked Solution - Question 6

Straight Line

Topic group

1. Find the gradient of l1

$3x - 4y + 20 = 0$ gives $y = \frac{3}{4}x + 5$, so the gradient is $\frac{3}{4}$.

2. Find l2 through R

Line l_2 is parallel to l_1 and passes through $R(8, 0)$, so $y = \frac{3}{4}(x - 8) = \frac{3}{4}x - 6$. Therefore $3x - 4y - 24 = 0$.

3. Find P and Q

For l_1, the x-intercept is $P\left(-\frac{20}{3}, 0\right)$ and the y-intercept is $Q = (0, 5)$.

4. Find the area of PQRS

Base $PR = 8 - \left(-\frac{20}{3}\right) = \frac{44}{3}$ and height $OQ = 5$. Area $= \frac{44}{3} \cdot 5 = \frac{220}{3}$.

5. Find S using a vector

In parallelogram $PQRS$, $\vec{PS} = \vec{QR} = (8, -5)$. Hence $S = P + (8, -5) = \left(-\frac{20}{3} + 8, -5\right) = \left(\frac{4}{3}, -5\right)$.

Final answer

$(a)\ 3x - 4y - 24 = 0.$

$(b)\ \dfrac{220}{3}.$

$(c)\ S = \left(\dfrac{4}{3}, -5\right).$

Question 8

8.

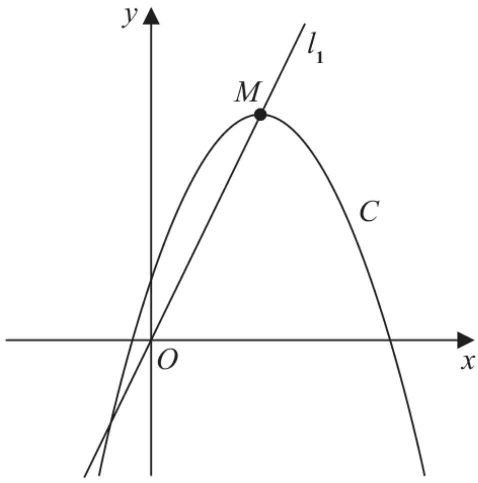


Figure 4

Figure 4 shows a sketch of the curve C with equation

$$y = 4 + 12x - 3x^2$$

The point M is the maximum turning point on C.

(a) (i) Write $4 + 12x - 3x^2$ in the form

$$a + b(x + c)^2$$

where a, b and c are constants to be found.

(ii) Hence, or otherwise, state the coordinates of M.

(5)

The line l_1 passes through O and M, as shown in Figure 4.

A line l_2 touches C and is parallel to l_1

(b) Find an equation for l_2

(5)

(Total 10 marks)

WMA11/01 OCTOBER 2021

Worked Solution - Question 8

Straight Line

Topic group

1. Complete the square

$4 + 12x - 3x^2 = -3(x^2 - 4x) + 4 = -3\big((x-2)^2 - 4\big) + 4 = 16 - 3(x-2)^2$

.

2. State the maximum point

The maximum occurs when $(x-2)^2 = 0$, so $M = (2, 16)$.

3. Find the gradient of l1

Line l_1 passes through $O(0,0)$ and $M(2,16)$, so its gradient is $\frac{16}{2} = 8$.

4. Use the parallel tangent gradient

Line l_2 is parallel to l_1, so its gradient is also 8. For $y = 4 + 12x - 3x^2$, $\frac{dy}{dx} = 12 - 6x$.

5. Find the point of contact

Set $12 - 6x = 8$, giving $x = \frac{2}{3}$. Then $y = 4 + 12\left(\frac{2}{3}\right) - 3\left(\frac{4}{9}\right) = \frac{32}{3}$.

6. Write l2

Using gradient 8 through $\left(\frac{2}{3}, \frac{32}{3}\right)$, $y - \frac{32}{3} = 8\left(x - \frac{2}{3}\right)$. Hence $y = 8x + \frac{16}{3}$.

Final answer

$(a)(i)\ 16 - 3(x-2)^2.$ $(a)(ii)\ M = (2, 16).$

$(b)\ y = 8x + \dfrac{16}{3}.$

Question 8

9 marks

Straight Line

8. The line l_1 has equation

$$2x - 5y + 7 = 0$$

(a) Find the gradient of l_1

(1)

Given that

- the point A has coordinates $(6, -2)$
- the line l_2 passes through A and is perpendicular to l_1

(b) find the equation of l_2 giving your answer in the form $y = mx + c$, where m and c are constants to be found.

(3)

The lines l_1 and l_2 intersect at the point M.

(c) Using algebra and showing all your working, find the coordinates of M.

(*Solutions relying on calculator technology are not acceptable.*)

(3)

Given that the diagonals of a square $ABCD$ meet at M,

(d) find the coordinates of the point C.

(2)

(Total 9 marks)

WMA11/01 JANUARY 2022

Worked Solution - Question 8

Straight Line

Topic group

1. Find the gradient of l1

$2x - 5y + 7 = 0$ gives $y = \frac{2}{5}x + \frac{7}{5}$, so the gradient is $\frac{2}{5}$.

2. Find l2

The gradient perpendicular to $\frac{2}{5}$ is $-\frac{5}{2}$. Through $A(6, -2)$, $y + 2 = -\frac{5}{2}(x - 6)$, so $y = -\frac{5}{2}x + 13$.

3. Find M

At the intersection, $\frac{2}{5}x + \frac{7}{5} = -\frac{5}{2}x + 13$. Multiplying by 10 gives $4x + 14 = -25x + 130$, so $x = 4$.

4. Find the y-coordinate of M

$y = \frac{2}{5}(4) + \frac{7}{5} = 3$, so $M = (4, 3)$.

5. Use midpoint of the diagonal

In a square, the diagonals bisect each other, so M is the midpoint of AC. Thus $C = 2M - A = (8, 6) - (6, -2) = (2, 8)$.

Final answer

(a) $\frac{2}{5}$.

(b) $y = -\frac{5}{2}x + 13$.

(c) $M = (4, 3)$.

(d) $C = (2, 8)$.

WMA11/01 MAY/JUNE 2023

Question 10

10 marks

Straight Line

10.

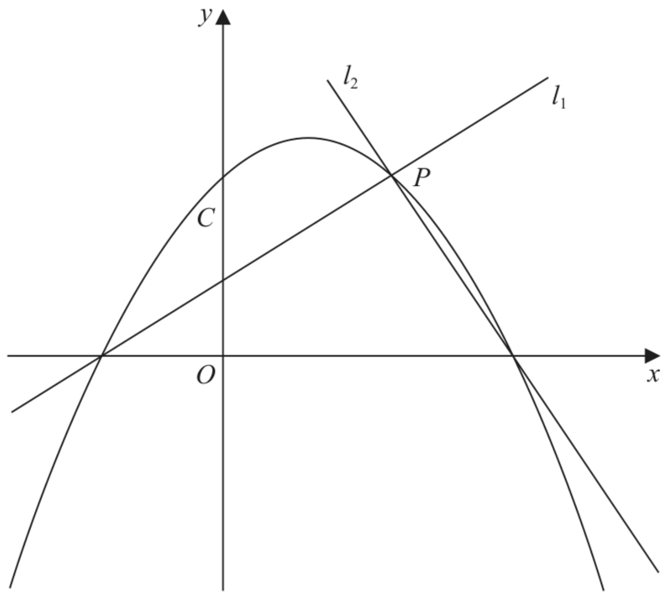


Figure 5

Figure 5 shows a sketch of the quadratic curve C with equation

$$y = -\frac{1}{4}(x+2)(x-b) \qquad \text{where } b \text{ is a positive constant}$$

The line l_1 also shown in Figure 5,

- has gradient $\frac{1}{2}$
- intersects C on the negative x-axis and at the point P

(a) (i) Write down an equation for l_1

(1)

(ii) Find, in terms of b, the coordinates of P

(3)

Given that the line l_2 is perpendicular to l_1 and intersects C on the positive x-axis,

(b) find, in terms of b, an equation for l_2

(2)

Given also that l_2 intersects C at the point P

(c) show that another equation for l_2 is

$$y = -2x + \frac{5b}{2} - 4$$

(2)

(d) Hence, or otherwise, find the value of b

(Total for Question 10 is 10 marks)

WMA11/01 MAY/JUNE 2023

Worked Solution - Question 10

Straight Line

Topic group

1. Find l1

Line l_1 has gradient $\frac{1}{2}$ and passes through the negative x-axis root $(-2, 0)$, so $y = \frac{1}{2}(x+2) = \frac{1}{2}x + 1$.

2. Find P

Set $-\frac{1}{4}(x+2)(x-b) = \frac{1}{2}(x+2)$.

3. Solve the intersection

Multiplying by -4 gives $(x+2)(x-b) = -2(x+2)$, so $(x+2)(x-b+2) = 0$.

4. Choose the second point

The root $x = -2$ is the x-axis point, so P has $x = b-2$. Then $y = \frac{1}{2}(b-2) + 1 = \frac{b}{2}$.

5. Find l2 from the positive root

Since l_2 is perpendicular to l_1, its gradient is -2. It passes through the positive x-axis root $(b, 0)$, so $y = -2(x-b) = -2x + 2b$.

6. Use P to write another l2

Through $P\left(b-2, \frac{b}{2}\right)$ with gradient -2: $y - \frac{b}{2} = -2(x-(b-2))$, so $y = -2x + \frac{5b}{2} - 4$.

7. Find b

Both equations represent l_2, so $2b = \frac{5b}{2} - 4$. Hence $b = 8$.

Final answer

$(a)(i)$ $y = \dfrac{1}{2}x + 1.$ $(a)(ii)$ $P = \left(b - 2, \dfrac{b}{2}\right).$

(b) $y = -2x + 2b.$

(d) $b = 8.$

Question 9

9 marks

Straight Line

9. Given that

- the point A has coordinates (4, 2)
- the point B has coordinates (15, 7)
- the line l_1 passes through A and B

(a) find an equation for l_1, giving your answer in the form $px + qy + r = 0$ where p, q and r are integers to be found.

(3)

The line l_2 passes through A and is parallel to the x-axis.

The point C lies on l_2 so that the length of BC is $5\sqrt{5}$

(b) Find both possible pairs of coordinates of the point C.

(4)

(c) Hence find the minimum possible area of triangle ABC.

(2)

(Total for Question 9 is 9 marks)

WMA11/01 MAY/JUNE 2024

Worked Solution - Question 9

Straight Line

Topic group

1. Find the gradient of l1

$m = \frac{7-2}{15-4} = \frac{5}{11}$.

2. Use point A

$y - 2 = \frac{5}{11}(x-4)$.

3. Rearrange

$11y - 22 = 5x - 20$, so $5x - 11y + 2 = 0$.

4. Use the horizontal line l2

Since l_2 passes through $A(4, 2)$ and is parallel to the x-axis, point C has coordinates $(c, 2)$.

5. Use BC equals 5 root 5

$(c-15)^2 + (2-7)^2 = (5\sqrt{5})^2$.

6. Solve for c

$(c-15)^2 + 25 = 125$, so $(c-15)^2 = 100$ and $c = 5$ or 25.

7. Find the smaller area

The height from B to $y = 2$ is 5. For $C = (5, 2)$, $AC = 1$, so the minimum area is $\frac{1}{2}(1)(5) = \frac{5}{2}$.

Final answer

(a) $5x - 11y + 2 = 0$.

(b) $C = (5, 2)$ or $(25, 2)$.

(c) $\dfrac{5}{2}$.

TOPIC

Radians

WMA11/01 MAY/JUNE 2021

Question 7

10 marks

Radians

7.

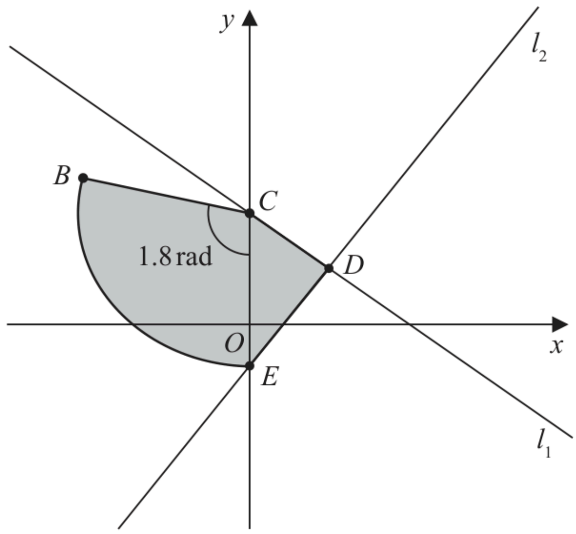


Figure 3

The line l_1 has equation $4y + 3x = 48$

The line l_1 cuts the y-axis at the point C, as shown in Figure 3.

(a) State the y coordinate of C.

(1)

The point $D(8, 6)$ lies on l_1

The line l_2 passes through D and is perpendicular to l_1

The line l_2 cuts the y-axis at the point E as shown in Figure 3.

(b) Show that the y coordinate of E is $-\frac{14}{3}$

(3)

A sector BCE of a circle with centre C is also shown in Figure 3.

Given that angle BCE is 1.8 radians,

(c) find the length of arc BE.

(3)

The region $CBED$, shown shaded in Figure 3, consists of the sector BCE joined to the triangle CDE.

(d) Calculate the exact area of the region $CBED$.

(3)

(Total 10 marks)

WMA11/01 MAY/JUNE 2021

Worked Solution - Question 7

Radians

Topic group

1. Find C

On the y-axis, $x = 0$. From $4y + 3x = 48$, $4y = 48$, so the y-coordinate of C is 12.

2. Find the gradient of l1

$4y + 3x = 48$ gives $y = -\frac{3}{4}x + 12$, so the gradient of l_1 is $-\frac{3}{4}$.

3. Use perpendicular gradients for l2

Since l_2 is perpendicular to l_1, its gradient is $\frac{4}{3}$. Through $D(8, 6)$, $y - 6 = \frac{4}{3}(x - 8)$.

4. Show the y-coordinate of E

$y = \frac{4}{3}x - \frac{32}{3} + 6 = \frac{4}{3}x - \frac{14}{3}$. Therefore when $x = 0$, $E_y = -\frac{14}{3}$.

5. Find the radius of the sector

$C = (0, 12)$ and $E = \left(0, -\frac{14}{3}\right)$, so $CE = 12 + \frac{14}{3} = \frac{50}{3}$.

6. Find arc BE

For sector BCE, $r = \frac{50}{3}$ and $\theta = 1.8 = \frac{9}{5}$. Hence arc $BE = r\theta = \frac{50}{3} \cdot \frac{9}{5} = 30$.

7. Find the sector area

Sector area $= \frac{1}{2}r^2\theta = \frac{1}{2}\left(\frac{50}{3}\right)^2\left(\frac{9}{5}\right) = 250$.

8. Find the triangle area and total area

Triangle CDE has vertical base $CE = \frac{50}{3}$ and horizontal height 8, so its area is $\frac{1}{2} \cdot \frac{50}{3} \cdot 8 = \frac{200}{3}$. Total area $= 250 + \frac{200}{3} = \frac{950}{3}$.

Final answer

(a) 12.

(b) $E_y = -\frac{14}{3}$.

(c) 30.

(d) $\frac{950}{3}$.

Question 7

10 marks

Radians

7.

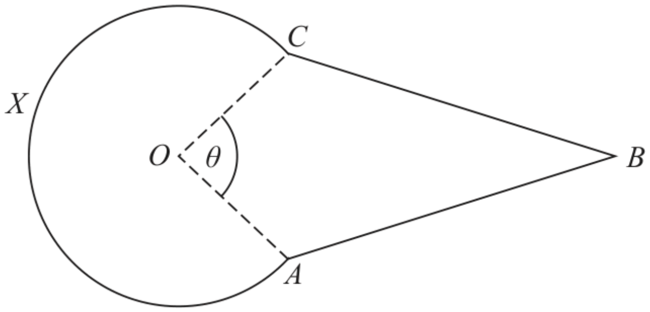


Figure 3

Figure 3 shows the design for a sign at a bird sanctuary.

The design consists of a kite $OABC$ joined to a sector $OCXA$ of a circle centre O.

In the design

- $OA = OC = 0.6\,\text{m}$
- $AB = CB = 1.4\,\text{m}$
- Angle OAB = Angle OCB = 2 radians
- Angle $AOC = \theta$ radians, as shown in Figure 3

Making your method clear,

(a) show that $\theta = 1.64$ radians to 3 significant figures,

(4)

(b) find the perimeter of the sign, in metres to 2 significant figures,

(2)

(c) find the area of the sign, in m^2 to 2 significant figures.

(4)

(Total 10 marks)

WMA11/01 OCTOBER 2021

Worked Solution - Question 7

Radians

Topic group

1. Work in triangle OAB

In triangle OAB, $OA = 0.6$, $AB = 1.4$, and $\angle OAB = 2$ radians.

2. Find OB

By the cosine rule, $OB^2 = 0.6^2 + 1.4^2 - 2(0.6)(1.4)\cos 2$, so $OB = 1.73756\ldots$.

3. Find half of theta

Using the cosine rule again, $\cos\angle AOB = \dfrac{0.6^2 + OB^2 - 1.4^2}{2(0.6)(OB)}$, giving $\angle AOB = 0.822199\ldots$ radians.

4. Use symmetry of the kite

The kite is symmetric, so $\theta = 2\angle AOB = 1.644398\ldots$, hence $\theta = 1.64$ radians to 3 significant figures.

5. Find the outside arc length

The sector $OCXA$ uses the major angle $2\pi - \theta$. Its arc length is $0.6(2\pi - 1.644398\ldots) = 2.78327\ldots$.

6. Find the perimeter

Perimeter $= AB + CB + \text{major arc} = 1.4 + 1.4 + 2.78327\ldots = 5.58327\ldots$, so the perimeter is 5.6 m.

7. Find the kite area

The kite consists of two congruent triangles like OAB, so area of kite $= 2\left(\frac{1}{2}(0.6)(1.4)\sin 2\right) = 0.76381\ldots$.

8. Find the sector area and total area

Major sector area $= \frac{1}{2}(0.6)^2(2\pi - 1.644398\ldots) = 0.83498\ldots$. Total area $= 0.76381\ldots + 0.83498\ldots = 1.59879\ldots$, so the area is $1.6\ \mathrm{m}^2$.

Final answer

$(a)\ \theta = 1.64\ \mathrm{rad}.$

$(b)\ 5.6\ \mathrm{m}.$

$(c)\ 1.6\ \mathrm{m}^2.$

WMA11/01 MAY/JUNE 2022

10 marks

Question 8

Radians

8.

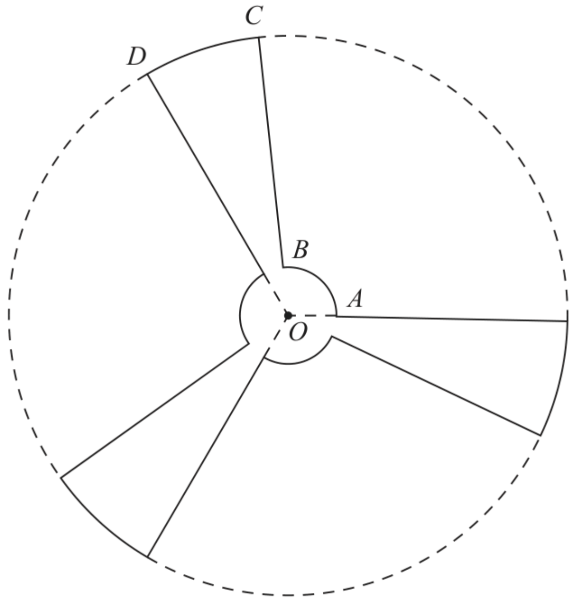


Figure 3

Figure 3 shows a sketch of the outline of the face of a ceiling fan viewed from below.

The fan consists of three identical sections congruent to *OABCDO*, shown in Figure 3, where

- *OABO* is a sector of a circle with centre *O* and radius 9 cm
- *OBCDO* is a sector of a circle with centre *O* and radius 84 cm
- angle $AOD = \dfrac{2\pi}{3}$ radians

Given that the length of the arc *AB* is 15 cm,

(a) show that the length of the arc *CD* is 35.9 cm to one decimal place.

(3)

The face of the fan is modelled to be a flat surface.

Find, according to the model,

(b) the perimeter of the face of the fan, giving your answer to the nearest cm,

(2)

(c) the surface area of the face of the fan.

Give your answer to 3 significant figures and make your units clear.

(5)

(Total 10 marks)

WMA11/01 MAY/JUNE 2022

Worked Solution - Question 8

Radians

Topic group

1. Find the small sector angle

For arc AB, $s = r\theta$, so $15 = 9\theta$ and $\theta = \dfrac{5}{3}$ radians.

2. Find the outer arc angle

Each identical section has total angle $\dfrac{2\pi}{3}$, so the angle for arc CD is $\dfrac{2\pi}{3} - \dfrac{5}{3} = \dfrac{2\pi - 5}{3}$.

3. Find CD

$CD = 84\left(\dfrac{2\pi - 5}{3}\right) = 28(2\pi - 5) = 35.9\ \mathrm{cm}$ to 3 significant figures.

4. Find the perimeter

There are 3 inner arcs, 3 outer arcs, and 6 straight radial edges of length $84 - 9 = 75$. Thus $P = 3(15) + 3(35.929\ldots) + 6(75) = 602.8\ldots\ \mathrm{cm}$.

5. Find the area of one section

One section has area $\dfrac{1}{2}(9^2)\left(\dfrac{5}{3}\right) + \dfrac{1}{2}(84^2)\left(\dfrac{2\pi - 5}{3}\right) = 1576.525\ldots\ \mathrm{cm}^2$.

6. Find the total area

The total area is $3(1576.525\ldots) = 4729.57\ldots\ \mathrm{cm}^2 = 4.73 \times 10^3\ \mathrm{cm}^2$ to 3 significant figures.

Final answer

(a) $CD = 35.9$ cm.

(b) 603 cm.

(c) 4.73×10^3 cm^2.

WMA11/01 OCTOBER 2022

Question 8

14 marks

Radians

8.

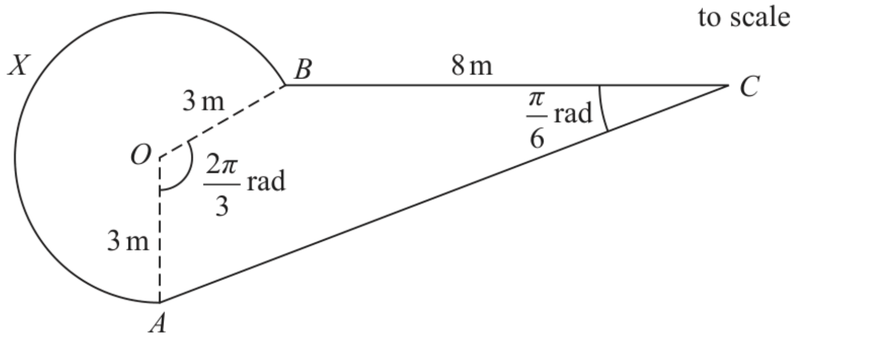


Figure 2

Figure 2 shows the plan view of a design for a pond.

The design consists of a sector $AOBX$ of a circle centre O joined to a quadrilateral $AOBC$.

- $BC = 8\,\text{m}$
- $OA = OB = 3\,\text{m}$
- angle AOB is $\frac{2\pi}{3}$ radians
- angle BCA is $\frac{\pi}{6}$ radians

(a) Calculate (i) the exact area of the sector $AOBX$,

(ii) the exact perimeter of the sector $AOBX$.

(5)

(b) Calculate the exact area of the triangle AOB.

(2)

(c) Show that the length AB is $3\sqrt{3}$ m.

(2)

(d) Find the total surface area of the pond. Give your answer in m^2 correct to 2 significant figures.

(5)

(Total for Question 8 is 14 marks)

WMA11/01 OCTOBER 2022

Worked Solution - Question 8

Radians

Topic group

1. Find the angle of the sector AOBX

The sector AOBX is the major sector, so its angle is $2\pi - \dfrac{2\pi}{3} = \dfrac{4\pi}{3}$.

2. Find the sector area

$\text{Area} = \dfrac{1}{2}r^2\theta = \dfrac{1}{2}(3^2)\left(\dfrac{4\pi}{3}\right) = 6\pi\ \mathrm{m}^2$.

3. Find the sector perimeter

The arc length is $r\theta = 3\left(\dfrac{4\pi}{3}\right) = 4\pi$. Adding the two radii gives $4\pi + 6\ \mathrm{m}$.

4. Find the area of triangle AOB

$\text{Area} = \dfrac{1}{2}(3)(3)\sin\left(\dfrac{2\pi}{3}\right) = \dfrac{9}{2}\cdot\dfrac{\sqrt{3}}{2} = \dfrac{9\sqrt{3}}{4}\ \mathrm{m}^2$.

5. Show AB

By the cosine rule, $AB^2 = 3^2 + 3^2 - 2(3)(3)\cos\left(\dfrac{2\pi}{3}\right) = 18 + 9 = 27$, so $AB = 3\sqrt{3}$.

6. Find AC

In triangle ABC, $AB^2 = AC^2 + 8^2 - 2(AC)(8)\cos\left(\dfrac{\pi}{6}\right)$, so $27 = AC^2 - 8\sqrt{3}\,AC + 64$.

7. Choose the correct length

Solving gives $AC = 4\sqrt{3} \pm \sqrt{11}$. From the diagram, AC is the longer value, so $AC = 4\sqrt{3} + \sqrt{11}$.

8. Find the total area

Area of triangle ABC is $\frac{1}{2}(8)(AC)\sin\left(\frac{\pi}{6}\right) = 2AC = 8\sqrt{3} + 2\sqrt{11}$. Total area is $6\pi + \frac{9\sqrt{3}}{4} + 8\sqrt{3} + 2\sqrt{11} = 43.236\ldots$, so $43\ \mathrm{m}^2$ to 2 significant figures.

Final answer

$(a)(i)\ 6\pi\ \mathrm{m}^2,\ (ii)\ 4\pi + 6\ \mathrm{m}.$

$(b)\ \frac{9\sqrt{3}}{4}\ \mathrm{m}^2.$

$(c)\ AB = 3\sqrt{3}\ \mathrm{m}.$

$(d)\ 43\ \mathrm{m}^2.$

Question 6

10 marks

Radians

6.

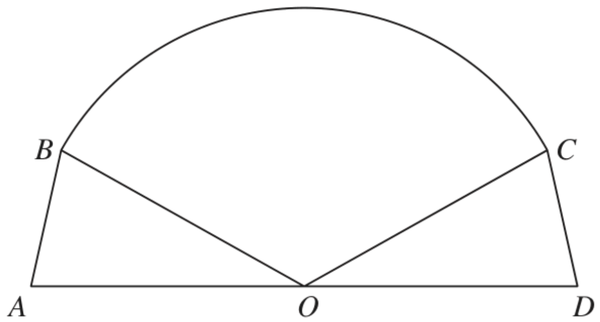


Diagram **NO**
accurately dr

Figure 1

Figure 1 shows the plan view for the design of a stage.

The design consists of a sector OBC of a circle, with centre O, joined to two congruent triangles OAB and ODC.

Given that

- angle $BOC = 2.4$ radians
- area of sector $BOC = 40\,\text{m}^2$
- AOD is a straight line of length $12.5\,\text{m}$

(a) find the radius of the sector, giving your answer, in m, to 2 decimal places,

(2)

(b) find the size of angle AOB, in radians, to 2 decimal places.

(1)

Hence find

(c) the total area of the stage, giving your answer, in m^2, to one decimal place,

(3)

(d) the total perimeter of the stage, giving your answer, in m, to one decimal place.

(4)

(Total for Question 6 is 10 marks)

WMA11/01 JANUARY 2023

Worked Solution - Question 6

Radians

Topic group

1. Find the radius

Area of sector BOC is $\frac{1}{2}r^2\theta$. So $40 = \frac{1}{2}r^2(2.4) = 1.2r^2$, giving $r = \sqrt{\frac{40}{1.2}} = 5.77$ m.

2. Find angle AOB

The triangles OAB and ODC are congruent, so $\angle AOB = \angle COD$. Since AOD is a straight line, $2\angle AOB + 2.4 = \pi$.

3. Calculate the angle

$\angle AOB = \dfrac{\pi - 2.4}{2} = 0.3708\ldots$, so 0.37 rad.

4. Find the total area

The two congruent triangle areas together are

$$2\left(\frac{1}{2}(6.25)(5.7735\ldots)\sin 0.3708\ldots\right) = 13.075\ldots.$$

5. Add the sector area

Total area $= 40 + 13.075\ldots = 53.1\ \mathrm{m}^2$ to one decimal place.

6. Find AB

Using the cosine rule, $AB^2 = 6.25^2 + r^2 - 2(6.25)(r)\cos 0.3708\ldots$, giving $AB = 2.265\ldots$ m.

7. Find the perimeter

Perimeter

$= AD + AB + CD + \text{arc } BC = 12.5 + 2(2.265\ldots) + 5.7735\ldots(2.4) = 30.9$ m.

Final answer

(a) $r = 5.77$ m.

(b) 0.37 rad.

(c) $53.1\ \mathrm{m}^2$.

(d) 30.9 m.

WMA11/01 OCTOBER 2023

Question 9

7 marks

Radians

9.

Diagram **NO** accurately dr

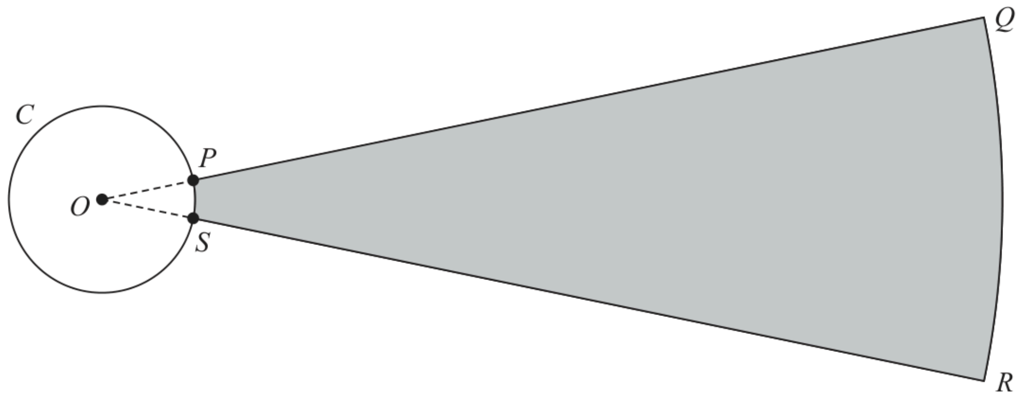


Figure 3

Figure 3 shows the plan view of the area being used for a ball-throwing competition.

Competitors must stand within the circle C and throw a ball as far as possible into the target area, $PQRS$, shown shaded in Figure 3.

Given that

- circle C has centre O
- P and S are points on C
- $OPQRSO$ is a sector of a circle with centre O
- the length of arc PS is 0.72 m
- the size of angle POS is 0.6 radians

(a) show that $OP = 1.2\,\text{m}$

(1)

Given also that

- the target area, $PQRS$, is $90\,\text{m}^2$
- length $PQ = x$ metres

(b) show that

$$5x^2 + 12x - 1500 = 0$$

(3)

(c) Hence calculate the total perimeter of the target area, $PQRS$, giving your answer to the nearest metre.

(3)

(Total for Question 9 is 7 marks)

WMA11/01 OCTOBER 2023

Worked Solution - Question 9

Radians

Topic group

1. Use arc length

$s = r\theta$, so $0.72 = OP \times 0.6$. Hence $OP = 1.2\text{ m}$.

2. Set up the target area

The outer sector has radius $x + 1.2$ and the inner sector has radius 1.2, with angle 0.6 radians.

3. Use sector area

$\frac{1}{2}(x+1.2)^2(0.6) - \frac{1}{2}(1.2)^2(0.6) = 90.$

4. Simplify to the given quadratic

$0.3\left((x+1.2)^2 - 1.44\right) = 90$, so $0.3x^2 + 0.72x - 90 = 0$. Multiplying by $\frac{50}{3}$ gives $5x^2 + 12x - 1500 = 0$.

5. Find the positive value of x

$$x = \frac{-12 + \sqrt{12^2 + 4(5)(1500)}}{10} = \frac{-6 + 4\sqrt{471}}{5} \approx 16.16.$$

6. Find the perimeter

$$\text{Perimeter} = PQ + SR + \text{arc } QR + \text{arc } PS = x + x + 0.6(x+1.2) + 0.72 \approx 4$$

.

7. Round to the nearest metre

The total perimeter is 43 m to the nearest metre.

Final answer

(a) $OP = 1.2$ m.

(b) $5x^2 + 12x - 1500 = 0$.

(c) 43 m.

Question 8

8.

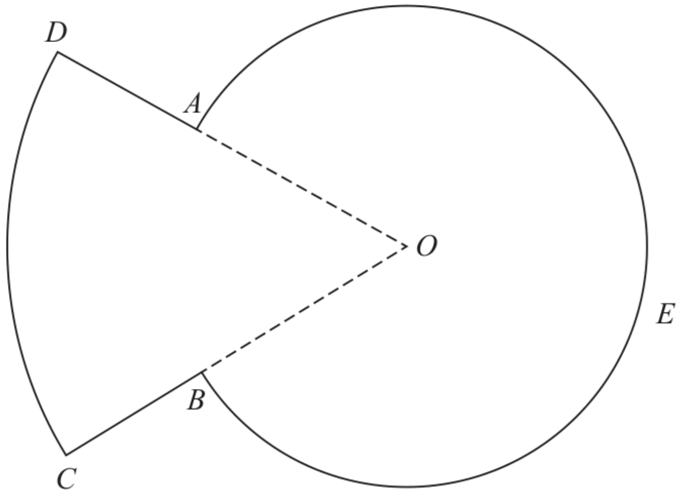


Figure 3

Figure 3 shows a sketch of the plan view of a platform.

The plan view of the platform consists of a sector DOC of a circle centre O joined to a sector $AOBEA$ of a different circle, also with centre O.

Given that

- angle $AOB = 0.8$ radians
- arc length $CD = 9\,\text{m}$
- $DA : AO = 3 : 5$

(a) show that $AO = 7.03\,\text{m}$ to 3 significant figures.

(3)

(b) Find the perimeter of the platform, in m, to 3 significant figures.

(3)

(c) Find the total area of the platform, giving your answer in m^2 to the nearest whole number.

(3)

(Total for Question 8 is 9 marks)

Worked Solution - Question 8

Radians

Topic group

1. Find the outer radius

For sector DOC, arc length $CD = 9$ and angle 0.8 radians, so $OD = \dfrac{9}{0.8} = 11.25 \text{ m}$.

2. Use the ratio DA to AO

$DA : AO = 3 : 5$, so $OD = DA + AO$ represents 8 parts. Hence $AO = \dfrac{5}{8}(11.25) = 7.03125 \text{ m} \approx 7.03 \text{ m}$.

3. Find the straight side lengths

$DA = CB = OD - AO = 11.25 - 7.03125 = 4.21875 \text{ m}$.

4. Find the major arc

The major sector $AOBEA$ has angle $2\pi - 0.8$, so arc $AEB = 7.03125(2\pi - 0.8)$.

5. Calculate the perimeter

$\text{Perimeter} = 9 + 2(4.21875) + 7.03125(2\pi - 0.8) \approx 56.0 \text{ m}$.

6. Calculate the total area

$\text{Area} = \dfrac{1}{2}(11.25)^2(0.8) + \dfrac{1}{2}(7.03125)^2(2\pi - 0.8) \approx 186 \text{ m}^2$ to the nearest whole number.

Final answer

(a) $AO = 7.03$ m.

(b) 56.0 m.

(c) $186\ \text{m}^2$.

Question 8

13 marks

Radians

8.

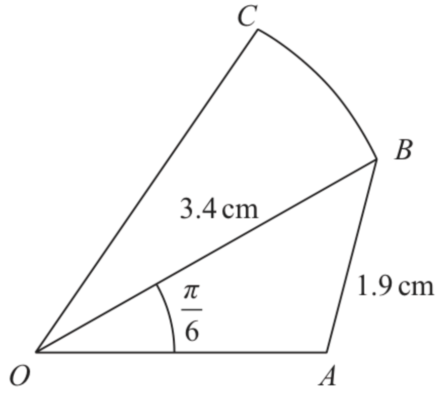


Figure 1

Figure 1 shows a sketch of a design for a badge.
The design consists of a triangle OAB joined to a sector OBC of a circle with centre O
In the design

- $OB = 3.4\,\text{cm}$
- $AB = 1.9\,\text{cm}$
- angle $AOB = \frac{\pi}{6}$ radians
- angle $OAB > \frac{\pi}{2}$ radians

Making your method clear,

(a) find the size of angle OAB, giving your answer in radians to 4 significant figures, **(3)**

(b) find the area of triangle OAB, in cm^2, giving your answer to 3 significant figures. **(2)**

Given that the ratio of the area of sector OBC to the area of triangle OAB is $3:2$

(c) show that angle BOC is 0.462 radians to 3 significant figures. **(3)**

(d) Hence find the perimeter of the badge, in cm, to the nearest integer. **(5)**

(Total for Question 8 is 13 marks)

WMA11/01 JANUARY 2025

Worked Solution - Question 8

Radians

Topic group

1. Use the sine rule

In triangle OAB, $AB = 1.9$, $OB = 3.4$ and $\angle AOB = \frac{\pi}{6}$.

2. Find sine of angle OAB

$\frac{\sin \angle OAB}{3.4} = \frac{\sin(\pi/6)}{1.9}$, so $\sin \angle OAB = \frac{17}{19}$.

3. Choose the obtuse angle

Since $\angle OAB > \frac{\pi}{2}$, $\angle OAB = \pi - \sin^{-1}\left(\frac{17}{19}\right) \approx 2.034$ radians.

4. Find OA

Using the sine rule again, $OA \approx 2.096$ cm.

5. Find the triangle area

$\text{Area } OAB = \frac{1}{2}(OA)(OB)\sin(\pi/6) \approx 1.78 \text{ cm}^2$.

6. Use the area ratio

$\text{Area sector } OBC : \text{Area triangle } OAB = 3 : 2$, so sector area $= \frac{3}{2}(1.78156\ldots)$.

7. Find angle BOC

$\frac{1}{2}(3.4)^2\theta = \frac{3}{2}(1.78156\ldots)$, so $\theta \approx 0.462$ radians.

8. Find the perimeter

The perimeter is $OA + AB + OC + \text{arc } BC = 2.096 + 1.9 + 3.4 + 3.4(0.46234\ldots) \approx 8.968$ cm.

9. Round

The perimeter is 9 cm to the nearest integer.

Final answer

$(a)\ \angle OAB = 2.034$ radians.

$(b)\ 1.78\ \text{cm}^2$.

$(c)\ \angle BOC = 0.462$ radians.

$(d)\ 9$ cm.

Question 6

9 marks

Radians

6.

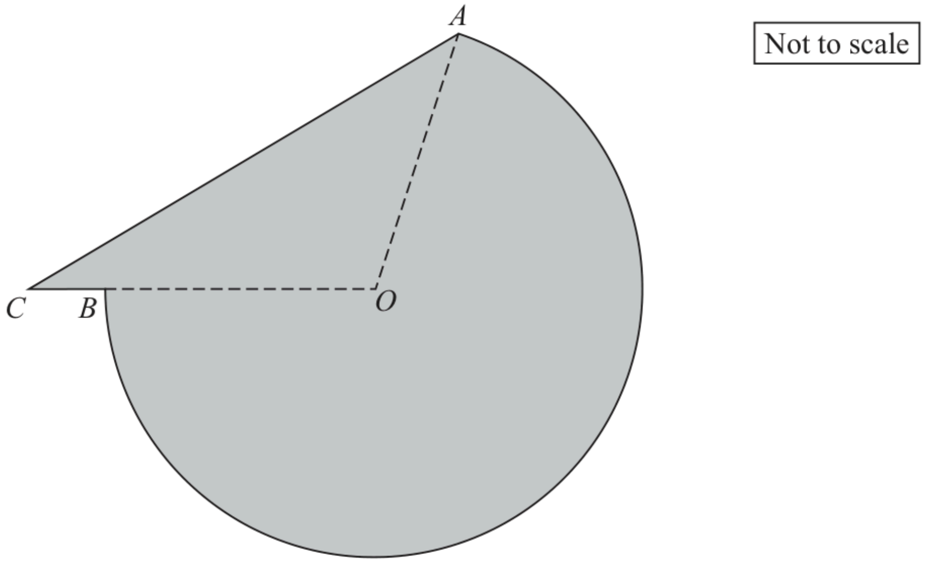


Figure 2

The shaded area in Figure 2 shows the plan view of a helicopter landing pad.

The area consists of the major sector AOB of a circle centre O joined to a triangle AOC.

Given that

- $AO = OB = 15\,\text{m}$
- $BC = 2\,\text{m}$
- CBO is a straight line
- angle $ACO = 0.6$ radians

(a) show that angle COA is 1.847 radians to 3 decimal places.

(3)

(b) Find the total area of the helicopter landing pad.
Give your answer in m^2 to 3 significant figures.

(3)

(c) Find the perimeter of the helicopter landing pad.
Give your answer in metres to 3 significant figures.

(3)

(Total for Question 6 is 9 marks)

WMA11/01 MAY/JUNE 2025

Worked Solution - Question 6

Radians

Topic group

1. Set up triangle AOC

$AO = 15$, $OC = OB + BC = 17$, and $\angle ACO = 0.6$.

2. Use the sine rule

$\dfrac{\sin \angle CAO}{17} = \dfrac{\sin 0.6}{15}$, so $\angle CAO \approx 0.6944$ radians.

3. Show angle COA

$\angle COA = \pi - 0.6 - 0.6944\ldots = 1.847\ldots$, so $\angle COA = 1.847$ radians to 3 decimal places.

4. Major sector angle

The shaded sector is the major sector AOB, so its angle is $2\pi - 1.847\ldots$.

5. Sector area

$$\text{Sector area} = \frac{1}{2}(15)^2(2\pi - 1.847\ldots) = 499.15\ldots$$

6. Triangle area

$$\text{Area } AOC = \frac{1}{2}(15)(17)\sin(1.847\ldots) = 122.73\ldots$$

7. Total area

$499.15\ldots + 122.73\ldots = 621.88\ldots$, so the total area is $622\ \mathrm{m}^2$ to 3 significant figures.

8. Find AC

By the cosine rule, $AC^2 = 15^2 + 17^2 - 2(15)(17)\cos(1.847\ldots)$, so $AC \approx 27.57$ m.

9. Major arc length

arc $AB = 15(2\pi - 1.847\ldots) = 66.54\ldots$ m.

10. Perimeter

$66.54\ldots + 27.57\ldots = 94.11\ldots$, so the perimeter is 94.1 m.

Final answer

(a) $\angle COA = 1.847$ rad.

(b) $622\ \text{m}^2$.

(c) 94.1 m.

Question 7

9 marks

Radians

7.

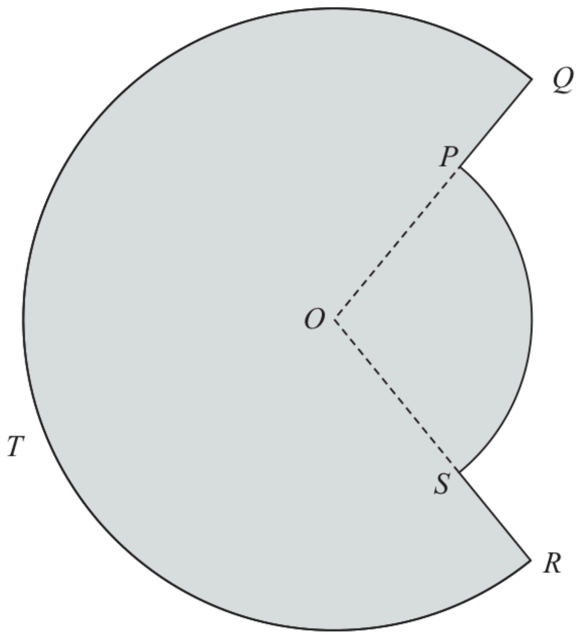


Figure 2

In this question you must show all stages of your working.
Solutions relying entirely on calculator technology are not acceptable.

Figure 2 shows the plan view of a design for a swimming pool.

The design consists of a sector *POS* of a circle centre *O* joined to a major sector *QORTQ* of a different circle, also with centre *O*.

Given that

- angle *POS* is 1.65 radians
- the area of sector *POS* is $30\,\text{m}^2$
- $PQ = 2.8\,\text{m}$

(a) show that, to 3 significant figures, $OQ = 8.83\,\text{m}$

(3)

(b) Find the total surface area of the swimming pool in m^2 to the nearest integer.

(3)

(c) Find the total perimeter of the swimming pool in metres to 2 significant figures.

(3)

(Total for Question 7 is 9 marks)

WMA11/01 OCTOBER 2025

Worked Solution - Question 7

Radians

Topic group

1. Find OP from the small sector

Let $OP = r$. The area of sector POS is 30, so $\frac{1}{2}r^2(1.65) = 30$.

2. Calculate OP

$r^2 = \frac{60}{1.65}$, so $r = 6.030\ldots$ m.

3. Show OQ

$OQ = OP + PQ = 6.030\ldots + 2.8 = 8.830\ldots$, so $OQ = 8.83$ m to 3 significant figures.

4. Major sector angle

The major sector $QORTQ$ has angle $2\pi - 1.65$ radians.

5. Total area

$\text{Area} = 30 + \frac{1}{2}(8.830\ldots)^2(2\pi - 1.65) = 210.6\ldots$.

6. Round area

The total surface area is $211\ \mathrm{m}^2$ to the nearest integer.

7. Arc lengths

The inner arc PS has length $6.030\ldots(1.65) = 9.95\ldots$, and the outer major arc has length $8.830\ldots(2\pi - 1.65) = 40.91\ldots$.

8. Add straight sides

The straight sides PQ and SR each have length 2.8 m.

9. Perimeter

$9.95...+40.91...+2.8+2.8=56.46...$, so the perimeter is $56\ \mathrm{m}$ to 2 significant figures.

Final answer

(a) $OQ = 8.83$ m.

(b) $211\ \mathrm{m}^2$.

(c) 56 m.

TOPIC

Trigonometric Functions

Question 9

9.

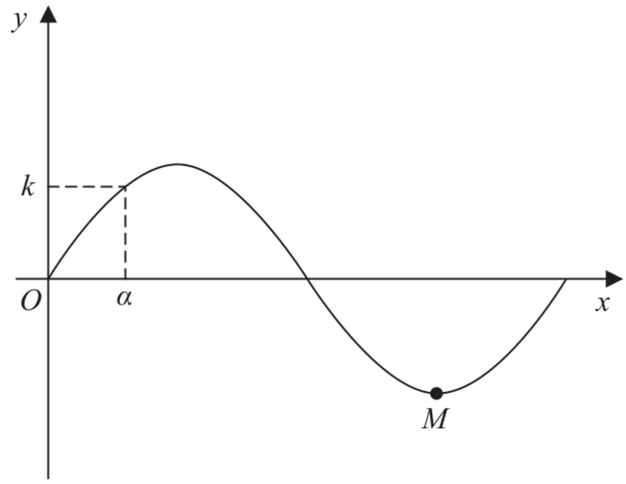


Figure 5

Figure 5 shows a sketch of part of the curve C with equation $y = \sin\left(\frac{x}{12}\right)$, where x is measured in radians. The point M shown in Figure 5 is a minimum point on C.

(a) State the period of C.

(1)

(b) State the coordinates of M.

(1)

The smallest positive solution of the equation $\sin\left(\frac{x}{12}\right) = k$, where k is a constant, is α.

Find, in terms of α,

(c) (i) the negative solution of the equation $\sin\left(\frac{x}{12}\right) = k$ that is closest to zero,

(ii) the smallest positive solution of the equation $\cos\left(\frac{x}{12}\right) = k$.

(2)

(Total 4 marks)

WMA11/01 OCTOBER 2019

Worked Solution - Question 9

Trigonometric Functions

Topic group

1. Use the scale factor inside the sine

For $y = \sin\left(\frac{x}{12}\right)$, one full sine cycle needs $\frac{x}{12}$ to increase by 2π, so the period is 24π.

2. Locate the minimum point

The sine function is -1 when its angle is $\frac{3\pi}{2}$. Therefore $\frac{x}{12} = \frac{3\pi}{2}$, so $x = 18\pi$ and $M = (18\pi, -1)$.

3. Use sine symmetry

If the smallest positive solution of $\sin\left(\frac{x}{12}\right) = k$ is α, then the closest negative solution is one half-period before the opposite sine position: $-12\pi - \alpha$.

4. Convert to cosine

Since $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$, the corresponding smallest positive cosine solution is $6\pi - \alpha$.

Final answer

$(a)\ 24\pi.$

$(b)\ (18\pi, -1).$

$(c)(i)\ \ -12\pi - \alpha. \quad (ii)\ 6\pi - \alpha.$

Question 7

5 marks

Trigonometric Functions

7.

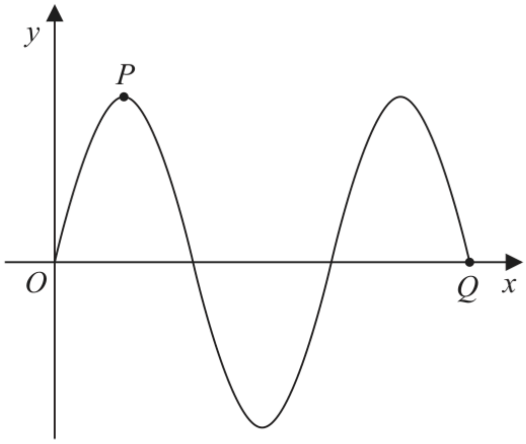


Figure 3

Figure 3 shows part of the curve C_1 with equation $y = 3\sin x$, where x is measured in degrees.

The point P and the point Q lie on C_1 and are shown in Figure 3.

(a) State

(i) the coordinates of P,

(ii) the coordinates of Q.

(3)

A different curve C_2 has equation $y = 3\sin x + k$, where k is a constant.

The curve C_2 has a maximum y value of 10

The point R is the minimum point on C_2 with the smallest positive x coordinate.

(b) State the coordinates of R.

(2)

(Total 5 marks)

WMA11/01 JANUARY 2020

Worked Solution - Question 7

Trigonometric Functions

Topic group

1. Read P from y = 3 sin x

The first maximum of $y = 3\sin x$ occurs at $x = 90°$, with $y = 3$. So $P = (90°, 3)$.

2. Read Q from the next shown zero

After the second maximum, the curve returns to the x-axis at $x = 540°$. So $Q = (540°, 0)$.

3. Find the vertical shift

For C_2, $y = 3\sin x + k$. Its maximum is $3 + k = 10$, so $k = 7$.

4. Find the minimum point R

The minimum value is $-3 + 7 = 4$. The smallest positive x-coordinate for a sine minimum is $270°$, so $R = (270°, 4)$.

Final answer

$(a)(i)\ P = (90°, 3).\quad (a)(ii)\ Q = (540°, 0).$

$(b)\ R = (270°, 4).$

WMA11/01 MAY/JUNE 2021

Question 9

7 marks

Trigonometric Functions

9.

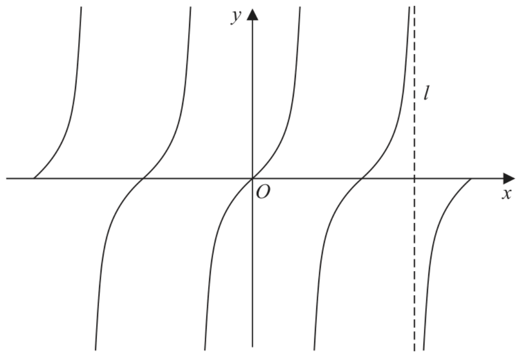


Figure 4

Figure 4 shows a sketch of the curve with equation

$$y = \tan x \qquad -2\pi \leqslant x \leqslant 2\pi$$

The line l, shown in Figure 4, is an asymptote to $y = \tan x$

(a) State an equation for l.

(1)

A copy of Figure 4, labelled Diagram 1, is shown on the next page.

(b) (i) On Diagram 1, sketch the curve with equation

$$y = \frac{1}{x} + 1 \qquad -2\pi \leqslant x \leqslant 2\pi$$

stating the equation of the horizontal asymptote of this curve.

(ii) Hence, **giving a reason**, state the number of solutions of the equation

$$\tan x = \frac{1}{x} + 1$$

in the region $-2\pi \leqslant x \leqslant 2\pi$

(4)

(c) State the number of solutions of the equation $\tan x = \frac{1}{x} + 1$ in the region

(i) $0 \leqslant x \leqslant 40\pi$

(ii) $-10\pi \leqslant x \leqslant \frac{5}{2}\pi$

(2)

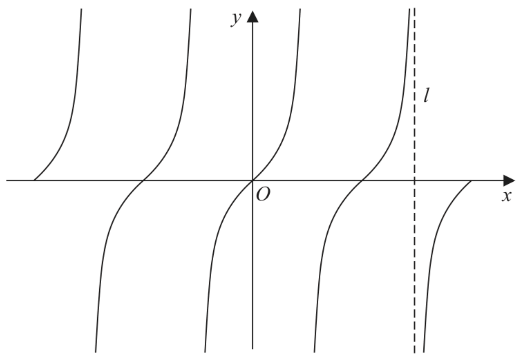


Diagram 1

(Total 7 marks)

WMA11/01 MAY/JUNE 2021

Worked Solution - Question 9

Trigonometric Functions

Topic group

1. State the shown tangent asymptote

The shown dashed asymptote is the right-hand positive asymptote before 2π, so l has equation $x = \dfrac{3\pi}{2}$.

2. Describe the reciprocal graph

For $y = \dfrac{1}{x} + 1$, the vertical asymptote is $x = 0$ and the horizontal asymptote is $y = 1$. The right branch lies above $y = 1$ and the left branch lies below $y = 1$.

3. Count intersections on -2pi to 2pi

The number of solutions of $\tan x = \dfrac{1}{x} + 1$ is the number of intersections of the two graphs. From the sketch on $-2\pi \le x \le 2\pi$, there are 5 intersections.

4. Count solutions for 0 <= x < 40pi

From the repeated tangent branches and the reciprocal curve, there is one intersection in each positive interval between consecutive tangent asymptotes. From 0 to 40π, this gives 40 solutions.

5. Count solutions for -10pi <= x <= 5pi/2

Extending the same intersection pattern across the stated interval gives 14 intersections, so there are 14 solutions.

Final answer

(a) $x = \dfrac{3\pi}{2}$.

$(b)(i)$ $y = 1$ is the horizontal asymptote.

$(b)(ii)$ 5.

$(c)(i)$ 40.

$(c)(ii)$ 14.

Question 9

4 marks

Trigonometric Functions

9.

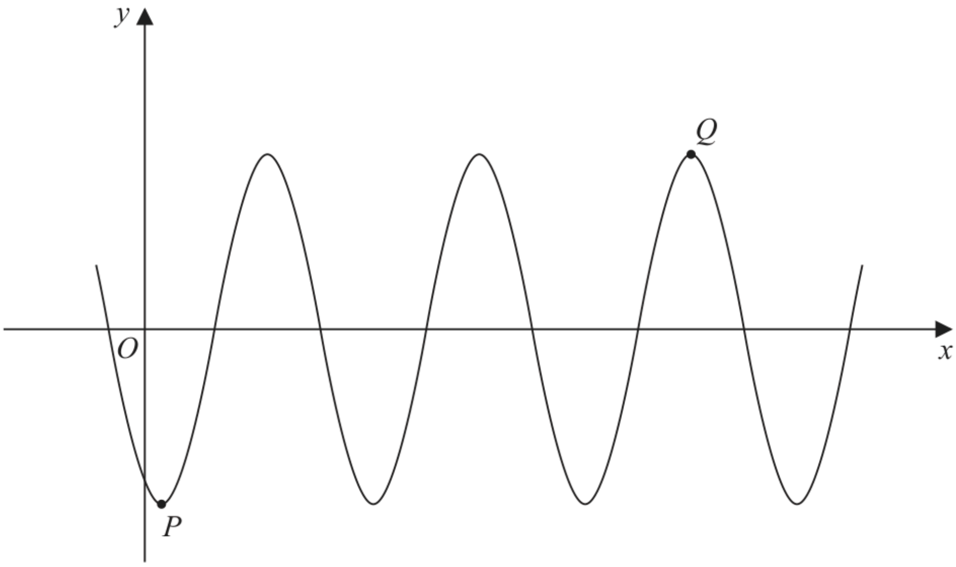


Figure 4

Figure 4 shows part of the curve with equation

$$y = A\cos(x - 30)°$$

where A is a constant.

The point P is a minimum point on the curve and has coordinates $(30, -3)$ as shown in Figure 4.

(a) Write down the value of A.

(1)

The point Q is shown in Figure 4 and is a maximum point.

(b) Find the coordinates of Q.

(3)

(Total 4 marks)

Worked Solution - Question 9

Trigonometric Functions

Topic group

1. Use the minimum point

At $P(30, -3)$, $x - 30 = 0$, so $\cos 0^\circ = 1$. Therefore $A = -3$.

2. Find maximum points

Since $A = -3$, maximum points occur when $\cos(x - 30)^\circ = -1$.

3. Use the shown maximum Q

$x - 30 = 180 + 360n$, so $x = 210 + 360n$. The maximum shown at Q is two full periods after the first maximum, so $x = 210 + 720 = 930$.

4. Write Q

The maximum y-value is 3, hence $Q = (930, 3)$.

Final answer

(a) $A = -3.$

(b) $Q = (930, 3).$

WMA11/01 MAY/JUNE 2022

Question 9

8 marks

Trigonometric Functions

9.

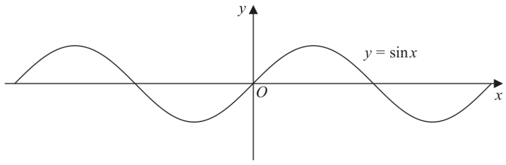


Figure 4

Figure 4 shows part of the graph of the curve with equation $y = \sin x$

Given that $\sin\alpha = p$, where $0 < \alpha < 90°$

(a) state, in terms of p, the value of

(i) $2\sin(180° - \alpha)$

(ii) $\sin(\alpha - 180°)$

(iii) $3 + \sin(180° + \alpha)$

(3)

A copy of Figure 4, labelled Diagram 1, is shown on page 27.

On Diagram 1,

(b) sketch the graph of $y = \sin 2x$

(2)

(c) Hence find, in terms of α, the x coordinates of any points in the interval $0 < x < 180°$ where

$$\sin 2x = p$$

(3)

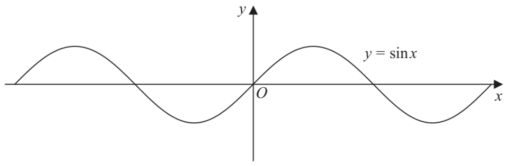


Diagram 1

(Total 8 marks)

WMA11/01 MAY/JUNE 2022

Worked Solution - Question 9

Trigonometric Functions

Topic group

1. Use sine symmetry

$\sin(180^\circ - \alpha) = \sin\alpha = p$, so $2\sin(180^\circ - \alpha) = 2p$.

2. Use odd symmetry

$\sin(\alpha - 180^\circ) = -\sin(180^\circ - \alpha) = -p$.

3. Use the third quadrant identity

$\sin(180^\circ + \alpha) = -\sin\alpha = -p$, so $3 + \sin(180^\circ + \alpha) = 3 - p$.

4. Sketch y = sin 2x

The graph has amplitude 1 and period 180°. It crosses the x-axis at $0^\circ, 90^\circ, 180^\circ$, has a maximum at 45°, and a minimum at 135°.

5. Solve sin 2x = p

Since $\sin\alpha = p$ and $0 < 2x < 360^\circ$, the possible values are $2x = \alpha$ or $2x = 180^\circ - \alpha$.

6. Find x

Therefore $x = \dfrac{\alpha}{2}$ or $x = 90^\circ - \dfrac{\alpha}{2}$.

Final answer

$(a)(i)\ 2p,\ (ii)\ -p,\ (iii)\ 3-p.$

$(c)\ x = \dfrac{\alpha}{2},\ 90^\circ - \dfrac{\alpha}{2}.$

Question 9

9.

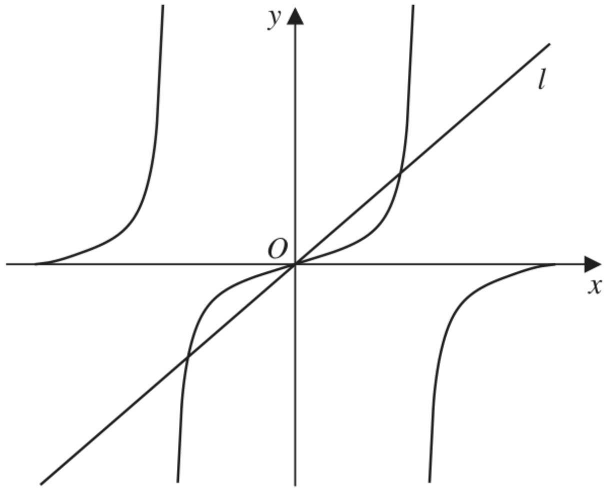


Figure 3

Figure 3 shows a sketch of

- the curve with equation $y = \tan x$
- the straight line l with equation $y = \pi x$

in the interval $-\pi < x < \pi$

(a) State the period of $\tan x$

(1)

(b) Write down the number of roots of the equation

(i) $\tan x = (\pi + 2)x$ in the interval $-\pi < x < \pi$

(1)

(ii) $\tan x = \pi x$ in the interval $-2\pi < x < 2\pi$

(1)

(iii) $\tan x = \pi x$ in the interval $-100\pi < x < 100\pi$

(1)

(Total for Question 9 is 4 marks)

WMA11/01 JANUARY 2023

Worked Solution - Question 9

Trigonometric Functions

Topic group

1. State the period

The period of $\tan x$ is π.

2. Count roots for the steeper line

In $-\pi

3. Extend to -2pi < x < 2pi

For $y = \pi x$, the interval $-2\pi

4. Extend to -100pi < x < 100pi

There are 99 positive outer roots, 99 negative outer roots, and 3 central roots, so the total is $99 + 99 + 3 = 201$.

Final answer

$(a)\ \pi.$

$(b)(i)\ 3,\ (ii)\ 5,\ (iii)\ 201.$

WMA11/01 MAY/JUNE 2023

Question 9

9 marks

Trigonometric Functions

9. (i)

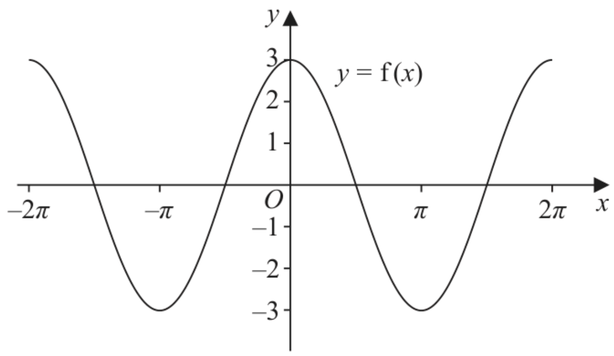


Figure 3

Figure 3 shows part of the graph of the trigonometric function with equation $y = \mathrm{f}(x)$

(a) Write down an expression for $\mathrm{f}(x)$

(2)

On a separate diagram,

(b) sketch, for $-2\pi < x < 2\pi$, the graph of the curve with equation $y = \mathrm{f}\left(x + \dfrac{\pi}{4}\right)$

Show clearly the coordinates of all the points where the curve intersects the coordinate axes.

(3)

(ii)

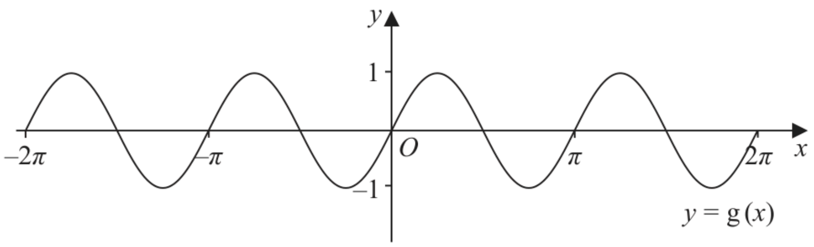


Figure 4

Figure 4 shows part of the graph of the trigonometric function with equation $y = \mathrm{g}(x)$

(a) Write down an expression for $\mathrm{g}(x)$

(2)

On a separate diagram,

(b) sketch, for $-2\pi < x < 2\pi$, the graph of the curve with equation $y = \mathrm{g}(x) - 2$

Show clearly the coordinates of the y intercept.

(2)

(Total for Question 9 is 9 marks)

WMA11/01 MAY/JUNE 2023

Worked Solution - Question 9

Trigonometric Functions

Topic group

1. Identify f(x)

The first graph has amplitude 3, period 2π, and maximum at $x = 0$, so $f(x) = 3\cos x$.

2. Transform f

$y = f\left(x + \frac{\pi}{4}\right) = 3\cos\left(x + \frac{\pi}{4}\right)$, a shift left by $\frac{\pi}{4}$.

3. Find the x-intercepts

$\cos\left(x + \frac{\pi}{4}\right) = 0$ gives $x = \frac{\pi}{4} + n\pi$. In $-2\pi

4. Find the y-intercept

At $x = 0$, $y = 3\cos\frac{\pi}{4} = \frac{3\sqrt{2}}{2}$.

5. Identify g(x)

The second graph has amplitude 1, period π, and passes through the origin with positive gradient, so $g(x) = \sin 2x$.

6. Transform g

$y = g(x) - 2 = \sin 2x - 2$, so the graph is shifted down by 2. Its y-intercept is $(0, -2)$.

Final answer

$(i)(a)$ $f(x) = 3\cos x$. $(i)(b)$ x-intercepts $-\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}$ and y-intercept $\frac{3\sqrt{2}}{2}$. $(ii)(a)$ $g(x) = \sin 2x$. $(ii)(b)$ y-intercept $(0, -2)$.

10.

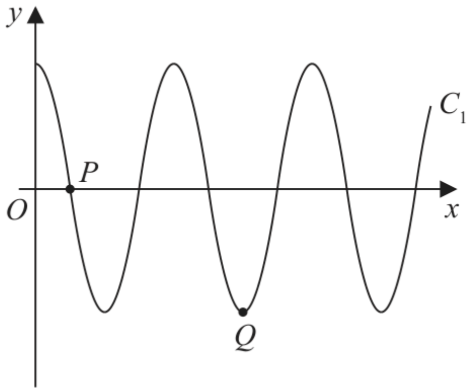


Figure 4

Figure 4 shows a sketch of part of the curve C_1 with equation

$$y = 3\cos\left(\frac{x}{n}\right)^{\circ} \qquad x \geqslant 0$$

where n is a constant.

The curve C_1 cuts the positive x-axis for the first time at point $P(270, 0)$, as shown in Figure 4.

(a) (i) State the value of n

(ii) State the period of C_1

(2)

The point Q, shown in Figure 4, is a minimum point of C_1

(b) State the coordinates of Q.

(2)

The curve C_2 has equation $y = 2\sin x° + k$, where k is a constant.

The point $R\left(a, \frac{12}{5}\right)$ and the point $S\left(-a, -\frac{3}{5}\right)$, both lie on C_2

Given that a is a constant less than 90

(c) find the value of k.

(2)

(Total for Question 10 is 6 marks)

WMA11/01 OCTOBER 2023

Worked Solution - Question 10

Trigonometric Functions

Topic group

1. Use the first positive x-intercept

For $y = 3\cos\left(\frac{x}{n}\right)^{\circ}$, the first positive zero occurs when $\frac{x}{n} = 90^{\circ}$.

2. Find n

Since $P = (270, 0)$, $\frac{270}{n} = 90$, so $n = 3$.

3. Find the period

The period is $360n = 360(3) = 1080^{\circ}$.

4. Read the shown minimum point

A minimum occurs when $\frac{x}{3} = 180^{\circ} + 360^{\circ}k$. The point Q shown is the next displayed minimum after one full period, so $x = 1620$ and $y = -3$.

5. Use the two points on C2

For $y = 2\sin x^{\circ} + k$, the points give $2\sin a + k = \frac{12}{5}$ and $-2\sin a + k = -\frac{3}{5}$.

6. Add the equations

Adding gives $2k = \frac{12}{5} - \frac{3}{5} = \frac{9}{5}$, so $k = \frac{9}{10}$.

Final answer

$(a)(i)\ n = 3. \quad (a)(ii)\ 1080^{\circ}.$

$(b)\ Q = (1620, -3).$

$(c)\ k = \frac{9}{10}.$

WMA11/01 JANUARY 2024

Question 6

6 marks

Trigonometric Functions

6.

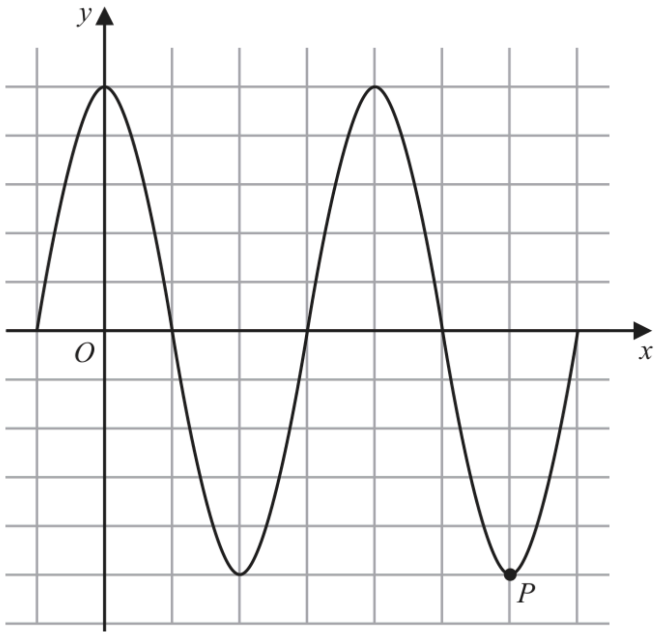


Figure 2

Figure 2 shows a plot of part of the curve C_1 with equation

$$y = 5\cos x$$

with x being measured in degrees.

The point P, shown in Figure 2, is a minimum point on C_1

(a) State the coordinates of P

(2)

The point Q lies on a different curve C_2

Given that point Q

- is a maximum point on the curve
- is the maximum point with the **smallest** x coordinate, $x > 0$

(b) find the coordinates of Q when

(i) C_2 has equation $y = 5\cos x - 2$

(ii) C_2 has equation $y = -5\cos x$

(4)

(Total for Question 6 is 6 marks)

WMA11/01 JANUARY 2024

Worked Solution - Question 6

Trigonometric Functions

Topic group

1. Use the cosine graph

For $y = 5\cos x$, the minimum value is -5 and it occurs at $x = 180° + 360°n$.

2. Read P from the displayed minimum

The plotted point P is the displayed minimum at $x = 540°$, so $P = (540°, -5)$.

3. Translate the graph down

For $y = 5\cos x - 2$, the maximum value is $5 - 2 = 3$. The smallest positive x-coordinate for a maximum shown is $x = 360°$, so $Q = (360°, 3)$.

4. Reflect in the x-axis

For $y = -5\cos x$, a maximum occurs where $\cos x = -1$, first at $x = 180°$. The maximum value is 5, so $Q = (180°, 5)$.

Final answer

$(a)\ P = (540°, -5)$.

$(b)(i)\ Q = (360°, 3)$.

$(b)(ii)\ Q = (180°, 5)$.

WMA11/01 MAY/JUNE 2024

Question 11

6 marks

Trigonometric Functions

11.

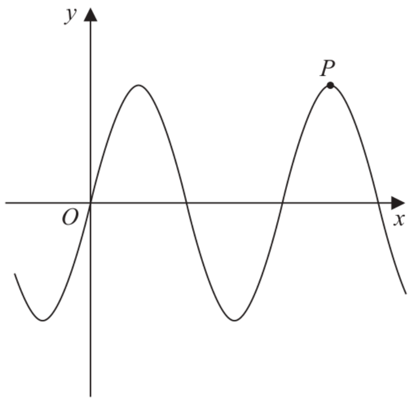


Figure 4

Figure 4 shows a sketch of part of the curve C_1 with equation

$$y = 12\sin x$$

where x is measured in radians.

The point P shown in Figure 4 is a maximum point on C_1

(a) Find the coordinates of P.

(2)

The curve C_2 has equation

$$y = 12\sin x + k$$

where k is a constant.

Given that the **maximum** value of y on C_2 is 3

(b) find the coordinates of the **minimum** point on C_2 which has the **smallest** positive x coordinate.

(2)

The curve C_3 has equation

$$y = 12\sin(x + B)$$

where B is a positive constant.

Given that $\left(\frac{\pi}{4}, A\right)$, where A is a constant, is the **minimum** point on C_3 which has the **smallest** positive x coordinate,

(c) find

(i) the value of A,

(ii) the smallest possible value of B.

(2)

(Total for Question 11 is 6 marks)

TOTAL FOR PAPER IS 75 MARKS

WMA11/01 MAY/JUNE 2024

Worked Solution - Question 11

Trigonometric Functions

Topic group

1. Use maxima of sine

For $y = 12\sin x$, maxima occur when $x = \frac{\pi}{2} + 2\pi n$, with value 12.

2. Read point P

The maximum point P shown is the second positive maximum, so $P = (\frac{5\pi}{2}, 12)$.

3. Find k for C2

For $y = 12\sin x + k$, the maximum value is $12 + k$. Given the maximum is 3, $k = -9$.

4. Find the smallest positive minimum on C2

The minimum value is $-12 - 9 = -21$, and it first occurs at $x = \frac{3\pi}{2}$. So the point is $(\frac{3\pi}{2}, -21)$.

5. Find A for C3

For $y = 12\sin(x + B)$, any minimum has value -12, so $A = -12$.

6. Find the smallest positive B

At the minimum $x = \frac{\pi}{4}$, so $\frac{\pi}{4} + B = \frac{3\pi}{2}$. Hence $B = \frac{5\pi}{4}$.

Final answer

$(a)\ P = (\frac{5\pi}{2}, 12).$

$(b)\ (\frac{3\pi}{2}, -21).$

$(c)(i)\ A = -12.$

$(c)(ii)\ B = \frac{5\pi}{4}.$

WMA11/01 OCTOBER 2024

Question 7

8 marks

Trigonometric Functions

7.

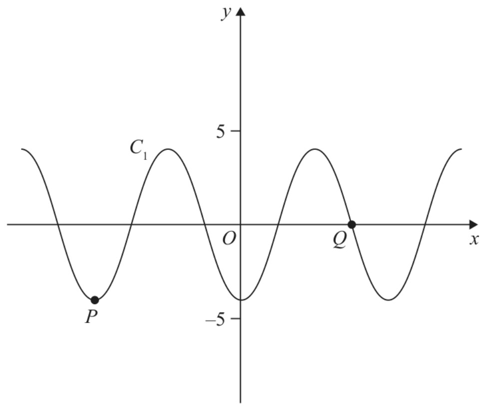


Figure 3

Figure 3 shows a plot of part of the curve C_1 with equation

$$y = -4\cos x$$

where x is measured in radians.

Points P and Q lie on the curve and are shown in Figure 3.

(a) State

(i) the coordinates of P

(ii) the coordinates of Q

(3)

The curve C_2 has equation $y = -4\cos x + k$ where x is measured in radians and k is a constant.

Given that C_2 has a maximum y value of 11

(b) (i) state the value of k

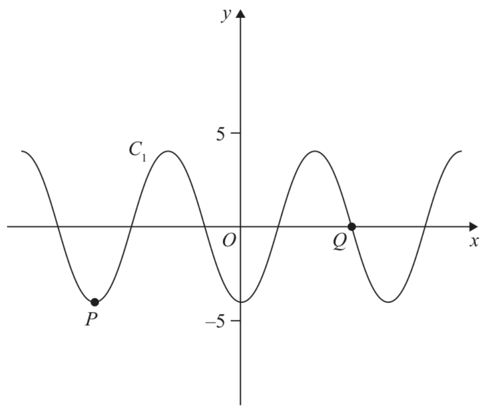
(ii) state the coordinates of the minimum point on C_2 with the smallest positive x coordinate.

(3)

On the opposite page there is a copy of Figure 3 labelled Diagram 1.

(c) Using Diagram 1, state the number of solutions of the equation

$$-4\cos x = 5 - \frac{10}{\pi}x$$



(Total for Question 7 is 8 marks)

WMA11/01 OCTOBER 2024

Worked Solution - Question 7

Trigonometric Functions

Topic group

1. Use y equals negative 4 cos x

For $y = -4\cos x$, minimum points occur at $x = 2\pi n$ with value -4.

2. Read P

The displayed minimum to the left of the y-axis is $P = (-2\pi, -4)$.

3. Read Q

The point Q is the positive x-intercept after the maximum at $x = \pi$, so $Q = (\frac{3\pi}{2}, 0)$.

4. Find k for C2

For $y = -4\cos x + k$, the maximum value is $4 + k$. Given the maximum is 11, $k = 7$.

5. Find the smallest positive minimum on C2

The minimum value is $-4 + 7 = 3$. The first positive minimum is at $x = 2\pi$, so the point is $(2\pi, 3)$.

6. Interpret the equation graphically

$-4\cos x = 5 - \frac{10}{\pi}x$ is the intersection of the cosine curve with the straight line $y = 5 - \frac{10}{\pi}x$.

7. Count intersections

The line passes through $(0, 5)$ and meets the displayed curve once, at the x-axis point, so there is 1 solution.

Final answer

$(a)(i)\ P = (-2\pi, -4)$. $(a)(ii)\ Q = (\frac{3\pi}{2}, 0)$.

$(b)(i)\ k = 7$.

$(b)(ii)\ (2\pi, 3)$.

(c) 1 solution.

WMA11/01 JANUARY 2025

Question 9

13 marks

Trigonometric Functions

9.

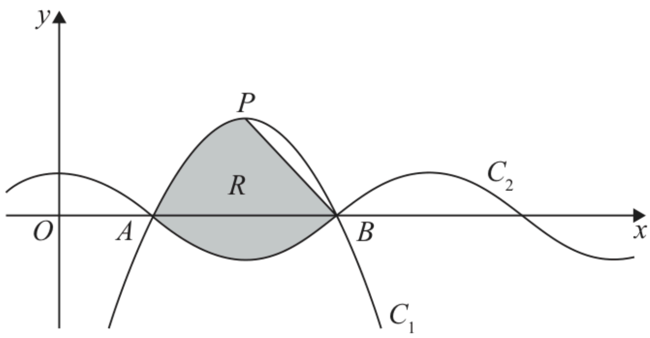


Figure 2

(a) Express $6x - \frac{27}{4} - x^2$ in the form $a + b(x + c)^2$ where a, b and c are constants to be found.

(3)

Figure 2 shows part of a sketch of curve C_1 with equation

$$y = 6x - \frac{27}{4} - x^2$$

Given that the point P is the maximum point on C_1

(b) state the coordinates of P

(2)

Figure 2 also shows part of a sketch of curve C_2 with equation

$$y = \cos(kx)$$

where k is a constant and x is measured in radians.

Given that C_1 and C_2 intersect on the x-axis at point A and at point B, as shown in Figure 2,

(c) (i) state the x coordinate of B

(ii) state the value of k

(iii) state the period of C_2

(3)

The line segment L joins P and B.
The region R, shown shaded in Figure 2, is bounded by L, C_1 and C_2

(d) Use inequalities to define R.

(5)

(Total for Question 9 is 13 marks)

WMA11/01 JANUARY 2025

Worked Solution - Question 9

Trigonometric Functions

Topic group

1. Complete the square

$6x - \frac{27}{4} - x^2 = -(x^2 - 6x) - \frac{27}{4} = -(x-3)^2 + 9 - \frac{27}{4}$.

2. State the completed-square form

$6x - \frac{27}{4} - x^2 = \frac{9}{4} - (x-3)^2$.

3. Find the maximum point

The maximum point is $P = (3, \frac{9}{4})$.

4. Find the x-intercepts of C1

$\frac{9}{4} - (x-3)^2 = 0$ gives $x = 3 \pm \frac{3}{2}$. Hence A has $x = \frac{3}{2}$ and B has $x = \frac{9}{2}$.

5. Use consecutive zeros of cosine

The distance between consecutive zeros of $\cos(kx)$ is $\frac{\pi}{k}$.

6. Find k and the period

Since $x_B - x_A = \frac{9}{2} - \frac{3}{2} = 3$, $\frac{\pi}{k} = 3$, so $k = \frac{\pi}{3}$. The period is $\frac{2\pi}{k} = 6$.

7. Find the line through P and B

The line through $P(3, \frac{9}{4})$ and $B(\frac{9}{2}, 0)$ has gradient $-\frac{3}{2}$, so L is $y = -\frac{3}{2}x + \frac{27}{4}$.

8. Define the x-range

The region lies between A and B, so $\frac{3}{2} \leq x \leq \frac{9}{2}$.

9. Define the lower boundary

The lower boundary is C_2, so $y \geq \cos\left(\frac{\pi}{3}x\right)$.

10. Define the upper boundaries

The region is below both C_1 and the line segment L, so $y \leq \frac{9}{4} - (x-3)^2$ and $y \leq -\frac{3}{2}x + \frac{27}{4}$.

Final answer

$(a)\ \frac{9}{4} - (x-3)^2.$

$(b)\ P = (3, \frac{9}{4}).$

$(c)(i)\ x_B = \frac{9}{2}.$

$(c)(ii)\ k = \frac{\pi}{3}.$

$(c)(iii)\ 6.$

$(d)\ \frac{3}{2} \leq x \leq \frac{9}{2},\ y \geq \cos(\frac{\pi}{3}x),\ y \leq \frac{9}{4} - (x-3)^2,\ y \leq -\frac{3}{2}x + \frac{27}{4}.$

WMA11/01 JANUARY 2026

Question 9

9 marks

Trigonometric Functions

9.

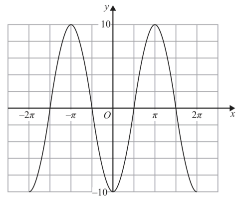


Figure 4

Figure 4 shows a sketch of part of the graph of the trigonometric function with equation $y = \mathrm{f}(x)$.

(a) Write down an expression for $\mathrm{f}(x)$.

(2)

Copies of Figure 4 (labelled Diagram 1 and Diagram 2) can be found on the following pages.

(b) (i) On Diagram 1 sketch a graph of the curve with equation

$$y = 10 - x^2$$

(ii) Hence find the **number** of solutions of the equation

$$\mathrm{f}(x) = 10 - x^2 \quad \text{in the interval} \quad -100\pi \leqslant x \leqslant 100\pi$$

(3)

(c) (i) On Diagram 2 sketch a graph of the curve with equation

$$y = \tan x \qquad -2\pi \leqslant x \leqslant 2\pi$$

(ii) Hence find the **number** of solutions of the equation

$$\mathrm{f}(x) = \tan x \quad \text{in the interval} \quad -100\pi \leqslant x \leqslant 100\pi$$

giving a reason for your answer.

(4)

(a) ____

____

(b)(i)

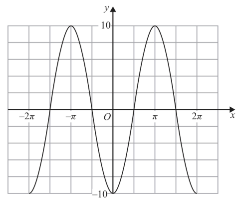
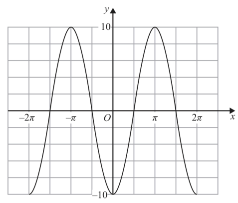


Diagram 1



Copy of Diagram 1

Only use this copy if you need to redraw your graph.

(b)(ii) ____

(c)(i)

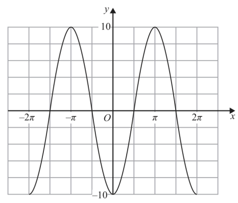
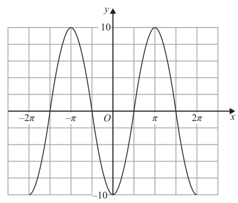


Diagram 2



Copy of Diagram 2

Only use this copy if you need to redraw your graph.

(c)(ii) ____

____

(Total for Question 9 is 9 marks)

Worked Solution - Question 9

Trigonometric Functions

Topic group

1. Identify f(x)

The graph has amplitude 10, period 2π, and value -10 at $x = 0$, so $f(x) = -10\cos x$.

2. Sketch y equals 10 minus x squared

The graph $y = 10 - x^2$ is a downward-opening parabola with vertex $(0, 10)$ and x-intercepts at $x = \pm\sqrt{10}$.

3. Count intersections with the parabola

On the given sketch, the parabola meets $f(x)$ twice. Outside the central part the parabola is below the range of $f(x)$, so there are 2 solutions in $-100\pi \leq x \leq 100\pi$.

4. Sketch y equals tan x

The graph $y = \tan x$ has zeros at integer multiples of π and vertical asymptotes at $x = \dfrac{\pi}{2} + n\pi$.

5. Count in the reference interval

In $-2\pi \leq x \leq 2\pi$, the graph $y = \tan x$ intersects $f(x)$ four times.

6. Scale to the required interval

The interval $-100\pi \leq x \leq 100\pi$ is 50 times as long as $-2\pi \leq x \leq 2\pi$, so the number of solutions is $50 \times 4 = 200$.

Final answer

$(a)\ f(x) = -10\cos x.$

$(b)(ii)\ 2.$

$(c)(ii)\ 200.$

TOPIC

Differentiation

Question 7

6 marks

Differentiation

7.

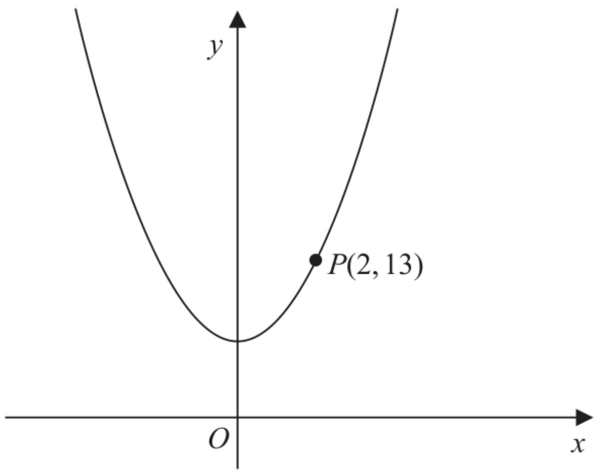


Figure 4

Figure 4 shows part of the curve with equation $y = 2x^2 + 5$

The point $P(2,13)$ lies on the curve.

(a) Find the gradient of the tangent to the curve at P.

(2)

The point Q with x coordinate $2 + h$ also lies on the curve.

(b) Find, in terms of h, the gradient of the line PQ. Give your answer in simplest form.

(3)

(c) Explain briefly the relationship between the answer to (b) and the answer to (a).

(1)

(Total 6 marks)

WMA11/01 OCTOBER 2019

Worked Solution - Question 7

Differentiation

Topic group

1. Differentiate for the tangent gradient

For $y = 2x^2 + 5$, $\dfrac{dy}{dx} = 4x$. At P, $x = 2$, so the tangent gradient is 8.

2. Find the coordinates of Q

If $x_Q = 2 + h$, then $y_Q = 2(2+h)^2 + 5 = 13 + 8h + 2h^2$.

3. Find the gradient of PQ

$$\text{gradient } PQ = \frac{(13 + 8h + 2h^2) - 13}{(2+h) - 2} = \frac{8h + 2h^2}{h} = 8 + 2h.$$

4. Explain the link

As h gets closer to 0, point Q gets closer to P, and the chord gradient $8 + 2h$ tends to the tangent gradient 8.

Final answer

(a) 8.

(b) $8 + 2h$.

(c) $As\ h \to 0,\ 8 + 2h \to 8.$

Question 9

6 marks

Differentiation

9. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

A curve has equation

$$y = \frac{4x^2 + 9}{2\sqrt{x}} \qquad x > 0$$

Find the x coordinate of the point on the curve at which $\frac{dy}{dx} = 0$

(6)

(Total 6 marks)

WMA11/01 JANUARY 2020

Worked Solution - Question 9

Differentiation

Topic group

1. Rewrite the function

$$y = \frac{4x^2 + 9}{2\sqrt{x}} = 2x^{3/2} + \frac{9}{2}x^{-1/2}.$$

2. Differentiate

$$\frac{dy}{dx} = 3x^{1/2} - \frac{9}{4}x^{-3/2}.$$

3. Set the derivative equal to zero

$3x^{1/2} - \frac{9}{4}x^{-3/2} = 0$, so $3\sqrt{x} = \frac{9}{4x^{3/2}}$.

4. Solve for x

Multiply by $4x^{3/2}$: $12x^2 = 9$. Hence $x^2 = \frac{3}{4}$. Since $x > 0$, $x = \frac{\sqrt{3}}{2}$.

Final answer

$$x = \frac{\sqrt{3}}{2}.$$

Question 7

9 marks

Differentiation

7. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

$$f(x) = 2x - 3\sqrt{x} - 5 \qquad x > 0$$

(a) Solve the equation

$$f(x) = 9$$

(4)

(b) Solve the equation

$$f''(x) = 6$$

(5)

(Total 9 marks)

WMA11/01 JANUARY 2021

Worked Solution - Question 7

Differentiation

Topic group

1. Solve f(x) = 9 without calculator methods

$2x - 3\sqrt{x} - 5 = 9$, so $2x - 3\sqrt{x} - 14 = 0$.

2. Use u = sqrt(x)

Let $u = \sqrt{x}$, where $u > 0$. Then $x = u^2$, so $2u^2 - 3u - 14 = 0$.

3. Factorise and reject the invalid root

$2u^2 - 3u - 14 = (2u - 7)(u + 2)$. Since $u > 0$, $u = \frac{7}{2}$, so $x = u^2 = \frac{49}{4}$.

4. Differentiate twice

$f(x) = 2x - 3x^{1/2} - 5$, so $f'(x) = 2 - \frac{3}{2}x^{-1/2}$ and $f''(x) = \frac{3}{4}x^{-3/2}$.

5. Solve f''(x) = 6

$\frac{3}{4}x^{-3/2} = 6$, so $x^{-3/2} = 8$. Therefore $x^{3/2} = \frac{1}{8}$.

6. Find x

Since $x^{3/2} = (\sqrt{x})^3$, we get $\sqrt{x} = \frac{1}{2}$, hence $x = \frac{1}{4}$.

Final answer

$(a)\ x = \frac{49}{4}$.

$(b)\ x = \frac{1}{4}$.

WMA11/01 JANUARY 2022

Question 7

11 marks

Differentiation

7.

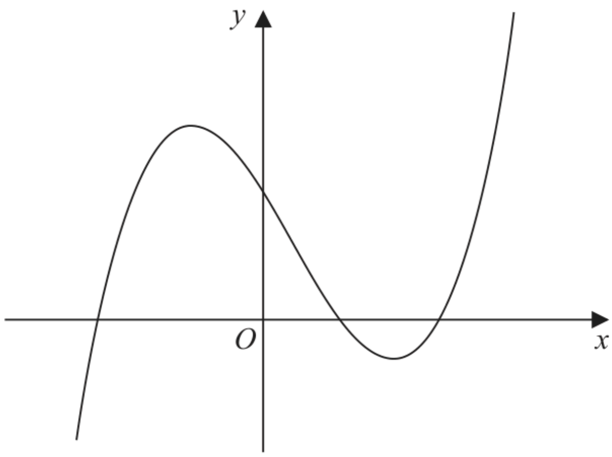


Figure 3

Figure 3 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = (x + 4)(x - 2)(2x - 9)$$

Given that the curve with equation $y = \mathrm{f}(x) - p$ passes through the point with coordinates (0, 50)

(a) find the value of the constant p.

(2)

Given that the curve with equation $y = \mathrm{f}(x + q)$ passes through the origin,

(b) write down the possible values of the constant q.

(2)

(c) Find $\mathrm{f}'(x)$.

(4)

(d) Hence find the range of values of x for which the gradient of the curve with equation $y = \mathrm{f}(x)$ is less than -18

(3)

(Total 11 marks)

WMA11/01 JANUARY 2022

Worked Solution - Question 7

Differentiation

Topic group

1. Find p from the translated curve

$f(0) = (4)(-2)(-9) = 72$. Since $y = f(x) - p$ passes through $(0, 50)$, $72 - p = 50$, so $p = 22$.

2. Use roots for f(x + q)

If $y = f(x + q)$ passes through the origin, then $f(q) = 0$. From $f(x) = (x + 4)(x - 2)(2x - 9)$, the possible values are $q = -4, 2, \frac{9}{2}$.

3. Expand f(x)

$(x + 4)(x - 2)(2x - 9) = 2x^3 - 5x^2 - 34x + 72$.

4. Differentiate

$f'(x) = 6x^2 - 10x - 34$.

5. Set the gradient less than -18

$6x^2 - 10x - 34 < -18$, so $6x^2 - 10x - 16 < 0$, or $3x^2 - 5x - 8 < 0$.

6. Solve the inequality

$3x^2 - 5x - 8 = (3x - 8)(x + 1)$. Since the quadratic opens upward, it is negative between the roots: $-1

Final answer

$(a)\ p = 22.$

$(b)\ q = -4, 2, \dfrac{9}{2}.$

$(c)\ f'(x) = 6x^2 - 10x - 34.$

$(d)\ -1

WMA11/01 MAY/JUNE 2022

Question 10

12 marks

Differentiation

10.

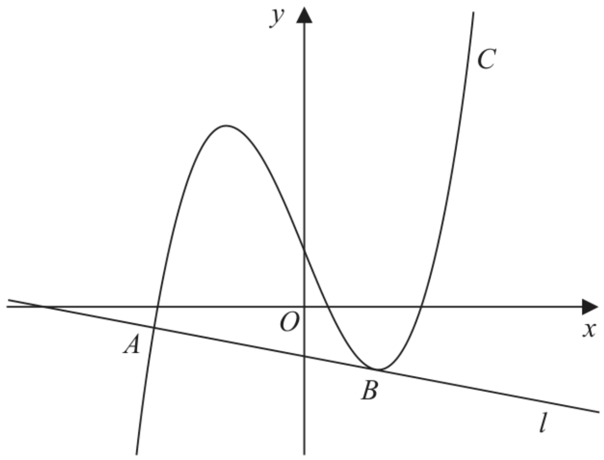


Figure 5

Figure 5 shows a sketch of the curve C with equation

$$y = \frac{2}{7}x^3 + \frac{1}{7}x^2 - \frac{5}{2}x + k$$

where k is a constant.

(a) Find $\frac{dy}{dx}$

(2)

The line l, shown in Figure 5, is the normal to C at the point A with x coordinate $-\frac{7}{2}$

Given that l is also a tangent to C at the point B,

(b) show that the x coordinate of the point B is a solution of the equation

$$12x^2 + 4x - 33 = 0$$

(4)

(c) Hence find the x coordinate of B, justifying your answer.

(2)

Given that the y intercept of l is -1

(d) find the value of k.

(4)

(Total 12 marks)

WMA11/01 MAY/JUNE 2022

Worked Solution - Question 10

Differentiation

Topic group

1. Differentiate the curve

For $y = \frac{2}{7}x^3 + \frac{1}{7}x^2 - \frac{5}{2}x + k$, $\frac{dy}{dx} = \frac{6}{7}x^2 + \frac{2}{7}x - \frac{5}{2}$.

2. Find the tangent gradient at A

At $x = -\frac{7}{2}$, $\frac{dy}{dx} = \frac{6}{7}\left(\frac{49}{4}\right) + \frac{2}{7}\left(-\frac{7}{2}\right) - \frac{5}{2} = \frac{21}{2} - 1 - \frac{5}{2} = 7$.

3. Find the gradient of l

Since l is the normal at A, its gradient is the negative reciprocal of 7, so $m_l = -\frac{1}{7}$.

4. Use that l is tangent at B

At B, the tangent gradient is also $-\frac{1}{7}$, so $\frac{6}{7}x^2 + \frac{2}{7}x - \frac{5}{2} = -\frac{1}{7}$.

5. Reach the given quadratic

Multiplying by 14 gives $12x^2 + 4x - 35 = -2$, hence $12x^2 + 4x - 33 = 0$.

6. Solve for the x-coordinate of B

$12x^2 + 4x - 33 = 0$ gives $x = \frac{-4 \pm 40}{24}$, so $x = \frac{3}{2}$ or $x = -\frac{11}{6}$. From the diagram, B is the point to the right of the y-axis, so $x_B = \frac{3}{2}$.

7. Use the y-intercept of l

Since the y-intercept is -1 and the gradient is $-\frac{1}{7}$, the line is $y = -\frac{1}{7}x - 1$.

8. Find the y-coordinate of A

At $x = -\frac{7}{2}$ on the line, $y = -\frac{1}{7}\left(-\frac{7}{2}\right) - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$.

9. Substitute A into the curve

$$-\frac{1}{2}=\frac{2}{7}\left(-\frac{7}{2}\right)^3+\frac{1}{7}\left(-\frac{7}{2}\right)^2-\frac{5}{2}\left(-\frac{7}{2}\right)+k=-\frac{7}{4}+k.$$

10. Find k

Therefore $k=\frac{5}{4}$.

Final answer

$(a)\ \frac{dy}{dx}=\frac{6}{7}x^2+\frac{2}{7}x-\frac{5}{2}.$

$(c)\ x_B=\frac{3}{2}.$

$(d)\ k=\frac{5}{4}.$

WMA11/01 OCTOBER 2022

Question 9

14 marks

Differentiation

9.

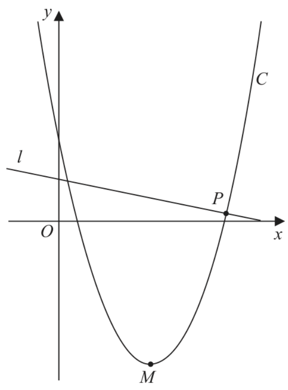


Figure 3

Figure 3 shows a sketch of the curve C with equation

$$y = \frac{1}{2}x^2 - 10x + 22$$

(a) Write $\frac{1}{2}x^2 - 10x + 22$ in the form

$$a(x + b)^2 + c$$

where a, b and c are constants to be found.

(3)

The point M is the minimum turning point of C, as shown in Figure 3.

(b) Deduce the coordinates of M

(2)

The line l is the normal to C at the point P, as shown in Figure 3.

Given that l has equation $y = k - \frac{1}{8}x$, where k is a constant,

(c) (i) find the coordinates of P

(ii) find the value of k

(6)

Question 9 continues on the next page

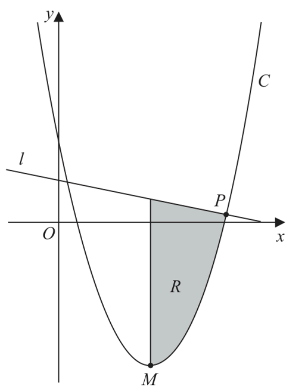


Figure 4

Figure 4 is a copy of Figure 3. The finite region R, shown shaded in Figure 4, is bounded by l, C and the line through M parallel to the y-axis.

(d) Identify the inequalities that define R.

(3)

(Total for Question 9 is 14 marks)

WMA11/01 OCTOBER 2022

Worked Solution - Question 9

Differentiation

Topic group

1. Complete the square

$$\frac{1}{2}x^2 - 10x + 22 = \frac{1}{2}(x^2 - 20x) + 22 = \frac{1}{2}\left((x-10)^2 - 100\right) + 22.$$

2. Simplify the completed square form

$$\frac{1}{2}\left((x-10)^2 - 100\right) + 22 = \frac{1}{2}(x-10)^2 - 28.$$

3. Read the minimum point

The minimum point is $M(10, -28)$.

4. Use the normal gradient

The line l has gradient $-\frac{1}{8}$, so the tangent gradient at P is 8.

5. Differentiate the curve

$\frac{dy}{dx} = x - 10$. At P, $x - 10 = 8$, so $x = 18$.

6. Find P

When $x = 18$, $y = \frac{1}{2}(18)^2 - 10(18) + 22 = 4$, so $P = (18, 4)$.

7. Find k

Since P lies on $y = k - \frac{1}{8}x$, $4 = k - \frac{1}{8}(18) = k - \frac{9}{4}$, so $k = \frac{25}{4}$.

8. Write the upper boundary

The line is $y = \frac{25}{4} - \frac{x}{8}$.

9. Write the lower boundary

The curve is $y = \dfrac{1}{2}x^2 - 10x + 22$.

10. Identify the region

The region lies to the right of the vertical line through M and between the curve and the normal line: $10 \le x \le 18$ and $\dfrac{1}{2}x^2 - 10x + 22 \le y \le \dfrac{25}{4} - \dfrac{x}{8}$.

Final answer

$(a)\ \dfrac{1}{2}(x - 10)^2 - 28.$

$(b)\ M(10, -28).$

$(c)\ P(18, 4),\ k = \dfrac{25}{4}.$

$(d)\ 10 \le x \le 18,\ \dfrac{1}{2}x^2 - 10x + 22 \le y \le \dfrac{25}{4} - \dfrac{x}{8}.$

Question 10

10.

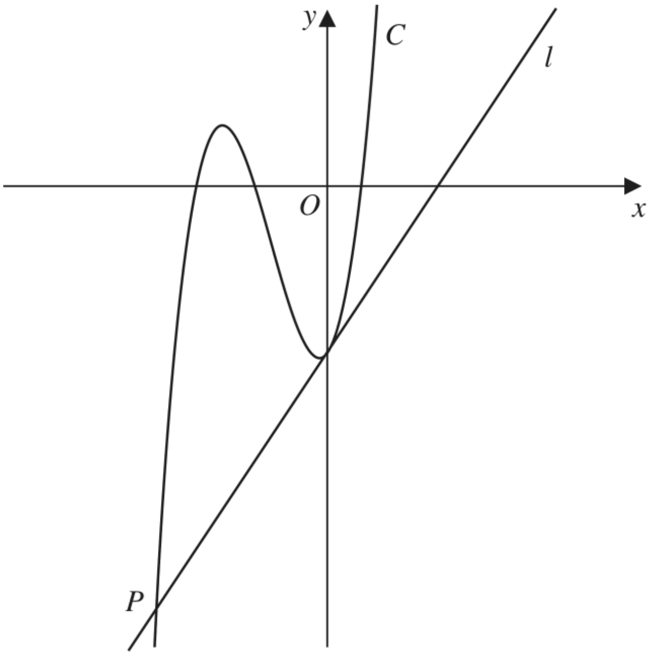


Figure 4

Figure 4 shows a sketch of part of the curve C with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = (3x + 20)(x + 6)(2x - 3)$$

(a) Use the given information to state the values of x for which

$$\mathrm{f}(x) > 0$$

(2)

(b) Expand $(3x + 20)(x + 6)(2x - 3)$, writing your answer as a polynomial in simplest form.

(3)

The straight line l is the tangent to C at the point where C cuts the y-axis.

Given that l cuts C at the point P, as shown in Figure 4,

(c) find, using algebra, the x coordinate of P

(*Solutions based on calculator technology are not acceptable*.)

(5)

(Total for Question 10 is 10 marks)

WMA11/01 JANUARY 2023

Worked Solution - Question 10

Differentiation

Topic group

1. Use the roots

$f(x) = (3x+20)(x+6)(2x-3)$ has roots $x = -\dfrac{20}{3}$, $x = -6$, and $x = \dfrac{3}{2}$.

2. State where f(x) is positive

Since the cubic has positive leading coefficient, $f(x) > 0$ for $-\dfrac{20}{3}\dfrac32$.

3. Expand the first two brackets

$(3x+20)(x+6) = 3x^2 + 38x + 120$.

4. Complete the expansion

$(3x^2+38x+120)(2x-3) = 6x^3 + 67x^2 + 126x - 360$.

5. Find the tangent at the y-axis

$f'(x) = 18x^2 + 134x + 126$, so at $x = 0$ the tangent gradient is 126. Also $f(0) = -360$.

6. Write the tangent equation

The tangent line is $y = 126x - 360$.

7. Find the other intersection

Set $6x^3 + 67x^2 + 126x - 360 = 126x - 360$. This gives $6x^3 + 67x^2 = 0$, so $x^2(6x+67) = 0$.

8. Choose point P

$x = 0$ is the tangent point on the y-axis, so the other intersection P has $x = -\dfrac{67}{6}$.

Final answer

$(a)\ -\dfrac{20}{3}\dfrac32.$

$(b)\ 6x^3 + 67x^2 + 126x - 360.$

$(c)\ x_P = -\dfrac{67}{6}.$

Question 7

8 marks

Differentiation

7. The curve C has equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 2x^3 - kx^2 + 14x + 24$$

and k is a constant.

(a) Find, in simplest form,

(i) $\mathrm{f}'(x)$

(ii) $\mathrm{f}''(x)$

(3)

The curve with equation $y = \mathrm{f}'(x)$ intersects the curve with equation $y = \mathrm{f}''(x)$ at the points A and B.

Given that the x coordinate of A is 5

(b) find the value of k.

(2)

(c) Hence find the coordinates of B.

(3)

(Total for Question 7 is 8 marks)

WMA11/01 MAY/JUNE 2024

Worked Solution - Question 7

Differentiation

Topic group

1. Differentiate once

$f'(x) = 6x^2 - 2kx + 14$.

2. Differentiate again

$f''(x) = 12x - 2k$.

3. Use the intersection at A

At an intersection of $y = f'(x)$ and $y = f''(x)$, $f'(x) = f''(x)$. Given $x_A = 5$, substitute $x = 5$.

4. Find k

$6(5)^2 - 2k(5) + 14 = 12(5) - 2k$, so $164 - 10k = 60 - 2k$ and $k = 13$.

5. Find the other x-coordinate

With $k = 13$, $6x^2 - 26x + 14 = 12x - 26$, so $6x^2 - 38x + 40 = 0$. Hence $(3x - 4)(x - 5) = 0$.

6. Use the point B

The other point has $x = \frac{4}{3}$. Then $y = f''(\frac{4}{3}) = 12(\frac{4}{3}) - 26 = -10$.

Final answer

$(a)(i)\ f'(x) = 6x^2 - 2kx + 14. \quad (a)(ii)\ f''(x) = 12x - 2k.$

$(b)\ k = 13.$

$(c)\ B = (\frac{4}{3}, -10).$

9.

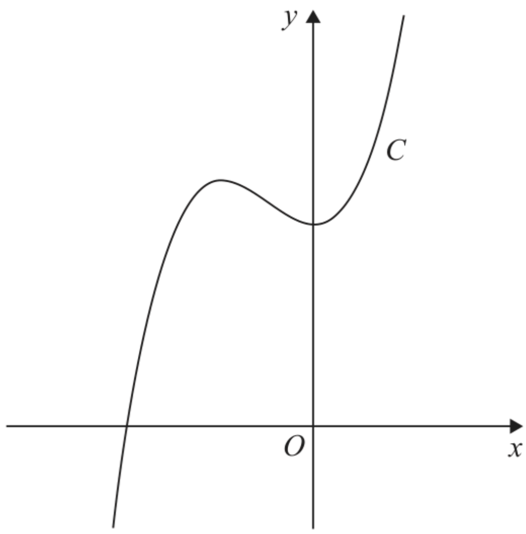


Figure 4

Figure 4 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = (x+5)(3x^2 - 4x + 20)$$

(a) Deduce the range of values of x for which $\mathrm{f}(x) \geqslant 0$

(1)

(b) Find $\mathrm{f}'(x)$ giving your answer in simplest form.

(3)

The point $R(-4, 84)$ lies on C.

Given that the tangent to C at the point P is parallel to the tangent to C at the point R

(c) find the x coordinate of P.

(4)

(d) Find the point to which R is transformed when the curve with equation $y = \mathrm{f}(x)$ is transformed to the curve with equation,

(i) $y = \mathrm{f}(x - 3)$

(ii) $y = 4\mathrm{f}(x)$

(2)

(Total for Question 9 is 10 marks)

WMA11/01 OCTOBER 2024

Worked Solution - Question 9

Differentiation

Topic group

1. Use the positive quadratic factor

$3x^2 - 4x + 20$ has discriminant $(-4)^2 - 4(3)(20) < 0$ and positive leading coefficient, so it is always positive.

2. Deduce the sign of f(x)

The sign of $f(x) = (x+5)(3x^2 - 4x + 20)$ is therefore the sign of $x + 5$. Hence $f(x) \geq 0$ for $x \geq -5$.

3. Expand f(x)

$f(x) = 3x^3 + 11x^2 + 100$.

4. Differentiate

$f'(x) = 9x^2 + 22x$.

5. Find the gradient at R

At $R(-4, 84)$, $f'(-4) = 9(16) + 22(-4) = 56$.

6. Use parallel tangents

For the tangent at P to be parallel, $f'(x) = 56$.

7. Solve for x

$9x^2 + 22x - 56 = 0$, so $x = \dfrac{-22 \pm 50}{18}$. The roots are $x = -4$ and $x = \dfrac{14}{9}$.

8. Choose the point P

$x = -4$ is the point R, so $x_P = \dfrac{14}{9}$.

9. Transform R under y equals f of x minus 3

$y = f(x-3)$ translates the curve 3 units to the right, so $(-4, 84)$ maps to $(-1, 84)$.

10. Transform R under y equals 4f(x)

$y = 4f(x)$ stretches vertically by scale factor 4, so $(-4, 84)$ maps to $(-4, 336)$.

Final answer

$(a)\ x \geq -5.$

$(b)\ f'(x) = 9x^2 + 22x.$

$(c)\ x_P = \dfrac{14}{9}.$

$(d)(i)\ (-1, 84).$

$(d)(ii)\ (-4, 336).$

WMA11/01 OCTOBER 2025

Question 10

10 marks

Differentiation

10.

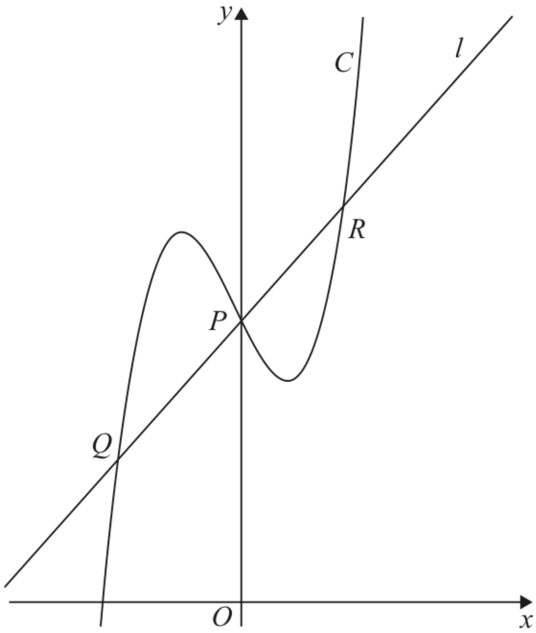


Figure 4

In this question you must show all stages of your working.
Solutions relying entirely on calculator technology are not acceptable.

Figure 4 shows a sketch of the curve C with equation

$$y = 2x^3 + \frac{1}{2}x^2 - 2x + 5$$

The line l is the normal to C at the point P where $x = 0$

The line l also intersects C at points Q and R as shown in Figure 4.

(a) Find, using algebra, the x coordinate of point Q.

(6)

The point T lies on C.

Given that

- the tangent to C at T is parallel to l
- the x coordinate of T is positive

(b) find, using algebra, the exact x coordinate of T.

(4)

(Total for Question 10 is 10 marks)

WMA11/01 OCTOBER 2025

Worked Solution - Question 10

Differentiation

Topic group

1. Differentiate C

$y = 2x^3 + \frac{1}{2}x^2 - 2x + 5$, so $\frac{dy}{dx} = 6x^2 + x - 2$.

2. Find the tangent gradient at P

At P, $x = 0$, so the tangent gradient is -2.

3. Equation of the normal

The normal gradient is $\frac{1}{2}$ and $P = (0, 5)$, so $l : y = \frac{1}{2}x + 5$.

4. Find intersections with the normal

Set $2x^3 + \frac{1}{2}x^2 - 2x + 5 = \frac{1}{2}x + 5$.

5. Form and factor the cubic

$2x^3 + \frac{1}{2}x^2 - \frac{5}{2}x = 0$. Multiplying by 2 gives
$4x^3 + x^2 - 5x = 0 = x(4x^2 + x - 5)$.

6. Solve for x

$x(4x + 5)(x - 1) = 0$, so the intersections have x-coordinates 0, $-\frac{5}{4}$ and 1.

7. Select point Q

The point Q is the left-hand intersection, so $x_Q = -\frac{5}{4}$.

8. Parallel tangent condition

The tangent at T is parallel to l, so its gradient is $\frac{1}{2}$.

9. Set derivative equal to one half

$6x^2 + x - 2 = \frac{1}{2}$, so $12x^2 + 2x - 5 = 0$.

10. Use the positive solution

$x = \frac{-2 \pm \sqrt{4 + 240}}{24} = \frac{-1 \pm \sqrt{61}}{12}$. Since x_T is positive, $x_T = \frac{-1 + \sqrt{61}}{12}$.

Final answer

(a) $x_Q = -\frac{5}{4}$.

(b) $x_T = \frac{-1 + \sqrt{61}}{12}$.

Question 7

10 marks

Differentiation

7. The curve C has equation

$$y = \frac{2}{3}x^3 - 8x^2 + 43x - \frac{20}{3}$$

(a) Show that $\frac{dy}{dx}$ can be written in the form

$$p(x+q)^2 + r$$

where p, q and r are constants to be found.

(5)

(b) Hence state

(i) the minimum value of $\frac{dy}{dx}$

(ii) the value of x at which this minimum value occurs.

(2)

Given that S is the point on C at which the gradient is a minimum,

(c) find the equation of the tangent to C at S, giving your answer in the form $y = mx + c$, where m and c are constants.

(3)

(Total for Question 7 is 10 marks)

WMA11/01 JANUARY 2026

Worked Solution - Question 7

Differentiation

Topic group

1. Differentiate

$y = \frac{2}{3}x^3 - 8x^2 + 43x - \frac{20}{3}$ gives $\frac{dy}{dx} = 2x^2 - 16x + 43$.

2. Complete the square

$2x^2 - 16x + 43 = 2(x^2 - 8x) + 43 = 2[(x-4)^2 - 16] + 43$.

3. Simplify

$\frac{dy}{dx} = 2(x-4)^2 + 11$.

4. Minimum gradient

Since $2(x-4)^2 \geq 0$, the minimum value of $\frac{dy}{dx}$ is 11.

5. Where it occurs

The minimum occurs when $x - 4 = 0$, so $x = 4$.

6. Find the point S

At $x = 4$, $y = \frac{2}{3}(4)^3 - 8(4)^2 + 43(4) - \frac{20}{3} = 80$.

7. Tangent equation

The gradient at S is 11, so $y - 80 = 11(x - 4)$.

8. Simplify

$y = 11x + 36$.

Final answer

(a) $\dfrac{dy}{dx} = 2(x-4)^2 + 11.$

(b) $\min \dfrac{dy}{dx} = 11$ at $x = 4.$

(c) $y = 11x + 36.$

WMA11/01 OCTOBER 2019

Question 10

10 marks

Differentiation

Also in Differentiation

Primary: Graphs of Functions

10.

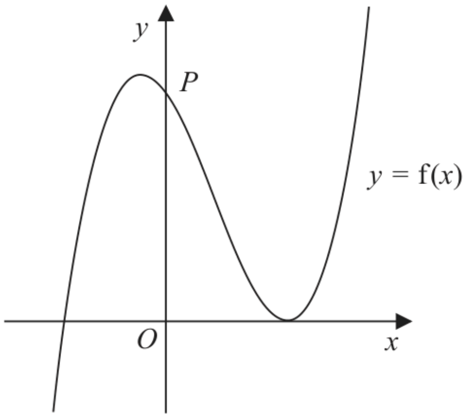


Figure 6

Figure 6 shows a sketch of part of the curve with equation $y = f(x)$, where

$$f(x) = (2x + 5)(x - 3)^2$$

(a) Deduce the values of x for which $f(x) \leqslant 0$

(2)

The curve crosses the y-axis at the point P, as shown.

(b) Expand $f(x)$ to the form

$$ax^3 + bx^2 + cx + d$$

where a, b, c and d are integers to be found.

(3)

(c) Hence, or otherwise, find

(i) the coordinates of P,

(ii) the gradient of the curve at P.

(2)

The curve with equation $y = f(x)$ is translated two units in the positive x direction to a curve with equation $y = g(x)$.

(d) (i) Find $g(x)$, giving your answer in a simplified factorised form.

(ii) Hence state the y intercept of the curve with equation $y = g(x)$.

(3)

(Total 10 marks)

WMA11/01 OCTOBER 2019

Worked Solution - Question 10

Differentiation

Primary group: Graphs of Functions

1. Use the factorised form for the inequality

$f(x) = (2x+5)(x-3)^2$. Since $(x-3)^2 \geq 0$, the sign is controlled by $2x+5$, with an extra zero at $x=3$. Hence $f(x) \leq 0$ for $x \leq -\frac{5}{2}$ or $x=3$.

2. Expand the cubic

$(x-3)^2 = x^2 - 6x + 9$, so
$f(x) = (2x+5)(x^2-6x+9) = 2x^3 - 7x^2 - 12x + 45$.

3. Find P and the gradient at P

At the y-axis, $x=0$, so $P=(0,45)$. Also $f'(x) = 6x^2 - 14x - 12$, so the gradient at P is $f'(0) = -12$.

4. Translate the graph

A translation 2 units in the positive x direction means $g(x) = f(x-2)$. Therefore $g(x) = (2(x-2)+5)((x-2)-3)^2 = (2x+1)(x-5)^2$.

5. Find the new y-intercept

Set $x=0$: $g(0) = (1)(25) = 25$, so the y-intercept is 25.

Final answer
$f(x) \leq 0$ for $x \leq -\frac{5}{2}$ or $x=3$; $f(x) = 2x^3 - 7x^2 - 12x + 45$;
$P=(0,45)$; gradient $= -12$; $g(x) = (2x+1)(x-5)^2$; y-intercept 25.

Question 11

10 marks

Differentiation

Also in Differentiation

Primary: Integration

11. A curve has equation $y = f(x)$.

The point $P\left(4, \frac{32}{3}\right)$ lies on the curve.

Given that

- $f''(x) = \frac{4}{\sqrt{x}} - 3$
- $f'(x) = 5$ at P

find

(a) the equation of the tangent to the curve at P, writing your answer in the form $y = mx + c$, where m and c are constants to be found,

(2)

(b) $f(x)$.

(8)

(Total 10 marks)

WMA11/01 OCTOBER 2019

Worked Solution - Question 11

Differentiation

Primary group: Integration

1. Use the given gradient for the tangent

At $P\left(4, \frac{32}{3}\right)$, $f'(4) = 5$, so the tangent has gradient 5.

2. Find the tangent equation

$y - \frac{32}{3} = 5(x - 4)$. Hence $y = 5x - \frac{28}{3}$.

3. Integrate f'' to find f'

$f''(x) = 4x^{-1/2} - 3$, so $f'(x) = 8x^{1/2} - 3x + C$.

4. Use f'(4)=5

$5 = 8(2) - 12 + C = 4 + C$, so $C = 1$. Therefore $f'(x) = 8\sqrt{x} - 3x + 1$.

5. Integrate again

$f(x) = \frac{16}{3}x^{3/2} - \frac{3}{2}x^2 + x + D$.

6. Use the point on the curve

Substitute $x = 4$, $f(4) = \frac{32}{3}$: $\frac{32}{3} = \frac{16}{3}(8) - \frac{3}{2}(16) + 4 + D = \frac{68}{3} + D$, so $D = -12$.

Final answer

(a) $y = 5x - \frac{28}{3}$.

(b) $f(x) = \frac{16}{3}x^{3/2} - \frac{3}{2}x^2 + x - 12$.

WMA11/01 JANUARY 2020

Question 11

11. A curve has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}''(x) = \frac{6}{\sqrt{x^3}} + x \qquad x > 0$$

The point $P(4, -50)$ lies on the curve.

Given that $\mathrm{f}'(x) = -4$ at P,

(a) find the equation of the normal at P, writing your answer in the form $y = mx + c$, where m and c are constants,

(3)

(b) find $\mathrm{f}(x)$.

(8)

(Total 11 marks)

WMA11/01 JANUARY 2020

Worked Solution - Question 11

Differentiation

Primary group: Integration

1. Find the normal gradient

At P, $f'(4) = -4$, so the tangent gradient is -4. The normal gradient is therefore $\frac{1}{4}$.

2. Find the normal equation

Using $P(4, -50)$: $y + 50 = \frac{1}{4}(x - 4)$, so $y = \frac{1}{4}x - 51$.

3. Integrate f'' to find f'

$f''(x) = 6x^{-3/2} + x$. Therefore $f'(x) = -12x^{-1/2} + \frac{1}{2}x^2 + k$.

4. Use f'(4) = -4

$-4 = -\frac{12}{2} + \frac{1}{2}(16) + k = -6 + 8 + k = 2 + k$, so $k = -6$.

5. Integrate again

$f(x) = -24x^{1/2} + \frac{x^3}{6} - 6x + d$.

6. Use the point on the curve

Substitute $P(4, -50)$:
$-50 = -24(2) + \frac{64}{6} - 24 + d = -72 + \frac{32}{3} + d = -\frac{184}{3} + d$. Hence $d = \frac{34}{3}$.

Final answer

$(a)\ y = \dfrac{1}{4}x - 51.$

$(b)\ f(x) = -24\sqrt{x} + \dfrac{x^3}{6} - 6x + \dfrac{34}{3}.$

WMA11/01 JANUARY 2021

Question 9

11 marks

Differentiation

Also in Differentiation

Primary: Integration

9. (i) Find

$$\int \frac{(3x+2)^2}{4\sqrt{x}}\,dx \qquad\qquad x > 0$$

giving your answer in simplest form.

(5)

(ii) A curve C has equation $y = f(x)$.

Given

- $f'(x) = x^2 + ax + b$ where a and b are constants
- the y intercept of C is -8
- the point $P(3, -2)$ lies on C
- the gradient of C at P is 2

find, in simplest form, $f(x)$.

(6)

(Total 11 marks)

WMA11/01 JANUARY 2021

Worked Solution - Question 9

Differentiation

Primary group: Integration

1. Expand and simplify the integrand

$$\frac{(3x+2)^2}{4\sqrt{x}} = \frac{9x^2+12x+4}{4x^{1/2}} = \frac{9}{4}x^{3/2} + 3x^{1/2} + x^{-1/2}.$$

2. Integrate term by term

$$\int \left(\frac{9}{4}x^{3/2} + 3x^{1/2} + x^{-1/2}\right) dx = \frac{9}{10}x^{5/2} + 2x^{3/2} + 2x^{1/2} + c.$$

3. Write in simplest form

Factoring out $\sqrt{x}$ gives $\dfrac{\sqrt{x}(9x^2+20x+20)}{10} + c$.

4. Use the gradient condition at P

Since $f'(x) = x^2 + ax + b$ and the gradient at $P(3,-2)$ is 2, $f'(3) = 2$. Hence $9 + 3a + b = 2$, so $3a + b = -7$.

5. Integrate f'(x)

$f(x) = \dfrac{x^3}{3} + \dfrac{a}{2}x^2 + bx + c$. The y-intercept is -8, so $c = -8$.

6. Use P(3,-2)

$-2 = \dfrac{27}{3} + \dfrac{9a}{2} + 3b - 8$, so $\dfrac{9a}{2} + 3b = -3$.

7. Solve for a and b

From $3a + b = -7$, $b = -7 - 3a$. Substitute into $\dfrac{9a}{2} + 3b = -3$ to get $a = -4$, and then $b = 5$.

8. Write f(x)

Therefore $f(x) = \dfrac{x^3}{3} - 2x^2 + 5x - 8$.

Final answer

$(i)\ \dfrac{\sqrt{x}(9x^2+20x+20)}{10}+c. \quad (ii)\ f(x)=\dfrac{x^3}{3}-2x^2+5x-8.$

Question 6

8 marks

Differentiation

Also in Differentiation

Primary: Integration

6. The curve C has equation $y = \mathrm{f}(x)$, $x > 0$

Given that

- C passes through the point $P(8, 2)$
- $\mathrm{f}'(x) = \dfrac{32}{3x^2} + 3 - 2\left(\sqrt[3]{x}\right)$

(a) find the equation of the tangent to C at P. Write your answer in the form $y = mx + c$, where m and c are constants to be found.

(3)

(b) Find, in simplest form, $\mathrm{f}(x)$.

(5)

(Total 8 marks)

WMA11/01 MAY/JUNE 2021

Worked Solution - Question 6

Differentiation

Primary group: Integration

1. Find the gradient at P

$f'(x) = \frac{32}{3x^2} + 3 - 2\sqrt[3]{x}$. At $x = 8$,

$f'(8) = \frac{32}{3(64)} + 3 - 2(2) = \frac{1}{6} - 1 = -\frac{5}{6}$.

2. Write the tangent equation

Using $P(8,2)$, $y - 2 = -\frac{5}{6}(x-8)$, hence $y = -\frac{5}{6}x + \frac{26}{3}$.

3. Integrate f'(x)

$f'(x) = \frac{32}{3}x^{-2} + 3 - 2x^{1/3}$, so $f(x) = -\frac{32}{3x} + 3x - \frac{3}{2}x^{4/3} + c$.

4. Use P(8,2) to find c

$2 = -\frac{32}{24} + 24 - \frac{3}{2}(16) + c = -\frac{4}{3} + c$, so $c = \frac{10}{3}$.

5. Write f(x)

$f(x) = -\frac{32}{3x} + 3x - \frac{3}{2}x^{4/3} + \frac{10}{3}$.

Final answer

$(a)\ y = -\frac{5}{6}x + \frac{26}{3}$.

$(b)\ f(x) = -\frac{32}{3x} + 3x - \frac{3}{2}x^{4/3} + \frac{10}{3}$.

WMA11/01 OCTOBER 2021

Question 6

10 marks

Differentiation

Also in Differentiation

Primary: Graphs of Functions

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6. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

A curve C has equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 2(x+1)(x-3)^2$$

(a) Sketch a graph of C.

Show on your graph the coordinates of the points where C cuts or meets the coordinate axes.

(3)

(b) Write $\mathrm{f}(x)$ in the form $ax^3 + bx^2 + cx + d$, where a, b, c and d are constants to be found.

(3)

(c) Hence, find the equation of the tangent to C at the point where $x = \frac{1}{3}$

(4)

(Total 10 marks)

WMA11/01 OCTOBER 2021

Worked Solution - Question 6

Differentiation

Primary group: Graphs of Functions

1. Use the factorised form for the sketch

$f(x) = 2(x+1)(x-3)^2$. The curve cuts the x-axis at $(-1, 0)$ and meets the x-axis at $(3, 0)$ because $x = 3$ is a repeated root.

2. Find the y-intercept

$f(0) = 2(1)(9) = 18$, so the y-intercept is $(0, 18)$.

3. Describe the cubic shape

The leading term is positive, so the cubic goes down to the left and up to the right. It crosses at $x = -1$ and touches at $x = 3$.

4. Expand f(x)

$(x+1)(x-3)^2 = (x+1)(x^2 - 6x + 9) = x^3 - 5x^2 + 3x + 9$, so $f(x) = 2x^3 - 10x^2 + 6x + 18$.

5. Differentiate

$f'(x) = 6x^2 - 20x + 6$.

6. Find the gradient at x = 1/3

$f'\left(\frac{1}{3}\right) = 6\left(\frac{1}{9}\right) - 20\left(\frac{1}{3}\right) + 6 = 0$, so the tangent is horizontal.

7. Find the y-coordinate

$$f\left(\frac{1}{3}\right) = 2\left(\frac{4}{3}\right)\left(-\frac{8}{3}\right)^2 = \frac{512}{27}.$$

8. Write the tangent

A horizontal tangent through this point has equation $y = \frac{512}{27}$.

Final answer

(a) x-intercepts $(-1, 0), (3, 0)$, y-intercept $(0, 18)$.

(b) $f(x) = 2x^3 - 10x^2 + 6x + 18$.

(c) $y = \dfrac{512}{27}$.

Question 8

10 marks

Differentiation

Also in Differentiation

Primary: Straight Line

8.

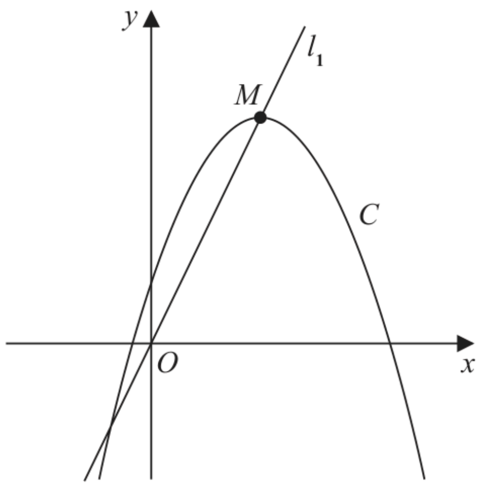

Figure 4

Figure 4 shows a sketch of the curve C with equation

$$y = 4 + 12x - 3x^2$$

The point M is the maximum turning point on C.

(a) (i) Write $4 + 12x - 3x^2$ in the form

$$a + b(x + c)^2$$

where a, b and c are constants to be found.

(ii) Hence, or otherwise, state the coordinates of M.

(5)

The line l_1 passes through O and M, as shown in Figure 4.

A line l_2 touches C and is parallel to l_1

(b) Find an equation for l_2

(5)

(Total 10 marks)

WMA11/01 OCTOBER 2021

Worked Solution - Question 8

Differentiation

Primary group: Straight Line

1. Complete the square

$4 + 12x - 3x^2 = -3(x^2 - 4x) + 4 = -3\big((x-2)^2 - 4\big) + 4 = 16 - 3(x-2)^2$

.

2. State the maximum point

The maximum occurs when $(x-2)^2 = 0$, so $M = (2, 16)$.

3. Find the gradient of l1

Line l_1 passes through $O(0,0)$ and $M(2,16)$, so its gradient is $\frac{16}{2} = 8$.

4. Use the parallel tangent gradient

Line l_2 is parallel to l_1, so its gradient is also 8. For $y = 4 + 12x - 3x^2$, $\frac{dy}{dx} = 12 - 6x$.

5. Find the point of contact

Set $12 - 6x = 8$, giving $x = \frac{2}{3}$. Then $y = 4 + 12\left(\frac{2}{3}\right) - 3\left(\frac{4}{9}\right) = \frac{32}{3}$.

6. Write l2

Using gradient 8 through $\left(\frac{2}{3}, \frac{32}{3}\right)$, $y - \frac{32}{3} = 8\left(x - \frac{2}{3}\right)$. Hence $y = 8x + \frac{16}{3}$.

Final answer

$(a)(i)\ 16 - 3(x-2)^2$. $\quad (a)(ii)\ M = (2, 16)$.

$(b)\ y = 8x + \dfrac{16}{3}$.

WMA11/01 OCTOBER 2021

Question 10

7 marks

Differentiation

Also in Differentiation

Primary: Integration

10. A curve has equation $y = \mathrm{f}(x),\ x > 0$

Given that

- $\mathrm{f}'(x) = ax - 12x^{\frac{1}{3}}$, where a is a constant
- $\mathrm{f}''(x) = 0$ when $x = 27$
- the curve passes through the point $(1, -8)$

(a) find the value of a.

(3)

(b) Hence find $\mathrm{f}(x)$.

(4)

(Total 7 marks)

WMA11/01 OCTOBER 2021

Worked Solution - Question 10

Differentiation

Primary group: Integration

1. Differentiate f'(x)

$f'(x) = ax - 12x^{1/3}$, so $f''(x) = a - 4x^{-2/3}$.

2. Use f''(27) = 0

Since $27^{2/3} = 9$, $0 = a - \frac{4}{9}$, so $a = \frac{4}{9}$.

3. Integrate f'(x)

With $a = \frac{4}{9}$, $f'(x) = \frac{4}{9}x - 12x^{1/3}$. Therefore $f(x) = \frac{2}{9}x^2 - 9x^{4/3} + c$.

4. Use the point (1, -8)

$-8 = \frac{2}{9} - 9 + c$, so $c = -8 + 9 - \frac{2}{9} = \frac{7}{9}$.

5. Write f(x)

$f(x) = \frac{2}{9}x^2 - 9x^{4/3} + \frac{7}{9}$.

Final answer

$(a)\ a = \frac{4}{9}$.

$(b)\ f(x) = \frac{2}{9}x^2 - 9x^{4/3} + \frac{7}{9}$.

11 marks

Differentiation

WMA11/01 JANUARY 2022

Question 6

Also in Differentiation

Primary: Integration

6. The curve C has equation $y = \mathrm{f}(x)$ where $x > 0$

Given that

- $\mathrm{f}'(x) = \dfrac{(x+3)^2}{x\sqrt{x}}$
- the point $P(4, 20)$ lies on C

(a) (i) find the value of the gradient at P

(ii) Hence find the equation of the tangent to C at P, giving your answer in the form $ax + by + c = 0$ where a, b and c are integers to be found.

(4)

(b) Find $\mathrm{f}(x)$, simplifying your answer.

(7)

(Total 11 marks)

WMA11/01 JANUARY 2022

Worked Solution - Question 6

Differentiation

Primary group: Integration

1. Simplify f'(x)

$$f'(x) = \frac{(x+3)^2}{x\sqrt{x}} = \frac{x^2+6x+9}{x^{3/2}} = x^{1/2} + 6x^{-1/2} + 9x^{-3/2}.$$

2. Find the gradient at P

At $x = 4$, $f'(4) = 2 + 3 + \frac{9}{8} = \frac{49}{8}$.

3. Find the tangent equation

Using $P(4, 20)$, $y - 20 = \frac{49}{8}(x-4)$. Multiplying out gives $49x - 8y - 36 = 0$.

4. Integrate f'(x)

$$f(x) = \int \left(x^{1/2} + 6x^{-1/2} + 9x^{-3/2}\right)dx = \frac{2}{3}x^{3/2} + 12x^{1/2} - 18x^{-1/2} + c.$$

5. Use P(4,20)

$20 = \frac{2}{3}(8) + 12(2) - \frac{18}{2} + c = \frac{61}{3} + c$, so $c = -\frac{1}{3}$.

6. Write f(x)

$$f(x) = \frac{2}{3}x\sqrt{x} + 12\sqrt{x} - \frac{18}{\sqrt{x}} - \frac{1}{3}.$$

Final answer

$(a)(i)\ \frac{49}{8}$. $\quad (a)(ii)\ 49x - 8y - 36 = 0.$

$(b)\ f(x) = \frac{2}{3}x\sqrt{x} + 12\sqrt{x} - \frac{18}{\sqrt{x}} - \frac{1}{3}.$

9 marks

Differentiation

Also in Differentiation

Primary: Integration

WMA11/01 MAY/JUNE 2022

Question 7

7. The curve C has equation $y = f(x)$, $x > 0$

Given that

- $f'(x) = \dfrac{2}{\sqrt{x}} + \dfrac{A}{x^2} + 3$, where A is a constant
- $f''(x) = 0$ when $x = 4$

(a) find the value of A.

(4)

Given also that

- $f(x) = 8\sqrt{3}$, when $x = 12$

(b) find $f(x)$, giving each term in simplest form.

(5)

(Total 9 marks)

WMA11/01 MAY/JUNE 2022

Worked Solution - Question 7

Differentiation

Primary group: Integration

1. Differentiate f'(x)

$f'(x) = 2x^{-1/2} + Ax^{-2} + 3$, so $f''(x) = -x^{-3/2} - 2Ax^{-3}$.

2. Use f''(4) = 0

$-4^{-3/2} - 2A(4^{-3}) = 0$, so $-\frac{1}{8} - \frac{A}{32} = 0$ and $A = -4$.

3. Substitute A into f'(x)

$f'(x) = \frac{2}{\sqrt{x}} - \frac{4}{x^2} + 3.$

4. Integrate to find f(x)

$f(x) = 4\sqrt{x} + \frac{4}{x} + 3x + c.$

5. Use f(12) = 8sqrt3

$f(12) = 8\sqrt{3} + \frac{1}{3} + 36 + c = 8\sqrt{3}$, so $c = -\frac{109}{3}$.

Final answer

$f(x) = 4\sqrt{x} + \frac{4}{x} + 3x - \frac{109}{3}.$

8 marks

Differentiation

Also in Differentiation

Primary: Integration

WMA11/01 JANUARY 2023

Question 11

11. A curve C has equation $y = \mathrm{f}(x), \quad x > 0$

Given that

- $\mathrm{f}''(x) = 4x + \dfrac{1}{\sqrt{x}}$
- the point P has x coordinate 4 and lies on C
- the tangent to C at P has equation $y = 3x + 4$

(a) find an equation of the normal to C at P

(2)

(b) find $\mathrm{f}(x)$, writing your answer in simplest form.

(6)

(Total for Question 11 is 8 marks)

WMA11/01 JANUARY 2023

Worked Solution - Question 11

Differentiation

Primary group: Integration

1. Find point P

Since P has x-coordinate 4 and lies on the tangent $y = 3x + 4$, $P = (4, 16)$.

2. Find the normal

The tangent gradient is 3, so the normal gradient is $-\frac{1}{3}$. Hence the normal is $y - 16 = -\frac{1}{3}(x - 4)$.

3. Integrate f''(x)

$f''(x) = 4x + x^{-1/2}$, so $f'(x) = 2x^2 + 2\sqrt{x} + A$.

4. Use the tangent gradient

At $x = 4$, $f'(4) = 3$. Thus $2(16) + 2(2) + A = 3$, so $A = -33$.

5. Integrate again

$f(x) = \frac{2}{3}x^3 + \frac{4}{3}x^{3/2} - 33x + B$.

6. Use P(4, 16)

$16 = \frac{2}{3}(64) + \frac{4}{3}(8) - 33(4) + B = \frac{160}{3} - 132 + B$.

7. Find B

$B = \frac{284}{3}$, so $f(x) = \frac{2}{3}x^3 + \frac{4}{3}x^{3/2} - 33x + \frac{284}{3}$.

Final answer

(a) $y - 16 = -\dfrac{1}{3}(x - 4)$.

(b) $f(x) = \dfrac{2}{3}x^3 + \dfrac{4}{3}x^{3/2} - 33x + \dfrac{284}{3}$.

10 marks

WMA11/01 MAY/JUNE 2023

Differentiation

Question 8

Also in Differentiation

Primary: Integration

8. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Find the equation of the tangent to the curve with equation

$$y = \frac{1}{4}x^3 - 8x^{-\frac{1}{2}}$$

at the point $P(4, 12)$

Give your answer in the form $ax + by + c = 0$ where a, b and c are integers.

(5)

The curve with equation $y = \mathrm{f}(x)$ also passes through the point $P(4, 12)$

Given that

$$\mathrm{f}'(x) = \frac{1}{4}x^3 - 8x^{-\frac{1}{2}}$$

(b) find $\mathrm{f}(x)$ giving the coefficients in simplest form.

(5)

(Total for Question 8 is 10 marks)

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Worked Solution - Question 8

Differentiation

Primary group: Integration

1. Differentiate the curve

For $y = \frac{1}{4}x^3 - 8x^{-1/2}$, $\frac{dy}{dx} = \frac{3}{4}x^2 + 4x^{-3/2}$.

2. Find the gradient at P

At $x = 4$, $\frac{dy}{dx} = \frac{3}{4}(16) + 4(4^{-3/2}) = 12 + \frac{1}{2} = \frac{25}{2}$.

3. Find the tangent

Using $P(4, 12)$, $y - 12 = \frac{25}{2}(x - 4)$.

4. Write in integer form

Multiplying by 2 gives $2y - 24 = 25x - 100$, hence $25x - 2y - 76 = 0$.

5. Integrate f'(x)

$f'(x) = \frac{1}{4}x^3 - 8x^{-1/2}$ gives $f(x) = \frac{x^4}{16} - 16\sqrt{x} + c$.

6. Use P(4, 12)

$12 = \frac{4^4}{16} - 16\sqrt{4} + c = 16 - 32 + c$, so $c = 28$.

Final answer

(a) $25x - 2y - 76 = 0$.

(b) $f(x) = \frac{x^4}{16} - 16\sqrt{x} + 28$.

10 marks

Differentiation

WMA11/01 OCTOBER 2023

Question 7

Also in Differentiation

Primary: Integration

7. The curve C has equation $y = \mathrm{f}(x)$ where $x > 0$

Given that

- $\mathrm{f}'(x) = \dfrac{4x^2 + 10 - 7x^{\frac{1}{2}}}{4x^{\frac{1}{2}}}$
- the point $P(4, -1)$ lies on C

(a) (i) find the value of the gradient of C at P

(ii) Hence find the equation of the normal to C at P, giving your answer in the form $ax + by + c = 0$ where a, b and c are integers to be found.

(4)

(b) Find $\mathrm{f}(x)$.

(6)

(Total for Question 7 is 10 marks)

WMA11/01 OCTOBER 2023

Worked Solution - Question 7

Differentiation

Primary group: Integration

1. Simplify f prime

$$f'(x) = \frac{4x^2 + 10 - 7x^{1/2}}{4x^{1/2}} = x^{3/2} + \frac{5}{2}x^{-1/2} - \frac{7}{4}.$$

2. Find the gradient at P

At $x = 4$, $f'(4) = 4^{3/2} + \frac{5}{2}(4^{-1/2}) - \frac{7}{4} = 8 + \frac{5}{4} - \frac{7}{4} = \frac{15}{2}$.

3. Find the normal gradient

The normal gradient is the negative reciprocal, so $m_n = -\frac{2}{15}$.

4. Use point P

Using $P(4, -1)$, $y + 1 = -\frac{2}{15}(x - 4)$.

5. Write the normal in integer form

$15y + 15 = -2x + 8$, hence $2x + 15y + 7 = 0$.

6. Integrate f prime

$$f(x) = \int \left(x^{3/2} + \frac{5}{2}x^{-1/2} - \frac{7}{4}\right)dx = \frac{2}{5}x^{5/2} + 5x^{1/2} - \frac{7}{4}x + c.$$

7. Use P to find c

$-1 = \frac{2}{5}(4^{5/2}) + 5(4^{1/2}) - \frac{7}{4}(4) + c = \frac{64}{5} + 10 - 7 + c$, so $c = -\frac{84}{5}$.

Final answer

$(a)(i)\ \dfrac{15}{2}$. $\quad (a)(ii)\ 2x + 15y + 7 = 0.$

$(b)\ f(x) = \dfrac{2}{5}x^{5/2} + 5x^{1/2} - \dfrac{7}{4}x - \dfrac{84}{5}.$

Question 10

8 marks

Differentiation

Also in Differentiation

Primary: Integration

10. **In this question you must show all stages of your working.**

The curve C has equation $y = \mathrm{f}(x),\ x > 0$

Given that

- the point $P\left(2, 8\sqrt{2}\right)$ lies on C
- $\mathrm{f}'(x) = 4\sqrt{x^3} + \dfrac{k}{x^2}$ where k is a constant
- $\mathrm{f}''(x) = 0$ at P

(a) find the exact value of k,

(4)

(b) find $\mathrm{f}(x)$, giving your answer in simplest form.

(4)

(Total for Question 10 is 8 marks)

WMA11/01 JANUARY 2024

Worked Solution - Question 10

Differentiation

Primary group: Integration

1. Differentiate f prime

$f'(x) = 4x^{3/2} + kx^{-2}$, so $f''(x) = 6x^{1/2} - 2kx^{-3}$.

2. Use f double prime equals zero at P

At P, $x = 2$ and $f''(2) = 0$, so $6\sqrt{2} - 2k(2^{-3}) = 0$.

3. Find k

$6\sqrt{2} - \dfrac{k}{4} = 0$, hence $k = 24\sqrt{2}$.

4. Integrate f prime

$f(x) = \int \left(4x^{3/2} + 24\sqrt{2}x^{-2}\right) dx = \dfrac{8}{5}x^{5/2} - 24\sqrt{2}x^{-1} + c$.

5. Use the point P

$P(2, 8\sqrt{2})$ lies on C, so $8\sqrt{2} = \dfrac{8}{5}(2^{5/2}) - \dfrac{24\sqrt{2}}{2} + c$.

6. Find c

$8\sqrt{2} = \dfrac{32\sqrt{2}}{5} - 12\sqrt{2} + c$, so $c = \dfrac{68\sqrt{2}}{5}$.

Final answer

(a) $k = 24\sqrt{2}$.

(b) $f(x) = \dfrac{8}{5}x^{5/2} - \dfrac{24\sqrt{2}}{x} + \dfrac{68\sqrt{2}}{5}$.

Question 10

10. The curve C has equation $y = \mathrm{f}(x)$ where $x > 0$

Given that

- $\mathrm{f}'(x) = 6x - \dfrac{(2x-1)(3x+2)}{2\sqrt{x}}$
- the point $P(4, 12)$ lies on C

(a) find the equation of the normal to C at P, giving your answer in the form $y = mx + c$ where m and c are integers to be found,

(4)

(b) find $\mathrm{f}(x)$, giving each term in simplest form.

(6)

(Total for Question 10 is 10 marks)

WMA11/01 MAY/JUNE 2024

Worked Solution - Question 10

Differentiation

Primary group: Integration

1. Expand the numerator

$(2x-1)(3x+2)=6x^2+x-2.$

2. Simplify f prime

$f'(x)=6x-\dfrac{6x^2+x-2}{2\sqrt{x}}=6x-3x^{3/2}-\dfrac{1}{2}x^{1/2}+x^{-1/2}.$

3. Find the tangent gradient at P

$f'(4)=24-3(8)-\dfrac{1}{2}(2)+\dfrac{1}{2}=-\dfrac{1}{2}.$

4. Use the normal gradient

The normal gradient is the negative reciprocal of $-\dfrac{1}{2}$, so it is 2.

5. Find the normal equation

Through $P(4,12)$, $y-12=2(x-4)$, hence $y=2x+4$.

6. Integrate f prime

$f(x)=3x^2-\dfrac{6}{5}x^{5/2}-\dfrac{1}{3}x^{3/2}+2x^{1/2}+c.$

7. Use P to find c

$12=3(4)^2-\dfrac{6}{5}(4^{5/2})-\dfrac{1}{3}(4^{3/2})+2(4^{1/2})+c.$

8. Simplify c

$12=48-\dfrac{192}{5}-\dfrac{8}{3}+4+c=\dfrac{164}{15}+c$, so $c=\dfrac{16}{15}$.

Final answer

(a) $y = 2x + 4$.

(b) $f(x) = 3x^2 - \frac{6}{5}x^{5/2} - \frac{1}{3}x^{3/2} + 2x^{1/2} + \frac{16}{15}$.

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Question 8

7 marks

Differentiation

Also in Differentiation

Primary: Integration

8. A curve C has equation $y = \mathrm{f}(x)$.

The point P with x coordinate 3 lies on C

Given

- $\mathrm{f}'(x) = 4x^2 + kx + 3$ where k is a constant
- the normal to C at P has equation $y = -\dfrac{1}{24}x + 5$

(a) show that $k = -5$

(3)

(b) Hence find $\mathrm{f}(x)$.

(4)

(Total for Question 8 is 7 marks)

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Worked Solution - Question 8

Differentiation

Primary group: Integration

1. Find the y-coordinate of P

The normal is $y = -\frac{1}{24}x + 5$. At $x = 3$, $y = -\frac{3}{24} + 5 = \frac{39}{8}$.

2. Use the normal gradient

The normal gradient is $-\frac{1}{24}$, so the tangent gradient is 24.

3. Use f prime at x equals 3

$f'(3) = 4(3)^2 + 3k + 3 = 39 + 3k$.

4. Find k

$39 + 3k = 24$, so $k = -5$.

5. Integrate f prime

With $k = -5$, $f'(x) = 4x^2 - 5x + 3$, so $f(x) = \frac{4}{3}x^3 - \frac{5}{2}x^2 + 3x + c$.

6. Use point P

$\frac{39}{8} = \frac{4}{3}(27) - \frac{5}{2}(9) + 3(3) + c = \frac{45}{2} + c$.

7. Find c

$c = \frac{39}{8} - \frac{45}{2} = -\frac{141}{8}$.

Final answer

$k = -5$ and $f(x) = \frac{4}{3}x^3 - \frac{5}{2}x^2 + 3x - \frac{141}{8}$.

WMA11/01 JANUARY 2025

Question 6

10 marks

Differentiation

Also in Differentiation

Primary: Integration

6. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The curve C has equation $y = \mathrm{f}(x)$, $x > 0$

Given that

- the point $P(4, -5)$ lies on C
- $\mathrm{f}'(x) = \dfrac{2x^2 + ax + b}{4\sqrt{x}}$, where a and b are constants
- the gradient of the tangent to C at P is 7

(a) show that

$$4a + b = 24$$

(2)

Given also that $a + b = -9$

(b) find, in simplest form, $\mathrm{f}(x)$

(7)

Curve C is transformed to the curve with equation $y = \mathrm{f}(x - 3)$

Given that point P is transformed to the point Q,

(c) state the coordinates of Q.

(1)

(Total for Question 6 is 10 marks)

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Worked Solution - Question 6

Differentiation

Primary group: Integration

1. Use the gradient at P

At $x = 4$, $f'(4) = 7$. Since $f'(x) = \dfrac{2x^2 + ax + b}{4\sqrt{x}}$, we get $\dfrac{32 + 4a + b}{8} = 7$.

2. Show the required equation

$32 + 4a + b = 56$, so $4a + b = 24$.

3. Solve for a and b

Together with $a + b = -9$, subtracting gives $3a = 33$, so $a = 11$ and $b = -20$.

4. Rewrite f prime

$f'(x) = \dfrac{2x^2 + 11x - 20}{4\sqrt{x}} = \dfrac{1}{2}x^{3/2} + \dfrac{11}{4}x^{1/2} - 5x^{-1/2}$.

5. Integrate

$f(x) = \dfrac{1}{5}x^{5/2} + \dfrac{11}{6}x^{3/2} - 10x^{1/2} + c$.

6. Use P to find c

$-5 = \dfrac{1}{5}(4^{5/2}) + \dfrac{11}{6}(4^{3/2}) - 10(4^{1/2}) + c = \dfrac{16}{15} + c$.

7. State f(x)

$c = -\dfrac{91}{15}$, so $f(x) = \dfrac{1}{5}x^{5/2} + \dfrac{11}{6}x^{3/2} - 10x^{1/2} - \dfrac{91}{15}$.

8. Transform P

$y = f(x - 3)$ translates the curve 3 units to the right, so $P(4, -5)$ becomes $Q(7, -5)$.

Final answer

(b) $f(x) = \frac{1}{5}x^{5/2} + \frac{11}{6}x^{3/2} - 10x^{1/2} - \frac{91}{15}$.

(c) $Q = (7, -5)$.

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Question 8

8. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

A curve has equation $y = f(x)$, $x > 0$

The point $P(4, 12)$ lies on the curve.

Given that

- $f'(x) = 3\sqrt{x} + kx^2$ where k is a constant
- the equation of the tangent to the curve at P has equation $y = 10x + c$ where c is a constant

(a) (i) show that $k = \frac{1}{4}$

(ii) find the value of c

(4)

(b) Hence find the value of $f''(x)$ at P.

(3)

(c) Find $f(x)$.

(4)

(Total for Question 8 is 11 marks)

WMA11/01 MAY/JUNE 2025

Worked Solution - Question 8

Differentiation

Primary group: Integration

1. Use the tangent gradient

The tangent at P has equation $y = 10x + c$, so the gradient at P is 10.

2. Show k

$f'(4) = 3\sqrt{4} + k(4)^2 = 6 + 16k = 10$, so $16k = 4$ and $k = \frac{1}{4}$.

3. Find c

Substitute $P(4, 12)$ into $y = 10x + c$: $12 = 40 + c$, so $c = -28$.

4. Differentiate f prime

With $k = \frac{1}{4}$, $f'(x) = 3x^{1/2} + \frac{1}{4}x^2$.

5. Find f double prime

$f''(x) = \frac{3}{2}x^{-1/2} + \frac{1}{2}x$.

6. Evaluate at P

$f''(4) = \frac{3}{2} \cdot \frac{1}{2} + 2 = \frac{3}{4} + 2 = \frac{11}{4}$.

7. Integrate f prime

$f(x) = \int \left(3x^{1/2} + \frac{1}{4}x^2\right) dx = 2x^{3/2} + \frac{1}{12}x^3 + d$.

8. Use P to find d

$12 = 2(4^{3/2}) + \frac{1}{12}(4^3) + d = 16 + \frac{16}{3} + d$.

9. State f(x)

$d = -\dfrac{28}{3}$, so $f(x) = 2x^{3/2} + \dfrac{1}{12}x^3 - \dfrac{28}{3}$.

Final answer

$(a)(i)\ k = \dfrac{1}{4}$. $\quad (a)(ii)\ c = -28.$

$(b)\ f''(P) = \dfrac{11}{4}$.

$(c)\ f(x) = 2x^{3/2} + \dfrac{1}{12}x^3 - \dfrac{28}{3}$.

Question 8

8. **In this question you must show all stages of your working.**
Solutions relying on calculator technology are not acceptable.

A curve has equation $y = \mathrm{f}(x)$, $x > 0$

Given that

- $\mathrm{f}'(x) = 2x + \dfrac{8}{x^2} + k$, where k is a constant
- the equation of the tangent to the curve at $x = \sqrt{2}$ is $y = 5x - 3\sqrt{2}$

(a) find the exact value of k.

(2)

(b) Find an equation of the normal to the curve at $x = \sqrt{2}$

(2)

(c) Find $\mathrm{f}(x)$, writing your answer in simplest form.

(4)

(Total for Question 8 is 8 marks)

WMA11/01 OCTOBER 2025

Worked Solution - Question 8

Differentiation

Primary group: Integration

1. Use tangent gradient

The tangent has equation $y = 5x - 3\sqrt{2}$, so the gradient at $x = \sqrt{2}$ is 5.

2. Find k

$f'(\sqrt{2}) = 2\sqrt{2} + \dfrac{8}{(\sqrt{2})^2} + k = 2\sqrt{2} + 4 + k = 5$, so $k = 1 - 2\sqrt{2}$.

3. Find the point on the curve

At $x = \sqrt{2}$, the tangent gives $y = 5\sqrt{2} - 3\sqrt{2} = 2\sqrt{2}$.

4. Normal gradient

The tangent gradient is 5, so the normal gradient is $-\dfrac{1}{5}$.

5. Equation of the normal

$y - 2\sqrt{2} = -\dfrac{1}{5}(x - \sqrt{2})$, hence $y = -\dfrac{1}{5}x + \dfrac{11\sqrt{2}}{5}$.

6. Integrate f prime

$f(x) = \int \left(2x + \dfrac{8}{x^2} + k\right) dx = x^2 - \dfrac{8}{x} + kx + c.$

7. Substitute k

$f(x) = x^2 - \dfrac{8}{x} + (1 - 2\sqrt{2})x + c.$

8. Use the point

$2\sqrt{2} = 2 - \dfrac{8}{\sqrt{2}} + (1 - 2\sqrt{2})\sqrt{2} + c = -2 - 3\sqrt{2} + c.$

9. Find c

$c = 2 + 5\sqrt{2}$, so $f(x) = x^2 - \frac{8}{x} + (1 - 2\sqrt{2})x + 2 + 5\sqrt{2}$.

Final answer

(a) $k = 1 - 2\sqrt{2}$.

(b) $y = -\frac{1}{5}x + \frac{11\sqrt{2}}{5}$.

(c) $f(x) = x^2 - \frac{8}{x} + (1 - 2\sqrt{2})x + 2 + 5\sqrt{2}$.

Question 8

12 marks

Differentiation

Also in Differentiation

Primary: Integration

8. **In this question you must show all stages of your working.**
Solutions relying on calculator technology are not acceptable.

A curve C has equation $y = \mathrm{f}(x), \quad x > 0$

A point P lies on C.

Given that

- $\mathrm{f}'(x) = 4x^2 + \dfrac{6}{x^2} - 4$
- the equation of the tangent to C at P is $y = 10x - 6\sqrt{3}$

(a) (i) verify that $\sqrt{3}$ is a possible x coordinate of P,

(ii) find, using algebra, the other possible x coordinate of P.

(6)

Given that the x coordinate of P is $\sqrt{3}$

(b) find an equation of the normal to C at P.

(2)

(c) Find $\mathrm{f}(x)$, writing your answer in simplest form.
You must show each stage of your working.

(4)

(Total for Question 8 is 12 marks)

WMA11/01 JANUARY 2026

Worked Solution - Question 8

Differentiation

Primary group: Integration

1. Verify x equals root 3

$f'(\sqrt{3}) = 4(\sqrt{3})^2 + \dfrac{6}{(\sqrt{3})^2} - 4 = 12 + 2 - 4 = 10$, which matches the tangent gradient.

2. Find all possible x-coordinates

Set $4x^2 + \dfrac{6}{x^2} - 4 = 10$.

3. Form a quadratic in x squared

Multiplying by x^2 gives $4x^4 - 14x^2 + 6 = 0$, so $2x^4 - 7x^2 + 3 = 0$.

4. Factor

$2x^4 - 7x^2 + 3 = (2x^2 - 1)(x^2 - 3) = 0$.

5. Use x greater than zero

$x = \sqrt{3}$ or $x = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$. The other possible x-coordinate is $\dfrac{\sqrt{2}}{2}$.

6. Find the point P

Given $x_P = \sqrt{3}$, the tangent gives $y_P = 10\sqrt{3} - 6\sqrt{3} = 4\sqrt{3}$.

7. Normal equation

The normal gradient is $-\dfrac{1}{10}$, so $y - 4\sqrt{3} = -\dfrac{1}{10}(x - \sqrt{3})$.

8. Integrate f prime

$f(x) = \int \left(4x^2 + \dfrac{6}{x^2} - 4\right) dx = \dfrac{4}{3}x^3 - \dfrac{6}{x} - 4x + c$.

9. Use P to find c

$$4\sqrt{3} = \frac{4}{3}(\sqrt{3})^3 - \frac{6}{\sqrt{3}} - 4\sqrt{3} + c = 4\sqrt{3} - 2\sqrt{3} - 4\sqrt{3} + c.$$

10. State f(x)

$c = 6\sqrt{3}$, so $f(x) = \frac{4}{3}x^3 - \frac{6}{x} - 4x + 6\sqrt{3}$.

Final answer

$(a)(i)$ verified.

$(a)(ii)$ $x = \frac{\sqrt{2}}{2}$.

(b) $y - 4\sqrt{3} = -\frac{1}{10}(x - \sqrt{3})$.

(c) $f(x) = \frac{4}{3}x^3 - \frac{6}{x} - 4x + 6\sqrt{3}$.

TOPIC

Integration

Question 11

11. A curve has equation $y = \mathrm{f}(x)$.

The point $P\left(4, \frac{32}{3}\right)$ lies on the curve.

Given that

- $\mathrm{f}''(x) = \dfrac{4}{\sqrt{x}} - 3$
- $\mathrm{f}'(x) = 5$ at P

find

(a) the equation of the tangent to the curve at P, writing your answer in the form $y = mx + c$, where m and c are constants to be found,

(2)

(b) $\mathrm{f}(x)$.

(8)

(Total 10 marks)

Worked Solution - Question 11

Integration

Topic group

1. Use the given gradient for the tangent

At $P\left(4, \frac{32}{3}\right)$, $f'(4) = 5$, so the tangent has gradient 5.

2. Find the tangent equation

$y - \frac{32}{3} = 5(x-4)$. Hence $y = 5x - \frac{28}{3}$.

3. Integrate f'' to find f'

$f''(x) = 4x^{-1/2} - 3$, so $f'(x) = 8x^{1/2} - 3x + C$.

4. Use f'(4)=5

$5 = 8(2) - 12 + C = 4 + C$, so $C = 1$. Therefore $f'(x) = 8\sqrt{x} - 3x + 1$.

5. Integrate again

$f(x) = \frac{16}{3}x^{3/2} - \frac{3}{2}x^2 + x + D$.

6. Use the point on the curve

Substitute $x = 4$, $f(4) = \frac{32}{3}$: $\frac{32}{3} = \frac{16}{3}(8) - \frac{3}{2}(16) + 4 + D = \frac{68}{3} + D$, so $D = -12$.

Final answer

(a) $y = 5x - \frac{28}{3}$.

(b) $f(x) = \frac{16}{3}x^{3/2} - \frac{3}{2}x^2 + x - 12$.

Question 11

11 marks

Integration

11. A curve has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}''(x) = \frac{6}{\sqrt{x^3}} + x \qquad x > 0$$

The point $P(4, -50)$ lies on the curve.

Given that $\mathrm{f}'(x) = -4$ at P,

(a) find the equation of the normal at P, writing your answer in the form $y = mx + c$, where m and c are constants,

(3)

(b) find $\mathrm{f}(x)$.

(8)

(Total 11 marks)

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Worked Solution - Question 11

Integration

Topic group

1. Find the normal gradient

At P, $f'(4) = -4$, so the tangent gradient is -4. The normal gradient is therefore $\frac{1}{4}$.

2. Find the normal equation

Using $P(4, -50)$: $y + 50 = \frac{1}{4}(x - 4)$, so $y = \frac{1}{4}x - 51$.

3. Integrate f'' to find f'

$f''(x) = 6x^{-3/2} + x$. Therefore $f'(x) = -12x^{-1/2} + \frac{1}{2}x^2 + k$.

4. Use f'(4) = -4

$-4 = -\frac{12}{2} + \frac{1}{2}(16) + k = -6 + 8 + k = 2 + k$, so $k = -6$.

5. Integrate again

$f(x) = -24x^{1/2} + \frac{x^3}{6} - 6x + d$.

6. Use the point on the curve

Substitute $P(4, -50)$:
$-50 = -24(2) + \frac{64}{6} - 24 + d = -72 + \frac{32}{3} + d = -\frac{184}{3} + d$. Hence $d = \frac{34}{3}$.

Final answer

$(a)\ y = \dfrac{1}{4}x - 51.$

$(b)\ f(x) = -24\sqrt{x} + \dfrac{x^3}{6} - 6x + \dfrac{34}{3}.$

Question 9

11 marks

Integration

9. (i) Find

$$\int \frac{(3x+2)^2}{4\sqrt{x}}\,dx \qquad x > 0$$

giving your answer in simplest form.

(5)

(ii) A curve C has equation $y = f(x)$.

Given

- $f'(x) = x^2 + ax + b$ where a and b are constants
- the y intercept of C is -8
- the point $P(3,-2)$ lies on C
- the gradient of C at P is 2

find, in simplest form, $f(x)$.

(6)

(Total 11 marks)

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Worked Solution - Question 9

Integration

Topic group

1. Expand and simplify the integrand

$$\frac{(3x+2)^2}{4\sqrt{x}} = \frac{9x^2+12x+4}{4x^{1/2}} = \frac{9}{4}x^{3/2} + 3x^{1/2} + x^{-1/2}.$$

2. Integrate term by term

$$\int \left(\frac{9}{4}x^{3/2} + 3x^{1/2} + x^{-1/2}\right) dx = \frac{9}{10}x^{5/2} + 2x^{3/2} + 2x^{1/2} + c.$$

3. Write in simplest form

Factoring out $\sqrt{x}$ gives $\dfrac{\sqrt{x}(9x^2+20x+20)}{10} + c.$

4. Use the gradient condition at P

Since $f'(x) = x^2 + ax + b$ and the gradient at $P(3,-2)$ is 2, $f'(3) = 2$. Hence $9 + 3a + b = 2$, so $3a + b = -7$.

5. Integrate f'(x)

$f(x) = \dfrac{x^3}{3} + \dfrac{a}{2}x^2 + bx + c$. The y-intercept is -8, so $c = -8$.

6. Use P(3,-2)

$-2 = \dfrac{27}{3} + \dfrac{9a}{2} + 3b - 8$, so $\dfrac{9a}{2} + 3b = -3$.

7. Solve for a and b

From $3a + b = -7$, $b = -7 - 3a$. Substitute into $\dfrac{9a}{2} + 3b = -3$ to get $a = -4$, and then $b = 5$.

8. Write f(x)

Therefore $f(x) = \dfrac{x^3}{3} - 2x^2 + 5x - 8.$

Final answer

$$(i)\ \frac{\sqrt{x}(9x^2+20x+20)}{10}+c. \quad (ii)\ f(x)=\frac{x^3}{3}-2x^2+5x-8.$$

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8 marks

Question 6

Integration

6. The curve C has equation $y = \mathrm{f}(x)$, $x > 0$

Given that

- C passes through the point $P(8, 2)$
- $\mathrm{f}'(x) = \dfrac{32}{3x^2} + 3 - 2\left(\sqrt[3]{x}\right)$

(a) find the equation of the tangent to C at P. Write your answer in the form $y = mx + c$, where m and c are constants to be found.

(3)

(b) Find, in simplest form, $\mathrm{f}(x)$.

(5)

(Total 8 marks)

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Worked Solution - Question 6

Integration

Topic group

1. Find the gradient at P

$f'(x) = \dfrac{32}{3x^2} + 3 - 2\sqrt[3]{x}$. At $x = 8$,

$f'(8) = \dfrac{32}{3(64)} + 3 - 2(2) = \dfrac{1}{6} - 1 = -\dfrac{5}{6}$.

2. Write the tangent equation

Using $P(8,2)$, $y - 2 = -\dfrac{5}{6}(x-8)$, hence $y = -\dfrac{5}{6}x + \dfrac{26}{3}$.

3. Integrate f'(x)

$f'(x) = \dfrac{32}{3}x^{-2} + 3 - 2x^{1/3}$, so $f(x) = -\dfrac{32}{3x} + 3x - \dfrac{3}{2}x^{4/3} + c$.

4. Use P(8,2) to find c

$2 = -\dfrac{32}{24} + 24 - \dfrac{3}{2}(16) + c = -\dfrac{4}{3} + c$, so $c = \dfrac{10}{3}$.

5. Write f(x)

$f(x) = -\dfrac{32}{3x} + 3x - \dfrac{3}{2}x^{4/3} + \dfrac{10}{3}$.

Final answer

$(a)\ y = -\dfrac{5}{6}x + \dfrac{26}{3}$.

$(b)\ f(x) = -\dfrac{32}{3x} + 3x - \dfrac{3}{2}x^{4/3} + \dfrac{10}{3}$.

Question 10

7 marks

Integration

10. A curve has equation $y = \mathrm{f}(x)$, $x > 0$

Given that

- $\mathrm{f}'(x) = ax - 12x^{\frac{1}{3}}$, where a is a constant
- $\mathrm{f}''(x) = 0$ when $x = 27$
- the curve passes through the point $(1, -8)$

(a) find the value of a.

(3)

(b) Hence find $\mathrm{f}(x)$.

(4)

(Total 7 marks)

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Worked Solution - Question 10

Integration

Topic group

1. Differentiate f'(x)

$f'(x) = ax - 12x^{1/3}$, so $f''(x) = a - 4x^{-2/3}$.

2. Use f''(27) = 0

Since $27^{2/3} = 9$, $0 = a - \frac{4}{9}$, so $a = \frac{4}{9}$.

3. Integrate f'(x)

With $a = \frac{4}{9}$, $f'(x) = \frac{4}{9}x - 12x^{1/3}$. Therefore $f(x) = \frac{2}{9}x^2 - 9x^{4/3} + c$.

4. Use the point (1, -8)

$-8 = \frac{2}{9} - 9 + c$, so $c = -8 + 9 - \frac{2}{9} = \frac{7}{9}$.

5. Write f(x)

$f(x) = \frac{2}{9}x^2 - 9x^{4/3} + \frac{7}{9}$.

Final answer

$(a)\ a = \frac{4}{9}$.

$(b)\ f(x) = \frac{2}{9}x^2 - 9x^{4/3} + \frac{7}{9}$.

Question 6

11 marks

Integration

6. The curve C has equation $y = \mathrm{f}(x)$ where $x > 0$

Given that

- $$\mathrm{f}'(x) = \frac{(x+3)^2}{x\sqrt{x}}$$
- the point $P(4, 20)$ lies on C

(a) (i) find the value of the gradient at P

(ii) Hence find the equation of the tangent to C at P, giving your answer in the form $ax + by + c = 0$ where a, b and c are integers to be found.

(4)

(b) Find $\mathrm{f}(x)$, simplifying your answer.

(7)

(Total 11 marks)

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Worked Solution - Question 6

Integration

Topic group

1. Simplify f'(x)

$$f'(x) = \frac{(x+3)^2}{x\sqrt{x}} = \frac{x^2+6x+9}{x^{3/2}} = x^{1/2} + 6x^{-1/2} + 9x^{-3/2}.$$

2. Find the gradient at P

At $x = 4$, $f'(4) = 2 + 3 + \frac{9}{8} = \frac{49}{8}$.

3. Find the tangent equation

Using $P(4, 20)$, $y - 20 = \frac{49}{8}(x - 4)$. Multiplying out gives $49x - 8y - 36 = 0$.

4. Integrate f'(x)

$$f(x) = \int \left(x^{1/2} + 6x^{-1/2} + 9x^{-3/2}\right)dx = \frac{2}{3}x^{3/2} + 12x^{1/2} - 18x^{-1/2} + c.$$

5. Use P(4,20)

$20 = \frac{2}{3}(8) + 12(2) - \frac{18}{2} + c = \frac{61}{3} + c$, so $c = -\frac{1}{3}$.

6. Write f(x)

$$f(x) = \frac{2}{3}x\sqrt{x} + 12\sqrt{x} - \frac{18}{\sqrt{x}} - \frac{1}{3}.$$

Final answer

$(a)(i)\ \frac{49}{8}$. $\quad (a)(ii)\ 49x - 8y - 36 = 0.$

$(b)\ f(x) = \frac{2}{3}x\sqrt{x} + 12\sqrt{x} - \frac{18}{\sqrt{x}} - \frac{1}{3}.$

Question 7

9 marks

Integration

7. The curve C has equation $y = f(x),\ x > 0$

Given that

- $f'(x) = \dfrac{2}{\sqrt{x}} + \dfrac{A}{x^2} + 3$, where A is a constant
- $f''(x) = 0$ when $x = 4$

(a) find the value of A.

(4)

Given also that

- $f(x) = 8\sqrt{3}$, when $x = 12$

(b) find $f(x)$, giving each term in simplest form.

(5)

(Total 9 marks)

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Worked Solution - Question 7

Integration

Topic group

1. Differentiate f'(x)

$f'(x) = 2x^{-1/2} + Ax^{-2} + 3$, so $f''(x) = -x^{-3/2} - 2Ax^{-3}$.

2. Use f''(4) = 0

$-4^{-3/2} - 2A(4^{-3}) = 0$, so $-\frac{1}{8} - \frac{A}{32} = 0$ and $A = -4$.

3. Substitute A into f'(x)

$f'(x) = \frac{2}{\sqrt{x}} - \frac{4}{x^2} + 3.$

4. Integrate to find f(x)

$f(x) = 4\sqrt{x} + \frac{4}{x} + 3x + c.$

5. Use f(12) = 8sqrt3

$f(12) = 8\sqrt{3} + \frac{1}{3} + 36 + c = 8\sqrt{3}$, so $c = -\frac{109}{3}$.

Final answer

$f(x) = 4\sqrt{x} + \frac{4}{x} + 3x - \frac{109}{3}.$

Question 11

8 marks

Integration

11. A curve C has equation $y = \mathrm{f}(x), \quad x > 0$

Given that

- $\mathrm{f}''(x) = 4x + \dfrac{1}{\sqrt{x}}$
- the point P has x coordinate 4 and lies on C
- the tangent to C at P has equation $y = 3x + 4$

(a) find an equation of the normal to C at P

(2)

(b) find $\mathrm{f}(x)$, writing your answer in simplest form.

(6)

(Total for Question 11 is 8 marks)

Worked Solution - Question 11

Integration

Topic group

1. Find point P

Since P has x-coordinate 4 and lies on the tangent $y = 3x + 4$, $P = (4, 16)$.

2. Find the normal

The tangent gradient is 3, so the normal gradient is $-\frac{1}{3}$. Hence the normal is $y - 16 = -\frac{1}{3}(x - 4)$.

3. Integrate f''(x)

$f''(x) = 4x + x^{-1/2}$, so $f'(x) = 2x^2 + 2\sqrt{x} + A$.

4. Use the tangent gradient

At $x = 4$, $f'(4) = 3$. Thus $2(16) + 2(2) + A = 3$, so $A = -33$.

5. Integrate again

$f(x) = \frac{2}{3}x^3 + \frac{4}{3}x^{3/2} - 33x + B$.

6. Use P(4, 16)

$16 = \frac{2}{3}(64) + \frac{4}{3}(8) - 33(4) + B = \frac{160}{3} - 132 + B$.

7. Find B

$B = \frac{284}{3}$, so $f(x) = \frac{2}{3}x^3 + \frac{4}{3}x^{3/2} - 33x + \frac{284}{3}$.

Final answer

(a) $y - 16 = -\frac{1}{3}(x - 4)$.

(b) $f(x) = \frac{2}{3}x^3 + \frac{4}{3}x^{3/2} - 33x + \frac{284}{3}$.

Question 8

10 marks

Integration

8. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Find the equation of the tangent to the curve with equation

$$y = \frac{1}{4}x^3 - 8x^{-\frac{1}{2}}$$

at the point $P(4, 12)$

Give your answer in the form $ax + by + c = 0$ where a, b and c are integers.

(5)

The curve with equation $y = \mathrm{f}(x)$ also passes through the point $P(4, 12)$

Given that

$$\mathrm{f}'(x) = \frac{1}{4}x^3 - 8x^{-\frac{1}{2}}$$

(b) find $\mathrm{f}(x)$ giving the coefficients in simplest form.

(5)

(Total for Question 8 is 10 marks)

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Worked Solution - Question 8

Integration

Topic group

1. Differentiate the curve

For $y = \frac{1}{4}x^3 - 8x^{-1/2}$, $\frac{dy}{dx} = \frac{3}{4}x^2 + 4x^{-3/2}$.

2. Find the gradient at P

At $x = 4$, $\frac{dy}{dx} = \frac{3}{4}(16) + 4(4^{-3/2}) = 12 + \frac{1}{2} = \frac{25}{2}$.

3. Find the tangent

Using $P(4, 12)$, $y - 12 = \frac{25}{2}(x - 4)$.

4. Write in integer form

Multiplying by 2 gives $2y - 24 = 25x - 100$, hence $25x - 2y - 76 = 0$.

5. Integrate f'(x)

$f'(x) = \frac{1}{4}x^3 - 8x^{-1/2}$ gives $f(x) = \frac{x^4}{16} - 16\sqrt{x} + c$.

6. Use P(4, 12)

$12 = \frac{4^4}{16} - 16\sqrt{4} + c = 16 - 32 + c$, so $c = 28$.

Final answer

(a) $25x - 2y - 76 = 0$.

(b) $f(x) = \frac{x^4}{16} - 16\sqrt{x} + 28$.

Question 7

10 marks

Integration

7. The curve C has equation $y = \mathrm{f}(x)$ where $x > 0$

Given that

- $\mathrm{f}'(x) = \dfrac{4x^2 + 10 - 7x^{\frac{1}{2}}}{4x^{\frac{1}{2}}}$
- the point $P(4, -1)$ lies on C

(a) (i) find the value of the gradient of C at P

(ii) Hence find the equation of the normal to C at P, giving your answer in the form $ax + by + c = 0$ where a, b and c are integers to be found.

(4)

(b) Find $\mathrm{f}(x)$.

(6)

(Total for Question 7 is 10 marks)

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Worked Solution - Question 7

Integration

Topic group

1. Simplify f prime

$f'(x) = \dfrac{4x^2 + 10 - 7x^{1/2}}{4x^{1/2}} = x^{3/2} + \dfrac{5}{2}x^{-1/2} - \dfrac{7}{4}.$

2. Find the gradient at P

At $x = 4$, $f'(4) = 4^{3/2} + \dfrac{5}{2}(4^{-1/2}) - \dfrac{7}{4} = 8 + \dfrac{5}{4} - \dfrac{7}{4} = \dfrac{15}{2}.$

3. Find the normal gradient

The normal gradient is the negative reciprocal, so $m_n = -\dfrac{2}{15}$.

4. Use point P

Using $P(4, -1)$, $y + 1 = -\dfrac{2}{15}(x - 4)$.

5. Write the normal in integer form

$15y + 15 = -2x + 8$, hence $2x + 15y + 7 = 0$.

6. Integrate f prime

$f(x) = \int \left(x^{3/2} + \dfrac{5}{2}x^{-1/2} - \dfrac{7}{4}\right) dx = \dfrac{2}{5}x^{5/2} + 5x^{1/2} - \dfrac{7}{4}x + c.$

7. Use P to find c

$-1 = \dfrac{2}{5}(4^{5/2}) + 5(4^{1/2}) - \dfrac{7}{4}(4) + c = \dfrac{64}{5} + 10 - 7 + c$, so $c = -\dfrac{84}{5}$.

Final answer

$(a)(i)\ \dfrac{15}{2}.\quad (a)(ii)\ 2x + 15y + 7 = 0.$

$(b)\ f(x) = \dfrac{2}{5}x^{5/2} + 5x^{1/2} - \dfrac{7}{4}x - \dfrac{84}{5}.$

Question 10

8 marks

Integration

10. **In this question you must show all stages of your working.**

The curve C has equation $y = \mathrm{f}(x)$, $x > 0$

Given that

- the point $P\left(2, 8\sqrt{2}\right)$ lies on C
- $\mathrm{f}'(x) = 4\sqrt{x^3} + \dfrac{k}{x^2}$ where k is a constant
- $\mathrm{f}''(x) = 0$ at P

(a) find the exact value of k,

(4)

(b) find $\mathrm{f}(x)$, giving your answer in simplest form.

(4)

(Total for Question 10 is 8 marks)

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Worked Solution - Question 10

Integration

Topic group

1. Differentiate f prime

$f'(x) = 4x^{3/2} + kx^{-2}$, so $f''(x) = 6x^{1/2} - 2kx^{-3}$.

2. Use f double prime equals zero at P

At P, $x = 2$ and $f''(2) = 0$, so $6\sqrt{2} - 2k(2^{-3}) = 0$.

3. Find k

$6\sqrt{2} - \dfrac{k}{4} = 0$, hence $k = 24\sqrt{2}$.

4. Integrate f prime

$f(x) = \int \left(4x^{3/2} + 24\sqrt{2}x^{-2}\right) dx = \dfrac{8}{5}x^{5/2} - 24\sqrt{2}x^{-1} + c$.

5. Use the point P

$P(2, 8\sqrt{2})$ lies on C, so $8\sqrt{2} = \dfrac{8}{5}(2^{5/2}) - \dfrac{24\sqrt{2}}{2} + c$.

6. Find c

$8\sqrt{2} = \dfrac{32\sqrt{2}}{5} - 12\sqrt{2} + c$, so $c = \dfrac{68\sqrt{2}}{5}$.

Final answer

$(a)\ k = 24\sqrt{2}$.

$(b)\ f(x) = \dfrac{8}{5}x^{5/2} - \dfrac{24\sqrt{2}}{x} + \dfrac{68\sqrt{2}}{5}$.

Question 10

10 marks

Integration

10. The curve C has equation $y = \mathrm{f}(x)$ where $x > 0$

Given that

- $\mathrm{f}'(x) = 6x - \dfrac{(2x-1)(3x+2)}{2\sqrt{x}}$
- the point $P(4, 12)$ lies on C

(a) find the equation of the normal to C at P, giving your answer in the form $y = mx + c$ where m and c are integers to be found,

(4)

(b) find $\mathrm{f}(x)$, giving each term in simplest form.

(6)

(Total for Question 10 is 10 marks)

Worked Solution - Question 10

Integration

Topic group

1. Expand the numerator

$(2x-1)(3x+2) = 6x^2 + x - 2.$

2. Simplify f prime

$f'(x) = 6x - \dfrac{6x^2+x-2}{2\sqrt{x}} = 6x - 3x^{3/2} - \dfrac{1}{2}x^{1/2} + x^{-1/2}.$

3. Find the tangent gradient at P

$f'(4) = 24 - 3(8) - \dfrac{1}{2}(2) + \dfrac{1}{2} = -\dfrac{1}{2}.$

4. Use the normal gradient

The normal gradient is the negative reciprocal of $-\dfrac{1}{2}$, so it is 2.

5. Find the normal equation

Through $P(4,12)$, $y - 12 = 2(x-4)$, hence $y = 2x + 4$.

6. Integrate f prime

$f(x) = 3x^2 - \dfrac{6}{5}x^{5/2} - \dfrac{1}{3}x^{3/2} + 2x^{1/2} + c.$

7. Use P to find c

$12 = 3(4)^2 - \dfrac{6}{5}(4^{5/2}) - \dfrac{1}{3}(4^{3/2}) + 2(4^{1/2}) + c.$

8. Simplify c

$12 = 48 - \dfrac{192}{5} - \dfrac{8}{3} + 4 + c = \dfrac{164}{15} + c$, so $c = \dfrac{16}{15}$.

Final answer

(a) $y = 2x + 4.$

(b) $f(x) = 3x^2 - \dfrac{6}{5}x^{5/2} - \dfrac{1}{3}x^{3/2} + 2x^{1/2} + \dfrac{16}{15}.$

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Question 8

7 marks

Integration

8. A curve C has equation $y = f(x)$.

The point P with x coordinate 3 lies on C

Given

- $f'(x) = 4x^2 + kx + 3$ where k is a constant
- the normal to C at P has equation $y = -\frac{1}{24}x + 5$

(a) show that $k = -5$

(3)

(b) Hence find $f(x)$.

(4)

(Total for Question 8 is 7 marks)

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Worked Solution - Question 8

Integration

Topic group

1. Find the y-coordinate of P

The normal is $y = -\frac{1}{24}x + 5$. At $x = 3$, $y = -\frac{3}{24} + 5 = \frac{39}{8}$.

2. Use the normal gradient

The normal gradient is $-\frac{1}{24}$, so the tangent gradient is 24.

3. Use f prime at x equals 3

$f'(3) = 4(3)^2 + 3k + 3 = 39 + 3k$.

4. Find k

$39 + 3k = 24$, so $k = -5$.

5. Integrate f prime

With $k = -5$, $f'(x) = 4x^2 - 5x + 3$, so $f(x) = \frac{4}{3}x^3 - \frac{5}{2}x^2 + 3x + c$.

6. Use point P

$\frac{39}{8} = \frac{4}{3}(27) - \frac{5}{2}(9) + 3(3) + c = \frac{45}{2} + c$.

7. Find c

$c = \frac{39}{8} - \frac{45}{2} = -\frac{141}{8}$.

Final answer

$k = -5$ and $f(x) = \frac{4}{3}x^3 - \frac{5}{2}x^2 + 3x - \frac{141}{8}$.

Question 6

10 marks

Integration

6. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The curve C has equation $y = \mathrm{f}(x)$, $x > 0$

Given that

- the point $P(4, -5)$ lies on C
- $\mathrm{f}'(x) = \dfrac{2x^2 + ax + b}{4\sqrt{x}}$, where a and b are constants
- the gradient of the tangent to C at P is 7

(a) show that

$$4a + b = 24$$

(2)

Given also that $a + b = -9$

(b) find, in simplest form, $\mathrm{f}(x)$

(7)

Curve C is transformed to the curve with equation $y = \mathrm{f}(x - 3)$

Given that point P is transformed to the point Q,

(c) state the coordinates of Q.

(1)

(Total for Question 6 is 10 marks)

WMA11/01 JANUARY 2025

Worked Solution - Question 6

Integration

Topic group

1. Use the gradient at P

At $x = 4$, $f'(4) = 7$. Since $f'(x) = \dfrac{2x^2 + ax + b}{4\sqrt{x}}$, we get $\dfrac{32 + 4a + b}{8} = 7$.

2. Show the required equation

$32 + 4a + b = 56$, so $4a + b = 24$.

3. Solve for a and b

Together with $a + b = -9$, subtracting gives $3a = 33$, so $a = 11$ and $b = -20$.

4. Rewrite f prime

$f'(x) = \dfrac{2x^2 + 11x - 20}{4\sqrt{x}} = \dfrac{1}{2}x^{3/2} + \dfrac{11}{4}x^{1/2} - 5x^{-1/2}$.

5. Integrate

$f(x) = \dfrac{1}{5}x^{5/2} + \dfrac{11}{6}x^{3/2} - 10x^{1/2} + c$.

6. Use P to find c

$-5 = \dfrac{1}{5}(4^{5/2}) + \dfrac{11}{6}(4^{3/2}) - 10(4^{1/2}) + c = \dfrac{16}{15} + c$.

7. State f(x)

$c = -\dfrac{91}{15}$, so $f(x) = \dfrac{1}{5}x^{5/2} + \dfrac{11}{6}x^{3/2} - 10x^{1/2} - \dfrac{91}{15}$.

8. Transform P

$y = f(x - 3)$ translates the curve 3 units to the right, so $P(4, -5)$ becomes $Q(7, -5)$.

Final answer

(b) $f(x) = \dfrac{1}{5}x^{5/2} + \dfrac{11}{6}x^{3/2} - 10x^{1/2} - \dfrac{91}{15}$.

(c) $Q = (7, -5)$.

Question 8

11 marks

Integration

8. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

A curve has equation $y = \mathrm{f}(x),\ x > 0$

The point $P(4, 12)$ lies on the curve.

Given that

- $\mathrm{f}'(x) = 3\sqrt{x} + kx^2$ where k is a constant
- the equation of the tangent to the curve at P has equation $y = 10x + c$ where c is a constant

(a) (i) show that $k = \dfrac{1}{4}$

(ii) find the value of c

(4)

(b) Hence find the value of $\mathrm{f}''(x)$ at P.

(3)

(c) Find $\mathrm{f}(x)$.

(4)

(Total for Question 8 is 11 marks)

WMA11/01 MAY/JUNE 2025

Worked Solution - Question 8

Integration

Topic group

1. Use the tangent gradient

The tangent at P has equation $y = 10x + c$, so the gradient at P is 10.

2. Show k

$f'(4) = 3\sqrt{4} + k(4)^2 = 6 + 16k = 10$, so $16k = 4$ and $k = \frac{1}{4}$.

3. Find c

Substitute $P(4, 12)$ into $y = 10x + c$: $12 = 40 + c$, so $c = -28$.

4. Differentiate f prime

With $k = \frac{1}{4}$, $f'(x) = 3x^{1/2} + \frac{1}{4}x^2$.

5. Find f double prime

$f''(x) = \frac{3}{2}x^{-1/2} + \frac{1}{2}x$.

6. Evaluate at P

$f''(4) = \frac{3}{2} \cdot \frac{1}{2} + 2 = \frac{3}{4} + 2 = \frac{11}{4}$.

7. Integrate f prime

$f(x) = \int \left(3x^{1/2} + \frac{1}{4}x^2\right) dx = 2x^{3/2} + \frac{1}{12}x^3 + d$.

8. Use P to find d

$12 = 2(4^{3/2}) + \frac{1}{12}(4^3) + d = 16 + \frac{16}{3} + d$.

9. State f(x)

$d = -\dfrac{28}{3}$, so $f(x) = 2x^{3/2} + \dfrac{1}{12}x^3 - \dfrac{28}{3}$.

Final answer

$(a)(i)\ k = \dfrac{1}{4}$. $(a)(ii)\ c = -28$.

$(b)\ f''(P) = \dfrac{11}{4}$.

$(c)\ f(x) = 2x^{3/2} + \dfrac{1}{12}x^3 - \dfrac{28}{3}$.

Question 8

8 marks

Integration

8. **In this question you must show all stages of your working.**
Solutions relying on calculator technology are not acceptable.

A curve has equation $y = \text{f}(x)$, $x > 0$

Given that

- $\text{f}'(x) = 2x + \dfrac{8}{x^2} + k$, where k is a constant
- the equation of the tangent to the curve at $x = \sqrt{2}$ is $y = 5x - 3\sqrt{2}$

(a) find the exact value of k.

(2)

(b) Find an equation of the normal to the curve at $x = \sqrt{2}$

(2)

(c) Find $\text{f}(x)$, writing your answer in simplest form.

(4)

(Total for Question 8 is 8 marks)

WMA11/01 OCTOBER 2025

Worked Solution - Question 8

Integration

Topic group

1. Use tangent gradient

The tangent has equation $y = 5x - 3\sqrt{2}$, so the gradient at $x = \sqrt{2}$ is 5.

2. Find k

$f'(\sqrt{2}) = 2\sqrt{2} + \dfrac{8}{(\sqrt{2})^2} + k = 2\sqrt{2} + 4 + k = 5$, so $k = 1 - 2\sqrt{2}$.

3. Find the point on the curve

At $x = \sqrt{2}$, the tangent gives $y = 5\sqrt{2} - 3\sqrt{2} = 2\sqrt{2}$.

4. Normal gradient

The tangent gradient is 5, so the normal gradient is $-\dfrac{1}{5}$.

5. Equation of the normal

$y - 2\sqrt{2} = -\dfrac{1}{5}(x - \sqrt{2})$, hence $y = -\dfrac{1}{5}x + \dfrac{11\sqrt{2}}{5}$.

6. Integrate f prime

$$f(x) = \int \left(2x + \frac{8}{x^2} + k\right) dx = x^2 - \frac{8}{x} + kx + c.$$

7. Substitute k

$$f(x) = x^2 - \frac{8}{x} + (1 - 2\sqrt{2})x + c.$$

8. Use the point

$$2\sqrt{2} = 2 - \frac{8}{\sqrt{2}} + (1 - 2\sqrt{2})\sqrt{2} + c = -2 - 3\sqrt{2} + c.$$

9. Find c

$c = 2 + 5\sqrt{2}$, so $f(x) = x^2 - \dfrac{8}{x} + (1 - 2\sqrt{2})x + 2 + 5\sqrt{2}$.

Final answer

(a) $k = 1 - 2\sqrt{2}$.

(b) $y = -\dfrac{1}{5}x + \dfrac{11\sqrt{2}}{5}$.

(c) $f(x) = x^2 - \dfrac{8}{x} + (1 - 2\sqrt{2})x + 2 + 5\sqrt{2}$.

Question 8

12 marks

Integration

8. **In this question you must show all stages of your working.**
Solutions relying on calculator technology are not acceptable.

A curve C has equation $y = \mathrm{f}(x)$, $x > 0$

A point P lies on C.

Given that

- $\mathrm{f}'(x) = 4x^2 + \dfrac{6}{x^2} - 4$
- the equation of the tangent to C at P is $y = 10x - 6\sqrt{3}$

(a) (i) verify that $\sqrt{3}$ is a possible x coordinate of P,

(ii) find, using algebra, the other possible x coordinate of P.

(6)

Given that the x coordinate of P is $\sqrt{3}$

(b) find an equation of the normal to C at P.

(2)

(c) Find $\mathrm{f}(x)$, writing your answer in simplest form.
You must show each stage of your working.

(4)

(Total for Question 8 is 12 marks)

WMA11/01 JANUARY 2026

Worked Solution - Question 8

Integration

Topic group

1. Verify x equals root 3

$f'(\sqrt{3}) = 4(\sqrt{3})^2 + \dfrac{6}{(\sqrt{3})^2} - 4 = 12 + 2 - 4 = 10$, which matches the tangent gradient.

2. Find all possible x-coordinates

Set $4x^2 + \dfrac{6}{x^2} - 4 = 10$.

3. Form a quadratic in x squared

Multiplying by x^2 gives $4x^4 - 14x^2 + 6 = 0$, so $2x^4 - 7x^2 + 3 = 0$.

4. Factor

$2x^4 - 7x^2 + 3 = (2x^2 - 1)(x^2 - 3) = 0$.

5. Use x greater than zero

$x = \sqrt{3}$ or $x = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$. The other possible x-coordinate is $\dfrac{\sqrt{2}}{2}$.

6. Find the point P

Given $x_P = \sqrt{3}$, the tangent gives $y_P = 10\sqrt{3} - 6\sqrt{3} = 4\sqrt{3}$.

7. Normal equation

The normal gradient is $-\dfrac{1}{10}$, so $y - 4\sqrt{3} = -\dfrac{1}{10}(x - \sqrt{3})$.

8. Integrate f prime

$f(x) = \int \left(4x^2 + \dfrac{6}{x^2} - 4\right) dx = \dfrac{4}{3}x^3 - \dfrac{6}{x} - 4x + c$.

9. Use P to find c

$$4\sqrt{3} = \frac{4}{3}(\sqrt{3})^3 - \frac{6}{\sqrt{3}} - 4\sqrt{3} + c = 4\sqrt{3} - 2\sqrt{3} - 4\sqrt{3} + c.$$

10. State f(x)

$c = 6\sqrt{3}$, so $f(x) = \dfrac{4}{3}x^3 - \dfrac{6}{x} - 4x + 6\sqrt{3}$.

Final answer

$(a)(i)$ verified.

$(a)(ii)$ $x = \dfrac{\sqrt{2}}{2}$.

(b) $y - 4\sqrt{3} = -\dfrac{1}{10}(x - \sqrt{3})$.

(c) $f(x) = \dfrac{4}{3}x^3 - \dfrac{6}{x} - 4x + 6\sqrt{3}$.